On the geometry of locally conformal almost cosymplectic manifolds and 2-order Lagrange spaces

A thesis submitted in fulfillment of the degree of Doctor of Philosophy at the University of KwaZulu-Natal, South Africa

By

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July 2017

[1]Supported by Simons Foundation through the Research and Graduate Studies in Mathematics and its Applications (RGSMA) Project (BIUST, Botswana).

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This thesis is submitted in fulfillment of the Academic requirement for the degree of Doctor of Philosophy in Mathematics to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and science, University of KwaZulu-Natal.

As the candidate's supervisor, I have approved this thesis for submission.

Prof. Fortuné Massamba

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Declaration

I declare that the contents of this thesis are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

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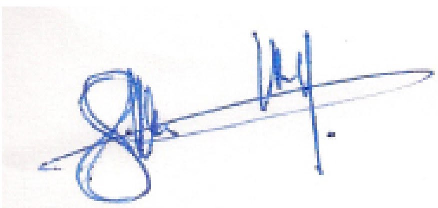

Declaration 2 - Publications

1. F. Massamba and A. Maloko Mavambou, A class of locally conformal almost cosymplectic manifolds, *Bulletin of the Malaysian Mathematical Sciences Society*, in press, DOI 10.1007/s40840-016-0309-3, 2016.

2. F. Massamba, A. Maloko Mavambou, S. Ssekajja, On Indefinite locally conformal cosymplectic manifolds, *Communications of the Korean Mathematical Society*, 32 no. 3 (2017), 725-743, DOI 10.4134/CKMS.c160197

3. A. Maloko Mavambou, F. Massamba, S. J. Mbatakou, Conformal geometry of 2-osculator bundles, under review.

4. A. Maloko Mavambou, F. Massamba, S. J. Mbatakou, On indefinite 2-Lagrangian spaces, under review.

To my Almighty God,
To my late mother Hélène Kialozua.

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Acknowledgements

I must confess that it was not easy to complete this thesis without some people.

Firstly, I owe my gratitude to my supervisor, Prof. Fortuné Massamba. He has offered me the best that I could not evaluate the price in so few words. An ideal research environment, a scholarship for daily needs throughout the preparation of this work, beyond that a moral support. I am deeply indebted to him for that. I thank Mr. Samuel Ssekajja, Dr. Joseph Mbatakou and Dr. Rodrigue Massoukou for their contribution in this work.

The staff members of the School of Mathematics, Statistics and Computer Science were very hospitable to me in person during the preparation of this work. I would like to express to each of them my sincere thanks.

To anyone who contributed directly or indirectly to the achievement of this thesis, in particular to my wife Bikoumou Rodeche and my child born Benesdras Maloko, I express my gratitude.

I would like to sincerely thank the Simons Foundation for their financial support through the RGSMA Project (BIUST, Botswana).

Abstract

A locally conformal (l.c.) almost cosymplectic manifold is a class of almost contact manifolds in Riemannian geometry, which is derived from a special class of conformal transformation such that the manifold has an open covering $\{U_\alpha\}_{\alpha \in I}$ and to any U_α there exists a function $\sigma_\alpha : U_\alpha \to \mathbb{R}$ such that the metric g restricted to each U_α is conformally related to an almost cosymplectic manifold. It has been proved that an almost cosymplectic and almost α-Kenmotsu manifolds belong to some subclasses of the class of locally conformal almost cosymplectic. In this thesis, we study a class of almost contact manifolds, namely, locally conformal almost cosymplectic manifolds. We investigate subclasses of such manifolds and prove that some of them contain the class of bundle-like metric structures. Under some conditions, we show that the class of conformal changes of almost cosymplectic structures is a subclass of (almost)-cosymplectic structure. Considering the indefinite l.c. almost cosymplectic, we prove that there exist foliations arising from Pfaffian equation $\omega = 0$ whose leaves are the maximal integral null manifolds immersed as submanifolds of indefinite locally conformal cosymplectic manifolds. Necessary and sufficient conditions for such leaves to be screen conformal, as well as possessing integrable distributions are given. Using Newton transformations, we show that any compact ascreen null leaf with a symmetric Ricci tensor admits a totally geodesic screen distribution.

Finally, we adapt the conformal transformation on 2-osculator bundle endowed with a Lagrangian function L. We investigate the behavior of the nonlinear connection under the conformal change of the fundamental tensor and study the conformal deformation of related geometrics objects like the canonical N-linear connection and associated curvatures. We also define a locally conformal almost cosymplectic structure on the 2-osculator bundle $Osc^2(M)$, where M is an n-dimensional smooth manifold. For a given almost contact structure on M, we obtain an almost n-contact structure depending only on the structure on M, by using the complete lift.

Key words and phrases: Locally conformal almost cosymplectic manifold; Newton transformation; 2-semispray vector field, 2-Lagrange space.

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Chapter One

Introduction

1.1 Introduction

Contact and almost contact structures are two of the most interesting examples of differential geometric structures. Indeed, their theory is a natural generalization of so-called contact geometry, which has important applications in classical and quantum mechanics. In addition, they were used in the classical description of time-dependent systems in mechanics such as regular Lagrangian systems and Hamiltonian systems (see [15, 28] and references therein for more details). Their study as differential geometric structures dates from works of the Gray [23] and Sasaki [45]. Almost contact structures provide a counterpart of almost complex structures in odd dimension and include several classes of special importance as contact, Sasakian and cosymplectic ones (see [6] and [8] and references therein).

The notion of almost cosymplectic manifolds was introduced by Goldberg and Yano in [22]. In fact, they extended earlier results on almost Kähler manifolds which says that if the curvature transformation of the almost Kähler metric commutes with the almost complex, then the latter is integrable. The simplest examples of such manifolds are those locally formed by the products of almost Kählerian manifolds and the real line $\mathbb{R}$ (or the circle S^1). In [24], the authors investigated a class of almost α-cosymplectic manifolds, with α constant. They studied canonical foliations of the same class of cosymplectic and proved that the foliation defined by the contact distribution is Riemannian and tangentially almost Kähler of codimension 1, and is tangentially Kählerian if the underlying manifold is normal. The indefinite case locally conformal almost cosymplectic secures the existence of a special subspace, namely, null (lightlike) space.

Null geometry of submanifolds of semi-Riemannian manifolds is remarkably different from the geometry of submanifolds immersed in a Riemannian manifold by the fact that the normal vector bundle of a null submanifold intersects with its tangent bundle. This aspect makes null geometry difficult to study despite having numerous applications in other fields like mathematical physics. The study of null submanifolds

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of a semi Riemannian manifold was initiated by Duggal-Bejancu [18] and Kupeli [26] and later by many authors [27], [36], [16] and references therein. Geometry of submanifolds in locally conformal cosymplectic manifolds as well as cosymplectic manifolds has also been studied by many authors, for instance [12], [22], [34], [43] and [44]. For the almost contact null geometry, see, for instance, the papers [30]- [32] and references therein.

The notion of Lagrange space of order k is introduced by means of regular non-degenerate Lagrangian defined on the total space of the k-accelerations bundle T^kM. In this case the Craig-Synge equations determine a k-semispray, which depend only on the considered Lagrangian. The importance of Lagrange geometry consists of the fact that variational problems for important Lagrangians have numerous applications in various fields, such as mathematics, the theory of dynamical systems, optimal control, biology, and economy (see [39, 40] for more details). In this respect, Antonelli remarked the following [4]: "There is now strong evidence that the symplectic geometry of Hamiltonian dynamical systems is deeply connected to Cartan geometry, the dual of Finsler geometry".

This thesis has two distinct parts. In the first part, we consider a class of almost contact metric manifolds which is called *locally conformal (l.c.) almost cosymplectic manifolds*. This part builds on the paper of Olszak [43] in which the author investigated properties of the curvature and of the pointwise constant φ-holomorphic sectional curvature condition in a certain class of locally conformal almost cosymplectic manifolds. This class was later studied by Chinea and Marrero [12] under the name of "conformal changes of almost cosymplectic manifolds". They characterized the structures in terms of its Lee form and obtained that the leaves of the contact foliation inherit a locally conformal (almost) Kähler structure. This part of the thesis also focuses on one of the classes in which the differentiable 1-form ω derived from the l.c. structure is proportional to the contact form structure η of the manifold under consideration. We also investigates the geometric conditions for which the class under study falls into the class of (almost) cosymplectic manifolds. Under some special conformal deformation, Olszak in [43] proved that such manifolds are almost α-Kemnotsu. In [34], the authors proved that the class of this deformations contain the one of bundle-like metric structures, in the Riemannian case.

We are also interested in indefinite locally conformal deformations of almost cosymplectic manifolds. We study the leaves (as submanifolds) of the foliations which are coming from the distributions generated by the Pfaffian equation $\omega = 0$, ω being the characteristic 1-form of the ambient manifold under consideration, $P : x \in M(c) \mapsto \mathbb{R}V_x \oplus \mathbb{R}B_x$, where $c = g(B_x, B_x)$ and $\mathbb{R}V_x$ and $\mathbb{R}B_x$ denotes line bundles locally spanned by V_x and B_x, respectively. In this case the Lee form ω is not required to be parallel as it is the case with locally conformal Kähler. But according to different positions of the Lee vector field B with respect to the structure vector ξ and due to the causal character of the Lee vector field, we obtain some rich information about the geometry of leaves in M.

The second part of the thesis deals with the Lagrange spaces of order 2. Given two Lagrange spaces of order 2, (M, L) and $(M, \widetilde{L})$, we establish a conformal transformation between their corresponding fundamental tensors when their fundamental tensors are conformally deformed. In this case, we will say that L and $\widetilde{L}$ are conformal-type. The Lagrange spaces of order 1 are the smooth manifolds endowed with a regular Lagrangian L of order 1. These spaces were introduced twenty years ago by Miron [40] and were studied due to their applications in Mechanics, Physics, Control theory etc. They lead to geometric models more general than those provided by Riemannian or Finslerian structures (see [40] and references therein). Such spaces can be extended to the higher order, in particular to order 2. In this case, the base manifold is called 2-osculator bundle and is denoted by $Osc^2(M)$. This is a natural extension of the notion of 1-osculator bundle. So, it is necessary to study the total space of the 2-osculator bundle $(Osc^2(M), \pi^2, M)$, where M is a smooth manifold (see [40]). The 2-osculator bundle has a profound geometrical meaning and is more suitable for the applications of Lagrange geometry in mechanics ([29], [46]), theoretical physics and biology [41].

The main results of this thesis are summarized in Section 1.2 and 1.3.

1.2 Main results on locally conformal geometry

Let (M, ϕ, ξ, η, g) be a locally conformal (l.c.) almost cosymplectic manifold. Then, there exists a 1-form ω on M such that

$$d\Phi = 2\omega \wedge \Phi, \quad d\eta = \omega \wedge \eta \quad \text{and} \quad d\omega = 0. \tag{1.1}$$

Let h be the $(1, 1)$-tensor field on M defined by

$$hX = \nabla_X \xi - \omega(\xi)X + \eta(X)B, \tag{1.2}$$

for any vector fields X and Y on M, where B is the dual vector field of ω. Then, we have one of the following results showing that, under some conditions, the class of conformal changes of almost cosymplectic structures is a subclass of (almost)-cosymplectic structures. Let r and r^* be the scalar curvature and scalar $*$-curvature defined, respectively, by $r = \sum_{i=0}^{2m} S(E_i, E_i)$ and $r^* = \sum_{i=0}^{2m} S^*(E_i, E_i)$, where $\{E_i\}_{0 \le i \le 2m}$ being an orthonormal frame with respect to g. Our main results are as follows:

(1) *If Let (M^{2m+1}, ϕ, ξ, g) is compact with $m > 1$ for which the function $\omega(\xi)$ is constant in the direction of ξ and the Chern-Hamilton τ is parallel and if $r^* = r$, then M is cosymplectic.*

Let $\widetilde{h}$ be another $(1,1)$-type tensor field associated to the structure (ϕ, ξ, η, g) as defined in [8, page 84] by

$$\widetilde{h} = \frac{1}{2}\mathcal{L}_{\xi}\phi. \tag{1.3}$$

We have the following result giving rise to a class of l.c. alomst cosymplectic manifolds admitting a Lee form proportional to the characteristic 1-form η.

(2) *The $(1,1)$-tensor field ϕ anticommutes with $\widetilde{h}$ and $\widetilde{h}$ is a symmetric operator if and only if there exists a smooth function f on M such that*

$$\omega = f\eta \ \text{ with } \ df \wedge \eta = 0 \ \text{ and } \ \widetilde{h} = \phi h.$$

Let $D := \ker \eta$ be the contact distribution and $D^{\perp}$ be the distribution spanned by the structure vector field ξ. Then, we have the following decomposition

$$TM = D \oplus D^{\perp}. \tag{1.4}$$

Here we have the following results:

(3) *The integral manifolds of the distribution D in (3.41) are l.c. almost Kähler manifolds with mean curvature vector field $H' = -\omega(\xi)\xi$. They are totally umbilical submanifolds of M if and only if the operator h vanishes.*

(4) *Let $\mathcal{F}$ be a foliation on M of codimension 1. If the metric g on M is bundle-like for the foliation $\mathcal{F}$, then the leaves of $\mathcal{F}$ are almost Kähler. Moreover, if M is normal, then the leaves of $\mathcal{F}$ are Kähler and totally umbilical.*

Let M be a $(2n+1)$-dimensional indefinite l. c. almost cosymplectic manifold of index q, $0 < q < 2n+1$. Set $c = g(B,B) \in \mathcal{C}^{\infty}(M)$ and $\mathrm{Sign}(B) = \{x \in M : B_x = 0\}$. Let $\mathcal{F}$ be the canonical foliation of codimension r whose leaves are the maximal connected integral manifolds of the Pfaffian equation $\omega = 0$ [10]. Then, we have the following fundamental theorem.

(5) *Assume that $0 < q < 2n+1$ and $\mathtt{Sign}(B) = \emptyset$. Then*

- *(i) If $c \neq 0$, then the index of each leaf L of $\mathcal{F}$ is given by $\mathrm{ind}(L) = q - s$, where $s = \mathrm{ind}((T\mathcal{F})^c)$ with $0 \leq s \leq r$. Moreover, L is totally geodesic r codimensional semi-Riemannian submanifold of (M,g) if and only if the Lee form ω is parallel.*
- *(ii) If $c = 0$, then each leaf of $\mathcal{F}$ is either a null hypersurface or a quasi generalized CR-null submanifold of (M,g).*

If the characteristic vector field ξ is decomposed as follows.

$$\xi = \xi_S + aB + bN,$$

where ξ_S denotes the component of ξ on $S(TL)$ while a and b are non-zero smooth functions on M, then the leaf L is *ascreen null hypersurface* [27] if $\xi_S = 0$. One of the cases of this theorem generates the followings.

(6) *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M such that $c = 0$ and $\omega(\xi) \neq 0$. Then L is an ascreen null hypersurface of $\mathcal{F}$ if and only if $\phi\mathrm{Rad}(T\mathcal{F}) = \phi l\mathrm{tr}(T\mathcal{F})$.*

For the higher order geodesibility of leaves of $\mathcal{F}$, we fix the pair of non-zero vector field on leaves of $\mathcal{F}$ as follows: Let $(L, g, c = 0)$ be an ascreen null hypersurface immersed in an l.c. almost cosymplectic manifold, with $\mathrm{Sign}(B) = \emptyset$, $\omega(\xi) \neq 0$ and a conformal vector field $\overline{Q} = e^{\sigma(t)}\xi$. We prove that:

(7) *If the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric, then there exists a pair $\{B, N\}$ on $\mathcal{U} \subset L$ such that the corresponding 1-form τ vanishes on any $\mathcal{U} \cap L$. Moreover, $g(\overline{Q}, B) \neq 0$ and $g(\overline{Q}, N) \neq 0$.*

Let dV_M be the volume element of M with respect to g and a given orientation. Then, we denote the volume form on $\mathcal{F}$ by

$$dV = i_N dV_M,$$

where i_N is the contraction with respect to the vector field N. Using the New transformations, we prove the following:

(8) *If L is a compact ascreen null hypersurface of $\mathcal{F}$ in an l.c. almost cosymeplectic of constant sectional curvature and the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric, then*

$$\int_L (B \cdot \overline{g}(T_r Q, N) + e^{\sigma(t)}\mathrm{tr}(T_r \circ h) + (-1)^r c_r \omega(\overline{Q})\{H_r + H_{r+1}\}) dV = 0,$$

where T_r are the Newton transformations with respect to the shape operator A_N defined by $T_r = (-1)^r S_r \mathbb{I} + A_N \circ T_{r-1}, \quad 0 \leq r \leq m$ and $H_r = \binom{m+1}{r}^{-1} S_r$ is the normalized mean curvature with respect to the shape operator A_N.

This leads to the following result:

(9) *Let a, b and σ be constants such that h is tangent to $\mathcal{F}$. If L is a compact ascreen null hypersurface of $\mathcal{F}$ in an l.c. almost cosymeplectic of constant sectional curvature and the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric and H_1 is constant, then the screen distribution $S(TL)$ of L is totally geodesic.*

1.3 Main results on 2-order Lagrange spaces

Let M be a real n-dimensional smooth manifold and L and $\widetilde{L}$ be two 2-Lagrangian on M with the fundamental tensors g and $\widetilde{g}$ of L and $\widetilde{L}$, respectively. One of the main results in this part of the thesis is stated as follows:

(10) *Assume that the fundamental tensors g and $\widetilde{g}$ of L and $\widetilde{L}$, respectively, are conformally deformed. If g is 0-homogeneous with respect to $y^{(2)}$, then L and $\widetilde{L}$ are related as*

$$\widetilde{L} = \psi L + A_i(x, y^{(1)})y^{(2)i} + U(x, y^{(1)}), \tag{1.5}$$

where A_i an arbitrary covector and U an arbitrary function on $Osc^1(M)$. Moreover, the coefficients of the d-tensor of the curvatures associated to D and $\widetilde{D}$ are given by

$$\widetilde{R}^h_{ijk} = R^h_{ijk} + r^h_{ijk},\ \underset{(1)}{\widetilde{P}}{}^h_{ijk} = \underset{(1)}{P}{}^h_{ijk} + \underset{(1)}{p}{}^h_{ijk}, \tag{1.6}$$

$$\underset{(2)}{\widetilde{P}}{}^h_{ijk} = \underset{(2)}{P}{}^h_{ijk} + \underset{(2)}{p}{}^h_{ijk},\ \underset{(21)}{\widetilde{S}}{}^h_{ijk} = \underset{(21)}{S}{}^h_{ijk} + \underset{(21)}{s}{}^h_{ijk}, \tag{1.7}$$

$$\underset{(11)}{\widetilde{S}}{}^h_{ijk} = \underset{(11)}{S}{}^h_{ijk} + \underset{(11)}{s}{}^h_{ijk}, \tag{1.8}$$

where

$$\begin{aligned} r^h_{ijk} &= \delta_i l^h_{jk} + \hat{\partial}_i \widetilde{L}^h_{jk} - \delta_j l^h_{ik} - \hat{\partial}_j \widetilde{L}^h_{ik} + L^s_{jk} l^h_{is} + l^s_{jk}\widetilde{L}^h_{is} - L^s_{ik} l^h_{js} \\ &\quad - l^s_{ik}\widetilde{L}^h_{js} - \underset{(01)}{R}{}^s_{ij}\underset{(1)}{c}{}^h_{sk} - \underset{(01)}{r}{}^s_{ij}\underset{(1)}{\widetilde{C}}{}^h_{sk} - \underset{(02)}{R}{}^s_{ij}\underset{01}{c}{}^h_{sk} - \underset{(02)}{r}{}^s_{ij}\underset{01}{\widetilde{C}}{}^h_{sk}, \end{aligned} \tag{1.9}$$

$$\begin{aligned} \underset{(1)}{p}{}^h_{ijk} &= \delta_{1i} l^h_{jk} + \hat{\partial}_{1i}\widetilde{L}^h_{jk} - \delta_{1j} l^h_{ik} - \hat{\partial}_{1j}\widetilde{L}^h_{ik} + L^s_{jk}\underset{(1)}{c}{}^h_{is} + l^s_{jk}\underset{(1)}{\widetilde{C}}{}^h_{is} - \underset{(1)}{C}{}^s_{ik} l^h_{js} \\ &\quad - \underset{(1)}{c}{}^s_{ik}\widetilde{L}^h_{js} + \underset{(11)}{B}{}^s_{ji}\underset{(1)}{c}{}^h_{sk} + \underset{(11)}{B}{}^s_{ji}\underset{(1)}{\widetilde{C}}{}^h_{sk} + \underset{(12)}{B}{}^s_{ji}\underset{(1)}{c}{}^h_{sk} + \underset{(12)}{b}{}^s_{ji}\underset{(1)}{\widetilde{C}}{}^h_{sk}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} \underset{(2)}{p}{}^h_{ijk} &= \delta_{2i} l^h_{jk} + \hat{\partial}_{2i}\widetilde{L}^h_{jk} - \delta_j \underset{(2)}{c}{}^h_{ik} - \hat{\partial}_j \underset{(2)}{\widetilde{C}}{}^h_{ik} + L^s_{jk}\underset{(2)}{c}{}^h_{is} + l^s_{jk}\underset{(2)}{\widetilde{C}}{}^h_{is} - \underset{(1)}{C}{}^s_{ik} l^h_{js} \\ &\quad - \underset{(1)}{c}{}^s_{ik}\widetilde{L}^h_{js} + \underset{(21)}{B}{}^s_{ji}\underset{(1)}{c}{}^h_{sk} + \underset{(21)}{b}{}^s_{ji}\underset{(1)}{\widetilde{C}}{}^h_{sk} + \underset{(22)}{B}{}^s_{ji}\underset{(2)}{c}{}^h_{sk} + \underset{(22)}{b}{}^s_{ji}\underset{(2)}{\widetilde{C}}{}^h_{sk}, \end{aligned} \tag{1.11}$$

$$\begin{aligned} \underset{(21)}{s}{}^h_{ijk} &= \delta_{2i}\underset{(1)}{c}{}^h_{jk} + \hat{\partial}_{2i}\underset{(1)}{\widetilde{C}}{}^h_{jk} - \delta_{1j}\underset{(2)}{c}{}^h_{ik} - \hat{\partial}_{1j}\underset{(2)}{\widetilde{C}}{}^h_{ik} + \underset{(1)}{C}{}^s_{jk}\underset{(2)}{c}{}^h_{is} + \underset{(1)}{c}{}^s_{jk}\underset{(2)}{\widetilde{C}}{}^h_{is} - \underset{(2)}{C}{}^s_{ik}\underset{(1)}{c}{}^h_{js} \\ &\quad - \underset{(2)}{c}{}^s_{ik}\underset{(1)}{\widetilde{C}}{}^h_{js} + \underset{(21)}{B}{}^s_{ji}\underset{(2)}{c}{}^h_{sk} + \underset{(21)}{b}{}^s_{ji}\underset{(2)}{\widetilde{C}}{}^h_{sk}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} \underset{(11)}{s}{}^h_{ijk} &= \delta_{1i}\underset{(1)}{c}{}^h_{jk} + \hat{\partial}_{1i}\underset{(1)}{\widetilde{C}}{}^h_{jk} - \delta_{1j}\underset{(1)}{c}{}^h_{ik} - \hat{\partial}_{1j}\underset{(1)}{\widetilde{C}}{}^h_{ik} + \underset{(1)}{C}{}^s_{jk}\underset{(1)}{c}{}^h_{is} + \underset{(1)}{c}{}^s_{jk}\underset{(1)}{\widetilde{C}}{}^h_{is} + \underset{(12)}{R}{}^s_{ij}\underset{(2)}{c}{}^h_{sk} \\ &\quad + \underset{(12)}{r}{}^s_{ij}\underset{(2)}{\widetilde{C}}{}^h_{sk}. \end{aligned} \tag{1.13}$$

Let $\{U_t\}_{t\in I}$ be a family of an open covering $Osc^2(M)$. Assume that at each U_t there exists a map $\rho_t : U_t \to \mathrm{R}$. Then we have the following.

(11) *The structure*

$$\widetilde{\mathbb{F}}, \quad \widetilde{\underset{s}{\xi}} = \exp(\frac{\rho_t}{2})\underset{s}{\xi}_t, \quad \widetilde{\overset{s}{\eta}} = \exp(\frac{-\rho_t}{2})\overset{s}{\eta}_t, \quad \widetilde{\mathbb{G}} = \exp(-\rho_t)\mathbb{G}_t, \tag{1.14}$$

is almost cosymplectic in U_t if and only if the following equations are satisfied

$$d\Phi = \omega \wedge \Phi + (g_{ij}\underset{(1)}{H}{}^i_k\omega - \underset{(1)}{H}{}^i_k dg_{ij} + g_{ij}d\underset{(1)}{H}{}^i_k) \wedge dx^j \wedge dx^k, \tag{1.15}$$

$$d\overset{i}{\eta} = \frac{1}{2}\overset{i}{\eta} \wedge\omega - \frac{1}{2}\underset{(1)}{H}{}^i_k\omega \wedge dy^{(1)k} - \frac{1}{2}K^i_k\omega \wedge dx^k - d\underset{(1)}{H}{}^i_k \wedge dy^{(1)k} - dK^i_k \wedge dx^k, \tag{1.16}$$

where $\omega = d\rho$ obtained by gluing up $d\rho_t$ on $Osc^2(M)$.

1.4 Organization of the Thesis

This thesis is organized as follows. Chapter 2 is devoted to some useful background concepts and definitions on almost contact metric manifolds, almost Kähler manifolds, and almost cosymplectic manifolds. We also provide the definition on 2-osculator bundle $Osc^2(M)$ and introduce necessary tools such as Liouville vector fields, 2-semispray vector fields and Homogeneity of functions of tangent bundle TM (see [40] and references therein for more details).

In Chapter 3, we give the definition of l.c. almost cosymplectic manifolds. We investigate the Chern-Hamilton tensor field τ which was introduced in the paper [11]. We prove that the geometric properties of the characteristic structure vector field ξ is closely related to the parallelism of the Chern-Hamilton tensor field τ. Theorem 3.1.1 gives another class of l.c. almost cosymplectic structures which are almost cosymplectic. This class contains the one given by Olszak in [43, Theorem 4.1]. A class of l.c. almost cosymplectic structures which are cosymplectic is also obtained. Examples are also given to support the results. We also discuss the proportionality of the locally conformal structure and prove that there are many classes that contain such a proportionality condition. Examples are also given. We also find some characterization theorems for a foliation to be Riemannian. Under some conditions, we prove that the foliation of the contact distribution has (almost) Kählerian leaves. Finally we end the section with characteristic remark.

In Chapter 4, we consider a locally conformal cosymplectic manifold endowed with an indefinite metric, and we study the leaves (as submanifolds) of the canonical foliations generated by the Pfaffian equation $\omega = 0$. We give the necessary and sufficient conditions for leaves to be screen conformal as well as some distributions on them to be integrable. Also, we give the necessary condition for the induced connection on the leaves to be a metric connection. By considering a suitable conformal vector field on M, we show that any ascreen null leaf, with a symmetric Ricci tensor, admits a totally geodesic screen distribution, using the concept of Newton transformations [2], [3] and [16].

In Chapter 5, we study the geometry of the conformal transformation on the bundle (T^2M, π^2, M) of accelerations of order 2. We establish the relationships between the geometric objects corresponding to L and $\widetilde{L}$, respectively, namely, Lie brackets, d-tensor of curvature tensors, and d-tensor of Ricci tensors. Some examples of Riemannian and Finsler manifolds are also given.

In Chapter 6, we are concerned with the study of structures on $Osc^2(M)$. We introduce the almost n-contact structure on $Osc^2(M)$. A similar characterization used by Vaisman on M, for l.c. almost cosymplectic manifolds is considered on $Osc^2(M)$. Making use of the complete lift of almost contact structures on M, we obtain $(1,1)$-tensors on $Osc^2(M)$, namely ι_ϕ and A, defined by $\iota_\phi = L \circ \phi \circ d\pi^2$ and $A(\delta_{2i}) = a_i^j \delta_j,\ A(\delta_i) = -a_i^j \delta_{2j},\ A(\delta_{1i}) = 0$, where $a_i^j = \phi_i^j \circ \pi^2$. Therefore, the pair $(Osc^2(M), A)$ is an almost n-contact manifold and for a given l.c. almost cosymplectic structure on M subject to some conditions, we obtain an l.c. almost cosymplectic on $Osc^2(M)$.

Chapter Two

Preliminaries

This chapter gives a brief exposition on almost complex, (almost) Kälher manifolds, almost contact manifolds with particular attention to almost cosymplectic manifolds. Furthermore, we give some definitions on 2-osculator bundles needed in the sequel.

We assume (unless otherwise stated) that all manifolds in this thesis are smooth and paracompact.

2.1 Almost contact structures

In [8], Blair defined an almost contact manifold as a $(2n+1)$-dimensional manifold M such that the structural group of its tangent bundle is reducible to $U(n)\times 1$. Several tensor fields are thereby distinguished, there are the linear transformation field ϕ which acting in each tangent space T_pM of M, $p\in M$, called fundamental singular collineation, the vector field ξ in M called the fundamental (or structure) vector field and the contact form η related by

$$\phi^2=-\mathbb{I}+\eta\otimes\xi,\quad \eta(\xi)=1,\quad \phi\xi=0,\quad \eta\circ\phi=0. \tag{2.1}$$

A smooth manifold M is called almost contact metric manifold if it admits a Riemannian metric g satisfying the compatible relation given by

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y). \tag{2.2}$$

Replacing respectively in (2.2) $Y=\phi Y$ and $Y=\xi$ and using (2.1), one gets

$$g(\phi X,Y)=-g(X,\phi Y)\quad\text{and}\quad \eta(X)=g(X,\xi). \tag{2.3}$$

A fundamental 2-form Φ of (M,ϕ,ξ,η,g) is given by

$$\Phi(X,Y)=g(\phi X,Y).$$

A *Nijenhius* tensor field with respect to ϕ is a $(1,2)$-tensor N_1 defined by

$$N_1=[\phi,\phi]+\mathrm{d}\eta\otimes\xi \tag{2.4}$$

where $[\phi, \phi]$ is the Nijenhuis torsion of the tensor field ϕ given by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y].$$

An almost contact manifold with a vanishing Nijenhius tensor field is a *Normal* almost contact manifold.
There are three other useful tensors in almost contact structure (ϕ, ξ, η) given by

$$N_2(X, Y) = (\mathcal{L}_{\phi X}\eta)Y - (L_{\phi Y}\eta)X, \tag{2.5}$$

$$N_3 = (\mathcal{L}_{\xi}\phi)X, \tag{2.6}$$

$$N_4 = (L_{\xi}\eta)X, \tag{2.7}$$

where $\mathcal{L}$ is the Lie derivative. It is known that the vanishing of N_1 implies the vanishing of N_2, N_3, N_4. (See [8, page 81] for more details). Therefore, for a normal almost contact metric manifold the following equation holds

$$\nabla_{\phi X}\xi = \phi\nabla_X\xi. \tag{2.8}$$

For more details see [8, Lemma 6.2].

2.2 Almost cosymplectic manifolds

An almost contact metric manifold M is called almost cosymplectic if its fundamental form and contact form are closed, that is

$$\mathrm{d}\Phi = 0 \quad \text{and} \quad \mathrm{d}\eta = 0. \tag{2.9}$$

A cosymplectic manifold is a normal almost cosymplectic manifold. For a cosymplectic manifold we have $\nabla\phi = 0$ (see [34] and references therein for more details).

Almost cosymplectic manifolds are viewed as odd-dimesional version of Almost Kälher manifolds.

Proposition 2.2.1. *Let (M, ξ, ϕ, η, g) be an almost cosymplectic manifold. The following relation holds*

$$(\nabla_{\phi X}\phi)\phi Y + (\nabla_X\phi)Y - \eta(Y)\nabla_{\phi X}\xi. \tag{2.10}$$

Proof. The details of the proof can be found in [44]. □

From (2.10), one gets

$$\nabla_{\xi}\phi = 0,\ \nabla_{\phi X}\xi = -\phi\nabla_X\xi \ \text{ and } \ \nabla_{\xi}\xi = 0, \tag{2.11}$$

which also imply

$$(\nabla_{\phi X}\eta)(Y) = (\nabla_X\eta)(\phi Y), \tag{2.12}$$

$$(\nabla_{\phi X}\eta)(\phi Y) = -(\nabla_X\eta)(Y). \tag{2.13}$$

Proposition 2.2.2. *Let (M, ξ, ϕ, η, g) be an almost cosymplectic manifold. The divergence and Ricci curvature satisfy*

$$\mathrm{div}(\xi) = 0, \tag{2.14}$$

$$S(\xi, \xi) + |\nabla \xi|^2 = 0. \tag{2.15}$$

Proof. Since, $\mathrm{div}(\xi) = \sum_{i=0}^{2n+1} g(\nabla_{E_i}\xi, E_i)$, then for a chosen ϕ-basis and by taking into account (2.11), one obtains

$$g(\nabla_{\phi E_i}\xi, \phi E_i) = -g(\nabla_{E_i}\xi, E_i),$$

then $\mathrm{div}(\xi) = 0$. The equation (2.15) derives from a straightforward calculation. Indeed, we have $S(X, Y) = \sum_{i=0}^{2n+1} R(E_i, X, Y, E_i)$, then

$$\begin{aligned} S(\xi,\xi) = \sum_{i=0}^{2n+1} R(E_i,\xi,\xi,E_i) &= \sum_{i=0}^{2n+1} g(\nabla_{E_i}\nabla_\xi\xi - \nabla_\xi\nabla_{E_i}\xi - \nabla_{[E_i,\xi]}\xi, E_i) \\ \sum_{i=0}^{2n+1} g(\nabla_{E_i}\nabla_\xi\xi, E_i) &+ \sum_{i=0}^{2n+1}(-g(\nabla_\xi\nabla_{E_i}\xi, E_i) - g(\nabla_{[E_i,\xi]}\xi, E_i) \\ &= \sum_{i=0}^{2n+1} g(\nabla_{E_i}\nabla_\xi\xi, E_i) - \xi\mathrm{div}(\xi) + \sum_{i=0}^{2n+1}(g(\nabla_{E_i}\xi, \nabla_\xi E_i) - (\nabla_{E_i}\eta)[E_i,\xi]) \\ &= \sum_{i=0}^{2n+1} g(\nabla_{E_i}\nabla_\xi\xi, E_i) - \xi\mathrm{div}(\xi) + \sum_{i=0}^{2n+1}(2g(\nabla_{E_i}\xi, \nabla_\xi E_i) - g(\nabla_{E_i}\xi, \nabla_{E_i}\xi)). \end{aligned} \tag{2.16}$$

By using (2.11), we have $g(\nabla_{E_i}\xi, \nabla_\xi E_i) + g(\nabla_{\phi E_i}\xi, \nabla_\xi \phi E_i) = 0$. Therefore, (2.16) gives

$$S(\xi,\xi) = -\sum_{i=0}^{2n+1} g(\nabla_{E_i}\xi, \nabla_{E_i}\xi) = -|\nabla\xi|^2,$$

which proves the assertion. □

2.3 Almost Kähler manifolds

An almost complex manifold is an even-dimensional manifold M endowed with a $(1,1)$-tensor field J such that $J^2 = -\mathbb{I}$, where $\mathbb{I}$ is the identity matrix. Such a manifold is orientable.

Denoting by N_J, or simply by N, the Nijennhuis tensor of J, one has

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

If $N = 0$ then the almost complex manifold (M, J) is called complex manifold.

An almost hermitian manifold (M, J, g) is an almost complex manifold in which J is skew-symmetric with respect to the metric g.

Denote by Ω the fundamental 2-form that is

$$\Omega(X, Y) = g(X, JY), \text{ for any } X, Y \in \Gamma(TM).$$

The manifold M is said to be almost Kähler if Ω is closed i.e., we have $d\Omega = 0$. It is easy to prove that a Hermitian manifold is a Kähler manifold if and only if the almost complex structure J is parallel with respect to ∇, i.e., we have $\nabla_X J = 0$ for any $X \in \Gamma(TM)$ (see [8] for more details).

2.4 The 2-Osculator bundles

Let M be a real n-dimensional smooth manifolds and $(Osc^2(M), \pi^2, M)$ its bundle of accelerations of order 2. For a local chart $(U, (x^i))$ in $p \in M$, its lifted local chart in $u \in (\pi^2)^{-1}(p)$ is

$$((\pi^2)^{-1}(U), (x^i, y^{(1)i}, y^{(2)i})).$$

For each point $u = (x, y^{(1)}, y^{(2)}) \in Osc^2(M)$, the natural basis of the tangent space $T_u Osc^2(M)$ is

$$\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^{(1)i}}|_u, \frac{\partial}{\partial y^{(2)i}}|_u\}.$$

In analytical mechanics, the manifold M is called the *space of configuration*, a mapping $c: \ t \to (x^i(t)) \in U \subset M$ is called *law of moving (evolution)* or a time-parametrized curve, a couple (t, x) is an *event* and the pair $(\frac{dx^i}{dt}, \frac{1}{2}\frac{d^2x^i}{dt^2})$ are the velocity and acceleration (see [40] for more details).

We have 2-canonical surjective submersions

$$\pi_0^2 = \pi^2 : (x, y^{(1)}, y^{(2)}) \in Osc^2(M) \to (x) \in M, \tag{2.17}$$
$$\pi_1^2 : (x, y^{(1)}, y^{(2)}) \in Osc^2(M) \to (x, y^{(1)}) \in Osc^1(M). \tag{2.18}$$

Thus π_0^2 and π_1^2 determine the vertical distributions

$$\begin{aligned} V_{u,1} &= \ker\ d\pi^2 = \text{Span}\{\frac{\partial}{\partial y^{(1)i}}|_u, \frac{\partial}{\partial y^{(2)i}}|_u\}_{i=1\cdots n}, \\ V_{u,2} &= \ker\ d\pi_1^2 = \text{Span}\{\frac{\partial}{\partial y^{(2)1}}|_u, \cdots, \frac{\partial}{\partial y^{(2)n}}|_u\}. \end{aligned} \tag{2.19}$$

Then we have the decomposition sum,

$$T_u Osc^2(M) = N_u \oplus V_{u,1}. \tag{2.20}$$

The paracompacteness of M guarantees the existence of the integrable distribution $V_{u,1} = \ker\ d\pi^2$ (see [40, theorem.3.3.3] for more details).

An extremely important structure on $Osc^2(M)$ is the so-called 2-tangent structure J introduced by Eliopoulous in [20] locally given by

$$J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i}, \tag{2.21}$$

and satisfies the following properties

$$\mathrm{Im}\ J = V_1,\ \ker\ J = V_2,\ J(V_1) = V_2,\ \mathrm{rank}\|J\| = 2n.$$

There are two important vectors fields, called Liouville vector fields in the study of geometry of $Osc^2(M)$:

$$\overset{1}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(2)i}} \text{ and } \overset{2}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}}. \tag{2.22}$$

We will also use the operator:

$$\Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}}. \tag{2.23}$$

A vector field S of $Osc^2(M)$ is called a 2-semispray on $Osc^2(M)$ if

$$JS = \overset{2}{\Gamma}$$

locally given by

$$\begin{aligned} S &= y^1\frac{\partial}{\partial x^i} + 2y^2\frac{\partial}{\partial y^{(1)i}} - 3G^i\frac{\partial}{\partial y^{(2)i}} \\ &= \Gamma - 3G^i\frac{\partial}{\partial y^{(2)i}}. \end{aligned} \tag{2.24}$$

The curve $\widetilde{c}:\ t \to \widetilde{c}(t) \in (\pi^2)^{-1}(U) \subset Osc^2(M)$ extension of c is given by

$$\widetilde{c} : t \in I \to (x^i(t), \frac{dx^i}{dt}, \frac{1}{2}\frac{d^2x^i}{dt^2}). \tag{2.25}$$

A 2-path on M is the curve $c : I \to M$ such that its extension on $Osc^2(M)$ is an integral curve of S. It is well known that the paths of the 2-semispray S are given by the differential equations

$$\frac{d^3x^i}{dt^3} + 3!G^i(x, \frac{dx}{dt}, \frac{1}{2}\frac{d^2x}{dx^2}) = 0. \tag{2.26}$$

The coefficients $G^i(x, y^1, y^2)$ allow us to obtain the system of functions

$$\underset{(1)}{N}{}^i_j = \frac{\partial G^i}{\partial y^{(2)j}}, \tag{2.27}$$

which are the coefficients of the nonlinear connection N in (2.20).

Using the 2-almost tangent structure (2.21) the decomposition (2.20) becomes

$$TOsc^2(M) = N_0 \oplus N_1 \oplus V_2, \tag{2.28}$$

where $N_0 = N$.

According to (2.28), there exists an adapted basis on $TOsc^2(M)$ denoted by

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}} \right\}, \tag{2.29}$$

given by

$$\begin{aligned} \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - \underset{(1)}{N_i}{}^j \frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N_i}{}^j \frac{\partial}{\partial y^{(2)j}}, \\ \frac{\delta}{\delta y^{(1)i}} &= \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N_i}{}^j \frac{\partial}{\partial y^{(2)j}}, \\ \frac{\delta}{\delta y^{(2)i}} &= \frac{\partial}{\partial y^{(2)i}}, \end{aligned} \tag{2.30}$$

with

$$\underset{(2)}{N_i}{}^j = \frac{1}{2}(S\underset{(1)}{N_i}{}^j - \underset{(1)}{N_i}{}^m \underset{(1)}{N_m}{}^j). \tag{2.31}$$

Its duals basis is given by:

$$\delta x^i = dx^i, \tag{2.32}$$

$$\delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M_m}{}^i dx^m, \tag{2.33}$$

$$\delta y^{(2)i} = dy^{(2)i} + \underset{(1)}{M_m}{}^i dy^{(1)m} + \underset{(2)}{M_m}{}^i dx^m, \tag{2.34}$$

where

$$\underset{(1)}{M_m}{}^i = \underset{(1)}{N_m}{}^i \quad \text{and} \quad \underset{(2)}{M_m}{}^i = \underset{(2)}{N_m}{}^i + \underset{(1)}{N_k}{}^i \underset{(1)}{N_m}{}^k. \tag{2.35}$$

As result, we have the following equations

$$\begin{aligned} &\frac{\partial}{\partial x^i} = \frac{\delta}{\delta x^i} + \underset{(1)}{M_i}{}^j \frac{\delta}{\delta y^{(1)j}} + \underset{(2)}{M_i}{}^j \frac{\delta}{\delta y^{(2)j}}, \frac{\partial}{\partial y^{(1)i}} = \frac{\delta}{\delta y^{(1)i}} + \underset{(1)}{M_i}{}^j \frac{\delta}{\delta y^{(2)j}}, \\ &\frac{\partial}{\partial y^{(2)i}} = \frac{\delta}{\delta y^{(2)i}}, \end{aligned} \tag{2.36}$$

$$dx^i = \delta x^i,\ dy^{(1)i} = \delta y^{(1)i} - \underset{(1)}{N_m}{}^i \delta x^m,\ dy^{(2)i} = \delta y^{(2)i} - \underset{(1)}{N_m}{}^i \delta y^{(1)m} - \underset{(2)}{N_m}{}^i \delta x^m. \tag{2.37}$$

Therefore, by using (2.36) we obtain the Liouville vector fields in adapted basis (2.29) as follows

$$\overset{1}{\Gamma} = z^{(1)i}\frac{\delta}{\delta y^{(2)i}}, \quad \overset{2}{\Gamma} = z^{(1)i}\frac{\delta}{\delta y^{(1)i}} + 2z^{(2)i}\frac{\delta}{\delta y^{(2)i}}, \tag{2.38}$$

$$\text{where} \quad z^{(1)i} = y^{(1)i}, \quad z^{(2)i} = y^{(2)i} + \frac{1}{2}\underset{(1)}{M^i_j} y^{(1)j}, \tag{2.39}$$

are the Liouville d-vectors.

Definition 2.4.1. [40] A distinguished tensor field (briefly: d-tensor field) on $Osc^2(M)$ of type (r,s) is a tensor field T of type (r,s) on $Osc^2(M)$ with the property:

$$T(\overset{1}{\omega}, \cdots, \overset{r}{\omega}, \underset{1}{X}, \cdots, \underset{s}{X}) = T(\overset{1}{\omega}^H, \cdots, \overset{r}{\omega}^{V_2}, \underset{1}{X}^H, \cdots, \underset{s}{X}^{V_2}) \tag{2.40}$$

for any $(\overset{1}{\omega}^H, \cdots, \overset{r}{\omega}^{V_2}) \in \mathfrak{X}^*(Osc^2(M))$ and $(\underset{1}{X}^H, \cdots, \underset{s}{X}^{V_2}) \in \mathfrak{X}(Osc^2(M))$. The tensor field T is locally defined by

$$T = T^{a_1\cdots a_r}_{b_1\cdots b_s}(x, y^{(1)}, y^{(2)})\frac{\delta}{\delta x^{a_1}} \otimes \cdots \frac{\partial}{\partial y^{(2)a_r}} \otimes dx^{b_1} \otimes \cdots \delta y^{(2)b_s}, \tag{2.41}$$

where the coefficients are given by

$$T^{a_1\cdots a_r}_{b_1\cdots b_s}(x, y^{(1)}, y^{(2)}) = T(dx^{a_1}, \cdots, \delta y^{(2)a_r}, \frac{\partial}{\partial x^{b_1}}, \cdots, \frac{\partial}{\partial y^{(2)b_s}}). \tag{2.42}$$

The dual basis $\{1, \delta x, \delta y^{(1)}, \delta y^{(2)}\}$ generates the algebra of the d-tensor fields over the ring of functions $\mathcal{F}(Osc^2(M))$.

Definition 2.4.2. [9] A function $f : TM \to R$ that is differentiable on $\widetilde{TM}$ and continuous only on the null section $0 : M \to TM$ is called homogeneous of order r $(r \in \mathbb{Z})$ on the fibres of TM *or r-homogeneous with respect to* y^i if:

$$f \circ h_a = a^r f, \forall a \in \mathbb{R}^+,$$

where $h_\lambda : TM \to TM$ *is given by* $h_\lambda(x,y) = (x, \lambda y)$.

The following Euler theorem holds:

Theorem 2.4.1. *[9] A function $f \in C^\infty(M)$ on $\widetilde{TM}$ and continuous only on the null section is homogeneous of order r if and only if*

$$L_\Gamma f = y^i \partial_i f = rf. \tag{2.43}$$

Proof. Assume that the function is r-homogeneous then $f(x^i, ry^i) = y^{ir}f(x^i, y^i)$, differentiating both side with respect to λ one gets $\frac{\partial f}{\partial \lambda} = \frac{\partial f}{\partial(\lambda y^i)}\frac{\partial(\lambda y^i)}{\partial \lambda} = y^i \frac{\partial f}{\partial(\lambda y^i)}$ and $\frac{\partial f}{\partial \lambda} = r\lambda^{r-1}f$ by putting $\lambda = 1$ one gets (2.43). Then the proof. $\square$

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The following properties hold:

(1) If f_1, f_2 are r-homogeneous functions, then the function $\lambda_1 f_1 + \lambda_2 f_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ is r-homogeneous too.

(2) If f_1 is r-homogeneous and f_2 is s-homogeneous, then the function $f_1.f_2$ is $(r+s)$-homogeneous.

Locally conformal almost Cosymplectic manifolds

In this chapter, we establish the equation which characterizes a locally conformal (l.c.) almost cosymplectic manifold. We also investigate the tensor field h and the Chern-Hamilton tensor field τ. By using some facts on foliations in [7], we study some subclasses of the class of l.c. almost cosymplectic manifolds in particular the subclass of bundle-like metric structures. We support this chapter with some examples.

3.1 L.c. almost cosymplectic manifolds

Let M be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is tensor field of type $(1,1)$ on M, a vector field ξ and a 1-form η satisfying the following relations

$$\phi^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{3.1}$$

$$\text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{3.2}$$

The fundamental 2-form of M is defined by $\Phi(X, Y) = g(X, \phi Y)$, for any vector fields X and Y on M.

M is said to be *almost cosymplectic* if the forms η and Φ are closed, that is, $d\eta = 0$ and $d\Phi = 0$, d being the operator of the exterior differentiation (see [22]). If M is almost cosymplectic and its almost contact structure (ϕ, ξ, η) is normal, then M is called *cosymplectic*. The normality condition says that the torsion tensor field

$$[\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{3.3}$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

It is well-known that a necessary and sufficient condition for the almost contact metric manifold M to be cosymplectic is $\nabla\phi = 0$, where ∇ is the Levi-Civita connection of M.

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Now, let (M, ϕ, ξ, η, g) be an almost contact metric manifold. Such a manifold is said to be *locally conformal (l.c.) almost cosymplectic* [43] if M has an open covering $\{U_t\}_{t\in I}$ endowed with smooth functions $\sigma_t : U_t \longrightarrow \mathbb{R}$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \quad \xi_t = \exp(\sigma_t)\xi, \quad \eta_t = \exp(-\sigma_t)\eta, \quad g_t = \exp(-2\sigma_t)g, \tag{3.4}$$

is *almost cosymplectic.* If the structures $(\phi_t, \xi_t, \eta_t, g_t)$ defined in (3.4) are cosymplectic, then M is called *l.c. cosymplectic.* L.c. conformal almost cosymplectic manifolds were characterized by Vaisman in [48]. This is stated as follows: *An almost contact metric manifold M is an l.c. almost cosymplectic manifold if and only if there exists a 1-form ω on M such that*

$$d\Phi = 2\omega \wedge \Phi, \quad d\eta = \omega \wedge \eta \quad \textit{and} \quad d\omega = 0. \tag{3.5}$$

Moreover, an l.c. almost cosymplectic (respectively, an l.c. cosymplectic) manifold M is *almost cosymplectic* (respectively, *cosymplectic*) if and only if $\omega = 0$. If ω has no singular points, M was termed, by Capursi and Dragomir in [10], *strongly non-cosymplectic.*

Assume that (M, ϕ, ξ, η, g) is an l.c. almost cosymplectic manifold. Then the relations in (3.5) are satisfied for a certain 1-form ω. For any t, over open set U_t, the structure $(\phi_t, \xi_t, \eta_t, g_t)$ given by (3.4) is almost cosymplectic and $d\sigma_t = \omega$.

Now, we give a proof to the formula (3.3) in [43]. Let ∇ and ∇^t be the Levi-Civita connections associated with the metrics g and g_t, respectively. Setting

$$\sigma_X Y = \nabla_X Y - \nabla^t_X Y.$$

Then, it is easy to see that σ is symmetric, that is, $\sigma_X Y = \sigma_Y X$.

Lemma 3.1.1. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. Let ∇ and ∇^t be the Levi-Civita connections associated with the metrics g and g_t, respectively. Then, for any vector fields X and Y on M,*

$$\nabla^t_X Y = \nabla_X Y - \omega(X)Y - \omega(Y)X + g(X, Y)B, \tag{3.6}$$

where B is the vector field defined by $g(B, X) = \omega(X)$.

Proof. Using (3.5), by direct calculation, we get, for any X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned} 0 = (\nabla^t_X g_t)(Y, Z) &= X(g_t(Y, Z)) - g_t(\nabla^t_X Y, Z) - g_t(Y, \nabla^t_X Z) \\ &= \exp(-2\sigma_t)\{-2X(\sigma_t)g(Y, Z) + g(\sigma_X Y, Z) + g(Y, \sigma_X Z)\}. \end{aligned} \tag{3.7}$$

The relation (3.7) becomes

$$g(\sigma_X Y, Z) + g(Y, \sigma_X Z) = 2\omega(X)g(Y, Z). \tag{3.8}$$

A circular permutation in (3.8) gives

$$g(\sigma_Y Z, X) + g(Z, \sigma_Y X) = 2\omega(Y)g(Z, X), \tag{3.9}$$
$$g(\sigma_Z X, Y) + g(X, \sigma_Z Y) = 2\omega(Z)g(X, Y). \tag{3.10}$$

Putting the pieces above using the operation (3.8) − (3.10) + (3.9), we have

$$g(\sigma_X Y, Z) = \omega(X)g(Y, Z) + \omega(Y)g(Z, X) - \omega(Z)g(X, Y),$$

which implies that $\sigma_X Y = \omega(X)Y + \omega(Y)X - g(X, Y)B$, where B is the vector field defined by $g(B, X) = \omega(X)$. This completes the proof. □

Note that the vector field B defined in Lemma 3.1.1 is explicitly given by $B = grad\,\sigma_t$, over any U_t.

Let us consider the following tensors N_1 and N_2 given in [8] by

$$N_1(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi, \tag{3.11}$$
$$N_2(X, Y) = (\mathcal{L}_{\phi X}\eta)Y - (\mathcal{L}_{\phi Y}\eta)X, \tag{3.12}$$

where $[\phi, \phi]$ is the Nijenhuis torsion of the tensor field ϕ and $\mathcal{L}$ the Lie derivative.

For a general almost contact metric structure (ϕ, ξ, η, g), the covariant derivative of ϕ is given by

$$\begin{aligned}
2g((\nabla_X\phi)Y, Z) = {} & -X(\Phi(Y, Z)) + \phi Y(\Phi(\phi Z, X) + \eta(X)\eta(Z)) - Z(\Phi(X, Y)) \\
& - \Phi([X, \phi Y], \phi Z) + \eta([X, \phi Y])\eta(Z) + \Phi([Z, X], Y) \\
& - g(\phi[\phi Y, Z], \phi X) + \eta(X)\eta(Z, \phi Y]) \\
& + X(\Phi(Y, Z)) + Y(\Phi(X, Z)) - \phi Z(\Phi(\phi Y, X) + \eta(X)\eta(Y)) \\
& + \Phi([X, Y], Z) + g(\phi[\phi Z, X], \phi Y) + \eta(Y)\eta([\phi Z, X]) \\
& - g(\phi[Y, \phi Z], \phi X) + \eta(X)\eta(\phi Z, Y).
\end{aligned} \tag{3.13}$$

Since

$$\begin{aligned}
2d\eta(\phi Y, X) &= \phi Y(\eta(X)) - \eta([\phi Y, X]), \\
g(N_1(Y, Z), \phi X) &= -\Phi([Y, Z], X) + \Phi([\phi Y, \phi Z], X) \\
&\quad - g(\phi[\phi Y, Z], \phi X) - g(\phi[Y, \phi Z], \phi X), \\
N_2(Y, Z) &= \phi Y(\eta(Z)) - \phi Z(\eta(Y)) - \eta([\phi Y, Z]) + \eta([\phi Z, Y]), \\
3d\Phi(X, Y, Z) &= X(\Phi(Y, Z)) + Y(\Phi(Z, X)) + Z(\Phi(X, Y)) - \Phi([X, Y], Z) \\
&\quad - \Phi([Z, X], Y) - \Phi([Y, Z], X), \\
3d\Phi(X, \phi Y, \phi Z) &= X(\Phi(\phi Y, \phi Z)) + \phi Y(\Phi(\phi Z, X)) + \phi Z(\Phi(X, \phi Y)) \\
&\quad - \Phi([X, \phi Y], \phi Z) - \Phi([\phi Z, X], \phi Y) - \Phi([\phi Y, \phi Z], X),
\end{aligned}$$

the relation (3.13) becomes

$$2g((\nabla_X\phi)Y,Z) = 3d\Phi(X,\phi Y,\phi Z) - 3d\Phi(X,Y,Z) + g(N_1(Y,Z),\phi X) + N_2(Y,Z)\eta(X) + 2d\eta(\phi Y,X)\eta(Z) - 2d\eta(\phi Z,X)\eta(Y). \tag{3.14}$$

Note that in view of (3.6), the covariant derivatives $\nabla^t\phi_t$ and $\nabla\phi$ are related by

$$(\nabla^t_X\phi_t)Y = (\nabla_X\phi)Y - \omega(\phi Y)X + \omega(Y)\phi X + g(X,\phi Y)B - g(X,Y)\phi B. \tag{3.15}$$

Lemma 3.1.2. *[35] For the structure $(\phi_t,\xi_t,\eta_t,g_t)$, we have on each U_t,*

$$N_{1t}(X,Y) = N_1(X,Y) - 2d\eta(X,Y)\xi, \tag{3.16}$$

$$N_{2t}(X,Y) = \exp(-\sigma_t)\{\omega(\phi Y)\eta(X) - \omega(\phi X)\eta(Y) + N_2(X,Y)\}. \tag{3.17}$$

If (M,ϕ,ξ,η,g) is an l.c. almost cosymplectic manifold, then $(\phi_t,\xi_t,\eta_t,g_t)$ is an almost cosymplectic. That is, $d\Phi_t = 0$ and $d\eta_t = 0$. These lead to

$$2g_t((\nabla^t_X\phi_t)Y,Z) = \exp(-2\sigma_t)\{g(N_1(Y,Z),\phi X) - \omega(\phi Y)\eta(X)\eta(Z) + \omega(\phi Z)\eta(X)\eta(Y) + N_2(Y,Z)\eta(X)\}, \tag{3.18}$$

which is equivalent to

$$2g((\nabla^t_X\phi_t)Y,Z) = g(N_1(Y,Z),\phi X) + N_2(Y,Z)\eta(X) - \omega(\phi Y)\eta(X)\eta(Z) + \omega(\phi Z)\eta(X)\eta(Y). \tag{3.19}$$

Using $d\eta = \omega\wedge\eta$ in (3.5), N_2 becomes $N_2(Y,Z) = \omega(\phi Y)\eta(Z) - \omega(\phi Z)\eta(Y)$, and (3.19) reduces to $2g((\nabla^t_X\phi_t)Y,Z) = g(N_1(Y,Z),\phi X)$. Therefore, we have the following lemma.

Lemma 3.1.3. *An almost contact metric manifold M is l.c. almost cosymplectic if and only if there exists a 1-form ω on M such that $d\omega = 0$ and*

$$2g((\nabla_X\phi)Y,Z) = g(N_1(Y,Z),\phi X) + 2\omega(\phi Y)g(X,Z) - 2\omega(\phi Z)g(X,Y) - 2\omega(Y)g(\phi X,Z) - 2\omega(Z)g(X,\phi Y), \tag{3.20}$$

for any vector fields X, Y and Z on M.

For the covariant derivative $\nabla\phi$ and using (3.20), we have

$$(\nabla_\xi\phi)\xi = \phi B \quad\text{and}\quad (\nabla_\xi\phi)X = \omega(\phi X)\xi + \eta(X)\phi B. \tag{3.21}$$

Let us consider a $(1,1)$-tensor field h on M by [43]

$$hX = \nabla_X\xi - \omega(\xi)X + \eta(X)B, \tag{3.22}$$

for any $X \in \Gamma(TM)$. This leads to

$$\nabla_\xi \xi = -B + \omega(\xi)\xi. \tag{3.23}$$

Using (3.4) and (3.1.1), we obtain on each U_t, $\exp(-\sigma_t)\nabla^t_X \xi_t = hX$. Note that the linear operator h is symmetric and satisfies (see [43] for details)

$$h\phi + \phi h = 0, \;\; h\xi = 0 \;\text{ and }\; \mathrm{trace}(h) = 0. \tag{3.24}$$

The divergence of ξ is given by

$$\mathrm{div}(\xi) = 2m\omega(\xi). \tag{3.25}$$

As an example of an l.c. almost cosymplectic manifold, we have the following.

Example 3.1.1. We consider the 5-dimensional manifold $M^5 = \{p \in \mathbb{R}^5 | x_1 \neq 0, \; z > 0\}$, where $p = (x_1, x_2, y_1, y_2, z)$ are the standard coordinates in $\mathbb{R}^5$. The vector fields,

$$X_i = z\frac{\partial}{\partial x_i}, \;\; Y_i = \frac{1}{z^3}\frac{\partial}{\partial y_i}, \;\; \xi = \frac{1}{x_1}\frac{\partial}{\partial z}, \;\; \text{for } \; i = 1, 2,$$

are linearly independent at each point of M. Let g be the Riemannian metric on M defined by $g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, $g(X_i, Y_j) = 0$ and $g(\xi, \xi) = 1$. That is, the form of the metric becomes

$$g = \frac{1}{z^2}(dx_1^2 + dx_2^2) + z^6(dy_1^2 + dy_2^2) + x_1^2 dz^2.$$

Let η be the 1-form on M defined by $\eta = x_1 dz$. Let ϕ be the $(1,1)$-tensor field defined by, $\phi X_1 = Y_1$, $\phi X_2 = -Y_2$, $\phi Y_2 = X_2$, $\phi Y_1 = -X_1$, $\phi\xi = 0$. By linearity of ϕ and g, the relations (4.1) and (4.2) are satisfied on M^5. Thus, (ϕ, ξ, η, g) defines an almost contact metric structure on M^5. We note that $d\eta = dx_1 \wedge dz = x_1(\frac{1}{x_1}dx_1 + \frac{1}{z}dz) \wedge dz$. The non-zero component of the fundamental 2-form Φ is $\Phi(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}) = -z^2$, and we have $\Phi = -z^2 dx_1 \wedge dy_1$. Its differential gives $d\Phi = -2z dx_1 \wedge dy_1 \wedge dz = -2z^2(\frac{1}{x_1}dx_1 + \frac{1}{z}dz) \wedge dx_1 \wedge dy_1$. By letting $\omega = \frac{1}{x_1}dx_1 + \frac{1}{z}dz$, we have, $d\eta = \omega \wedge \eta$ and $d\Phi = 2\omega \wedge \Phi$. It is easy to see that $d\omega = 0$ and the dual vector field B is given by $B = \frac{z}{x_1}X_1 + \frac{1}{x_1 z}\xi$. Let us consider the open neighborhood U of M given by $U = \{p \in M^5 | x_1 > 0\}$, and there exists a differentiable function σ on U such that $\omega = d\sigma$, where $\sigma = \ln(x_1 z)$. By Vaisman's characterization above-mentioned, $(M^5, \phi, \xi, \eta, g)$ is an l.c. almost cosymplectic manifold. Let ∇ be the Levi-Civita connection with respect to the metric g. Then, the non-zero Lie brackets are $[X_i, \xi] = -\frac{1}{x_1 z}X_i - (2-i)\frac{z}{x_1}\xi$ and $[Y_i, \xi] = \frac{3}{x_1 z}Y_i$, for $i = 1, 2$. These lead to $\nabla_\xi \xi = -\frac{z}{x_1}X_1$, $\nabla_{X_i}\xi = -\frac{1}{x_1 z}X_i$, $\nabla_{Y_i}\xi = \frac{3}{x_1 z}Y_i$, for $i = 1, 2$. The components of the tensor h defined in (3.22) are given by $h\xi = 0$, $hX_i = 0$, $hY_i = \frac{2}{x_1 z}Y_i$, for $i = 1, 2$. Using the fact that $g(\nabla_{(\cdot)}\xi, \xi) = 0$, it is easy to check that its trace vanishes, that is, $\mathrm{trace}(h) = 0$.

Next, we investigate the *torsion tensor* τ for an l.c. almost cosymplectic manifold. This tensor was introduced by Chern and Hamilton [11] and is defined by

$$g(\tau X, Y) = (\mathcal{L}_\xi g)(X, Y),$$

for vector fields X, Y on a contact metric manifold (see [21] for details).

The Lie derivative $\mathcal{L}_\xi$ of g with respect to the vector field ξ is given by

$$(\mathcal{L}_\xi g)(X, Y) = 2g(hX, Y) + 2\omega(\xi)g(X, Y) - \omega(X)\eta(Y) - \omega(Y)\eta(X), \qquad (3.26)$$

where h is given by (3.22). Thus, an l.c. almost cosymplectic manifold M has a tensor τ such that $g(\tau X, Y) = (\mathcal{L}_\xi g)(X, Y)$, $\forall\, X, Y \in \Gamma(TM)$. By (3.26), have

$$\tau X = 2hX + 2\omega(\xi)X - \omega(X)\xi - \eta(X)B. \qquad (3.27)$$

Lemma 3.1.4. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. Then, the following assertions are equivalent:*

(a) The structure vector field ξ is Killing.

(b) The differential 1-form ω and the operator h vanish.

Proof. Suppose the structure vector field ξ is Killing. Then, the Lie derivative

$$(\mathcal{L}_\xi g)(X, Y) = 0,$$

for any vector fields X and Y on M. The latter implies that the Chern-Hamilton tensor τ vanishes identically on M. Its trace, with respect to an adapted frame $\{E_i\}_{1\leq i\leq 2m+1}$ in TM, gives

$$2trace(h) + 4m\omega(\xi) = 0.$$

Since $trace(h) = 0$ and $m \geq 1$. Also, for $X = \xi$ in (3.27), we have $B = 0$. Hence $\omega = 0$ and $h = 0$. This means that (a) implies (b), and the conserve is obvious, using the equation (3.26). □

Using Example 3.1.1, the components of the Chern-Hamilton tensor on M^5 are given by $\tau X_1 = \frac{2}{x_1 z}X_1 - \frac{z}{x_1}\xi$, $\tau X_2 = \frac{2}{x_1 z}X_2$, $\tau Y_1 = \frac{2}{x_1 z}Y_1$, $\tau Y_2 = \frac{2}{x_1 z}Y_2$ and $\tau\xi = -\frac{z}{x_1}\xi$. This means that the vector field structure $\xi = \frac{1}{x_1}\frac{\partial}{\partial z}$ of the l.c. almost cosymplectic manifold M^5 in Example 3.1.1 is Killing.

As a consequence to Lemma 3.1.4, we have the following.

Theorem 3.1.1. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold for which the structure vector field ξ is Killing. Then M is almost cosymplectic and h vanishes.*

Contrary to Theorem 4.1 in [43] in which compactness and other conditions were added to derive the result, the Theorem 3.1.1 leads to the same conclusion with only one condition. The condition, in fact, contains the ones on the Ricci tensor, the scalar curvatures and the function $\omega(\xi)$. This can be seen as follows. Let S and r be the Ricci curvature tensor and the scalar curvature defined, respectively, by $S(X,Y) = \sum_{i=0}^{2m} g(R(E_i,X)Y,E_i)$, $r = \sum_{i=0}^{2m} S(E_i,E_i)$, respectively, $\{E_i\}_{0\leq i\leq 2m}$ being an orthonormal frame with respect to g. In addition, the Ricci $*$-curvature tensor S^* and scalar $*$-curvature r^*, are given by $S^*(X,Y) = \sum_{i=0}^{2m} g(R(E_i,X)\phi Y,\phi E_i)$, $r^* = \sum_{i=0}^{2m} S^*(E_i,E_i)$. The Ricci curvature tensor S and the scalar curvatures r and r^* of an l.c. almost cosymplectic manifold satisfies the identities [43, Proposition 4.1]:

$$S(\xi,\xi) + |\nabla\xi|_g^2 + \mathrm{div}(B) + (2m-1)\xi(\omega(\xi)) - |B|_g^2 - (2m-1)(\omega(\xi))^2 = 0, \quad (3.28)$$

$$\begin{aligned} &r - r^* + |\nabla\xi|^2 + \frac{1}{2}|\nabla\phi|^2 + (4m-2)\mathrm{div}(B) + 2\xi(\omega(\xi)) - (4m^2-2)|B|_g^2 \\ &- 2(\omega(\xi))^2 = 0. \end{aligned} \quad (3.29)$$

If ξ is Killing, by Lemma 3.1.4, $B = 0$. Hence $\omega = 0$ and this implies $S(\xi,\xi)+|\nabla\xi|_g^2 = 0$ and $r - r^* + |\nabla\xi|_g^2 + \frac{1}{2}|\nabla\phi|_g^2 = 0$ (see [43] for details).

Now, we explore the parallelism of the Chern-Hamilton tensor τ. The covariant derivative of τ is given by $(\nabla_X\tau)Y = \nabla_X\tau Y - \tau\nabla_X Y$. We have the following.

Lemma 3.1.5. *Let (M,ϕ,ξ,η,g) be an l.c. almost cosymplectic manifold. If the tensor τ is parallel, then the identity*

$$(2m-1)(\omega(\xi))^2 + 2\,\mathrm{trace}(h^2) + |B|_g^2 = \xi(\omega(\xi)) - \mathrm{div}(B), \quad (3.30)$$

where $|B|_g^2 = g(B,B)$, holds.

Proof. Let τ be the parallel tensor field. Then, for any X, $Y \in \Gamma(TM)$, $(\nabla_X\tau)Y = 0$. That is, $\nabla_X\tau Y = \tau\nabla_X Y$. Putting $Y = \xi$, one obtains, $\nabla_X\tau\xi = \tau\nabla_X\xi$, together with the following pieces,

$$\begin{aligned} \nabla_X\tau\xi &= \nabla_X(\omega(\xi)\xi - B) = X(\omega(\xi))\xi + \omega(\xi)\nabla_X\xi - \nabla_X B, \\ \tau\nabla_X\xi &= 2h\nabla_X\xi + 2\omega(\xi)\nabla_X\xi - \omega(\nabla_X\xi)\xi - \eta(\nabla_X\xi)B \\ &= 2h^2X + 2\omega(\xi)hX - 2\eta(X)hB + 2\omega(\xi)\nabla_X\xi - \omega(\nabla_X\xi)\xi, \end{aligned}$$

one gets

$$\begin{aligned} X(\omega(\xi))\xi - \omega(\xi)\nabla_X\xi - \nabla_X B = 2h^2X &+ 2\omega(\xi)hX - 2\eta(X)hB \\ &- \omega(\nabla_X\xi)\xi. \end{aligned} \quad (3.31)$$

Using (3.23), one of the properties of the tensor h in (3.24), the relation (3.25) and an adapted frame in TM, $\{E_i\}_i$ with $i = 1, 2, \cdots, (2m+1)$ and contracting the above equation with respect to X, we have,

$$
\begin{aligned}
0 &= \sum_{i=1}^{2m+1} E_i(\omega(\xi))g(\xi, E_i) - \omega(\xi)\sum_{i=1}^{2m+1} g(\nabla_{E_i}\xi, E_i) - \sum_{i=1}^{2m+1} g(\nabla_{E_i}B, E_i) \\
&- 2\sum_{i=1}^{2m+1} g(h^2E_i, E_i) - 2\omega(\xi)\sum_{i=1}^{2m+1} g(hE_i, E_i) + 2\sum_{i=1}^{2m+1} \eta(E_i)g(hB, E_i) \\
&+ \sum_{i=1}^{2m+1} \omega(\nabla_{E_i}\xi)g(\xi, E_i) \\
&= \xi(\omega(\xi)) - (2m-1)(\omega(\xi))^2 - \operatorname{div}(B) - 2\operatorname{trace}(h^2) - |B|_g^2,
\end{aligned}
$$

where $|B|_g^2 = g(B, B)$. That is, $(2m-1)(\omega(\xi))^2 + 2\operatorname{trace}(h^2) + |B|_g^2 = \xi(\omega(\xi)) - \operatorname{div}(B)$, which completes the proof. $\square$

Suppose τ is parallel. Then, putting $X = \xi$ into (3.31), we have

$$\xi(\omega(\xi))\xi - \omega(\xi)\nabla_\xi\xi - \nabla_\xi B = -2hB - \omega(\nabla_\xi\xi)\xi.$$

g-doting this equation with ξ gives $\omega(\nabla_\xi\xi) = 0$. This implies that $B = \omega(\xi)\xi$. Its divergence is $\operatorname{div}(B) = \xi(\omega(\xi)) - 2m(\omega(\xi))^2$ and the relation (3.30) reduces to $\operatorname{trace}(h^2) = 0$. That is $h = 0$. Therefore, we have the following.

Lemma 3.1.6. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. If the Chern-Hamilton tensor τ is parallel. Then, the dual vector field B of ω is proportional to ξ and $h = 0$.*

Theorem 3.1.2. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a compact l.c. almost cosymplectic manifold with $m > 1$ for which the function $\omega(\xi)$ is constant in the direction of ξ. If the tensor τ is parallel, then, the structure vector field is Killing.*

Proof. Let $\int$ be the integral over M with respect to the natural volume element arising from the metric g. Integrating the relation (3.30), using Green's Theorem and $\xi(\omega(\xi)) = 0$, we have $\int_M\{(2m-1)\omega(\xi)^2 + 2\,trace(h^2) + |B|_g^2\} = 0$, which implies $B = 0$ and $h^2 = 0$. Hence $\omega = 0$ and $h = 0$. Putting these into (3.26), we complete the proof. $\square$

Theorem 3.1.3. *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a compact l.c. almost cosymplectic manifold with $m > 1$ for which the function $\omega(\xi)$ is constant in the direction of ξ and the Chern-Hamilton tensor τ is parallel. If $r^* = r$, then M is cosymplectic.*

Proof. By Theorem 3.1.2, $\omega = 0$ and $h = 0$. Taking into account this, integrating the relation (3.29) and using Green's Theorem, we have $\int_M \{r^* - r\} = \int_M \{|\nabla\xi|^2 + \frac{1}{2}|\nabla\phi|^2\}$. Hence, under our assumption, we obtain $\nabla\phi = 0$ and $\nabla\xi = 0$. By Lemma 3.1.3, M is normal and therefore M is cosymplectic. □

The natural example of compact cosymplectic manifold is given by the product of a compact Kähler manifold (V, J, g_V) with the circle S^1. The cosymplectic structure (ϕ, ξ, η, g) on the product manifold $M = V \times S^1$ is defined $\phi = J \circ (\mathrm{pr}_1)_*$, $\xi = \frac{E}{c}$, $\eta = c(\mathrm{pr}_2)^*(\theta)$, $g = (\mathrm{pr}_1)^*(h) + c^2(\mathrm{pr}_2)^*(\theta \otimes \theta)$, where $\mathrm{pr}_1 : M \longrightarrow V$ and $\mathrm{pr}_2 : M \longrightarrow S^1$ are projections of $V \times S^1$ onto the first and the second factor respectively, θ is the length element of S^1, E is its dual vector field and c is a real number, $c \neq 0$ (see [14] for more details).

3.2 Classes of l.c. almost cosymplectic Manifolds

This section deals with some subclasses of the class of l.c. almost cosymplectic manifolds by particularly paying attention to those in which the smooth 1-form ω is proportional to the contact structure η.

Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. Then, by identities (2.7) and (2.11) from [43], on each U_t, we have,

$$(\nabla^t_{\phi_t X}\phi_t)\phi_t Y + (\nabla^t_X \phi_t)Y = -\eta_t(Y)\phi_t \nabla^t_X \xi_t. \tag{3.32}$$

In [43], Olszak observed the following:

Theorem 3.2.1 (Olszak [43]). *For an almost contact metric manifold M, the following conditions are mutually equivalent:*

(a) the manifold is normal l.c. almost cosymplectic,

(b) the manifold is l.c. cosymplectic with $\omega = f\eta$,

(c) $(\nabla_X \phi)Y = f\{g(\phi X, Y)\xi - \eta(Y)\phi X\}$,

where f is function such $df \wedge \eta = 0$.

Now, let us comment on the Theorem 3.2.1. If M is normal, then the structure $(\phi_t, \xi_t, \eta_t, g_t)$ given in (3.4) is normal for each t. In this theorem, the condition "ω is proportional to the contact form η (i.e. $\omega = f\eta$)" plays an important role. This has permitted Olszak to observe that the tensors N_{1t} and N_1 of structures $(\phi_t, \xi_t, \eta_t, g_t)$ and (ϕ, ξ, η, g), respectively, in (3.16), are equal. If M is normal l.c. almost cosymplectic, then $h = 0$ and $B = f\xi$. In fact, he made use of the identity

$$\nabla_{\phi X}\xi = \phi\nabla_X \xi, \quad \forall\, X \in \Gamma(TM), \tag{3.33}$$

which is satisfied by a normal almost contact metric manifold. The function f can specifically be found as follows. From the equation (3.33), we have

$$\omega(\xi)\phi X = \omega(\xi)\phi X + \eta(X)\phi B.$$

This implies that $B = \eta(B)\xi$, i.e., $f = \eta(B) = \omega(\xi)$.

Now, suppose $\omega \neq f\eta$. If M is normal l.c. almost cosymplectic, then $N_{1t} = 0$ and using (3.11), the relation (3.16) reduces to $[\phi, \phi] = 0$. This leads to

$$0 = [\phi, \phi](X, \xi) = \phi^2[X, \xi] - \phi[\phi X, \xi] = -\nabla_X \xi - \phi(\nabla_{\phi X}\xi) + \phi(\nabla_\xi \phi)X. \quad (3.34)$$

That is

$$\phi(\nabla_{\phi X}\xi) = -\nabla_X \xi + \phi(\nabla_\xi \phi)X.$$

Applying ϕ to this and using the second equation in (3.21), one obtains $\nabla_{\phi X}\xi = \phi\nabla_X \xi + \eta(X)\phi B$. This generalizes the relations given in (3.33), and together with (3.22), we obtain $hX = 0$. The latter confirms what Olszak said in [43, p. 76], that is, $h = 0$ if M is l.c. cosymplectic, that is if M is normal l.c. almost cosymplectic.

Can an l.c. almost cosymplectic admit a 1-form ω that is proportional to η? The answer is affirmative. Next we list some cases where this occurs.

First of all we start with a remark. Let $[X, Y]_t = \nabla^t_X Y - \nabla^t_Y X$ on each U_t. Using (3.6), one obtains

$$[X, Y]_t = \nabla^t_X Y - \nabla^t_Y X = \nabla_X Y - \nabla_Y X = [X, Y].$$

This means that, on each U_t, $[\ ,\ ]_t = [\ ,\ ]$.

Let $\widetilde{h}$ be the tensor field associated to the structure (ϕ, ξ, η, g) and defined in [8, page 84] by

$$\widetilde{h} = \frac{1}{2}\mathcal{L}_\xi \phi. \quad (3.35)$$

Its $(\phi_t, \xi_t, \eta_t, g_t)$-structure associated tensor field is denoted by $\widetilde{h}^t$ and given, on each U_t, by

$$\widetilde{h}^t X = \frac{1}{2}(\mathcal{L}_{\xi_t}\phi)X.$$

Then, we have

$$2\widetilde{h}^t X = (\mathcal{L}_{\xi_t}\phi)X = [\xi_t, \phi X]_t - \phi[\xi_t, X]_t. \quad (3.36)$$

A direct calculation of the right-hand side of (3.36) gives

$$\begin{aligned} [\xi_t, \phi X]_t - \phi[\xi_t, X]_t &= \nabla^t_{\xi_t}\phi X - \nabla^t_{\phi X}\xi_t - \phi\nabla^t_{\xi_t}X + \phi\nabla^t_X \xi_t \\ &= (\nabla^t_{\xi_t}\phi)X + 2\exp(\sigma_t)\phi h X. \end{aligned} \quad (3.37)$$

From (3.36) and (3.37), we obtain

$$\widetilde{h}^t X = \frac{1}{2}(\nabla^t_{\xi_t}\phi)X + \exp(\sigma_t)\phi h X. \quad (3.38)$$

By the definition of l.c. almost cosymplectic, the structure $(\phi_t, \xi_t, \eta_t, g_t)$ satisfies the relations (2.10) and (2.11) given in [44]. That is,

$$\nabla^t_{\xi_t}\phi = 0 \text{ and } \nabla^t_{\phi X}\xi_t = -\phi\nabla^t_X\xi_t.$$

The relation (3.38) becomes $\widetilde{h}^t X = \exp(\sigma_t)\phi h X$. It is easy to check that $\widetilde{h}^t$ inherits the properties of h, that is, it is a symmetric operator and satisfies

$$\widetilde{h}^t\phi + \phi\widetilde{h}^t = 0, \;\; \widetilde{h}^t\xi_t = 0 \text{ and } \operatorname{trace}(\widetilde{h}^t) = 0.$$

A direct calculation of (3.36) gives

$$\begin{aligned} 2\widetilde{h}^t X &= \xi_t(\phi X) - \phi X(\xi_t) - \phi(\xi_t(X) - X(\xi_t)) \\ &= -\omega(\phi X)\xi_t + 2\exp(\sigma_t)\widetilde{h}X. \end{aligned} \tag{3.39}$$

From (3.38) and (3.39), one obtains $\widetilde{h}X = \frac{1}{2}\omega(\phi X)\xi + \phi h X$. It is easy to see that $trace(\widetilde{h}) = 0$, $\widetilde{h}\xi = 0$ and

$$\widetilde{h}\phi X + \phi\widetilde{h}X = \frac{1}{2}\{\omega(\xi)\eta(X) - \omega(X)\}\xi, \tag{3.40}$$

$$g(\widetilde{h}X, Y) - g(X, \widetilde{h}Y) = \frac{1}{2}\{\omega(\phi X)\eta(Y) - \omega(\phi Y)\eta(X)\}.$$

Lemma 3.2.1. *On an almost contact metric manifold M, $\widetilde{h}$ is not a symmetric operator,*

$$\nabla_X\xi = \omega(\xi)X - \eta(X)B + \phi\widetilde{h}X,$$

for any vector field X on M, $\widetilde{h}$ does not anticommute with ϕ, and $trace(\widetilde{h}) = 0$.

We have the following.

Lemma 3.2.2. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. Then $\widetilde{h}\phi + \phi\widetilde{h} = 0$ and $\widetilde{h}$ is a symmetric operator if and only if there exists a smooth function f on M such that $\omega = f\eta$ with $df \wedge \eta = 0$ and $\widetilde{h} = \phi h$.*

Another condition in which an l.c. almost cosymplectic manifold admits a 1-form ω that is proportional to η is as follows.

Let $D := \ker\eta$ be the contact distribution and $D^\perp$ be the distribution spanned the structure vector field ξ. Then, we have the following decomposition

$$TM = D \oplus D^\perp, \tag{3.41}$$

where $\oplus$ denotes the orthogonal direct sum. By the decomposition (3.41), any $X \in \Gamma(TM)$ is written as

$$X = QX + Q^\perp X, \tag{3.42}$$

where Q and $Q^\perp$ are the projection morphisms of TM into D and $D^\perp$, respectively. Here, it is easy to see that $Q^\perp X = \eta(X)\xi$ and $X = QX + \eta(X)\xi$.

Lemma 3.2.3. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. Then the contact distribution D defines on M a foliation $\mathcal{F}$ of codimension 1.*

Proof. Let X, $Y \in \Gamma(D)$. Then, $\eta(X) = \eta(Y) = 0$ and

$$\eta([X, Y]) = -2(\omega \wedge \eta)(X, Y) = \omega(Y)\eta(X) - \omega(X)\eta(Y) = 0.$$

This means that $[X, Y] \in \Gamma(D)$, i.e, the contact distribution D is integrable. □

Let $\mathcal{F}$ be a foliation on an l.c. almost cosymplectic manifold (M, ϕ, ξ, η, g) of codimension 1. The metric g is said to be *bundle-like* for the foliation $\mathcal{F}$ if the induced metric on the transversal distribution $D^\perp$ is parallel with respect to the intrinsic connection on $D^\perp$. This is true if and only if the Levi-Civita connection ∇ of (M, ϕ, ξ, η, g) satisfies (see [7] and [47] for more details):

$$g(\nabla_{Q^\perp Y} QX, Q^\perp Z) + g(\nabla_{Q^\perp Z} QX, Q^\perp Y) = 0, \tag{3.43}$$

for any X, Y, $Z \in \Gamma(TM)$. If for a given foliation $\mathcal{F}$, the Riemannian metric g on M is bundle-like for $\mathcal{F}$, then we say that $\mathcal{F}$ is a *Riemannian foliation on* (M, ϕ, ξ, η, g).

Let $\mathcal{F}^\perp$ be the orthogonal complementary foliation generated by ξ. Now we provide necessary and sufficient conditions for the metric on an l.c. almost cosymplectic manifold to be bundle-like for foliations $\mathcal{F}$ and $\mathcal{F}^\perp$.

Theorem 3.2.2. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold and let $\mathcal{F}$ be a foliation on M of codimension 1. Then the following assertions are equivalent:*

(i) The metric g on M is bundle-like for the foliation $\mathcal{F}$.

(ii) The dual vector field B of ω has a no components along D.

Proof. Using (3.23), for any $X, Y, Z \in \Gamma(TM)$, we have $Q^\perp Y = \eta(Y)\xi$, $Q^\perp Z = \eta(Z)\xi$ and the left-hand side of (3.43) gives

$$g(\nabla_{Q^\perp Y} QX, Q^\perp Z) + g(\nabla_{Q^\perp Z} QX, Q^\perp Y) = 2\eta(Y)\eta(Z)\omega(QX),$$

for which the equivalence follows. □

As an example, we have the following.

Example 3.2.1. We consider the 5-dimensional manifold $\widetilde{M}^5 = \{p \in \mathbb{R}^5 | z \neq 0\}$, where $p = (x_1, x_2, y_1, y_2, z)$ are the standard coordinates in $\mathbb{R}^5$. The vector fields,

$$e_i = z\frac{\partial}{\partial x_i}, \quad \varepsilon_i = \frac{1}{z^3}\frac{\partial}{\partial y_i}, \quad \xi = \frac{\partial}{\partial z}, \quad \text{for } \ i = 1, 2,$$

are linearly independent at each point of $\widetilde{M}^5$. Let g be the Riemannian metric on $\widetilde{M}^5$ defined by $g(e_i, e_j) = g(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, $g(e_i, \varepsilon_i) = 0$ and $g(\xi, \xi) = 1$. That is, the form of the metric becomes

$$g = \frac{1}{z^2}(dx_1^2 + dx_2^2) + z^6(dy_1^2 + dy_2^2) + dz^2.$$

Let η be the 1-form on $\widetilde{M}^5$ defined by $\eta = dz$. Then, $d\eta = 0$. Let ϕ be the $(1,1)$-tensor field defined by, $\phi e_1 = \varepsilon_1$, $\phi e_2 = -\varepsilon_2$, $\phi\varepsilon_2 = e_2$, $\phi\varepsilon_1 = -e_1$, $\phi\xi = 0$. By linearity of ϕ and g, (4.1) and (4.2) are satisfied on $\widetilde{M}^5$. Thus, (ϕ, ξ, η, g) defines an almost contact metric structure on $\widetilde{M}^5$. The non-zero component of the 2-form Φ is

$$\Phi(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}) = -z^2,$$

and we have $\Phi = -z^2 dx_1 \wedge dy_1$. Its differential gives

$$d\Phi = -2z dx_1 \wedge dy_1 \wedge dz = -2z^2(\frac{1}{z}dz) \wedge dx_1 \wedge dy_1.$$

Letting $\omega = \frac{1}{z}dz = \frac{1}{z}\eta$, we have, $d\omega = 0$ and $d\Phi = 2\omega \wedge \Phi$. It is easy to see that $d\omega = 0$ and the dual vector field B is given by $B = \frac{1}{z}\xi$. That is, $\omega(\xi) = \frac{1}{z}$. Let us consider the open neighborhood U of $\widetilde{M}^5$ given by $U = \{p \in M^5 | z > 0\}$, and there exists a differentiable function σ on U such that $\omega = d\sigma$, where $\sigma = \ln(z)$. Again, by Vaisman's characterization above-mentioned, $(\widetilde{M}^5, \phi, \xi, \eta, g)$ is an l.c. almost cosymplectic manifold with $\omega = \frac{1}{z}\eta$. Let ∇ be the Levi-Civita connection with respect to the metric g. Then, the non-zero Lie brackets are $[e_i, \xi] = -\frac{1}{z}e_i$ and $[\varepsilon_i, \xi] = \frac{3}{z}\varepsilon_i$, for $i = 1, 2$, and the action of ∇ on $\{e_i, \varepsilon, \xi\}$ is given by $\nabla_\xi\xi = 0$, $\nabla_{e_i}\xi = -\frac{1}{z}e_i$ and $\nabla_{\varepsilon_i}\xi = \frac{3}{z}\varepsilon_i$, for $i = 1, 2$. The components of the tensor h defined in (3.22) are given by $h\xi = 0$, $he_i = -\frac{2}{z}e_i$, $h\varepsilon_i = \frac{2}{z}\varepsilon_i$, for $i = 1, 2$. Let D be the contact distribution of $\widetilde{M}^5$. Then $D = \ker\eta$. Since we need vector fields which are orthogonal to ξ, and $g(e_i, \xi) = g(\varepsilon_i, \xi) = 0$, for $i = 1, 2$, we have $D = \mathrm{Span}\{e_i, \varepsilon_i\}_{i=1,2}$. Let X and Y be two vector fields of D. Then $X = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_1^*\varepsilon_1 + \alpha_2^*\varepsilon_2$ and $Y = \beta_1 e_1 + \beta_2 e_2 + \beta_1^*\varepsilon_1 + \beta_2^*\varepsilon_2$. We have $g(\nabla_X\xi, Y) = g(\nabla_Y\xi, X) = -\frac{1}{z}(\alpha_1\beta_1 + \alpha_1\beta_1) + \frac{3}{z}(\alpha_1^*\beta_1^* + \alpha_2^*\beta_2^*)$. Since $\eta([X, Y]) = g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X) = 0$, that is, $[X, Y] \in \Gamma(D)$. This means that the distribution D is integrable and therefore admits a foliation $\mathcal{F}$. Since $g(\nabla_\xi X, \xi) = -g(X, \nabla_\xi\xi) = 0$, then the metric g on $\widetilde{M}^5$ is bundle-like for the foliation $\mathcal{F}$.

For any X, Y, $Z \in \Gamma(TM)$, using (3.26) and the fact that h is symmetric and

$$g(\nabla_{QY}Q^\perp X, QZ) = \eta(X)\{g(hQY, QZ) + \omega(\xi)g(QY, QZ)\}, \tag{3.44}$$

we have

$$\begin{aligned} g(\nabla_{QY}Q^\perp X, QZ) + g(\nabla_{QZ}Q^\perp X, QY) = 2\eta(X)\{g(hQY, QZ) \\ + 2\omega(\xi)g(QY, QZ). \end{aligned} \tag{3.45}$$

Using the Lie derivative in (3.26), one obtains

$$g(\nabla_{QY}Q^{\perp}X, QZ) + g(\nabla_{QZ}Q^{\perp}X, QY) = 2\eta(X)(\mathcal{L}_{\xi}g)(QY, QZ). \tag{3.46}$$

We have therefore the following.

Theorem 3.2.3. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold and let $\mathcal{F}$ be a foliation on M of codimension 1. Then the following assertions are equivalent:*

(a) The metric g on M is bundle-like for the canonical totally real foliation $\mathcal{F}^{\perp}$.

(b) The structure vector field ξ is D-Killing (i.e. $D^{\perp}$ is D-Killing distribution).

Let M' be a leaf of the distribution D. Since M' is a submanifold of M and for any X, $Y \in \Gamma(TM')$, we have

$$\nabla_X Y = \nabla'_X Y + \alpha(X, Y), \tag{3.47}$$

$$\nabla_X \xi = -A_{\xi}X + \nabla'^{\perp}_X \xi, \tag{3.48}$$

where ∇' and α are the Levi-Civita connection and the second fundamental form of M', respectively. On the other hand, since ξ is a unit normal vector field, we have $g(\nabla_X \xi, \xi) = 0$, hence $\nabla'^{\perp}_X \xi = 0$, for any $X \in \Gamma(TM')$. Therefore, the Weingartem formula (3.48) becomes

$$\nabla_X \xi = -A_{\xi}X.$$

Proposition 3.2.1. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. Then, integral manifolds of the distribution D in (3.41) are l.c. almost Kähler manifolds with mean curvature vector field $H' = -\omega(\xi)\xi$. They are totally umbilical submanifolds of M if and only if the operator h vanishes.*

Proof. Let M' be an integral manifold of D. The tensor fields ϕ_t and g_t induce an almost complex structure $J_t = J$ and a Hermitian metric g'_t on M'. Then, for any X, $Y \in \Gamma(TM')$, we have $\Phi'_t(X, Y) = g'_t(X, J_tY) = g_t(X, \phi_tY) = \Phi_t(X, Y)$ and $d\Phi'_t = (d\Phi_t)_{|M'} = 0$, so M' is an l.c. almost Kähler. Using (3.47), the second fundamental form of M' gives

$$\alpha(X, Y) = g(A_{\xi}X, Y)\xi = -g(hX, Y)\xi - \omega(\xi)g(X, Y)\xi. \tag{3.49}$$

Fixing a local orthonormal frame $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n\}$ in TM' and applying the properties on h, one has,

$$H = \frac{1}{\mathrm{rank}(D)}\{\sum_{i=1}^{n}\alpha(e_i, e_i) + \sum_{i=1}^{n}\alpha(\phi e_i, \phi e_i)\} = -\omega(\xi)\xi.$$

The last assertion follows and this completes the proof. □

This result can be extended to the foliation $\mathcal{F}^{\perp}$. That is, if $h = \omega(\xi) = 0$, $g(\nabla_X Y, \xi) = 0$. This means that the foliation $\mathcal{F}^{\perp}$ is Riemannian. Therefore, $h = 0$, the leaves of $\mathcal{F}$ are totally geodesic if and only if the orthogonal complementary foliation $\mathcal{F}^{\perp}$ generated by ξ is Riemannian.

On each $U_t \cap M'$, the Gauss and Weingartem formulas are given by

$$\nabla^t_X Y = \nabla'^t_X Y + \alpha^t(X, Y), \tag{3.50}$$

$$\nabla^t_X \xi_t = -A_{\xi_t} X, \tag{3.51}$$

where $g_t(\alpha^t(X, Y), \xi_t) = g_t(A_{\xi_t} X, Y)$, that is, $\alpha^t(X, Y) = g_t(A_{\xi_t} X, Y)\xi_t$. However,

$$\alpha^t(X, Y) = g_t(A_{\xi_t} X, Y)\xi_t = g(A_\xi X, Y)\xi = \alpha(X, Y). \tag{3.52}$$

For any X, $Y \in \Gamma(TM')$, and using (3.6) and (3.47), we have

$$\begin{aligned}(\nabla'^t_X J)Y &= \nabla'^t_X JY - J(\nabla'^t_X Y) = \nabla^t_X \phi Y - \alpha(X, \phi Y) - \phi(\nabla^t_X Y)\\ &= (\nabla_X \phi)Y - \omega(\phi Y)X + \omega(Y)\phi X + g(X, \phi Y)B\\ &- g(X, Y)\phi B - g(A_\xi X, \phi Y)\xi.\end{aligned} \tag{3.53}$$

If the integral manifold M' is l.c. Kähler, then, $(\nabla'^t_X J)Y = 0$ and we have

$$\begin{aligned}(\nabla_X \phi)Y &= \omega(\phi Y)X - \omega(Y)\phi X - g(X, \phi Y)B + g(X, Y)\phi B\\ &+ g(A_\xi X, \phi Y)\xi,\end{aligned} \tag{3.54}$$

for any X, $Y \in \Gamma(TM')$. Therefore, if the foliation $\mathcal{F}$ has locally conformal Kähler leaves, then for any X, $Y \in \Gamma(TM)$, the vector fields $X - \eta(X)\xi$, $Y - \eta(Y)\xi$ and $B - \eta(B)\xi$ belong to D and using (3.21) and (3.23), we have

$$\begin{aligned}(\nabla_{X-\eta(X)\xi}\phi)(Y - \eta(Y)\xi) &= (\nabla_X \phi)Y - \eta(Y)\phi A_\xi X - \eta(X)\omega(\phi Y)\xi,\\ g(A_\xi(X - \eta(X)\xi), \phi Y) &= g(A_\xi X, \phi Y) - \eta(X)\omega(\phi Y).\end{aligned}$$

Putting these pieces into (3.54) and taking into account the following relations

$$\begin{aligned}\omega(\phi(Y - \eta(Y)\xi))(X - \eta(X)\xi) &= \omega(\phi Y)X - \eta(X)\omega(\phi Y)\xi,\\ \omega(Y - \eta(Y)\xi)\phi(X - \eta(X)\xi) &= \omega(Y)\phi X - \eta(Y)\omega(\xi)\phi X,\\ g(X - \eta(X)\xi, \phi(Y - \eta(Y)\xi)) &= g(X, \phi Y),\\ g(X - \eta(X)\xi, Y - \eta(Y)\xi) &= g(X, Y) - \eta(X)\eta(Y),\end{aligned}$$

one obtains,

$$\begin{aligned}(\nabla_X \phi)Y = &-g(\phi A_\xi X, Y)\xi + \eta(Y)\phi A_\xi X + \omega(\phi Y)X + \{\eta(Y)\omega(\xi) - \omega(Y)\}\phi X\\ &- g(X, \phi Y)\{B - \omega(\xi)\xi\} - \eta(X)\omega(\phi Y)\xi.\end{aligned}$$

Proposition 3.2.2. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. Then the distribution D in (3.41) has locally conformal Kähler leaves if and only if*

$$\begin{aligned}(\nabla_X \phi)Y = &-g(\phi A_\xi X, Y)\xi + \eta(Y)\phi A_\xi X + \omega(\phi Y)X + \{\eta(Y)\omega(\xi) - \omega(Y)\}\phi X \\ &- g(X, \phi Y)\{B - \omega(\xi)\xi\} - \eta(X)\omega(\phi Y)\xi, \end{aligned} \tag{3.55}$$

for any X, $Y \in \Gamma(TM)$.

When the differential 1-form ω is reduced to $\omega = f\eta$, where f is a function such that $df \wedge \eta = 0$, then M becomes an almost f-cosymplectic manifold [1] and the relation (3.55) for any leaves of M to be Kählerian becomes

$$(\nabla_X \phi)Y = -g(\phi A_\xi X, Y)\xi + \eta(Y)\phi A_\xi X,$$

for any X, $Y \in \Gamma(TM)$. The latter relation is exactly the one find by Aktan *et al* in [1, Proposition 6]. We have the following.

Theorem 3.2.4. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold and let $\mathcal{F}$ be a foliation on M of codimension 1. If the metric g on M is bundle-like for the foliation $\mathcal{F}$, then the leaves of $\mathcal{F}$ are almost Kähler.*

Moreover, if M is normal, then the leaves of $\mathcal{F}$ are Kähler and totally umbilical.

Proof. Let $\mathcal{F}$ be a foliation on an l.c. almost cosymplectic manifold M of codimension 1. If the metric g on M is bundle-like for the foliation $\mathcal{F}$, then, by Theorem 3.2.2, the dual vector field B of ω is proportional to ξ, that is, $B = \omega(\xi)\xi$. This means that M becomes an almost f-cosymplectic manifold with $f = \omega(\xi)$ and the leaves of $\mathcal{F}$ are almost Kähler.

If the structure is normal, then $h = 0$ and the tensor N_1 in (3.11) vanishes. By the equality

$$\begin{aligned}N_1(X, Y) = [\phi, \phi](X, Y) &= \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \\ &= [J, J](X, Y),\end{aligned}$$

and by Proposition 3.2.1, we complete the proof. □

We finally have the following result.

Theorem 3.2.5. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold and let $\mathcal{F}$ be a foliation on M of codimension 1. If the Chern-Hamilton tensor τ is parallel. Then,*

(i) (M, ϕ, ξ, η, g) *is an almost $\omega(\xi)$- cosymplectic manifold.*

(ii) (M, ϕ, ξ, η, g) *is a $\omega(\xi)$- cosymplectic manifold if and only if leaves of $\mathcal{F}$ are Kähler.*

(iii) The foliation $\mathcal{F}$ is Riemannian and totally geodesic.

(iv) The foliation $\mathcal{F}^{\perp}$ is Riemannian

As an example, we have the following.

Example 3.2.2. Let $\widetilde{M}^5$ be the 5-dimensional manifold defined in Example 3.2.1. That is $\widetilde{M}^5 = \{p \in \mathbb{R}^5 | z \neq 0\}$, where $p = (x_1, x_2, y_1, y_2, z)$ are the standard coordinates in $\mathbb{R}^5$. The vector fields,

$$e_i = z\frac{\partial}{\partial x_i}, \quad \vartheta_i = z\frac{\partial}{\partial y_i}, \quad \xi = \frac{\partial}{\partial z}, \quad \text{for } i = 1, 2,$$

are linearly independent at each point of $\widetilde{M}^5$. Let g be the Riemannian metric on $\widetilde{M}^5$ defined by $g(e_i, e_j) = g(\vartheta_i, \vartheta_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, $g(e_i, \vartheta_i) = 0$ and $g(\xi, \xi) = 1$. That is, the form of the metric becomes

$$g = \frac{1}{z^2}(dx_1^2 + dx_2^2 + dy_1^2 + dy_2^2) + dz^2.$$

Let η be the 1-form on $\widetilde{M}^5$ defined by $\eta = dz$. Obviously, $d\eta = 0$. Let ϕ be the $(1,1)$-tensor field defined by, $\phi e_1 = \vartheta_1$, $\phi e_2 = -\vartheta_2$, $\phi\vartheta_2 = e_2$, $\phi\vartheta_1 = -e_1$, $\phi\xi = 0$. By linearity of ϕ and g, the relations (4.1) and (4.2) are satisfied on $\widetilde{M}^5$. Thus, (ϕ, ξ, η, g) defines an almost contact metric structure on $\widetilde{M}^5$. The non-zero component of the fundamental 2-form Φ is $\Phi(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}) = -\frac{1}{z^2}$, and we have $\Phi = -\frac{1}{z^2}dx_1 \wedge dy_1$. Its differential gives

$$d\Phi = 2\frac{1}{z^3}dx_1 \wedge dy_1 \wedge dz = -2\frac{1}{z^2}dx_1 \wedge dy_1 \wedge (-\frac{1}{z}dz). \tag{3.56}$$

Letting $\omega = -\frac{1}{z}dz = -\frac{1}{z}\eta$, we have, $d\omega = 0$ and $d\Phi = 2\omega \wedge \Phi$. It is easy to see that $d\omega = 0$ and the dual vector field B is given by $B = -\frac{1}{z}\xi$. That is, $\omega(\xi) = -\frac{1}{z}$. Let us consider the open neighborhood U of $\widetilde{M}^5$ given by $U = \{p \in M^5 | z > 0\}$, and there exists a differentiable function σ on U such that $\omega = d\sigma$, where $\sigma = -\ln(z)$. By Vaisman's characterization above-mentioned, $(\widetilde{M}^5, \phi, \xi, \eta, g)$ is an l.c. almost cosymplectic manifold with $\omega = -\frac{1}{z}\eta$. Let ∇ be the Levi-Civita connection with respect to the metric g. Then, the non-zero Lie brackets are $[e_i, \xi] = -\frac{1}{z}e_i$ and $[\vartheta_i, \xi] = -\frac{1}{z}\varepsilon_i$, for $i = 1, 2$, and the action of ∇ on the elements of the basic $\{e_i, \varepsilon_i, \xi\}$ is given by

$$\nabla_{e_i}e_i = -\frac{1}{z}\xi, \quad \nabla_{\vartheta_i}\vartheta_i = -\frac{1}{z}\xi, \quad \nabla_{\xi}\xi = 0, \quad \nabla_{e_i}\xi = -\frac{1}{z}e_i \text{ and } \nabla_{\varepsilon_i}\xi = -\frac{1}{z}\varepsilon_i,$$

for $i = 1, 2$. The components of the $(1,1)$-tensor h defined in (3.22) are given by $h\xi = 0$, $he_i = h\vartheta_i = 0$, for $i = 1, 2$. It can be noted that Nijenhuis torsion tensor of ϕ defined in (3.56) is zero. By Theorem 3.2.1, $(\widetilde{M}^5, \phi, \xi, \eta, g)$ is an l.c. cosymplectic manifold with $\omega = -\frac{1}{z}\eta$. Let D be the contact distribution of $\widetilde{M}^5$. Then $D = \ker\eta$. Since we need vector fields which are orthogonal to ξ, and $g(e_i, \xi) = g(\vartheta_i, \xi) = 0$, for $i = 1, 2$, we have $D = \mathrm{Span}\{e_i, \vartheta_i\}_{i=1,2}$. It is easy to see that D is integrable and $(\nabla_{e_i}\phi)e_i = (\nabla_{\vartheta_i}\phi)\vartheta_i = (\nabla_{e_i}\phi)\vartheta_i = 0$. This means that the leaves of D are Kähler and totally umbilical with mean curvature vector field $H = \frac{1}{z}\xi$. Since $g(\nabla_\xi X, \xi) = -g(X, \nabla_\xi\xi) = 0$, then the metric g on $\widetilde{M}^5$ is bundle-like for the foliation $\mathcal{F}$. Therefore $\mathcal{F}$ is Riemannian and totally geodesic, and $\mathcal{F}^\perp$ is Riemannian.

Indefinite locally conformal almost Cosymplectic manifolds

This chapter is focused on foliations of indefinite l.c. almost cosymplectic manifolds. In the first section we define the canonical foliations $\mathcal{F}$ which generally derived from the Pfaffian equation $\omega = 0$. The rest of the chapter is, respectively devoted to the geometry of non-tangential leaves of foliation $\mathcal{F}$ and the higher order geodesibility of leaves of $\mathcal{F}$ via the Newton transformations.

4.1 Indefinite l.c. almost Cosymplectic manifolds

Let M be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact metric structure (ϕ, ξ, η), where ϕ is tensor field of type $(1,1)$ on M, a vector field ξ and a 1-form η satisfying the following relations

$$\phi^2 = -\mathbb{I} + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta\phi = 0. \tag{4.1}$$

Then the structure $(\overline{\phi}, \xi, \eta, g)$ is called an *indefinite almost contact metric structure* on M if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on M and g is a semi-Riemannian metric on M such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{4.2}$$

for any vector fields X and Y on M. In this case, we call M an *indefinite almost contact manifold.*

As an example of an indefinite l. c. almost cosymplectic manifold, we have the following.

Example 4.1.1. Consider M^9 a 9-dimensional semi-Riemannian manifold

$$M^9 = \{p \in \mathbb{R}^9 | x_1 > 1, y_1 > 1\},$$

where $p = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z)$ are the standard coordinates in $\mathbb{R}^9$. The vectors fields

$$X_1 = e^{-z-x_1y_1}\{\partial x_1 + \partial y_1\},\quad X_2 = e^{-z-x_1y_1}\{\partial x_2 + \partial y_2\},\quad X_3 = e^{-z-x_1y_1}\partial x_3,$$
$$X_4 = e^{-z-x_1y_1}\partial x_4,\quad Y_1 = e^{-z-x_1y_1}\{\partial x_1 - \partial y_1\},\quad Y_2 = e^{-z-x_1y_1}\{\partial x_2 - \partial y_2\},$$
$$Y_3 = -e^{-z-x_1y_1}\partial y_3,\quad Y_4 = -e^{-z-x_1y_1}\partial y_4,\quad Z = e^{-z-x_1y_1}\partial z,$$

are linearly independent at each point of M^9. Let g be the indefinite metric on M^9 defined by $g(X_i, X_i) = g(Y_i, Y_i) = -1$, for $i = 1, 2$ and $g(X_i, X_i) = g(Y_i, Y_i) = 1$, for $i = 3, 4$, $g(X_i, Y_j) = 0$ and $g(\xi, \xi) = 1$. Let η be the 1-form on M^9 defined by $\eta = e^{z+x_1y_1}dz$, then the structure vector field is $\xi = e^{-z-x_1y_1}\partial z$. Let ϕ be the $(1, 1)$-tensor field defined by,

$$\phi X_1 = -Y_1,\quad \phi Y_1 = X_1,\quad \phi X_2 = -Y_2,\quad \phi Y_2 = X_2,\quad \phi X_3 = Y_3,$$
$$\phi Y_3 = -X_3,\quad \phi X_4 = Y_4,\quad \phi Y_4 = -X_4,\quad \phi\xi = 0.$$

By linearity of ϕ and g, the relations (4.2) are satisfied thus, (ϕ, ξ, η, g) defines an almost contact metric structure on M^9. We have also

$$d\eta = e^{z+x_1y_1}\{y_1dx_1 \wedge dz + x_1dy_1 \wedge dz\}.$$

By straightforward calculations we obtain

$$\Phi = e^{2(z+x_1y_1)}\{\frac{1}{2}dx_1 \wedge dy_1 + \frac{1}{2}dx_2 \wedge dy_2 + dx_3 \wedge dy_3 + dx_4 \wedge dy_4\}.$$

By letting $\omega = y_1dx_1 + x_1dy_1 + dz$, we have $d\eta = \omega \wedge \eta$ and $d\Phi = 2\omega \wedge \Phi$ and $d\omega = 0$, which show that $(M^9, \phi, \xi, \eta, g)$ is an l.c. almost cosymplectic manifold with the dual vector field B of ω given by $B = e^{-2(z+x_1y_1)}\{\frac{y_1}{2}\partial x_1 + x_1\partial y_1 + \partial z\}$.

Let M be a $(2n + 1)$-dimensional indefinite l. c. almost cosymplectic manifold of index q, $0 < q < 2n + 1$. Let us set $c = g(B, B) \in \mathcal{C}^\infty(M)$ and $\mathrm{Sign}(B) = \{x \in M : B_x = 0\}$. Note that c and $\mathrm{Sing}(B)$ determine the causal character of B, so it may be $c = 0$ and $\mathrm{Sing}(B) = \emptyset$ when B is null.

From now on, the characteristic 1-form ω given in (3.5) does not vanish, *unless otherwise started.*

Since M is an l.c. almost cosymplectic, it admits a canonical foliation $\mathcal{F}$ of codimension r whose leaves are the maximal connected integral manifolds of the Pfaffian equation $\omega = 0$ (see [10] for details and references therein).

Let $(T\mathcal{F})^c$ be the complementary distribution to $T\mathcal{F}$ in TM. Then, its dimension is r.

First, assume that $c = g(B, B) \neq 0$. Then, it is easy to see that the index of each leaf L of $\mathcal{F}$ is given by

$$\mathrm{ind}(L) = q - s,$$

where $s = \text{ind}((T\mathcal{F})^c)$ with $0 \leq s \leq r$.

Now we assume that $c = 0$. Then $B \in T\mathcal{F}$. Set

$$\text{Rad}(T\mathcal{F})_x = (T\mathcal{F})_x \cap (T\mathcal{F}^\perp)_x, \quad x \in M.$$

It is easy to see that $B \in \text{Rad}(T\mathcal{F})$. Let $S(T\mathcal{F})$ be a distribution on M^9 such that

$$T\mathcal{F} = S(T\mathcal{F}) \perp \text{Rad}(T\mathcal{F}). \tag{4.3}$$

The screen distribution $S(T\mathcal{F})$ is seen as the complementary bundle of $\text{Rad}(T\mathcal{F})$ in $T\mathcal{F}$. It is then a rank $(n - p - \dim_{\mathbb{R}} \text{Rad}(T\mathcal{F}))$ non-degenerate distribution over $\mathcal{F}$. In fact, there are infinitely many possibilities of choices for such a distribution provided the foliation $\mathcal{F}$ is paracompact, but each of them is canonically isomorphic to the factor vector bundle $T\mathcal{F}/\text{Rad}(T\mathcal{F})$.

Case 1: If $\omega(\xi) = 0$, i.e., $\xi \in T\mathcal{F}$ and using (4.2), one has $g(\phi B, \phi B) = g(B, B) - \omega(\xi)^2 = 0$, and since $g(\phi B, B) = 0$, the vector field ϕB belongs to $T\mathcal{F}$ and is also null and it may be in the radical distribution or not.

As the structure vector field ξ belongs to $T\mathcal{F}$, we assume that $\xi \in S(T\mathcal{F})$.

If $r = 1$, then by Proposition 2.2 in [18] $\dim_{\mathbb{R}}(\text{Rad}(T\mathcal{F}))_x = 1$, for any $x \in M$. Let $\mathbb{R}B$ be the line bundle spanned by the vector field B. Since $\texttt{Sign}(B) = \emptyset$, we have $\text{Rad}(T\mathcal{F}) = \mathbb{R}B$. Also $\phi B \notin \text{Rad}(T\mathcal{F})$ which means that $\phi B \in S(T\mathcal{F})$. Therefore L is a null hypersurface immersed in (M, g). Let $S(T\mathcal{F})^\perp$ be an orthogonal complementary vector bundle to $S(T\mathcal{F})$ in $TM|_{\mathcal{F}}$. Consider a complementary vector bundle F of $\mathbb{R}B$ in $S(T\mathcal{F})^\perp$ and take $V \in \Gamma(F|_{\mathcal{U}})$ a locally non-zero section defined on the open subset $\mathcal{U} \subset M$. Then $\omega(V) \neq 0$, otherwise $S(T\mathcal{F})^\perp$ would be degenerate at a point of $\mathcal{U}$ (see [18, p. 79] for more details). We define on $\mathcal{U}$ a vector field

$$N_V = \frac{1}{\omega(V)} \left\{ V - \frac{g(V, V)}{2\omega(V)} B \right\}. \tag{4.4}$$

It is easy to see that

$$\omega(N_V) = 1 \ \text{ and } \ g(N_V, N_V) = g(N_V, W) = 0, \tag{4.5}$$

for any $W \in \Gamma(S(T\mathcal{F})|_{\mathcal{U}})$. If we consider another coordinate neighborhood $\mathcal{U}^* \subseteq M$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \emptyset$. As both $\mathbb{R}B$ and F are vector bundles over $\mathcal{F}$ of rank 1, we have $B^* = \beta B$ and $V^* = \gamma V$, where β and γ are non-zero smooth functions on $\mathcal{U} \cap \mathcal{U}^*$. It follows that $N^*_{V^*}$ is related with N_V on $\mathcal{U} \cap \mathcal{U}^*$ by $N^*_{V^*} = (1/\beta) N_V$. Therefore, the vector bundle F induces a vector bundle $\text{tr}(T\mathcal{F})$ of rank 1 over $\mathcal{F}$ such that, locally, the equations in (4.5) are satisfied. Finally, we consider another complementary vector bundle E to $\mathbb{R}B$ in $S(T\mathcal{F})^\perp$ and by using (4.4), for both F and E, we obtain the same $\text{tr}(T\mathcal{F})$. As $g(\phi N_V, N_V) = 0$, we have $\phi N_V \in S(T\mathcal{F})$. From (4.2), we have $g(\phi N_V, \phi B) = 1$. Therefore, $\{\phi\mathbb{R}B \oplus \phi\mathbb{R}N_V\}$ (direct sum but not orthogonal) is a non-degenerate vector subbundle of $S(T\mathcal{F})$ of rank 2. Since $\xi \in S(T\mathcal{F})$ and $g(\phi N_V, \xi) = g(\phi B, \xi) = 0$, there exists a non-degenerate invariant distribution D_0 of rank $2n-4$ such that

$$S(T\mathcal{F}) = \{\phi\mathbb{R}B \oplus \phi\mathbb{R}N_V\} \perp D_0 \perp \mathbb{R}\xi, \tag{4.6}$$

and the tangent space of $\mathcal{F}$ is decomposed as follows:

$$T\mathcal{F} = \{\phi\mathbb{R}B \oplus \phi\mathbb{R}N_V\} \perp D_0 \perp \mathbb{R}\xi \perp \mathbb{R}B. \tag{4.7}$$

If $r > 1$, then the radical $\mathrm{Rad}(T\mathcal{F})$ is of rank p with $1 \leq p < \min\{2n+1-r, r\}$ and L is a p-null submanifold.

Case 2: If $\omega(\xi) \neq 0$, i.e., $\xi \notin T\mathcal{F}$. Therefore, L is a null submanifold immersed in M. This holds even when $\phi B \notin \mathrm{Rad}(T\mathcal{F})$. In this case, ξ takes the form

$$\xi = \xi_{T\mathcal{F}} + \xi_{\mathrm{tr}(T\mathcal{F})},$$

where $\xi_{T\mathcal{F}}$ and $\xi_{\mathrm{tr}(T\mathcal{F})}$ are the tangential and transversal components of ξ in M, respectively. But if $\phi B \in \mathrm{Rad}(T\mathcal{F})$, then $r \geq 2$ and there exists a distribution D_2 of k with $0 \leq k < \min\{2n+1-r, r\}$ in $T\mathcal{F}$ such that

$$\mathrm{Rad}(T\mathcal{F}) = D_1 \oplus D_2, \tag{4.8}$$

where $D_1 = \{B, \phi B\}$. This means D_1 is invariant under ϕ. By Lemma 1.2 given in [18, p. 142], we have the following. Choose a screen transversal bundle $S(T\mathcal{F}^\perp)$, which is semi-Riemannian and complementary to $\mathrm{Rad}(\mathcal{F})$ in $T\mathcal{F}^\perp$. Since, for any local basis $\{E_0 = B, E_1 = \phi B, E_k\}$ of $\mathrm{Rad}(T\mathcal{F})$, there exists a local null frame $\{N_0, N_1 = \phi N_0, N_k\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $S(TM)^\perp$ such that $g(E_i, N_j) = \delta_{ij}$, it follows that there exists a null transversal vector bundle $l\mathrm{tr}(T\mathcal{F})$ locally spanned by $\{N_0, N_1 = \phi N_0, N_k\}$ [18]. Then,

$$\mathrm{tr}(T\mathcal{F}) = l\mathrm{tr}(T\mathcal{F}) \perp S(T\mathcal{F}^\perp), \tag{4.9}$$
$$TM = S(T\mathcal{F}) \perp S(T\mathcal{F}^\perp) \perp \{\mathrm{Rad}(T\mathcal{F}) \oplus l\mathrm{tr}(T\mathcal{F})\}. \tag{4.10}$$

It is easy to check that $\phi D_2 \subseteq S(T\mathcal{F})$. The latter means there exists a subbundle L_2 of rank k in $l\mathrm{tr}(T\mathcal{F})$ such that $\phi L_2 \subseteq S(T\mathcal{F})$. Also there exists a subbundle $\mathcal{S}$ in $S(T\mathcal{F}^\perp)$ such that $\phi\mathcal{S} \subseteq S(T\mathcal{F})$. The bundle $\{\phi D_2 \oplus \phi L_2 \oplus \phi\mathcal{S}\}$ is a subbundle of $S(T\mathcal{F})$ of rank at least 2. Therefore there exists a non-degenerate invariant distribution $\mathcal{D}_0$ of even rank such that

$$S(T\mathcal{F}) = \{\phi D_2 \oplus \phi L_2 \oplus \phi\mathcal{S}\} \perp D_0. \tag{4.11}$$

Thus, in this case, L is a quasi generalized CR-null submanifold immersed in M (see [36] for more details of quasi generalized CR concept). Therefore, we have the following theorem.

Theorem 4.1.1. *Let M be a $(2n+1)$-dimensional indefinite l.c. almost cosymplectic manifold of index q, where $0 < q < 2n+1$ with $\mathtt{Sign}(B) = \emptyset$. Then*

(i) If $c \neq 0$, then the index of each leaf L of $\mathcal{F}$ is given by $\operatorname{ind}(L) = q - s$, where $s = \operatorname{ind}((T\mathcal{F})^c)$ with $0 \leq s \leq r$. Moreover, L is totally geodesic r codimensional semi-Riemannian submanifold of (M, g) if and only if the Lee form ω is parallel.

(ii) If $c = 0$, then each leaf of $\mathcal{F}$ is either a null hypersurface or a quasi generalized CR-null submanifold of (M, g).

The Example 4.1.1 shows that $c = g(B, B) = \frac{1}{2}e^{-2(z+x_1y_1)}\{-2x_1^2 - y_1^2 + 2\}$, which is always different from zero, since $-2x_1^2 - y_1^2 + 2 \neq 0$, for $x_1 > 0$ and $y_1 > 0$. The item (ii) in Theorem 4.1.1 is supported by the following example.

Example 4.1.2. Consider M^7 a 7-dimensional semi-Riemannian manifold

$$M^7 = \{p \in \mathbb{R}^7 | x_1 > 0, y_3 > 0\},$$

where $p = (x_1, x_2, x_3, y_1, y_2, y_3, z)$ are the standard coordinates in $\mathbb{R}^7$. The vectors fields

$$X_1 = \frac{1}{x_1 + y_3}\partial x_1, \quad Y_1 = \frac{1}{x_1 + y_3}\partial y_1, \quad X_2 = \frac{1}{x_1 + y_3}\partial x_2, \quad Y_2 = \frac{1}{x_1 + y_3}\partial y_2,$$
$$X_3 = \frac{1}{x_1 + y_3}\partial x_3, \quad Y_3 = -\frac{1}{x_1 + y_3}\partial y_3, \quad Z = \frac{1}{x_1 + y_3}\partial z,$$

are linearly independent at each point of M^7. Let g be the indefinite metric on M^7 defined by $g(X_i, X_j) = g(Y_i, Y_j) = -\delta_{i,j}$ for any $i, j = 1, 2$, $g(X_3, X_3) = g(Y_3, Y_3) = 1$, $g(\xi, \xi) = 1$, $g(X_l, X_k) = g(Y_l, Y_k) = 0$, for all $l \neq k, l, k = 1, 2, \cdots, 7$. Let η be the 1-form on M^7 defined by $\eta = (x_1 + y_3)dz$ and the structure vector field given by $\xi = \frac{1}{x_1+y_3}\partial z$. Let ϕ be the $(1, 1)$-tensor field defined by, $\phi X_1 = -Y_1$, $\phi Y_1 = X_1$, $\phi X_2 = -Y_2$, $\phi Y_2 = X_2, \phi X_3 = Y_3$, $\phi Y_3 = -X_3$, $\phi X_4 = Y_4$, $\phi Y_4 = -X_4$, $\phi\xi = 0$. By linearity of ϕ and g the quadruplet (ϕ, ξ, η, g) defines an almost contact metric structure on M^7. Take $\sigma = \ln(x_1 + y_3)$. It follows that $\omega = \frac{1}{x_1+y_3}(dx_1 + dy_3)$, then clearly we have $d\eta = \omega \wedge \eta$. The 2-form fundamental is given by

$$\Phi = (x_1 + y_3)^2\{-dx_1 \wedge dy_1 - dx_2 \wedge dy_2 + dx_3 \wedge dy_3\},$$

which satisfies $d\Phi = 2\omega \wedge \Phi$. The Lee vector field (i.e. the dual vector field of ω) is given by $B = \frac{1}{(x_1+y_3)^2}(X_1 + Y_3)$. It follows that $c = g(B, B) = 0$ and thus B is a null vector field. It is easy to see that $\omega(\xi) = 0$ and for $p \in M^7$, the distribution $D_p = \{X \in T_pM^7 : \omega(X) = 0\}$ is spanned by $\{X_2, X_3, Y_1, Y_2, B, \xi\}$. The non-vanishing components of the Lie brackets are $[X_{2,3}, B] = \frac{2}{(x_1+y_3)^4}X_{2,3}$ and $[Y_{1,2}, B] = \frac{2}{(x_1+y_3)^4}Y_{1,2}$, which prove that the distribution D is integrable and therefore admits a foliation $\mathcal{F}$

whose leaves are null hypersurfaces immersed in M^7. In this case the anti-Lee vector field $V = -\phi B = \frac{1}{(x_1+y_3)^2}\{X_3 - Y_1\} \in T\mathcal{F}$. The transversal vector field is given by $N = \frac{1}{2}(x_1 + y_3)^2\{-X_1 + Y_3\}$.

Note that if the ambient space M is an indefinite l.c. cosymplectic manifold, then $h = 0$ and $B = \omega(\xi)\xi$ (see [34] and [43]). In this case the condition $c = 0$ implies $\omega(\xi) = 0$. Therefore, we have the following.

Lemma 4.1.1. *There exist no null hypersurfaces immersed in an indefinite l.c. cosymplectic manifold with* $\mathrm{Sign}(B) = \emptyset$ *such that* $c = 0$ *and* $\omega(\xi) \neq 0$.

From now on, we consider the *leaf L of the foliation $\mathcal{F}$ to be a null hypersurface immersed in M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$ (Theorem 4.1.1)*.

According to the terminology in [18, p. 79], the portion of $\mathrm{tr}(T\mathcal{F})$ over a leaf L of $\mathcal{F}$ is the null transversal vector bundle of L with respect to the screen distribution $S(T\mathcal{F})|_L$ (see [17] for more details). By definition of null hypersurface, (4.9) and (4.10), we obtain the decomposition

$$TM = S(T\mathcal{F}) \perp \{T\mathcal{F}^{\perp} \oplus \mathrm{tr}(T\mathcal{F})\} = T\mathcal{F} \oplus \mathrm{tr}(T\mathcal{F}). \tag{4.12}$$

Let $\tan : TM \longrightarrow T\mathcal{F}$ and $\mathrm{tra} : TM \longrightarrow \mathrm{tr}(T\mathcal{F})$ be the projections associated with (4.12). We set

$$\begin{aligned} \nabla^{\mathcal{F}}_X Y &= \tan(\nabla_X Y), \quad \mathcal{H}(X,Y) = \mathrm{tra}(\nabla_X Y) \\ A_V X &= -\tan(\nabla_X V), \quad \nabla^{\mathrm{tr}}_X V = \mathrm{tra}(\nabla_X V), \end{aligned}$$

for any X, $Y \in T\mathcal{F}$ and any $V \in \mathrm{tr}(T\mathcal{F})$. Then $\nabla^{\mathcal{F}}$ is a connection in $T\mathcal{F} \longrightarrow M$, $\mathcal{H}$ is a symmetric $\mathrm{tr}(T\mathcal{F})$-valued bilinear form on $T\mathcal{F}$, A_V is an endomorphism of $T\mathcal{F}$, and ∇^{tra} is a connection in $\mathrm{tr}(T\mathcal{F}) \longrightarrow M$. Then, the Gauss and Weingarten formulas of $\mathcal{F}$ in (M, g) are giving by

$$\nabla_X Y = \nabla^{\mathcal{F}}_X Y + \mathcal{H}(X,Y), \quad \nabla_X V = -A_V X + \nabla^{\mathrm{tr}}_X V. \tag{4.13}$$

Similarly, if P denotes the projection morphism of $T\mathcal{F}$ onto $S(T\mathcal{F})$ with respect to the decomposition (4.3), we obtain

$$\nabla^{\mathcal{F}}_X PY = \nabla^{*\mathcal{F}}_X PY + \mathcal{H}^*(X, PY), \quad \nabla^{\mathcal{F}}_X U = -A^*_U X - \nabla^{*\mathrm{tr}}_X U. \tag{4.14}$$

The details given in [18, p. 83 and 85] show clearly that the pointwise restrictions of $\nabla^{\mathcal{F}}$, ∇^{tr}, $\mathcal{H}$ and A_V to a leaf L of the foliation $\mathcal{F}$ are respectively the induced connections, the second fundamental form and the shape operator of L in (M, g). The pointwise restrictions of $\nabla^{*\mathcal{F}}$, $\mathcal{H}^*$ and A^*_U to L are respectively the linear connection, the second fundamental form and the shape operator on the vector bundle $S(TL) \longrightarrow$

L, while the pointwise restriction of $\nabla^{*\mathrm{tr}}$ to L is linear of connection on the vector bundle $TL^{\perp} \longrightarrow L$.

Keeping the same notations of geometric objects above for the pointwise restrictions to a leaf L of $\mathcal{F}$, and locally supposing $\{B, N\}$ is a pair of sections on a coordinate neighborhood $\mathcal{U} \cap L \subset L$ (see [18, Theorem 1.1, p. 79], then the local Gauss-Weingarten equations of $\mathcal{F}$ are given by

$$\nabla_X Y = \nabla^{\mathcal{F}}_X Y + \mathcal{B}(X,Y)N, \quad \nabla_X N = -A_N X + \tau(X)N, \tag{4.15}$$

$$\nabla^{\mathcal{F}}_X PY = \nabla^{*\mathcal{F}}_X PY + \mathcal{C}(X,PY)B, \quad \nabla^{\mathcal{F}}_X B = -A^*_B X - \tau(X)B, \tag{4.16}$$

for all $B \in \Gamma(TL^{\perp})$, $N \in \Gamma(\mathrm{tr}(TL))$, where $\mathcal{B}$ and $\mathcal{C}$ are the local second fundamental forms of L and $S(TL)$, respectively, and τ is a differential 1-form on L. Notice that $\nabla^{*\mathcal{F}}$ is a metric connection on $S(TL)$ while $\nabla^{\mathcal{F}}$ is generally not a metric connection and satisfies the following relation

$$(\nabla^{\mathcal{F}}_X g)(Y,Z) = \mathcal{B}(X,Y)\lambda(Z) + \mathcal{B}(X,Z)\lambda(Y), \tag{4.17}$$

for all $X, Y, Z \in \Gamma(T\mathcal{F})$, where λ is a 1-form on L given $\lambda(\cdot) = \overline{g}(\cdot, N)$. It is well-known from [18] that $\mathcal{B}$ is independent of the choice of $S(TL)$ and it satisfy

$$\mathcal{B}(X,B) = 0, \quad X \in \Gamma(TL). \tag{4.18}$$

The local second fundamental forms $\mathcal{B}$ and $\mathcal{C}$ are related to their shape operators by the following equations $g(A^*_B X, Y) = \mathcal{B}(X,Y)$, $\overline{g}(A^*_B X, N) = 0$, $g(A_N X, PY) = \mathcal{C}(X, PY)$ and $\overline{g}(A_N X, N) = 0$, for all $X, Y \in \Gamma(TL)$. Note that A^*_B is $S(T\mathcal{F})$-valued, self-adjoint and satisfies $A^*_B B = 0$.

In this case, ξ is decomposed as follows.

$$\xi = \xi_S + aB + bN, \tag{4.19}$$

where ξ_S denotes the component of ξ on $S(TL)$ while a and b are non-zero smooth functions on M. If $\xi_S = 0$, then L is called an *ascreen null hypersurface* [27].

Theorem 4.1.2. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M such that $c = 0$ and $\omega(\xi) \neq 0$. Then L is an ascreen null hypersurface of $\mathcal{F}$ if and only if $\phi\mathrm{Rad}(T\mathcal{F}) = \phi l\mathrm{tr}(T\mathcal{F})$.*

Proof. The proof follows from a straightforward calculation. □

Example 4.1.3. Consider M a 7-dimensional semi-Riemannian manifold

$$M = \{p \in \mathbb{R}^7 | x_1 > 0, y_1 > 0, \, z > 0\},$$

with a metric of signature $(-,+,+,-,+,+,+)$ with respect to the canonical basis $\{\partial x_i, \partial y_i, \partial z\}$, for $i = 1,2,3$. The vectors fields $X_1 = e^{-\sigma}\partial x_1$, $Y_1 = e^{-\sigma}\partial y_1$, $X_2 =$

$e^{-\sigma}\partial x_2$, $Y_2 = e^{-\sigma}\partial y_2$, $X_3 = e^{-\sigma}\partial x_3$, $Y_3 = -e^{-\sigma}\partial y_3$, $Z = e^{-\sigma}\partial z$, where $\sigma = x_1 + y_1 + \sqrt{2}z$, are linearly independent at each point of M. Let g be the indefinite metric on M defined by $g(X_1, X_1) = g(Y_1, Y_1) = -1$, $g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}$, for $i, j = 2, 3$, $g(X_l, X_k) = g(Y_l, Y_k) = 0$, for any $l \neq k, l, k = 1, 2, 3$ and $g(\xi, \xi) = 1$. Let η be the 1-form on M defined by $\eta = e^{\sigma}dz$ and the structure vector field given by $\xi = e^{-\sigma}\partial z$. Let ϕ be the $(1,1)$-tensor field defined by, $\phi X_1 = -Y_1$, $\phi Y_1 = X_1$, $\phi X_2 = -Y_2$, $\phi Y_2 = X_2, \phi X_3 = Y_3$, $\phi Y_3 = -X_3$, $\phi X_4 = Y_4$, $\phi Y_4 = -X_4$, $\phi\xi = 0$. By linearity of ϕ and g the quadruplet (ϕ, ξ, η, g) defines an almost contact metric structure on M. The smooth 1-form ω is locally given by $\omega = d\sigma = dx_1 + dy_1 + \sqrt{2}dz$ and satisfies $d\eta = \omega \wedge \eta$. The 2-form fundamental Φ is given by

$$\Phi = e^{-2\sigma}\{-dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3\},$$

and verifies $d\Phi = 2\omega \wedge \Phi$. The Lee vector field is given by $B = \partial x_1 + \partial y_1 + \sqrt{2}\partial z$ and satisfies $c = g(B, B) = 0$. Thus B is a null vector field. It is easy to see $\omega(\xi) = e^{-\sigma}\sqrt{2} \neq 0$. The distribution $D_p = \ker \omega_p$ with $p \in M^7$ is spanned by $\{X_2, X_3, Y_2, Y_3, B\}$. The non-vanishing components of the Lie brackets are $[X_{2,3}, B] = 4X_{2,3}$ and $[Y_{2,3}, B] = 4Y_{2,3}$, which prove that the distribution D is integrable and therefore admits a foliation $\mathcal{F}$ of codimension 1 and its leaves are null hypersurfaces immersed in M^7. The transversal vector field is given by $N = -\frac{1}{4}\{\partial x_1 + \partial y_1 - \sqrt{2}\partial z\}$. We can easily see that $\xi = \frac{e^{-\sigma}}{2\sqrt{2}}(B + 4N)$ and also $\phi B = -4\phi N$. Hence, the leaves of $\mathcal{F}$ are ascreen null hypersurfaces of M.

From Theorem 4.1.2, we notice that if L is an ascreen null hypersurface of $\mathcal{F}$ then $\dim(\phi\mathbb{R}B \oplus \phi\mathbb{R}N_V) = 1$ and hence TL decomposes as follows

$$TL = \mathbb{R}B \perp \phi\mathbb{R}B \perp D_0, \tag{4.20}$$

where D_0 is a non-degenerate ϕ invariant distribution, i.e., $\phi D_0 = D_0$.

As the geometry of null hypersurfaces depends on the vector bundles $S(TL)$ and $\mathrm{tr}(TL)$, it is important to investigate the relationship between geometric objects induced by two screen distributions. The components of the structural vector field ξ in (4.19) depends on both the screen distribution $S(TL)$ and the transversal bundle $\mathrm{tr}(TL)$ and this is proven as follows. Suppose a screen $S(TL)$ changes to another screen $S(TL)'$. The following are some of the local transformation equations due to this change (see [18] for details):

$$K_i' = \sum_{j=1}^{2n-1} K_i^j \left(K_j - \epsilon_j c_j B\right), \tag{4.21}$$

$$N'(X) = N - \frac{1}{2}g(K, K)B + K, \tag{4.22}$$

$$\nabla_X^{\mathcal{F}'} Y = \nabla_X^{\mathcal{F}} Y + \mathcal{B}(X, Y)\{\frac{1}{2}g(K, K)B - K\}, \tag{4.23}$$

for any X, $Y \in \Gamma(TL|_{\mathcal{U}\cap L})$, where $K = \sum_{i=1}^{2n-1} c_i K_i$, $\{K_i\}$ and $\{K_i'\}$ are the local orthonormal bases of $S(T\mathcal{F})$ and $S(T\mathcal{F})'$ with respective transversal sections N and N' for the same null section B. Here c_i and K_i^j are smooth functions on $\mathcal{U}$ and $\{\epsilon_1, \cdots, \epsilon_{2n-1}\}$ is the signature of the basis $\{K_1, \cdots, K_{2n-1}\}$. Denote by κ is the dual 1-form of K, characteristic vector field of the screen change, with respect to the induced metric $g = g|_L$ of $L \hookrightarrow M$ [18], that is,

$$\kappa(X) = g(X, K), \quad \forall\, X \in \Gamma(TL). \tag{4.24}$$

Suppose that the structure vector field ξ in (4.19) is written for a given screen distribution $S(TL)$. Let $\xi = \xi_{S'} + a'B + b'N'$ be another form of the structure vector field ξ in the change distribution $S(TL)'$. Then we have the following.

Lemma 4.1.2. *If the screen distribution $S(TL)$ changes to another screen $S(TL)'$, then $b' = b$ and $\xi_{S'} = \xi_S + \{a - a' + \frac{1}{2}g(K,K)b\}B - bK$. Moreover, the combination in (4.19) is independent of $S(TL)$ if and only if 1-form κ vanishes identically on L.*

4.2 Geometry of non-tangential leaves of $\mathcal{F}$

This section deals with the geometry of the leaves of the foliations $\mathcal{F}$. First of all, we define the following.

A leaf L of $\mathcal{F}$ is called *non-tangential* if ξ satisfies relation (4.19). From (3.22), we can set

$$\nabla_X \xi = hX + \mathcal{A}X, \quad \forall\, X \in \Gamma(T\mathcal{F}), \tag{4.25}$$

where A is a $(1,1)$-tensor field defined by $\mathcal{A}X := \omega(\xi)X - \eta(X)B$. It is easy to see that $\mathcal{A}$ is symmetric with respect to g, i.e., $g(\mathcal{A}X, Y) = g(X, \mathcal{A}Y)$, for any $X, Y \in \Gamma(T\mathcal{F})$, $\mathcal{A}\xi = \omega(\xi)\xi - B$, $\mathcal{A}B = 0$ and $\mathcal{A}\phi X - \phi\mathcal{A}X = \eta(X)\phi B$.

A null hypersurface L of $\mathcal{F}$ with $c = 0$ is said to be *screen conformal* [18] if there exists a non-vanishing smooth function φ such that $A_N = \varphi A_B^*$, and screen homothetic if φ is a constant function.

Theorem 4.2.1. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is a non-tangential null hypersurface. Then L is screen conformal if h satisfies*

$$h = \nabla^{*\mathcal{F}} \xi_S + \{2\eta - b\lambda - \lambda \circ h\} \otimes B - (\omega \circ h) \otimes N - b\mathbb{I},$$

where $\mathbb{I}$ denotes the identity on $\mathcal{F}$.

Proof. By straightforward calculations using (4.25), (4.19) and Gauss-Weingetein formulas for L one gets, for any $X \in \Gamma(T\mathcal{F})$,

$$\begin{aligned} aA^*_B X + bA_N X = \nabla^{*\mathcal{F}}_X \xi_S + \{X(a) - a\tau(X) + \mathcal{C}(X, \xi_S)\}B \\ + \{X(b) + b\tau(X) + \mathcal{B}(X, \xi_S)\}N - \mathcal{A}X - hX. \end{aligned} \tag{4.26}$$

Then taking the g-product of (4.26) with B and N in turn, we get

$$X(b) + b\tau(X) + \mathcal{B}(X, \xi_S) = -g(\mathcal{A}X, B) - g(hX, B) \tag{4.27}$$

$$\text{and } \; X(a) - a\tau(X) + \mathcal{C}(X, \xi_S) = -\overline{g}(\mathcal{A}X, N) - g(hX, N), \tag{4.28}$$

for any $X \in \Gamma(T\mathcal{F})$. Applying the definition of A to (4.27) and (4.28), we get $g(\mathcal{A}X, B) = 0$ and $g(\mathcal{A}X, N) = b\lambda(X) - \eta(X)$. Hence, (4.26) reduces to

$$\begin{aligned} aA^*_B X + bA_N X = \nabla^{*\mathcal{F}}_X \xi_S + \{\eta(X) - b\lambda(X) - g(hX, N)\}B \\ - g(hX, B)N - AX - hX, \end{aligned} \tag{4.29}$$

from which our assertion follows and $\varphi = -\frac{a}{b}$, which completes the proof. □

Theorem 4.2.2. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is non-tangential null hypersurface in M. Then $S(TL)$ is integrable if and only if $g(\nabla^{*\mathcal{F}}_X \xi_S, Y) = g(\nabla^{*\mathcal{F}}_Y \xi_S, X)$ for all $X, Y \in (S(TL))$.*

Proof. By straightforward calculations using (4.29) and the fact that h is symmetric, we have $g([X, Y], N) = \frac{1}{b}\{g(\nabla^{*\mathcal{F}}_X \xi_S, Y) - g(\nabla^{*\mathcal{F}}_Y \xi_S, X)\}$, for any $X, Y \in \Gamma(S(TL))$, which completes the proof. □

The following corollary is obvious.

Corollary 4.2.1. *Let L be a leaf of a foliation $\mathcal{F}$ in an l. c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. If L is an ascreen null hypersurface, then $S(TL)$ is integrable.*

Using the Koszul's formula, the non-vanishing components of the covariant derivatives on the basis of the TL defined in Example 4.1.3 are given by $\nabla_{X_i} X_i = -4N$ and $\nabla_{Y_i} Y_i = -4N$ for $i = 2, 3$, from which we deduce $\mathcal{B}(X_i, X_i) = -4$ and $\mathcal{B}(Y_i, Y_i) = -4$ and zero otherwise. Also, $g(\nabla_U B, N) = 0$ for all $U \in \Gamma(T\mathcal{F})$ which means $\nabla^{\mathcal{F}}_U B$ has no component along $\mathrm{Rad}\, TL$ and hence $\mathcal{C} = 0$ on $\mathcal{F}$. This means that $S(TL)$ is totally geodesic and therefore integrable.

Next, we study the geometry of distribution D_0 in (4.20). Suppose that $\xi_S = 0$, that is L is an ascreen null hypersurface immersed in M. First, we notice that if

$Y \in \Gamma(D_0)$ then $\omega(Y) = \omega(\phi Y) = 0$. Let F be the projection of TL on to D_0. Then by decomposition (4.20) we have

$$X = FX + \lambda(X)B - \frac{1}{b^2}g(X, \phi B)\phi B, \quad \forall\, X \in \Gamma(T\mathcal{F}). \tag{4.30}$$

Applying ϕ to (4.30) we get

$$\phi X = fX + \frac{1}{b^2}g(X, \phi B)B + \lambda(X)\phi B - \frac{1}{b}g(X, \phi B)\xi, \tag{4.31}$$

for all $X \in \Gamma(T\mathcal{F})$, where $fX = \phi FX$.

Theorem 4.2.3. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is an ascreen null hypersurface. Then D_0 is integrable if and only if, for any $X, Y \in \Gamma(D_0)$ and $Z \in \Gamma(S(TL))$, $2g((\nabla^{\mathcal{F}}_X f)Y - (\nabla^{\mathcal{F}}_Y f)X, Z) = g(N_1(Y, Z), fX) - g(N_1(X, Z), fY)$, and in this case f is anti-symmetric on $S(TL)$.*

Proof. Let $X, Y \in \Gamma(D_0)$, then $\nabla_X \phi Y = \nabla_X fY$. Then, using this equation together with (3.20) we derive

$$\begin{aligned}
&2g((\nabla^{\mathcal{F}}_X f)Y, Z)\\
&= -2\lambda(Z)\mathcal{B}(X, fY) - 2\lambda(\nabla^{\mathcal{F}}_X Y)\omega(\phi Z) + \frac{2}{b^2}g(\nabla^{\mathcal{F}}_X Y, \phi B)\omega(Z)\\
&- \frac{2}{b}g(\nabla^{\mathcal{F}}_X Y, \phi B)\eta(Z) - 2g(X, \phi Y)\omega(Z)B - g(X, Y)\omega(\phi Z)\\
&g(N_1(Y, Z), \phi X), \quad \forall\, Z \in \Gamma(T\mathcal{F}).
\end{aligned} \tag{4.32}$$

Then from (4.32) we get

$$\begin{aligned}
&2g((\nabla^{\mathcal{F}}_X f)Y - (\nabla^{\mathcal{F}}_Y f)X, Z) + 2\lambda(Z)\{\mathcal{B}(X, fY) - \mathcal{B}(Y, fX)\}\\
&= \frac{2}{b^2}g([X, Y], \phi B)\omega(Z) - 2\lambda([X, Y])\omega(\phi Z) - \frac{2}{b}g([X, Y], \phi B)\eta(Z)\\
&+ g(N_1(Y, Z), \phi X) - g(N_1(X, Z), \phi Y) + 4g(\phi X, Y)\omega(Z).
\end{aligned} \tag{4.33}$$

Hence, from(4.33) we can see that if D_0 is integrable then

$$2g((\nabla^{\mathcal{F}}_X f)Y - (\nabla^{\mathcal{F}}_Y f)X, Z) = g(N_1(Y, Z), fX) - g(N_1(X, Z), fY),$$

for all $Z \in \Gamma(S(T\mathcal{F}))$. Conversely, using this relation and (4.33) we can easily see that $g([X, Y], \phi B) = 0$ and $\lambda([X, Y]) = 0$, which together shows that D_0 is integrable. $\square$

Corollary 4.2.2. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is an ascreen null hypersurface. Then D_0 is integrable if and only if, $\mathcal{B}(X, fY) - \mathcal{B}(Y, fX) = \frac{1}{2\lambda(Z)}\{g(N_1(Y, Z), fX) - g(N_1(X, Z), fY)\}$, $\forall X, Y \in \Gamma(D_0)$, $Z \in \Gamma((TL^{\perp}))$.*

A leaf L of $\mathcal{F}$ will be called D_0-totally geodesic if for any $X, Y \in \Gamma(D_0)$ we have $h(X,Y) = 0$, or equivalently, $\mathcal{B}(X,Y) = 0$.

Theorem 4.2.4. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is an ascreen null hypersurface. Then L is D_0-totally geodesic if and only if $h + \mathcal{A} = -\omega(\xi)A_N$ on D_0.*

Proof. By straightforward calculations, using (4.2), (4.15) and (4.25), we have

$$\overline{g}(\mathcal{H}(X,Y), B) = \overline{g}(\phi\nabla_X Y, \phi B) - \omega(\xi) g((h+\mathcal{A})X, Y), \tag{4.34}$$

for any $X, Y \in \Gamma(D_0)$. Now, applying (4.15) to (4.34) we get

$$\overline{g}(\mathcal{H}(X,Y), B) = \frac{1}{2}\mathcal{B}(X,Y) - b^2\lambda(\nabla^{\mathcal{F}}_X Y) - \omega(\xi) g((h+\mathcal{A})X, Y),$$

from which we deduce that $\mathcal{B}(X,Y) = -2b^2\lambda(\nabla^{\mathcal{F}}_X Y) - 2\omega(\xi) g((h+\mathcal{A})X, Y)$, which completes the proof. □

It is important to investigate the relationship between some geometric objects induced, studied above, with the change of the screen distributions. We know that the local second fundamental form $\mathcal{B}$ of L on $\mathcal{U} \cap L$ is independent of the vector bundles $(S(TL), S(TL^{\perp}))$ and $\mathrm{tr}(TL)$. This means that all results above depending only on $\mathcal{B}$ are stable with respect to any change of those vector bundles. Let P and P' be projections of TL on $S(TL)$ and $S(TL)'$, respectively, with respect to the orthogonal decomposition of TL. Any vector field X on $L \hookrightarrow M$ can be written as $X = PX + \lambda(X)B = P'X + \lambda'(X)B$ with $\lambda'(X) = \lambda(X) + \kappa(X)$. Then we have $P'X = PX - \kappa(X)B$ and $\mathcal{C}'(X, P'X) = \mathcal{C}'(X, PY)$. The relationship between the local second fundamental forms $\mathcal{C}$ and $\mathcal{C}'$ of the screen distributions $S(TL)$ and $S(TL)'$, respectively is given using (4.22) by $\mathcal{C}'(X, PY) = \mathcal{C}(X, PY) - \frac{1}{2}\kappa(\nabla^{\mathcal{F}}_X PY + \mathcal{B}(X,Y)K)$. All equations in this section depending only on the local second fundamental form $\mathcal{C}$ (making equations non unique), are independent of $S(TL)$ if and only if $\kappa(\nabla^{\mathcal{F}}_X PY + \mathcal{B}(X,Y)K) = 0$.

Using the changes $\tau'(X) = \tau(X) + \mathcal{B}(X, K)$ and $A^{*'}_B X = A^*_B X - \mathcal{B}(X,K)B$, the linear connections $\nabla^{*\mathcal{F}}$ and $\nabla^{*\mathcal{F}'}$ associated to the change are related by

$$\begin{aligned}\nabla^{*\mathcal{F}'}_X P'Y = {} & \nabla^{*\mathcal{F}}_X PY - \mathcal{B}(X, PY)K - \kappa(Y)A^*_B X - X(\kappa(Y))B \\ & - \{\kappa(Y)\tau(X) + \frac{1}{2}\kappa(\nabla^{\mathcal{F}}_X PY + \mathcal{B}(X,Y)K) \\ & - \frac{1}{2}\mathcal{B}(X, PY) g(K,K)\}B. \end{aligned} \tag{4.35}$$

4.3 Higher order geodesibility of leaves of $\mathcal{F}$

Let L be a leaf of the foliation. In this section L is considered to be an ascreen null hypersurface immersed in an l.c. almost cosymplectic manifold M with $\mathrm{Sign}(B) = \emptyset$ such that $c = 0$ and $\omega(\xi) \neq 0$. Denote the vector ξ_t given in (3.4) by $\overline{Q}$. Since L is an ascreen null hypersurface, we have

$$\overline{Q} = \xi_t = e^{\sigma(t)}\xi = e^{\sigma(t)}(aB + bN). \tag{4.36}$$

Let denote the tangential part $ae^{\sigma_t}B$ of $\overline{Q}$ by Q. Then

$$Q = \overline{Q} - be^{\sigma(t)}N. \tag{4.37}$$

Now, we study the umbilicity of L via the divergence of T_rQ, where T_r denotes the Newton transformation with respect to the operator A_N. Applying ∇_X to $\overline{Q}$ and using (4.25), we have

$$\nabla_X\overline{Q} = X(\sigma(t))\overline{Q} + e^{\sigma(t)}(h + \mathcal{A})X, \quad \forall\, X \in \Gamma(T\mathcal{F}). \tag{4.38}$$

In a similar way using (4.37) and (4.38), we have

$$\begin{aligned}\nabla^{\mathcal{F}}_X Q &= e^{\sigma(t)}(h + \mathcal{A})X + be^{\sigma(t)}A_NX + X(\sigma(t))\overline{Q}\\ &\quad - \{X(b)e^{\sigma(t)} + X(\sigma(t))be^{\sigma(t)} + be^{\sigma(t)}\tau(X) + \mathcal{B}(X,Q)\}N,\end{aligned} \tag{4.39}$$

for any $X \in \Gamma(T\mathcal{F})$. Then from (4.39) we deduce that

$$g(\nabla^{\mathcal{F}}_X Q, Y) = e^{\sigma(t)}g((h + \mathcal{A})X, Y) + be^{\sigma(t)}g(A_NX, Y), \tag{4.40}$$

$$\text{and}\quad g(\nabla^{\mathcal{F}}_X Q, N) = X(\sigma(t))g(\overline{Q}, N) + e^{\sigma(t)}g(hX, N), \tag{4.41}$$

for any $X \in \Gamma(T\mathcal{F})$ and $Y \in \Gamma(S(T\mathcal{F}))$.

Proposition 4.3.1. *[35] Let $(L, g, c = 0)$ be an ascreen null hypersurface immersed in an l.c. almost cosymplectic manifold, with $\mathrm{Sign}(B) = \emptyset$, $\omega(\xi) \neq 0$ and a conformal vector field $\overline{Q} = e^{\sigma(t)}\xi$. If the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric, then there exists a pair $\{B, N\}$ on $\mathcal{U} \subset L$ such that the corresponding 1-form τ vanishes on any $\mathcal{U} \cap L$. Moreover, $g(\overline{Q}, B) \neq 0$ and $g(\overline{Q}, N) \neq 0$.*

Proof. Since L is ascreen, then $\overline{Q} = e^{\sigma(t)}\xi = e^{\sigma(t)}(aB + bN)$ and thus, $g(\overline{Q}, B) = be^{\sigma(t)} \neq 0$ and $g(\overline{Q}, N) = ae^{\sigma(t)} \neq 0$. Furthermore, since the Ricci tensor with respect $\nabla^{\mathcal{F}}$ is symmetric, then the induced 1-form τ is closed [18]. That is $d\tau = 0$; so we can set $\tau = d\alpha$. Thus, $\tau(X) = X(\alpha)$. If we take $\overline{B} = fB$ and $\overline{N} = \frac{1}{f}N$, then the corresponding 1-form $\overline{\tau}$ is given by

$$\overline{\tau}(X) = \overline{g}(\nabla^t_X\overline{N}, \overline{B}) = -X(\log f) + \tau(X),$$

where f is a smooth function. Then, one can choose $f = e^{\alpha}$ and hence $\overline{\tau}(X) = 0$ for any $X \in \Gamma(T\mathcal{F}|_{\mathcal{U}})$. Since $\overline{g}(\overline{Q}, B) \neq 0$ and $\overline{g}(\overline{Q}, N) \neq 0$, then $\{\overline{B}, \overline{N}\}$ are the corresponding vectors which satisfies Proposition 4.3.1. $\square$

We have seen that if L is ascreen null hypersurface immersed in an l.c. almost cosymplectic manifold M with $\mathrm{Sign}(B) = \emptyset$, $c = 0$ and $\omega(\xi) \neq 0$, then $S(TL)$ is integrable (see Corollary 4.2.1). Further still, A is screen-valued and $AB = 0$, which leads to $A_N B = 0$. The operator A_N is also symmetric on $S(TL)$ and hence diagonalizable. Let $l_0 = 0, l_1, \cdots, l_m$ its principal curvatures with respect to the quasi-orthonormal basis $\{B, Z_1, \cdots, Z_m\}$, where $\{Z_1, \cdots, Z_m\}$ is the basis $S(TL)$. Associated to the operator A_N are the m algebraic invariants $S_r = e_r(l_0, l_1, \cdots, l_m)$, where $e_r : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}$ denotes the r-th symmetric polynomial in variables $l_0, l_1, \cdots, l_m$. We usually set $S_0 = 1$ and it is also easy to see that $S_1 = \mathrm{tr}(A_N)$, the mean curvature. Furthermore, S_r is called the r-th mean curvature with respect to A_N. Then the Newton transformations T_r with respect to the operator A_N are defined by $T_r : TL \longrightarrow TL$ and explicitly given by the recurrence relation

$$T_r = (-1)^r S_r \mathbb{I} + A_N \circ T_{r-1}, \quad 0 \leq r \leq m. \tag{4.42}$$

It is important to know that T_r is also symmetric and commutes with A_N. Let $H_r = \binom{m+1}{r}^{-1} S_r$ denote the normalized mean curvature with respect to A_N and let further $c_r = (m+1-r)\binom{m+1}{r}$. The following properties of T_r can be deduced from (4.42).

$$\mathrm{tr}(T_r) = (-1)^r (m+1-r) S_r = (-1)^r c_r H_r, \tag{4.43}$$

$$\mathrm{tr}(A_N \circ T_r) = (-1)^r (r+1) S_{r+1} = (-1)^r c_r H_{r+1}. \tag{4.44}$$

Details on Newton transformations can be found in [3], [16] and many more references therein.

Note that the interrelation between the second fundamental forms of the null hypersurface L and its screen distribution and their respective shape operators indicates that the null geometry depends on the choice of a screen distribution. By [18, p. 87], A_N and $A'_{N'}$ are related by

$$\begin{aligned} A'_{N'} X = A_N X + \delta(X) B + \sum_{j=1}^{2n-1} \mu_j(X) K_j - \sum_{j=1}^{2n-1} c_j \nabla^{\mathcal{F}}_X K_j \\ - \frac{1}{2} g(K, K) A^*_B X, \end{aligned} \tag{4.45}$$

where $\delta = \sum_{j=1}^{2n-1} \{\epsilon_j c_j X(c_j) - \tau(X)\epsilon_j (c_j)^2 + \frac{1}{2}\epsilon_j (c_j)^2 \mathcal{B}(X, K) - c_j \mathcal{C}(X, K_j)\}$ and $\mu_j = c_j(\tau(X) + \mathcal{B}(X, K)) - X(c_j)$.

The dependence of T_r on $S(TL)$ is as follows. Let $Z_i \in \Gamma(S(TL))$ be an eigenvector of A_N, then it is easy to show that $T_r Z_i = (-1)^r S^i_r Z_i$. Notice that $(-1)^r S^i_r$ is an eigenvalue of T_r corresponding to eigenvector Z_i. Then by direct calculations we have

$$T_r Z_i = (-1)^r S_r \mathbb{I} + (-1)^{r-1} S^i_{r-1} A_N Z_i, \tag{4.46}$$

$$\text{and} \quad T'_r Z_i = (-1)^r S'_r \mathbb{I} + (-1)^{r-1} S'^i_{r-1} A'_{N'} Z_i. \tag{4.47}$$

Subtracting (4.46) from (4.47) we deduce that

$$\begin{aligned} T'_r &= T_r + (-1)^r (S'_r - S_r)\mathbb{I} + \mathfrak{S}_{r-1} A_N \\ &+ (-1)^{r-1} S'^i_{r-1} \left\{ \delta B + \sum_{j=1}^{2n-1} \mu_j K_j - \sum_{j=1}^{2n-1} c_j \nabla^{\mathcal{F}} K_j - \frac{1}{2} g(K,K) A^*_B \right\}, \end{aligned} \tag{4.48}$$

where $\mathfrak{S}_r = (-1)^r (S'^i_r - S^i_r)$. Hence, from (4.48) we can see that the operators T_r depends on a chosen section N and on $S(TL)$. Note that T_r is unique if and only if M is r-maximal (i.e., $S_r = 0$, for all r).

Next, the divergence of T_r on the screen distribution will be denoted by $\mathrm{div}^{\nabla^*}(T_r)$ and given by

$$\mathrm{div}^{\nabla^*}(T_r) = \sum_{i=1}^{m} (\nabla^{\mathcal{F}}_{Z_i} T_r) Z_i. \tag{4.49}$$

Since L is null, the divergence $\mathrm{div}^{\nabla^{\mathcal{F}}}(Y)$ of a vector $Y \in \Gamma(T\mathcal{F})$ with respect to the degenerate metric g on L is intrinsically defined by (see [19, p. 136], for more details and references therein)

$$\mathrm{div}^{\nabla^{\mathcal{F}}}(Y) = \mathrm{div}^{\nabla^*}(Y) + g(\nabla^{\mathcal{F}}_B Y, N). \tag{4.50}$$

Let dV_M be the volume element of M with respect to g and a given orientation. Then, we denote the volume form on $\mathcal{F}$ by

$$dV = i_N dV_M,$$

where i_N is the contraction with respect to the vector field N. We have the following.

Theorem 4.3.1. *Let $(L, g, c = 0)$ be a compact ascreen null hypersurface of a $\mathcal{F}$ in an l.c. almost cosymplectic manifold M of constant sectional curvature, with* $\mathrm{Sign}(B) = \emptyset$, $\omega(\xi) \neq 0$ *and a conformal vector field $\overline{Q} = e^{\sigma(t)}\xi$. If the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric, then*

$$\int_L (B \cdot \overline{g}(T_r Q, N) + e^{\sigma(t)} \mathrm{tr}(T_r \circ h) + (-1)^r c_r \omega(\overline{Q}) \{H_r + H_{r+1}\}) dV = 0.$$

Proof. Our proof follows by computation of the divergence of the vector field $T_r Q$ from (4.50). That is;

$$\mathrm{div}^{\nabla^{\mathcal{F}}}(T_r Q) = \mathrm{div}^{\nabla^*}(T_r Q) + \overline{g}(\nabla^{\mathcal{F}}_B T_r Q, N). \tag{4.51}$$

Applying (4.49) to (4.51) we obtain

$$\mathrm{div}^{\nabla^{\mathcal{F}}}(T_rQ) = \overline{g}(\mathrm{div}^{\nabla^*}(T_r), Q) + \sum_{i=1}^{m} \epsilon_i \overline{g}(\nabla^{\mathcal{F}}_{Z_i} Q, T_r Z_i) + \overline{g}(\nabla_B T_r Q, N),$$

from which, after applying (4.40), Proposition 4.3.1 and the fact that M is a space form of constant sectional curvature, we get

$$\begin{aligned} \mathrm{div}^{\nabla^{\mathcal{F}}}(T_rQ) &= e^{\sigma(t)}\mathrm{tr}(T_r \circ h) + e^{\sigma(t)}\mathrm{tr}(T_r \circ \mathcal{A}) \\ &\quad + be^{\sigma(t)}\mathrm{tr}(T_r \circ A_N) + B \cdot \overline{g}(T_rQ, N). \end{aligned} \tag{4.52}$$

When L is ascreen, we see from (4.29) that A is screen-valued operator and in fact $AX = \omega(\xi)X$ for any $X \in \Gamma(S(T\mathcal{F}))$. Thus, (4.51) reduces to

$$\begin{aligned} \mathrm{div}^{\nabla^{\mathcal{F}}}(T_rQ) &= e^{\sigma(t)}\mathrm{tr}(T_r \circ h) + e^{\sigma(t)}\omega(\xi)\mathrm{tr}(T_r) \\ &\quad + be^{\sigma(t)}\mathrm{tr}(T_r \circ A_N) + B \cdot \overline{g}(T_rQ, N). \end{aligned} \tag{4.53}$$

Finally, our result follows from (4.53) by considering (4.49) and the fact that L is compact. $\square$

Next we look at some applications of Theorem 4.3.1 in which the functions $a = \eta(N)$, $b = \eta(B) = \omega(\xi)$ and $\sigma(t)$ are all constants. Hypersurfaces with constant higher order mean curvatures are of great importance to modern differential geometry and have been a focal point of study for the past decades. For instance, in the analysis of minimal surfaces (surfaces with zero mean curvatures) and in the study of physical interfaces between fluids, which are assumed to have constant mean curvatures (see [2] and many more references therein). We suppose that L is of constant higher order mean curvature in the rest of the paper.

Theorem 4.3.2. *Under the assumptions of Theorem 4.3.1, if the functions a, b and σ are all constant, then*

$$\int_L (aB(S_r) + (-1)^{r-1}\mathrm{tr}(T_r \circ h) + \omega(\xi)c_r\{H_r + H_{r+1}\})dV = 0. \tag{4.54}$$

Proof. By Proposition 4.3.1 and the fact that $\mathcal{B}(\overline{g}(T_rQ, N)) = (-1)^r B(S_r\lambda(Q)) = (-1)^r ae^{\sigma(t)}B(S_r)$, we complete the proof. $\square$

Theorem 4.3.3. *[35] Let $(L, g, c = 0)$ be a compact ascreen null hypersurface of a $\mathcal{F}$ in an l.c. almost cosymplectic manifold M of constant sectional curvature, with $\mathrm{Sign}(B) = \emptyset$, $\omega(\xi) \neq 0$ and a conformal vector field $\overline{Q} = e^{\sigma(t)}\xi$. Let a, b and σ be constant such that h is tangent to $\mathcal{F}$. If the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric and H_1 is constant, then $S(TL)$ is totally geodesic.*

Proof. By considering $r = 0$ in (4.54) and multiplying the resultant equation by H_1, we get

$$\int_L (H_1 + H_1^2)dV = 0. \tag{4.55}$$

Then substituting $r = 1$, (4.54) and using T_1, properties of h, the fact that $B(S_1) = 0$, we get

$$\int_L (H_1 + H_2)dV = 0. \tag{4.56}$$

Then, from (4.55) and (4.56) we have $\int_L (H_1^2 - H_2)dV = 0$. But, for $l_1 = \cdots = l_m$ we have

$$H_1^2 - H_2 = \frac{1}{m(m-1)}\left(\frac{m-1}{m}\left(\sum_{i=1}^m l_i\right)^2 - 2\sum_{i=1}^m l_i^2\right). \tag{4.57}$$

Using Cauchy-Schwarz inequality on (4.57) we get that

$$H_1^2 - H_2 \geq \frac{m-2}{m(m-1)}\sum_{i=1}^m l_i^2 \geq 0, \tag{4.58}$$

with equality if $l_1 = \cdots = l_m = 0$. Hence, $S(T\mathcal{F})$ is totally geodesic. □

Corollary 4.3.1. *Under the assumptions of Theorem 4.3.3, if H_2 is a positive constant (or H_{r-1} and H_r, for $r = 1, \cdots, m$, are both constant) and $\mathrm{tr}(T_r \circ h) = 0$, then $S(TL)$ is also totally geodesic.*

Note that all results above depending only on the local second fundamental form $\mathcal{B}$ are independent of any change of screen distributions.

Chapter Five

Geometry of 2-Lagrange space and conformal change

In this chapter, we study the effect of the conformal deformation of fundamental tensors on 2-Lagrange spaces.

5.1 Conformal deformation and 2-Lagrange spaces

Definition 5.1.1. [39]

(a) A differential Lagrangian of order 2, is a mapping $L : Osc^2(M) \to \mathbb{R}$ of C^∞-class on $Osc^2(M)_0 = Osc^2(M) - \{0\}$ and continuous on the null section of the projection $\pi^2 : Osc^2(M) \to M$, such that

$$g_{ij}(x, y^{(1)}, y^{(2)}) = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(2)i}\partial y^{(2)j}}, \tag{5.1}$$

is a $(0,2)$-type symmetric d-tensor field on $Osc^2(M)$.

(b) A differential Lagrangian is said to be regular (or non-degenerate) if

$$\operatorname{rank}(g_{ij}) = n, \tag{5.2}$$

on $Osc^2(M)$.

c) A Lagrange space is a pair $L^{(2)n} = (M, L)$ formed by a smooth real n-dimensional manifold M and a regular differentiable Lagrangian L on M, for which the d-tensor field g_{ij} from (5.1) has constant signature on $Osc^2(M)$.

Note that the null section $0 : M \to T^2M$ of the projection π^2 is defined by $0 : (x) \in M \to (x, 0, y^{(2)}) \in T^2M$. Constant signature means that the signature is either $(n, 0)$ or $(0, n)$.

The change of local coordinates on $Osc^2(M)$: $(x^i, y^{(i)1}, y^{(2)i}) \to (\overline{x}^i, \overline{y}^{(1)i}, \overline{y}^{(2)i})$ is given by

$$\overline{x}^i = \overline{x}^i(x^1, \cdots, x^n), \ \mathrm{rank}\|\frac{\partial \overline{x}^i}{\partial x^j}\| = n,$$
$$\overline{y}^{(1)i} = \frac{\partial \overline{x}^i}{\partial x^j} y^{(1)j}, \ 2\overline{y}^{(2)i} = \frac{\partial \overline{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\frac{\partial \overline{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j}. \tag{5.3}$$

We have,

$$\frac{\partial \overline{x}^i}{\partial x^j} = \frac{\partial \overline{y}^{(1)i}}{\partial y^{(1)j}} = \frac{\partial \overline{y}^{(2)i}}{\partial y^{(2)j}}. \tag{5.4}$$

Let L be a 2-Lagrangian and c a time-parametrized curve, c^* its extension $Osc^2(M)$, the integral for the action of the Lagrangian L on the curve c is given by the functional

$$I(c) = \int_0^1 L(x, \frac{dx^i}{dt}, \frac{1}{2}\frac{d^2x^i}{dt^2})dt. \tag{5.5}$$

The variational problem we obtain allows us to get the Euler-Lagrange equations given by (see [40, p. 110] for more details)

$$\overset{(0)}{E}_j(L) = 0, \ \ y^{(1)} = \frac{dx^i}{dt}, \ \ y^{(2)i} = \frac{1}{2}\frac{d^2x^i}{dt^2}, \tag{5.6}$$
$$\text{where } \overset{(0)}{E}_i(L) = \frac{\partial}{\partial x^i} - \frac{d}{dt}\frac{\partial}{\partial y^{(1)i}} + \frac{1}{2}\frac{d^2}{dt^2}\frac{\partial}{\partial y^{(2)i}} \tag{5.7}$$

on the curve c^*.

Using the 2-almost tangent structure J we obtain

$$\overset{(1)}{E}_i = -J\overset{(0)}{E}_i = -\frac{\partial}{\partial y^{(1)i}} + \frac{d}{dt}\frac{\partial}{\partial y^{(2)i}}, \ \overset{(2)}{E}_i = -\frac{1}{2}J\overset{(1)}{E}_i = \frac{1}{2}\frac{\partial}{\partial y^{(2)i}}, \tag{5.8}$$

where $\overset{(0)}{E}_i$, $\overset{(1)}{E}_i$, $\overset{(2)}{E}_i$ are covectors called by Craig-Synge covectors on the curve c^*.

Let M be a smooth manifold and L and $\widetilde{L}$ be two 2-Lagrangian on M. The fundamental tensors g and $\widetilde{g}$ of L and $\widetilde{L}$, respectively, are *conformally deformed* if there exists a smooth positive function ψ on $Osc^2(M)$ such that

$$\widetilde{g} = \psi \otimes g. \tag{5.9}$$

Lemma 5.1.1. *[37] For the 2-Langrange space $L^{(2)n} = (M, L)$, the following properties hold:*

(1) The functions $f_i := \frac{\partial L}{2\partial y^{(2)i}}$ are the components of d-covector field.

(2) *The functions*

$$C_{ijk} := \frac{1}{4}\frac{\partial^3 L}{\partial y^{(2)i}\partial y^{(2)j}\partial y^{(2)k}} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^{(2)k}} \tag{5.10}$$

are the components of a third order symmetric d-tensor field.

Proof. (1) The transformation of coordinates (5.3) and (5.4) on $Osc^2(M)$ produces the transformation

$$\tilde{f}_i = \frac{1}{2}\frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial L}{\partial y^{(2)j}} = \frac{1}{2}\frac{\partial x^j}{\partial \tilde{x}^i} f_j.$$

Which proves that f_i are the components of d-covector field on $Osc^2(M)$.

(2) In the same manner, we have

$$\tilde{C}_{ijk} = \frac{\partial x^r}{\partial \tilde{x}^i}\frac{\partial x^p}{\partial \tilde{x}^j}\frac{\partial x^q}{\partial \tilde{x}^k} C_{ijk}. \tag{5.11}$$

then we obtain that C_{ijk} are the components of a symmetric d-tensor field of $(0,3)$ type.

□

The tensor field $\tilde{C}_{ijk}$ defined in (5.10) is called *Cartan tensor field* of the 2-osculator bundle.

Proposition 5.1.1. *[37] Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two 2-Lagrangian on M. If the fundamental tensors g and $\widetilde{g}$ of L and $\widetilde{L}$, respectively, are conformally deformed, then the factor of proportionality is constant in the $y^{(2)}$-direction.*

Proof. Assume the fundamental tensors g and $\widetilde{g}$ of L and $\widetilde{L}$, respectively, are conformally deformed. Then, there exists a positive function ψ on $Osc^2(M)$ such that $\tilde{g}_{ij} = \psi g_{ij}$. By differentiating the latter with respect to $y^{(2)}$, we have

$$\frac{\partial \tilde{g}_{ij}}{\partial y^{(2)k}} = \frac{\partial \psi}{\partial y^{(2)k}} g_{ij} + \frac{\partial g_{ij}}{\partial y^{(2)k}}\psi \tag{5.12}$$

It follows by (5.12) that,

$$2\tilde{C}_{ijk} = 2\psi C_{ijk} + \frac{\partial \psi}{\partial y^{(2)k}} g_{ij}, \tag{5.13}$$

since C_{ijk} is completely symmetric. Then we have,

$$2\tilde{C}_{ijk} = 2\psi C_{ikj} + \frac{\partial \psi}{\partial y^{(2)j}} g_{ik}, \tag{5.14}$$

Thus by (5.13) and (5.14) we obtain,

$$\frac{\partial \psi}{\partial y^{(2)k}} g_{ij} = \frac{\partial \psi}{\partial y^{(2)j}} g_{ik}. \tag{5.15}$$

It follows from the relation (5.15), that

$$\frac{\partial \psi}{\partial y^{(2)k}}\delta_j^k = n\frac{\partial \psi}{\partial y^{(2)j}}, \tag{5.16}$$

and one obtain

$$\frac{\partial \psi}{\partial y^{(2)j}} = 0 \tag{5.17}$$

Thus the factor of proportionality is constant in the $y^{(2)}$-direction. □

Lemma 5.1.2. *[37] Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two 2-Lagrangian on M with fundamental tensors g and $\widetilde{g}$, respectively, such that (5.9) is satisfied. If g is 0-homogeneous with respect to $y^{(2)}$, so is $\widetilde{g}$.*

Proof. As ψ is constant in $y^{(2)}$-direction then it is 0-homogeneous with respect to $y^{(2)}$, then $\widetilde{g} = \psi \otimes g$ is 0-homogeneous with respect to $y^{(2)}$. □

Note that if g_{ij} is the fundamental tensor of L and 0-homogeneous with respect to $y^{(2)}$, then one of the solutions of the partial differential equation

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(2)i}\partial y^{(2)j}}, \tag{5.18}$$

is given by

$$L = g_{ij}y^{(2)i}y^{(2)j} + B_i(x, y^{(1)})y^{(2)i} + V, \tag{5.19}$$

where $B_i(x, y^{(1)})$ is an arbitrary covector and V an arbitrary function in $Osc^1(M)$ (see [40] for more details).

Therefore, we have the following.

Proposition 5.1.2. *Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two 2-Lagrangian on M with fundamental tensors g and $\widetilde{g}$, respectively, such that (5.9) is satisfied. If g is 0-homogeneous with respect to $y^{(2)}$, then L and $\widetilde{L}$ are related as*

$$\widetilde{L} = \psi L + A_i(x, y^{(1)})y^{(2)i} + U(x, y^{(1)}), \tag{5.20}$$

where A_i an arbitrary covector and U an arbitrary function on $Osc^1(M)$.

Proof. By lemma 5.1.2, $\widetilde{g}$ is 0-homogeneous since g and then from the equation

$$\widetilde{g}_{ij} = \frac{1}{2}\frac{\partial^2 \widetilde{L}}{\partial y^{(2)i}\partial y^{(2)j}}, \tag{5.21}$$

we have

$$\widetilde{L} = \widetilde{g}_{ij}y^{(2)i}y^{(2)j} + a_i(x, y^{(1)})y^{(2)i} + u, \tag{5.22}$$

and also

$$L = g_{ij}y^{(2)i}y^{(2)j} + b_i(x, y^{(1)})y^{(2)i} + v, \tag{5.23}$$

since g_{ij} is also 0-homogeneous. From (5.22), one obtains

$$\begin{aligned}\widetilde{L} &= \psi L + (a_i - \psi b_i)y^{(2)i} + (u - \psi v) \\ &= \psi L + A_i y^{(2)i} + U,\end{aligned} \tag{5.24}$$

where $A_i = a_i - \psi b_i$ and $U = u - \psi v$. This completes the proof. □

In this case, we say that L and $\widetilde{L}$ are *conformal-type.*

Let us recall by the following theorem the coefficients of 2-semispray in terms of Lagrangian (see [41, theorem 7.3] for more details).

Theorem 5.1.1. *[41] If L is the fundamental function of a Lagrange space of second order $L^{(2)n}$, then on a curve c^*, the system of differential equation*

$$g^{ij} \overset{(1)}{E_j}(L) = 0 \tag{5.25}$$

determines a 2-spray S, with components

$$3G^i = \frac{1}{2}g^{ij}\{\Gamma(\frac{\partial L}{\partial y^{(2)j}}) - \frac{\partial L}{\partial y^{(1)j}}\} \tag{5.26}$$

where $\overset{(1)}{E_j}$ is a Craig-Synge covector given in equation (5.8).

Corollary 5.1.1. *The coefficients G^i of the semispray are obtained along every extremal curve of Euler-equations $\overset{(0)}{E_j}(L) = 0$.*

Proof. We have $g^{ij} \overset{(1)}{E_j}(L) = 0$, using the first equation in (5.8), one gets

$$g^{ij} J \overset{(0)}{E_j}(L) = J g^{ij} \overset{(0)}{E_j}(L) = 0$$

wich implies $\overset{(0)}{E_j}(L) = 0$. This completes the proof. □

The objects G^i given by (5.26) allows us to obtain the coefficients of the nonlinear connection given by

$$\underset{(1)}{N^i_j} = \frac{\partial G^i}{\partial y^{(2)j}} \quad \text{and} \quad \underset{(2)}{N^i_j} = \frac{1}{2}(S\underset{(1)}{N^i_j} - \underset{(1)}{N^i_m}\underset{(1)}{N^m_j}). \tag{5.27}$$

From now on, let us denote the conformal factor by e^ρ, where $\rho \in C^\infty(Osc^2(M))$.

Lemma 5.1.3. *The coefficients G^i and $\widetilde{G}^i$ of the 2-semispray with respect to L and $\widetilde{L}$, respectively, are related as*

$$\widetilde{G}^i = G^i + H^i, \tag{5.28}$$

where

$$H^i = \frac{1}{6} g^{ij} (\Gamma(\rho) \frac{\partial L}{\partial y^{(2)j}} - \frac{\partial \rho}{\partial y^{(1)j}} L - e^{-\rho} (\Gamma(A_j) + \frac{\partial A_k}{\partial y^{(1)j}} y^{(2)k} + \frac{\partial U}{\partial y^{(1)j}})). \tag{5.29}$$

Proof. Replacing (5.20) in (5.26) then by a direct calculation one gets the assertion. □

Corollary 5.1.2. *According to $\widetilde{L}$ on $Osc^2(M)$ the 2-semispray is given by*

$$\widetilde{S} = S - 3H^i \frac{\partial}{\partial y^{(2)i}} \tag{5.30}$$

where S is the 2-semispray arised from the Lagrangian L.

Proposition 5.1.3. *[37] Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two 2-Lagrangians on M with fundamental tensors g and $\widetilde{g}$, respectively, such that (5.9) is satisfied. Then, the coefficients of the nonlinear connection are given by*

$$\underset{(1)}{\widetilde{N_j^i}} = \underset{(1)}{N_j^i} + \underset{(1)}{H_j^i} \quad \textit{and} \quad \underset{(2)}{\widetilde{N}_j^i} = \underset{(2)}{N_j^i} + \underset{(2)}{H_j^i}, \tag{5.31}$$

where

$$\begin{aligned}
\underset{(1)}{H_j^i} &= \frac{\partial H^i}{\partial y^{(2)j}}, \\
\underset{(2)}{H_j^i} &= \frac{1}{2} \{ S \underset{(1)}{H_j^i} - \underset{(1)}{H_r^i} \underset{(1)}{H_j^r} - 3H^m \frac{\partial}{\partial y^{(2)m}} (\underset{(1)}{N_j^i} + \underset{(1)}{H_j^i}) - \underset{(1)}{N_r^i} \underset{(1)}{H_j^r} - \underset{(1)}{H_r^i} \underset{(1)}{N_j^r} \}.
\end{aligned}$$

Notations

In the sequel, we adopt the following notations

$$\begin{aligned}
&\frac{\delta}{\delta x^i} = \delta_i, \ \frac{\delta}{\delta y^{(1)i}} = \delta_{1i}, \ \frac{\delta}{\delta y^{(2)i}} = \delta_{2i}, \\
&\frac{\partial}{\partial x^i} = \partial_i, \ \frac{\partial}{\partial y^{(1)i}} = \partial_{1i}, \ \frac{\partial}{\partial y^{(2)i}} = \partial_{2i}, \\
&\hat{\partial}_i = -\underset{(1)}{H_i^m} \partial_{1m} - \underset{(2)}{H_i^m} \partial_{2m}, \ \hat{\partial}_{1i} = -\underset{(1)}{H_i^m} \partial_{2m}
\end{aligned}$$

Note that the adapted basis changes also under the conformal transformation. That is described by

$$\widetilde{\delta}_i = \delta_i + \hat{\partial}_i,\ \widetilde{\delta}_{1i} = \delta_{1i} + \hat{\partial}_{1i},\ \widetilde{\delta}_{2i} = \delta_{2i}. \tag{5.32}$$

The 1-forms from the adapted co-basis are given by

$$dx^i,\ \widetilde{\delta}y^{(1)i} = dy^{(1)i} + \underset{(1)}{\widetilde{M}^i_j}dx^j,\ \widetilde{\delta}y^{(2)i} = dy^{(2)i} + \underset{(1)}{\widetilde{M}^i_j}dy^{(1)j} + \underset{(2)}{\widetilde{M}^i_j}dx^j \tag{5.33}$$

where $(\underset{(1)}{\widetilde{M}^i_j}, \underset{(2)}{\widetilde{M}^i_j})$ are the dual coefficients to $(\underset{(1)}{\widetilde{N}^i_j}, \underset{(2)}{\widetilde{N}^i_j})$ satisfying

$$\underset{(1)}{\widetilde{M}^i_j} = \underset{(1)}{\widetilde{N}^i_j},\ \ \underset{(2)}{\widetilde{M}^i_j} = \underset{(2)}{M^i_j} + K^i_j, \tag{5.34}$$

whith $\underset{(2)}{M^i_j} = \dfrac{\partial G^i}{\partial y^{(1)j}}$ and $K^i_j = \dfrac{\partial H^i}{\partial y^{(1)j}}$.

Therefore, the change of the adapted co-basis can be described by

$$dx^i,\ \widetilde{\delta}y^{(1)i} = \delta y^{(1)i} + \underset{(1)}{H^i_j}dx^j,\ \widetilde{\delta}y^{(2)i} = \delta y^{(2)i} + \underset{(1)}{H^i_j}dy^{(1)j} + K^i_jdx^j. \tag{5.35}$$

5.2 Examples in Riemannian and Finsler geometry

Example 5.2.1. Let (M, γ_{ij}) be a (pseudo) Riemannian space. We denote by γ^i_{jk} its Christoffel symbol, The Liouville d-vector field is given by

$$z^{(2)i} = y^{(2)i} + \frac{1}{2}\gamma^i_{jk}(x)y^{(1)j}y^{(1)k} \tag{5.36}$$

where γ^i_{jk} is the christoffel symbol of the metric γ_{ij} define on M. $z^{(2)i}$ is globally defined on $\widetilde{Osc}^2(M)$ and depends only on the metric γ_{ij}, then the function given by

$$L(x, y^{(1)}, y^{(2)}) = \gamma_{ij}z^{(2)i}z^{(2)j} \tag{5.37}$$

is a differentiable Lagrangian globally defined on $\widetilde{Osc}^2(M)$, depending only on the metric γ_{ij} and it is regular. By differentiating (5.37), one gets

$$\frac{1}{2}\partial_{2i}\partial_{2j}L = \gamma_{ij}, \tag{5.38}$$

and the coefficients of the canonical 2-spray are given by

$$3G^i = \gamma^{ij}(x)\{\Gamma(\gamma_{jm}z^{(2)m}) - \gamma_{mi}z^{(2)m}\partial_{1j}z^{(2m)}\} \tag{5.39}$$

the canonical nonlinear connection determined by the Lagrange (5.37) has the dual coefficients

$$\underset{(1)}{M^i_j} = \gamma^i_{jk} y^{(1)k}, \ \underset{(2)}{M^i_j} = \frac{1}{2}\{\Gamma(\gamma^i_{jk} y^{(1)k}) + \underset{(1)}{M^i_r}\underset{(1)}{M^r_j}\}. \tag{5.40}$$

They are globally defined on $\widetilde{Osc}^2(M)$ and depend only on the structure γ_{ij}.

Let $\rho : M \to \mathrm{R}$ be a function over M and

$$\widetilde{\gamma}_{ij} = \exp(\rho)\gamma_{ij} \tag{5.41}$$

be the conformal metric to the metric γ_{ij}.

Proposition 5.2.1. *Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two 2-Lagrangians on M with fundamental tensors γ_{ij} and $\widetilde{\gamma}_{ij}$, respectively, such that (5.41) is satisfied, The following statement hold*

(a) The Liouville d-vector field $\widetilde{z}^{(2)i}$ is written as

$$\widetilde{z}^{(2)i} = z^{(2)i} + \Omega_i, \tag{5.42}$$

where

$$\Omega_i = \frac{1}{2}(\delta_i\rho\delta^s_j + \delta_j\rho\delta^s_k - \delta_s\rho g_{jk}g^{is})y^{(1)j}y^{(1)k}. \tag{5.43}$$

(b) The fundamental functions L and $\widetilde{L}$ are related as

$$\widetilde{L}(x, y^{(1)}, y^{(2)}) = e^\rho L(x, y^{(1)}, y^{(2)}) + e^\rho(\Omega_j z^{(2)i} + \Omega_i z^{(2)j} + \Omega_i\Omega_j). \tag{5.44}$$

(c) The coefficients of the canonical spray related to $\widetilde{\gamma}_{ij}$ are given by

$$\widetilde{G}^i = G^i + H^i, \tag{5.45}$$

where

$$\begin{aligned} H^i = \frac{1}{3}\gamma^{ij}(x)\{&\Gamma(\rho)\gamma_{jm}\widetilde{z}^{(2)m} - \gamma_{mi}z^{(2)m}\partial_{1j}\Omega_m - \gamma_{mi}\Omega_m\partial_{1j}z^{(2)m} + \Gamma(\Omega_m\gamma_{jm}) \\ &- \gamma_{mi}\Omega_m\partial_{1j}\Omega_m\}. \end{aligned} \tag{5.46}$$

Proof. Under the conformal change the Christofell symbol $\widetilde{\gamma}^i_{jk}$ in terms of γ^i_{jk} is given by

$$\widetilde{\gamma}^i_{jk} = \gamma^i_{jk} + \frac{1}{2}(\partial_i\rho\delta^i_k + \partial_j\rho\delta^i_j - \partial_s\rho\gamma_{jk}\gamma^{is}) \tag{5.47}$$

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then we get the Liouville d-vector field using (5.36), following equation (5.37), one has

$$\widetilde{L}(x, y^{(1)}, y^{(2)}) = \widetilde{\gamma}_{ij}\widetilde{z}^{(2)i}\widetilde{z}^{(2)j}. \tag{5.48}$$

By a direct expansion of (5.48) and taking into account (5.42) one gets .

$$\begin{aligned}\widetilde{L} &= e^{\rho}\gamma_{ij}(z^{(2)i} + \Omega_i)(z^{(2)j} + \Omega_j)\\ &= e^{\rho}L + e^{\rho}(\Omega_j z^{(2)i} + \Omega_i z^{(2)j} + \Omega_i\Omega_j).\end{aligned} \tag{5.49}$$

Then one gets the assertion (b). Under the conformal change we have

$$3\widetilde{G}^i = \widetilde{\gamma}^{ij}(x)\{\Gamma(\widetilde{\gamma}_{jm}\widetilde{z}^{(2)m}) - \widetilde{\gamma}_{mi}\widetilde{z}^{(2)m}\partial_{1j}\widetilde{z}^{(2m)}\}, \tag{5.50}$$

then substituting (5.42), (5.41) and (5.47) in (5.50), one gets the assertion (c). □

Definition 5.2.1. *[9]* A Finsler space is a pair $F^n = (M, F)$ formed by a real n-dimensional M and a scalar positive function F on TM, differentiable on TM_0 and continuous on the null section, which has the properties:

(1) $F(x, y)$ is positively homogeneous of degree 1, with respect to y^i on TM_0.

(2) The pair (M, F^2) is a Lagrange space.

The function F is called the fundamental or metric function and the d-tensor field

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial F^2}{\partial y^{(1)i}\partial y^{(1)j}} \tag{5.51}$$

is the fundamental tensor or metric of the Finsler space F^n, it is 0-homogeneous with respect to y^i and it is non-degenerate. The Cartan tensor field

$$C_{ijk} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i\partial y^j\partial y^k} \tag{5.52}$$

is completely symmetric, using the homogeneity of F and g_{ij},

$$y^m C_{mij} = C_{0ij} = 0. \tag{5.53}$$

Then one obtains

$$F^2 = g_{ij}(x, y)y^{(1)i}y^{(1)j} \tag{5.54}$$

(see [40] and references therein for details).

By making use of Theorem 5.1.1 and equation (2.27) we obtain the coefficients of nonlinear connection $N^i_j(x, y^{(1)})$ of the Lagrange space (M, F^2), called *Cartan* nonlinear connection of the Finsler space $F^n = (M, F)$.

According to Miron [40, Theorem 4.10.1], the function $L : Osc^2(M) \to$ R given by

$$L(x, y^{(1)}, y^{(2)}) = g_{ij}(x, y^{(1)}) z^{(2)i} z^{(2)j}, \tag{5.55}$$

where

$$z^{(2)i} = y^{(2)i} + \frac{1}{2} N^i_j(x, y^{(1)}) y^{(1)j} \tag{5.56}$$

is a regular Lagrangian on the manifold $Osc^2(M)$. It depends only on the fundamental function F of the Finsler space F^n.

Proposition 5.2.2. *[37] Let $\rho : Osc^2(M) \to R$ be a 0-homogeneous function constante on $y^{(2)}$-direction, if g_{ij} is the fundamental tensor of the Finsler space F^n then $\widetilde{g}_{ij} = e^{2\rho} g_{ij}$ is a fundamental tensor of the Finsler space having $\widetilde{F} = e^{\rho} F$ as fundamental function.*

Proof. It is known that F is positively 1-homogeneous. Then we have

$$e^{\rho(x, \lambda y^{(1)})} F(x, \lambda y^{(1)}) = \lambda e^{\rho(x, y^{(1)})} F(x, y^{(1)}),$$

since ρ is 0-homogeneous. This means $\widetilde{F}$ is positively homogeneous of degree 1. As F is fundamental function of Finsler space F^n one has

$$\widetilde{F}^2 = e^{2\rho} g_{ij}(x, y) y^{(1)i} y^{(1)j}.$$

□

Corollary 5.2.1. *The pair $(M, \widetilde{F})$ is a Finsler space and the function $\widetilde{L} : Osc^2(M) \to R$ given by*

$$\widetilde{L}(x, y^{(1)}, y^{(2)}) = e^{2\rho} L + \psi, \tag{5.57}$$

where

$$\psi = e^{2\rho} g_{ij} (z^{(2)i} \Omega_j + \Omega_i z^{(2)i} + \Omega_i \Omega_j),$$

and Ω_i follows from (5.43), is a regular Lagrangian depending on the fundamental function $\widetilde{F}$ of the Finsler space $(M, \widetilde{F})$.

Note that in proposition 5.2.2, the fact that ρ is 0-homogeneous allows the function $\widetilde{F}$ to satisfy the assumptions of the definition 5.2.1.

5.3 Canonical metrical N-linear connections under the conformal change

We start this section by establishing the expressions of Lie-brackets under the conformal change.

5.3.1 Lie brackets and conformal deformation

Proposition 5.3.1. *[40] The Lie bracket of the vector fields of the adapted basis are given by*

$$[\delta_j, \delta_k] = \underset{(01)}{R^i_{jk}}\delta_{1i} + \underset{(02)}{R^i_{jk}}\delta_{2i},\ [\delta_j, \delta_{1k}] = \underset{(11)}{B^i_{jk}}\delta_{1i} + \underset{(12)}{B^i_{jk}}\delta_{2i} \tag{5.58}$$

$$[\delta_j, \delta_{2k}] = \underset{(21)}{B^i_{jk}}\delta_{1i} + \underset{(22)}{B^i_{jk}}\delta_{2i},\ [\delta_{1j}, \delta_{2k}] = \underset{(12)}{R^i_{jk}}\delta_{2i}, \tag{5.59}$$

$$[\delta_{2j}, \delta_{2k}] = \underset{(21)}{B^i_{jk}}\delta_{2i}, \tag{5.60}$$

where

$$\begin{aligned}
&\underset{(01)}{R^i_{jk}} = \delta_k \underset{(1)}{N^i_j} - \delta_j \underset{(1)}{N^i_k}, \underset{(02)}{R^i_{jk}} = \underset{(1)}{N^i_m}\underset{(01)}{R^m_{jk}} + \delta_k \underset{(2)}{N^i_j} - \delta_j \underset{(2)}{N^i_j},\\
&\underset{(11)}{B^i_{jk}} = \delta_{1k}\underset{(1)}{N^i_j},\ \underset{(12)}{B^i_{jk}} = \underset{(1)}{N^i_m}\underset{(11)}{B^m_{jk}} + \delta_{1k}\underset{(2)}{N^i_j} - \delta_{1j}\underset{(2)}{N^i_j},\\
&\underset{(21)}{B^i_{jk}} = \delta_{2k}\underset{(1)}{N^i_j},\ \underset{(22)}{B^i_{jk}} = \underset{(1)}{N^i_m}\underset{(21)}{B^m_{jk}} + \delta_{2k}\underset{(2)}{N^i_j},\ \underset{(12)}{R^i_{jk}} = \delta_{1k}\underset{(1)}{N^i_j} - \delta_{1j}\underset{(1)}{N^i_k}.
\end{aligned} \tag{5.61}$$

Remark 5.3.1. Note that the N-linear connection N is integrable if and only if

$$\underset{(01)}{R^i_{jk}} = \underset{(02)}{R^i_{jk}} = 0,$$

and the vertical distribution V_1 is integrable if and only if

$$\underset{(12)}{R^i_{jk}} = 0.$$

The analogue result to proposition 5.3.1 for the conformal adapted basis is given as follows.

Theorem 5.3.1. *[37] Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two 2-Lagrangian on M with fundamental tensors g and $\widetilde{g}$, respectively, such that (5.9) is satisfied. Then the Lie brackets of the conformal adapted basis is given by*

$$[\widetilde{\delta}_j, \widetilde{\delta}_k] = [\delta_j, \delta_k] + \underset{(01)}{r^i_{jk}}\delta_{1i} - \underset{(01)}{r^i_{jk}}\hat{\partial}_{1m} + \underset{(02)}{r^i_{jk}}\delta_{2i}, \tag{5.62}$$

$$[\widetilde{\delta}_j, \widetilde{\delta}_{1k}] = [\delta_j, \delta_{1k}] + \underset{(11)}{b^i_{jk}}\delta_{1i} + \underset{(11)}{b^i_{jk}}\hat{\partial}_{1m} + \underset{(12)}{b^i_{jk}}\partial_{2m}, \tag{5.63}$$

$$[\widetilde{\delta}_j, \widetilde{\delta}_{2k}] = [\delta_j, \delta_{2k}] - \underset{(21)}{b^i_{jk}}\delta_{1i} - \underset{(21)}{b^i_{jk}}\hat{\partial}_{1m} + \underset{(22)}{b^i_{jk}}\partial_{2i}, \tag{5.64}$$

$$[\widetilde{\delta}_{1j}, \widetilde{\delta}_{1k}] = [\delta_{1j}, \delta_{1k}] - \underset{(12)}{r^i_{jk}}\delta_{2i}, \tag{5.65}$$

$$[\widetilde{\delta}_{1j}, \widetilde{\delta}_{2k}] = [\delta_{1j}, \delta_{2k}] + \underset{(21)}{b^i_{jk}}\partial_{2i}, \tag{5.66}$$

where

$$\underset{(01)}{r^i_{jk}} = \delta_k \underset{(1)}{H^j_i} - \delta_j \underset{(1)}{H^k_i} + \hat{\partial}_m \underset{(1)}{N^i_j} - \hat{\partial}_m \underset{(1)}{N^i_k} + \hat{\partial}_m \underset{(1)}{H^k_i} - \hat{\partial}_m \underset{(1)}{H^i_k}, \tag{5.67}$$

$$\underset{(02)}{r^i_{jk}} = \underset{(1)}{N^i_m}\underset{(01)}{r^m_{jk}} + \underset{(1)}{H^i_m}\underset{(01)}{R^m_{jk}} + \delta_k \underset{(2)}{H^i_j} + \hat{\partial}_k \underset{(2)}{N^i_j} - \delta_j \underset{(2)}{H^i_k} - \hat{\partial}_j \underset{(2)}{N^i_k}, \tag{5.68}$$

$$\underset{(11)}{b^i_{jk}} = \delta_{1k} \underset{(1)}{H^j_i} + \hat{\partial}_{1k} \underset{(1)}{N^j_i}, \tag{5.69}$$

$$\underset{(12)}{b^i_{jk}} = \underset{(1)}{N^i_m}\underset{(11)}{b^m_{jk}} + \underset{(1)}{H^i_m}\underset{(11)}{B^m_{jk}} + \delta_{1k} \underset{(2)}{H^i_j} + \hat{\partial}_{1k} \underset{(2)}{N^i_j} - \delta_{1j} \underset{(2)}{H^i_k} - \hat{\partial}_{1j} \underset{(2)}{N^i_k}, \tag{5.70}$$

$$\underset{(21)}{b^i_{jk}} = \delta_{2k} \underset{(1)}{H^j_i}, \tag{5.71}$$

$$\underset{(22)}{b^i_{jk}} = \underset{(1)}{N^i_m}\underset{(21)}{b^m_{jk}} + \delta_{2k} \underset{(2)}{H^i_k}, \tag{5.72}$$

$$\underset{(12)}{r^i_{jk}} = \delta_{1k} \underset{(1)}{H^j_i} - \delta_{1j} \underset{(1)}{H^k_i} + \hat{\partial}_{1m} \underset{(1)}{N^i_j} - \hat{\partial}_{1m} \underset{(1)}{N^i_k} + \hat{\partial}_{1m} \underset{(1)}{H^k_i} - \hat{\partial}_{1m} \underset{(1)}{H^i_k}, \tag{5.73}$$

Proof. Indeed, we have $\underset{(01)}{\widetilde{R}^i_{jk}} = \widetilde{\delta}_k \underset{(1)}{\widetilde{N}^i_j} - \widetilde{\delta}_j \underset{(1)}{\widetilde{N}^i_k}$ then by using the equations (5.32) and (5.31) one gets $\underset{(01)}{\widetilde{R}^i_{jk}} = \delta_k \underset{(1)}{N^i_j} - \delta_j \underset{(1)}{N^i_k} + \delta_k \underset{(1)}{H^j_i} - \delta_j \underset{(1)}{H^k_i} + \hat{\partial}_m \underset{(1)}{N^i_j} - \hat{\partial}_m \underset{(1)}{N^i_k} + \hat{\partial}_m \underset{(1)}{H^k_i} - \hat{\partial}_m \underset{(1)}{H^i_k} = \underset{(01)}{R^i_{jk}} + \underset{(01)}{r^i_{jk}}$.

Likewise, we obtain $\underset{(02)}{\widetilde{R}^i_{jk}} = \underset{(02)}{R^i_{jk}} + \underset{(02)}{r^i_{jk}}$, then we obtain (5.62). In the same way the relations (5.63)-(5.66) are obtained. □

Now, let us endow the osculator bundle $Osc^2(M)$ with the Riemannian metric (4.5) given in [5, p.72] by:

$$\mathbb{G} = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^{(1)i} \otimes \delta y^{(1)j} + g_{ij}\delta y^{(2)i} \otimes \delta y^{(2)j}, \tag{5.74}$$

where

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(2)i}\partial y^{(2)j}}.$$

A connection D on $L^{(2)n}$ is an N-linear connection if

(1) D preserves by parallelism the horizontal distribution N,

(2) $DJ = 0$.

D is called canonical N-linear connection of a second order Lagrange space $L^{(2)n}$ if it satisfies the properties of the following theorem

Theorem 5.3.2. *[39] The following properties hold:*

(1) There exists a unique N-linear connection D on $Osc_0^2(M)$ verifying the axioms

$$g_{ij|k} = 0,\ g_{ij} \overset{(1)}{|}_k,\ g_{ij} \overset{(2)}{|}_k \tag{5.75}$$

$$\underset{(0)}{T^i_{jk}} = 0,\ \underset{(1)}{S^i_{jk}} = 0,\ \underset{(2)}{S^i_{jk}} = 0 \tag{5.76}$$

(2) This connection has the coefficients

$$\begin{aligned} L^i_{jk} &= \frac{1}{2} g^{is}(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}), \\ \underset{(\alpha)}{C^i_{jk}} &= \frac{1}{2} g^{is}(\delta_{\alpha j} g_{js} + \delta_{\alpha k} g_{js} - \delta_{\alpha s} g_{jk}),\ \alpha = 1, 2, \end{aligned} \tag{5.77}$$

where $\underset{(0)}{T^i_{jk}}$, $\underset{(1)}{S^i_{jk}}$ and $\underset{(2)}{S^i_{jk}}$ are the coefficients of the $d-$tensor of torsion respectively along the directions of the distributions N, V_1 and V_2 given by

$$\underset{(0)}{T^i_{jk}} = L^i_{jk} - L^i_{kj},\ \underset{(\alpha)}{S^i_{jk}} = \underset{(\alpha)}{C^i_{jk}} - \underset{(\alpha)}{C^i_{kj}},\ \alpha = 1, 2. \tag{5.78}$$

Denote by $\widetilde{D}$ the canonical metrical N-linear connection under the conformal change, it is clear that it is given by means of the coefficients

$$C\Gamma(N) = \left\{ \widetilde{L}^i_{jk}(x, y^{(1)}, y^{(2)}), \underset{(1)}{\widetilde{C}^i_{jk}}(x, y^{(1)}, y^{(2)}), \underset{(2)}{\widetilde{C}^i_{jk}}(x, y^{(1)}, y^{(2)}) \right\}. \tag{5.79}$$

The covariant derivative of the conformal adapted basis is given by

$$\widetilde{D}_{\widetilde{\delta}_j} \widetilde{\delta}_{si} = \widetilde{L}^m_{ij} \widetilde{\delta}_{sm},\ \widetilde{D}_{\widetilde{\delta}_{\alpha j}} \widetilde{\delta}_{si} = \underset{(\alpha)}{\widetilde{C}^m_{ij}} \widetilde{\delta}_{sm},\ \alpha = 1, 2,\ s = 0, 1, 2. \tag{5.80}$$

It follows then the relations

$$\widetilde{g}_{ij|_h} = 0,\ \widetilde{g}_{ij} \overset{(1)}{|}_h = \widetilde{g}_{ij} \overset{(2)}{|}_h = 0,\ \widetilde{L}^i_{jk} = \widetilde{L}^i_{kj},\ \underset{(\alpha)}{\widetilde{C}^i_{jk}} = \underset{(\alpha)}{\widetilde{C}^i_{kj}},\ (\alpha = 1, 2) \tag{5.81}$$

hold.

Lemma 5.3.1. *Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two 2-Lagrangians on M with fundamental tensors g and $\widetilde{g}$, respectively, such that (5.9) is satisfied. Then, the coefficients of the N-linear connections D and $\widetilde{D}$ are related as*

$$\widetilde{L}^i_{jk} = L^i_{jk} + l^i_{jk}, \tag{5.82}$$

$$\underset{(1)}{\widetilde{C}^i_{jk}} = \underset{(1)}{C^i_{jk}} + \underset{(1)}{c^i_{jk}}, \tag{5.83}$$

$$\underset{(2)}{\widetilde{C}^i_{jk}} = \underset{(2)}{C^i_{jk}}, \tag{5.84}$$

where

$$
\begin{aligned}
l^i_{jk} &= \frac{1}{2}(\delta_j\rho\delta^i_k + \delta_k\rho\delta^i_j - \delta_s\rho g_{jk}g^{is}) + \frac{1}{2}g^{is}(\hat{\partial}_j g_{sk} + \hat{\partial}_k g_{js} - \hat{\partial}_s g_{jk}) \\
&+ \frac{1}{2}(\hat{\partial}_j\rho\delta^i_k + \hat{\partial}_k\rho\delta^i_j - \hat{\partial}_s\rho g_{jk}g^{is}), \\
\textit{and}\ \ \underset{(1)}{c^i_{jk}} &= \frac{1}{2}(\delta_{1j}\rho\delta^i_k + \delta_{1k}\rho\delta^i_j - \delta_{1s}\rho g_{jk}g_{is}) + \frac{1}{2}g^{is}(\hat{\partial}_{1j} g_{sk} + \hat{\partial}_{1k} g_{js} - \hat{\partial}_{1s} g_{jk}). \quad (5.85)
\end{aligned}
$$

Proof. Indeed, under the conformal change, the equations (5.77) become

$$
\begin{aligned}
\widetilde{L}^i_{jk} &= \frac{1}{2}\widetilde{g}^{is}(\widetilde{\delta}_j\widetilde{g}_{sk} + \widetilde{\delta}_k\widetilde{g}_{js} - \widetilde{\delta}_s\widetilde{g}_{jk}), \\
\underset{(\alpha)}{\widetilde{C}^i_{jk}} &= \frac{1}{2}\widetilde{g}^{is}(\widetilde{\delta}_{\alpha j}g_{js} + \widetilde{\delta}_{\alpha k}\widetilde{g}_{js} - \widetilde{\delta}_{\alpha s}\widetilde{g}_{jk}),\ \alpha = 1, 2, \quad (5.86)
\end{aligned}
$$

then by using equations (5.32) in (5.86), we get

$$
\begin{aligned}
\widetilde{L}^i_{jk} &= \frac{1}{2}\widetilde{g}^{is}((\delta_j + \hat{\partial}_j)(e^\rho g_{sk}) + (\delta_k + \hat{\partial}_k)(e^\rho g_{js}) - (\delta_s + \hat{\partial}_s)(e^\rho g_{jk})) \\
&= L^i_{jk} + \frac{1}{2}(\delta_j\rho\delta^i_k + \delta_k\rho\delta^i_j - \delta_s\rho g_{jk}g^{is}) + \frac{1}{2}g^{is}(\hat{\partial}_j g_{sk} + \hat{\partial}_k g_{js} - \hat{\partial}_s g_{jk}) \\
&+ \frac{1}{2}(\hat{\partial}_j\rho\delta^i_k + \hat{\partial}_k\rho\delta^i_j - \hat{\partial}_s\rho g_{jk}g^{is}),
\end{aligned}
$$

$$
\begin{aligned}
\underset{(1)}{\widetilde{C}^i_{jk}} &= \frac{1}{2}\widetilde{g}^{is}((\delta_{1j} + \hat{\partial}_{1j})(e^\rho g_{sk}) + (\delta_{1k} + \hat{\partial}_{1k})(e^\rho g_{js}) - (\delta_{1s} + \hat{\partial}_{1s})(e^\rho g_{jk})) \\
&= \underset{(1)}{C^i_{jk}} + \frac{1}{2}(\delta_{1j}\rho\delta^i_k + \delta_{1k}\rho\delta^i_j - \delta_{1s}\rho g_{jk}g_{is}) + \frac{1}{2}g^{is}(\hat{\partial}_{1j} g_{sk} + \hat{\partial}_{1k} g_{js}) - \hat{\partial}_{1s} g_{jk}),
\end{aligned}
\quad (5.87)
$$

which complete the proof. □

5.3.2 Conformal change of d-tensors of Curvature

In this section, we investigate the relationship between the d-tensor of curvature associated to D and $\widetilde{D}$. For a N-linear connection D in $L^{(2)n}$ the coefficients of the

d-tensor of the curvature are given by (see [40, theorem 3.5.3] for more details)

$$R^h_{ijk} = \delta_i L^h_{jk} - \delta_j L^h_{ik} + L^s_{jk} L^h_{is} - L^s_{ik} L^h_{js} - \underset{(01)}{R}{}^s_{ij} \underset{(1)}{C}{}^h_{sk} - \underset{(02)}{R}{}^s_{ij} \underset{01}{C}{}^h_{sk}, \tag{5.88}$$

$$\underset{(1)}{P}{}^h_{ijk} = \delta_{1i} L^h_{jk} - \delta_j \underset{(1)}{C}{}^h_{ik} + L^s_{jk} \underset{(1)}{C}{}^h_{is} - \underset{(1)}{C}{}^s_{ik} L^h_{js} + \underset{(11)}{B}{}^s_{ji} \underset{(1)}{C}{}^h_{sk} + \underset{(02)}{B}{}^s_{ji} \underset{(2)}{C}{}^h_{sk}, \tag{5.89}$$

$$\underset{(2)}{P}{}^h_{ijk} = \delta_{2i} L^h_{jk} - \delta_j \underset{(2)}{C}{}^h_{ik} + L^s_{jk} \underset{(2)}{C}{}^h_{is} - \underset{(1)}{C}{}^s_{ik} L^h_{js} + \underset{(21)}{B}{}^s_{ji} \underset{(1)}{C}{}^h_{sk} + \underset{(22)}{B}{}^s_{ji} \underset{(2)}{C}{}^h_{sk}, \tag{5.90}$$

$$\underset{(21)}{S}{}^h_{ijk} = \delta_{2i} \underset{(1)}{C}{}^h_{jk} - \delta_{1j} \underset{(2)}{C}{}^h_{jk} + \underset{(1)}{C}{}^s_{jk} \underset{(2)}{C}{}^h_{is} - \underset{(2)}{C}{}^s_{ik} \underset{(1)}{C}{}^h_{js} + \underset{(21)}{B}{}^s_{ji} \underset{(2)}{C}{}^h_{sk} \tag{5.91}$$

$$\underset{(11)}{S}{}^h_{ijk} = \delta_{1i} \underset{(1)}{C}{}^h_{jk} - \delta_{1j} \underset{(1)}{C}{}^h_{ik} + \underset{(1)}{C}{}^s_{jk} \underset{(1)}{C}{}^h_{is} - \underset{(1)}{C}{}^s_{ik} \underset{(1)}{C}{}^h_{js} + \underset{(12)}{R}{}^s_{ij} \underset{(2)}{C}{}^h_{sk}, \tag{5.92}$$

$$\underset{(22)}{S}{}^h_{ijk} = \underset{(22)}{S}{}^h_{ijk}, \tag{5.93}$$

with

$$R^h_{ijk} = g(R(\delta_i, \delta_j)\delta_k, \delta_h),\ \underset{(\alpha)}{P}{}^h_{ijk} = g(R(\delta_{\alpha i}, \delta_j)\delta_k, \delta_h),$$

$$\underset{(11)}{S}{}^h_{ijk} = g(R(\delta_{1i}, \delta_{1j})\delta_k, \delta_h),\ \underset{(2\alpha)}{S}{}^h_{ijk} = g(R(\delta_{2i}, \delta_{\alpha j})\delta_k, \delta_h),\ \alpha = 1, 2.$$

In the next proposition, we give the d-tensor of curvature under the conformal change of the fundamental tensor of the Lagrange space (M, L) in terms of d-tensor of curvature of the Lagrange space (M, L). Thus, in view of equations (5.32), (5.82), (5.83) and (5.84), we have the following

Proposition 5.3.2. *Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two fundamental functions on M with fundamental tensors g and $\widetilde{g}$, respectively, such that (5.9) is satisfied. Then, the coefficients of the d-tensor of the curvatures associated to D and $\widetilde{D}$ are given by*

$$\widetilde{R}^h_{ijk} = R^h_{ijk} + r^h_{ijk},\ \underset{(1)}{\widetilde{P}}{}^h_{ijk} = \underset{(1)}{P}{}^h_{ijk} + \underset{(1)}{p}{}^h_{ijk}, \tag{5.94}$$

$$\underset{(2)}{\widetilde{P}}{}^h_{ijk} = \underset{(2)}{P}{}^h_{ijk} + \underset{(2)}{p}{}^h_{ijk},\ \underset{(21)}{\widetilde{S}}{}^h_{ijk} = \underset{(21)}{S}{}^h_{ijk} + \underset{(21)}{s}{}^h_{ijk}, \tag{5.95}$$

$$\underset{(11)}{\widetilde{S}}{}^h_{ijk} = \underset{(11)}{S}{}^h_{ijk} + \underset{(11)}{s}{}^h_{ijk}, \tag{5.96}$$

where

$$\begin{aligned}
r^h_{ijk} &= \delta_i l^h_{jk} + \hat{\partial}_i \widetilde{L}^h_{jk} - \delta_j l^h_{ik} - \hat{\partial}_j \widetilde{L}^h_{ik} + L^s_{jk} l^h_{is} + l^s_{jk} \widetilde{L}^h_{is} - L^s_{ik} l^h_{js} \\
&\quad - l^s_{ik} \widetilde{L}^h_{js} - \underset{(01)}{R^s_{ij}} \underset{(1)}{c^h_{sk}} - \underset{(01)}{r^s_{ij}} \underset{(1)}{\widetilde{C}^h_{sk}} - \underset{(02)}{R^s_{ij}} \underset{01}{c^h_{sk}} - \underset{(02)}{r^s_{ij}} \underset{01}{\widetilde{C}^h_{sk}}, \qquad (5.97)\\
\underset{(1)}{p^h_{ijk}} &= \delta_{1i} l^h_{jk} + \hat{\partial}_{1i} \widetilde{L}^h_{jk} - \delta_{1j} l^h_{ik} - \hat{\partial}_{1j} \widetilde{L}^h_{ik} + L^s_{jk} \underset{(1)}{c^h_{is}} + l^s_{jk} \underset{(1)}{\widetilde{C}^h_{is}} - \underset{(1)}{C^s_{ik}} l^h_{js} \\
&\quad - \underset{(1)}{c^s_{ik}} \widetilde{L}^h_{js} + \underset{(11)}{B^s_{ji}} \underset{(1)}{c^h_{sk}} + \underset{(11)}{B^s_{ji}} \underset{(1)}{\widetilde{C}^h_{sk}} + \underset{(12)}{B^s_{ji}} \underset{(1)}{c^h_{sk}} + \underset{(12)}{b^s_{ji}} \underset{(1)}{\widetilde{C}^h_{sk}}, \qquad (5.98)\\
\underset{(2)}{p^h_{ijk}} &= \delta_{2i} l^h_{jk} + \hat{\partial}_{2i} \widetilde{L}^h_{jk} - \delta_j \underset{(2)}{c^h_{ik}} - \hat{\partial}_j \underset{(2)}{\widetilde{C}^h_{ik}} + L^s_{jk} \underset{(2)}{c^h_{is}} + l^s_{jk} \underset{(2)}{\widetilde{C}^h_{is}} - \underset{(1)}{C^s_{ik}} l^h_{js} \\
&\quad - \underset{(1)}{c^s_{ik}} \widetilde{L}^h_{js} + \underset{(21)}{B^s_{ji}} \underset{(1)}{c^h_{sk}} + \underset{(21)}{b^s_{ji}} \underset{(1)}{\widetilde{C}^h_{sk}} + \underset{(22)}{B^s_{ji}} \underset{(2)}{c^h_{sk}} + \underset{(22)}{b^s_{ji}} \underset{(2)}{\widetilde{C}^h_{sk}}, \qquad (5.99)
\end{aligned}$$

$$\begin{aligned}
\underset{(21)}{s^h_{ijk}} &= \delta_{2i} \underset{(1)}{c^h_{jk}} + \hat{\partial}_{2i} \underset{(1)}{\widetilde{C}^h_{jk}} - \delta_{1j} \underset{(2)}{c^h_{ik}} - \hat{\partial}_{1j} \underset{(2)}{\widetilde{C}^h_{ik}} + \underset{(1)}{C^s_{jk}} \underset{(2)}{c^h_{is}} + \underset{(1)}{c^s_{jk}} \underset{(2)}{\widetilde{C}^h_{is}} - \underset{(2)}{C^s_{ik}} \underset{(1)}{c^h_{js}} \\
&\quad - \underset{(2)}{c^s_{ik}} \underset{(1)}{\widetilde{C}^h_{js}} + \underset{(21)}{B^s_{ji}} \underset{(2)}{c^h_{sk}} + \underset{(21)}{b^s_{ji}} \underset{(2)}{\widetilde{C}^h_{sk}}, \qquad (5.100)\\
\underset{(11)}{s^h_{ijk}} &= \delta_{1i} \underset{(1)}{c^h_{jk}} + \hat{\partial}_{1i} \underset{(1)}{\widetilde{C}^h_{jk}} - \delta_{1j} \underset{(1)}{c^h_{ik}} - \hat{\partial}_{1j} \underset{(1)}{\widetilde{C}^h_{ik}} + \underset{(1)}{C^s_{jk}} \underset{(1)}{c^h_{is}} + \underset{(1)}{c^s_{jk}} \underset{(1)}{\widetilde{C}^h_{is}} + \underset{(12)}{R^s_{ij}} \underset{(2)}{c^h_{sk}} \\
&\quad + \underset{(12)}{r^s_{ij}} \underset{(2)}{\widetilde{C}^h_{sk}}. \qquad (5.101)
\end{aligned}$$

Proposition 5.3.3. *Let M be a real n-dimensional manifold, L and $\widetilde{L}$ be two 2-Lagrangians on M with fundamental tensors g and $\widetilde{g}$, respectively, such that (5.9) is satisfied. Then, the d-tensors of the Ricci and the scalar curvature are given by*

$$\begin{aligned}
\widetilde{Ric}_R(\widetilde{\delta}_i, \widetilde{\delta}_k) &= Ric_R(\delta_i, \delta_k) + \hat{\partial}_i \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) - l^s_{ik} \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) + \delta_i \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) \\
&\quad - \delta_i \mathrm{div}^D(\delta_k) - L^s_{ik} \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) + L^s_{ik} \mathrm{div}^D(\delta_k) \\
&\quad - \sum_{j=1}^n (\delta_j l^j_{ik} + \hat{\partial}_j \widetilde{L}^j_{ik} - L^s_{jk} l^j_{is} - l^s_{jk} \widetilde{L}^j_{is} + \underset{(01)}{R^s_{ij}} \underset{(1)}{c^j_{sk}} + \underset{(01)}{r^s_{ij}} \underset{(1)}{\widetilde{C}^j_{sk}} \\
&\quad + \underset{(02)}{R^s_{ij}} \underset{01}{c^j_{sk}} + \underset{(02)}{r^s_{ij}} \underset{01}{\widetilde{C}^j_{sk}}), \qquad (5.102)
\end{aligned}$$

$$\begin{aligned}
\widetilde{Ric}_{P_\alpha}(\widetilde{\delta}_i, \widetilde{\delta}_k) &= Ric_{P_\alpha}(\widetilde{\delta}_i, \widetilde{\delta}_k) + \delta_{\alpha i} \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) - \delta_{\alpha i} \mathrm{div}^D(\delta_k) \\
&\quad + \hat{\partial}_{\alpha i} \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) - \underset{(\alpha)}{\widetilde{C}^s_{ik}} (\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) - \mathrm{div}^D(\delta_k)) - \underset{(\alpha)}{\widetilde{c}^s_{ik}} \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s \\
&\quad - \sum_{j=1}^n (\delta_{\alpha j} l^j_{ik} + \hat{\partial}_{\alpha j} \widetilde{L}^j_{ik} - L^s_{jk} \underset{(\alpha)}{c^j_{is}} - l^s_{jk} \underset{(\alpha)}{\widetilde{C}^j_{is}} + \underset{(\alpha 1)}{B^s_{ij}} \underset{(\alpha)}{c^j_{sk}} + \underset{(\alpha 1)}{B^s_{ij}} \underset{(\alpha)}{\widetilde{C}^j_{sk}} \\
&\quad + \underset{(\alpha 2)}{B^s_{ij}} \underset{(\alpha)}{\widetilde{c}^j_{sk}} + \underset{(\alpha 2)}{b^s_{ij}} \underset{(\alpha)}{\widetilde{C}^j_{sk}}), \qquad (5.103)
\end{aligned}$$

$$\begin{aligned}\widetilde{Ric}_{S_{21}}(\widetilde{\delta}_i,\widetilde{\delta}_k) &= Ric_{S_{21}}(\widetilde{\delta}_i,\widetilde{\delta}_k) + \delta_{2i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) - \delta_{2i}\mathrm{div}^{D}(\delta_k)\\ &+ \hat{\partial}_{2i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) - \underset{(2)}{\widetilde{C}^s_{ik}}(\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) - \mathrm{div}^{D}(\delta_s)) - \underset{(2)}{\widetilde{c}^s_{ik}}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s)\\ &- \sum_{j=1}^{n}(\delta_{1j}\underset{(2)}{c^j_{ik}} + \hat{\partial}_{1j}\underset{(2)}{\widetilde{C}^j_{ik}} - \underset{(1)}{C^s_{jk}}\underset{(2)}{c^j_{is}} - \underset{(1)}{c^s_{jk}}\underset{(2)}{\widetilde{C}^j_{is}} - \underset{(21)}{B^s_{ij}}\underset{(2)}{c^j_{sk}} - \underset{(21)}{b^s_{ij}}\underset{(2)}{\widetilde{C}^j_{sk}}),\end{aligned} \tag{5.104}$$

$$\begin{aligned}\widetilde{Ric}_{S_{11}}(\widetilde{\delta}_i,\widetilde{\delta}_k) &= Ric_{S_{11}}(\widetilde{\delta}_i,\widetilde{\delta}_k) + \delta_{1i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) - \delta_{1i}\mathrm{div}^{D}(\delta_k) + \hat{\partial}_{1i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k)\\ &- \sum_{j=1}^{n}(\delta_{1j}c^j_{ik} + \hat{\partial}_{1j}\underset{(1)}{\widetilde{C}^j_{ik}} + \underset{(1)}{C^s_{jk}}\underset{(1)}{c^j_{is}} - \underset{(1)}{c^s_{jk}}\underset{(1)}{\widetilde{C}^j_{is}} - \underset{(12)}{R^s_{ij}}\underset{(2)}{c^j_{sk}} - \underset{(12)}{r^s_{ij}}\underset{(2)}{\widetilde{C}^j_{sk}}),\end{aligned} \tag{5.105}$$

$$\begin{aligned}\widetilde{K}_R &= K_R + \sum_{k=1}^{n}(\hat{\partial}_i\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) - l^s_{ii}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) + \delta_i\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) - \delta_i\mathrm{div}^{D}(\delta_i)\\ &- L^s_{ii}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) + L^s_{ii}\mathrm{div}^{D}(\delta_i)) + \sum_{j=1,k=1}^{n,n}(L^s_{ji}l^j_{is} - \delta_j l^j_{ii} - \hat{\partial}_j\widetilde{L}^j_{ii}\\ &+ l^s_{ji}\widetilde{L}^j_{is} - \underset{(01)}{R^s_{ij}}\underset{(1)}{c^j_{si}} - \underset{(01)}{r^s_{ij}}\underset{(1)}{\widetilde{C}^j_{si}} - \underset{(02)}{R^s_{ij}}\underset{01}{c^j_{si}} - \underset{(02)}{r^s_{ij}}\underset{01}{\widetilde{C}^j_{si}}),\end{aligned} \tag{5.106}$$

$$\begin{aligned}\widetilde{K}_{P_\alpha} &= K_{P_\alpha} + \sum_{i,j=1}^{n}(\delta_{\alpha i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) - \delta_{\alpha i}\mathrm{div}^{D}(\delta_i) + \hat{\partial}_{\alpha i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i)\\ &- \underset{(\alpha)}{\widetilde{C}^s_{ii}}(\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) - \mathrm{div}^{D}(\delta_i) - \underset{(\alpha)}{\widetilde{c}^s_{ii}}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s))\\ &- \sum_{i,j=1}^{n}(\delta_{\alpha j}l^j_{ii} + \hat{\partial}_{\alpha j}\widetilde{L}^j_{ii} - L^s_{ji}\underset{(\alpha)}{c^j_{is}} - l^s_{ji}\underset{(\alpha)}{\widetilde{C}^j_{is}} + \underset{(\alpha1)}{B^s_{ij}}\underset{(\alpha)}{c^j_{si}} + \underset{(\alpha1)}{B^s_{ij}}\underset{(\alpha)}{\widetilde{C}^j_{si}}\\ &+ \underset{(\alpha2)}{B^s_{ij}}\underset{(\alpha)}{\widetilde{c}^j_{si}} + \underset{(\alpha2)}{b^s_{ij}}\underset{(\alpha)}{\widetilde{C}^j_{si}}),\end{aligned} \tag{5.107}$$

$$\begin{aligned}\widetilde{K}_{S_{21}} &= K_{S_{21}} + \sum_{i=1}^{n}(\delta_{2i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) - \delta_{2i}\mathrm{div}^{D}(\delta_i) + \hat{\partial}_{2i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i)\\ &- \underset{(2)}{\widetilde{C}^s_{ii}}(\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) - \mathrm{div}^{D}(\delta_s)) - \underset{(2)}{\widetilde{c}^s_{ii}}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s))\\ &- \sum_{i,j=1}^{n}(\delta_{1j}\underset{(2)}{c^j_{ii}} + \hat{\partial}_{1j}\underset{(2)}{\widetilde{C}^j_{ii}} - \underset{(1)}{C^s_{ji}}\underset{(2)}{c^j_{is}} - \underset{(1)}{c^s_{ji}}\underset{(2)}{\widetilde{C}^j_{is}} - \underset{(21)}{B^s_{ij}}\underset{(2)}{c^j_{si}} - \underset{(21)}{b^s_{ij}}\underset{(2)}{\widetilde{C}^j_{si}}),\end{aligned} \tag{5.108}$$

$$\begin{aligned}\widetilde{K}_{S_{11}} &= K_{S_{11}} + \sum_{i=1}^{n}(\delta_{1i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) - \delta_{1i}\mathrm{div}^{D}(\delta_i) + \hat{\partial}_{1i}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i))\\ &- \sum_{i,j=1}^{n}(\delta_{1j}c^j_{ii} + \hat{\partial}_{1j}\underset{(1)}{\widetilde{C}^j_{ii}} + \underset{(1)}{C^s_{ji}}\underset{(1)}{c^j_{is}} - \underset{(1)}{c^s_{ji}}\underset{(1)}{\widetilde{C}^j_{is}} - \underset{(12)}{R^s_{ij}}\underset{(2)}{c^j_{si}} - \underset{(12)}{r^s_{ij}}\underset{(2)}{\widetilde{C}^j_{si}}),\end{aligned} \tag{5.109}$$

where $\mathrm{div}^D(\delta_k) = \sum_{j=1}^{n} g^{jj} g(D_{\delta_j}\delta_k, \delta_j)$ *and* $\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) = \sum_{j=1}^{n} \widetilde{g}^{jj}\widetilde{g}(\widetilde{D}_{\widetilde{\delta}_j}\widetilde{\delta}_k, \widetilde{\delta}_j)$.

Proof. We have $\widetilde{R}^h_{ijk} = R^h_{ijk} + r^h_{ijk}$, and contracting j by h in the term R^h_{ijk}, we obtain $R^j_{ijk} = Ric(\delta_i, \delta_k)$, and in term r^h_{ijk},

$$\sum_{j=1}^{n} r^j_{ijk} = \sum_{j=1}^{n}(\delta_i l^j_{jk} + \hat{\partial}_i \widetilde{L}^j_{jk} - \delta_j l^j_{ik} - \hat{\partial}_j \widetilde{L}^j_{ik} + L^s_{jk} l^j_{is} + l^s_{jk}\widetilde{L}^j_{is} - L^s_{ik} l^j_{js} - l^s_{ik}\widetilde{L}^j_{js}$$
$$- \underset{(01)}{R^s_{ij}} \underset{(1)}{c^j_{sk}} - \underset{(01)}{r^s_{ij}} \underset{(1)}{\widetilde{C}^j_{sk}} - \underset{(02)}{R^s_{ij}} \underset{01}{c^j_{sk}} - \underset{(02)}{r^s_{ij}} \underset{01}{\widetilde{C}^j_{sk}}). \tag{5.110}$$

But $\sum_{j=1}^{n} \widetilde{L}^j_{jk} = \sum_{j=1}^{n} \widetilde{L}^j_{jk} = \sum_{j=1}^{n} \widetilde{g}(\widetilde{D}_{\widetilde{\delta}_j}\widetilde{\delta}_k, \widetilde{\delta}_j) = \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k)$, so we have

$$\sum_{j=1}^{n} r^j_{ijk} = \hat{\partial}_i \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) - l^s_{ik}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) + \delta_i \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) - \delta_i \mathrm{div}^{D}(\delta_k)$$
$$- L^s_{ik}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) + L^s_{ik}\mathrm{div}^{D}(\delta_k) + \sum_{j=1}^{n}(-\delta_j l^j_{ik} - \hat{\partial}_j \widetilde{L}^j_{ik} + L^s_{jk} l^j_{is}$$
$$+ l^s_{jk}\widetilde{L}^j_{is} - \underset{(01)}{R^s_{ij}} \underset{(1)}{c^j_{sk}} - \underset{(01)}{r^s_{ij}} \underset{(1)}{\widetilde{C}^j_{sk}} - \underset{(02)}{R^s_{ij}} \underset{01}{c^j_{sk}} - \underset{(02)}{r^s_{ij}} \underset{01}{\widetilde{C}^j_{sk}}), \tag{5.111}$$

which leads to (5.115). In the same manner for the scalar curvature, contracting k by i in the relation (5.115), one gets

$$\sum_{j=1,k=1}^{n,n} r^j_{iji} = \sum_{k=1}^{n}(\hat{\partial}_i \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) - l^s_{ii}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) + \delta_i \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i)$$
$$- \delta_i \mathrm{div}^{D}(\delta_i) - L^s_{ii}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) + L^s_{ii}\mathrm{div}^{D}(\delta_i)) + \sum_{j=1,k=1}^{n,n}(-\delta_j l^j_{ii} - \hat{\partial}_j \widetilde{L}^j_{ii}$$
$$+ L^s_{ji} l^j_{is} + l^s_{ji}\widetilde{L}^j_{is} - \underset{(01)}{R^s_{ij}} \underset{(1)}{c^j_{si}} - \underset{(01)}{r^s_{ij}} \underset{(1)}{\widetilde{C}^j_{si}} - \underset{(02)}{R^s_{ij}} \underset{01}{c^j_{si}} - \underset{(02)}{r^s_{ij}} \underset{01}{\widetilde{C}^j_{si}}), \tag{5.112}$$

which leads to (5.103). Similarly, we obtain the others expressions. $\square$

Using Example 5.2.1, the components of the d-tensor fields of curvatures are given as follows

$$L^i_{jk} = \gamma^i_{jk},\ l^i_{jk} = \frac{1}{2}(\delta_j \rho \delta^i_k + \delta_k \rho \delta^i_j - \delta_s \rho g_{jk} g^{is}),\ C^i_{jk} = 0,\ c^i_{jk} = 0. \tag{5.113}$$

Then the non-vanishing d-tensor fields are

$$\begin{aligned}
\widetilde{Ric}_R(\widetilde{\delta}_i, \widetilde{\delta}_k) &= Ric_R(\delta_i, \delta_k) - l^s_{ik}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_s) + \delta_i \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) \\
&- \delta_i \mathrm{div}^{D}(\delta_k) - \gamma^s_{ik}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_k) + \gamma^s_{ik}\mathrm{div}^{D}(\delta_k) \\
&- \sum_{j=1}^{n}(\delta_j l^j_{ik} - \gamma^s_{jk} l^j_{is} - l^s_{jk}\widetilde{\gamma}^j_{is}), \qquad (5.114) \\
\widetilde{K}_R &= K_R + \delta_i \mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) - \delta_i \mathrm{div}^{D}(\delta_i) - \gamma^s_{ii}\mathrm{div}^{\widetilde{D}}(\widetilde{\delta}_i) + \gamma^s_{ii}\mathrm{div}^{D}(\delta_i) \\
&- \sum_{i=1}^{n}(\delta_j l^j_{ii} - \gamma^s_{ji} l^j_{is} - l^s_{ji}\widetilde{\gamma}^j_{is}). \qquad (5.115)
\end{aligned}$$

Chapter Six

Prolongation of structures to $Osc^2(M)$

In this chapter the 2-osculator bundle $Osc^2(M)$ is endowed with an almost n-contact structure, we define an locally conformal almost cosymplectic structure on $Osc^2(M)$. We suppose M is endowed with an l.c. almost cosymplectic structure then we prolong it to $Osc^2(M)$.

6.1 Riemannian almost n-contact structures

Let us consider the linear mapping $\mathbb{F} : \mathfrak{X}(Osc^2(M)) \longrightarrow \mathfrak{X}(Osc^2(M))$ defined on the adapted basis by

$$\mathbb{F}(\delta_i) = -\delta_{1i}, \ \ \mathbb{F}(\delta_{1i}) = \delta_i, \ \ \mathbb{F}(\delta_{2i}) = 0, \ \ (i = 1, \cdots, n). \tag{6.1}$$

We have the following properties:

$$\ker\ \mathbb{F} = V_2, \ \ \mathrm{Im}\,\mathbb{F} = N_0 + N_1\ , \ \mathrm{rank}|\mathbb{F}| = 2n, \ \text{and}\ \mathbb{F}^3 + \mathbb{F} = 0.$$

Denote by $\{\underset{s}{\xi}\}$, $(s = 1, \cdots, n)$ the adapted vector fields of the distribution V_2 and their duals by $(\overset{s}{\eta})$.

Proposition 6.1.1. *The 2-osculator bundle $Osc^2(M)$ endowed with the structure $(\mathbb{F}, \underset{s}{\xi}, \overset{s}{\eta}, \mathbb{G})$ with $\mathbb{G}$ given in (5.74) is an almost n-contact Riemannian structure.*

Proof. Indeed, we have $\mathbb{F}(\underset{s}{\xi}) = 0$, $\overset{s}{\eta}\,(\underset{t}{\xi}) = \delta^s_t$ and $TOsc^2(M) = N_0 \oplus N_1 \oplus V_2$. For any $X \in TOsc^2(M)$,

$$\mathbb{F}^2 X = Y + \sum_{s=1}^{n} \alpha_s \underset{s}{\xi} \quad \text{where} \quad Y \in \mathrm{Im}(\mathbb{F}) \tag{6.2}$$

with α_s the 1-forms in $TOsc^2(M)$. The equation (6.2) leads to $\mathbb{F}^3X = \mathbb{F}Y$ then $Y = -X$ since we have the property $\mathbb{F}^3X = -\mathbb{F}X$, then we have

$$\mathbb{F}^2X = -X + \sum_{s=1}^{n} \overset{s}{\eta}(X)\underset{s}{\xi}.$$

Which implies

$$\mathbb{G}(\mathbb{F}X, \mathbb{F}Y) = \mathbb{G}(X,Y) - \sum_{s=1}^{n} \overset{s}{\eta}(X)\overset{s}{\eta}(Y) \tag{6.3}$$

This completes the proof. □

6.1.1 L.c. almost cosymplectic structure on $Osc^2(M)$

Let $\{U_t\}_{t\in I}$ be a family of an open covering $Osc^2(M)$. Assume that at each U_t there exist a map $\rho_t : U_t \to$ R. The requirement on ρ_t is that it is constant on $y^{(2)}$-direction.

Proposition 6.1.2. *The structure*

$$\widetilde{\mathbb{F}}, \quad \underset{s}{\widetilde{\xi}} = \exp(\frac{\rho_t}{2})\underset{s}{\xi_t}, \quad \overset{s}{\widetilde{\eta}} = \exp(\frac{-\rho_t}{2})\overset{s}{\eta_t}, \quad \widetilde{\mathbb{G}} = \exp(-\rho_t)\mathbb{G}_t, \tag{6.4}$$

is almost cosymplectic in U_t if and only if the following equations are satisfied

$$d\Phi = \omega \wedge \Phi + (g_{ij}\underset{(1)}{H^i_k}\omega - \underset{(1)}{H^i_k}dg_{ij} + g_{ij}d\underset{(1)}{H^i_k}) \wedge dx^j \wedge dx^k, \tag{6.5}$$

$$d\overset{i}{\eta} = \frac{1}{2}\overset{i}{\eta} \wedge \omega - \frac{1}{2}\underset{(1)}{H^i_k}\omega \wedge dy^{(1)k} - \frac{1}{2}K^i_k\omega \wedge dx^k - d\underset{(1)}{H^i_k} \wedge dy^{(1)k} - dK^i_k \wedge dx^k, \tag{6.6}$$

where $\omega = d\rho$ obtained by gluing up $d\rho_t$ on $Osc^2(M)$.

Proof. The fundamental 2-form given by $\widetilde{\Phi}(X,Y) = \widetilde{\mathbb{G}}(X, \mathbb{F}Y)$ is locally written as

$$\widetilde{\Phi} = \widetilde{g}_{ij}dx^i \wedge \widetilde{\delta}y^{(1)j}, \tag{6.7}$$

where $\widetilde{g}_{ij} = e^{-\rho}g_{ij}$ and the n-contact forms are given by

$$\overset{i}{\widetilde{\eta}} = \exp(\frac{-\rho}{2})\widetilde{\delta}y^{(2)i}. \tag{6.8}$$

By differentiating equations (6.7), (6.8) and using (5.35) together with the definition of almost cosymplectic structure i.e., $d\widetilde{\Phi} = d\overset{s}{\widetilde{\eta}} = 0$, one completes the proof. □

The normality condition is given by

$$N_{\widetilde{\mathbb{F}}}(X,Y) + \sum_{a=1}^{n} d(\widetilde{\delta y}^{(2)a})(X,Y) = 0, \ \forall\, X,Y \in \mathfrak{X}(Osc^2(M)). \tag{6.9}$$

Note that if a such open covering $\{U_t\}_{t\in I}$ exists on $Osc^2(M)$ then the 2-osculator bundle $Osc^2(M)$ endowed with the almost n-contact metric structure $(\mathbb{F}, \underset{s}{\xi}, \overset{s}{\eta}, \mathbb{G})$ is said to be locally conformal almost cosymplectic manifold.

The almost n-contact structure $\mathbb{F}$ associated to the nonlinear connection N satisfies

$$D_X\mathbb{F} = 0, \ \ \forall\, X \in \mathfrak{X}(Osc^2(M)). \tag{6.10}$$

Thus, the coefficients of the N-linear connection D on $(Osc^2(M), \mathbb{F}, \underset{s}{\xi}, \overset{s}{\eta}, \mathbb{G})$ satisfy

$$L^i_{jk} = \underset{(1)}{C^i_{jk}}.$$

6.2 Prolongation to $Osc^2(M)$ of l.c. almost cosymplectic structures

In this section the base manifold M is endowed with an almost contact structure (ϕ, ξ, η, g).

Complete lift of tensor fields

Let X be a vector field over M, and let $\{\phi_t\}$ be the 1-parameter group of local transformations of M induced by X its extension on $Osc^2(M)$ $\{\phi_t^*\}$ is again a 1-parameter group of local transformations of $Osc^2(M)$, and hence it defines a unique vector field over $Osc^2(M)$ which will be denoted by X^c and called *complete lift* of X (see [13] for more details). In [39], the authors expressed it locally by

$$X^c = (X^i \circ \pi^2)\partial_i + S(X^i \circ \pi^2)\partial_{1i} + \frac{1}{2}S^2(X^i \circ \pi^2)\partial_{2i}, \tag{6.11}$$

for any $X = X^i\partial_i \in \mathfrak{X}(M)$ and S a 2-semispray. For two semisprays S and S', we have $S^\alpha(X^i) = S'^\alpha(X^i)$ so the complete lift of a vector field X is independent on the choice of the semispray.

The complete lift of a function $f \in \mathcal{C}^\infty(M)$ is the function $f^c \in \mathcal{C}^\infty(Osc^2(M))$ given by

$$f^c = S(f) + \frac{1}{2}S^2(f). \tag{6.12}$$

Properties 6.2.1. The following holds

(i) $J^k(X^c) = X^{v_k}$, where X^{v_k} is the vertical v_k-lift of X given by $X^{v_k} = X^{(k)i}\delta_{ki}$.

(ii) $(fX)^c = \sum_{\alpha=0}^{k} \frac{1}{\alpha!} S^\alpha(f) J^\alpha(X^c)$, $f \in \mathcal{F}(M)$, $X \in \mathfrak{X}(M)$.

(iii) X and X^c are π^2-related i.e.,

$$X \circ \pi^2 = d\pi^2 \circ X^c.$$

(iv) The complete lift of a linear connection D on M is given by:

$$D^c_{X^c} Y^c = (D_X Y)^c$$

and we also have

(v) $D^c_{J^\alpha(X^c)} Y^c = D^c_{X^c} J^\alpha(Y^c) = J^\alpha(D_X Y)^c$, $\alpha = \overline{0,2}$.

Now M is endowed with an almost contact Riemannian structure (ϕ, ξ, η, g). Recall that for any point $(x^i, y^{(1)i}, y^{(2)i}) \in Osc^2(M)$ the canonical submersion π^2 is given by $\pi^2(x^i, y^{(1)i}, y^{(2)i}) = (x^i)$. The kernel of its differential $d\pi^2 : TOsc^2(M) \to TM$ is $\ker d\pi^2 = \mathrm{Span}\{\partial_{1i}, \partial_{2i}\}$. Hence, from equation (6.11) and by taking into account the property 6.2.1(iii), we have

$$d\pi^2(X^c) = (X^i \circ \pi^2)\partial_i = X. \tag{6.13}$$

Then we obtain

$$\phi X = \phi d\pi^2(X^c).$$

Let $L : TM \to TOsc^2(M)$ be a linear map given by $L(X) = X^c$. Such a map is unique (see [42, lemma 1.9] for more details). Let ι_ϕ be a $\mathcal{F}(Osc^2(M))$-linear map given by

$$\iota_\phi = L \circ \phi \circ d\pi^2 : TOsc^2(M) \to TOsc^2(M). \tag{6.14}$$

Locally in adapted basis, we have

$$d\pi^2(\delta_i) = \partial_i,\ d\pi^2(\delta_{1i}) = 0,\ d\pi^2(\delta_{2i}) = 0,$$

which implies

$$\phi d\pi^2(\delta_i) = \phi\partial_i = \phi_i^j \partial_j,\ \phi \circ d\pi^2(\delta_{1i}) = \phi \circ d\pi^2(\delta_{2i}) = 0.$$

Therefore we have $\iota_\phi(\delta_i) = L(\phi_i^j \partial_j)$, which implies

$$\iota_\phi(\delta_i) = (\phi_i^j \circ \pi^2)\partial_j + S(\phi_i^j \circ \pi^2)\partial_{1j} + \frac{1}{2} S^2(\phi_i^j \circ \pi^2)\partial_{2j}. \tag{6.15}$$

Clearly, we have

$$\ker \iota_\phi = \mathrm{Span}\{\delta_1, \delta_2\}.$$

In virtue of equations (2.36) the equation (6.15) may be written as

$$\iota_\phi(\delta_i) = a_i^j \delta_j + b_i^j \delta_{1j} + c_i^j \delta_{2j}, \tag{6.16}$$

where

$$a_i^j = \phi_i^j \circ \pi^2, \;\; b_i^j = S(\phi_i^j \circ \pi^2) + (\phi_i^j \circ \pi^2) \underset{(1)}{M_j^k} \delta_k^j,$$

$$c_i^j = \frac{1}{2} S^2(\phi_i^j \circ \pi^2) + S(\phi_i^j \circ \pi^2) \underset{(1)}{M_j^k} \delta_k^j + (\phi_i^j \circ \pi^2) \underset{(2)}{M_j^k} \delta_k^j. \tag{6.17}$$

It follows that

$$\iota_\phi^2(\delta_i) = a_i^j a_j^k \delta_j + a_i^j b_j^k \delta_{1j} + a_i^j c_j^k \delta_{2j}. \tag{6.18}$$

Let A be a $(1,1)$-tensor in $Osc^2(M)$ defined by

$$A(\delta_{2i}) = a_i^j \delta_j, \; A(\delta_i) = -a_i^j \delta_{2j}, \; A(\delta_{1i}) = 0. \tag{6.19}$$

It is clear that A satisfies

$$\ker A = \mathrm{Span}\{\delta_{1i}\}, \;\; \mathrm{Im}\, A = \mathrm{Span}\{\delta_i, \delta_{2i}\} \;\text{ and }\; A^3 + A = 0.$$

Indeed, $A^2(\delta_{2i}) = a_i^j A\delta_j = -a_i^j a_j^k \delta_{2k} = -(\phi_i^j \circ \pi^2)(\phi_j^k \circ \pi^2)\delta_{2k}$ then for an orthonormal basis $\{\partial_i, \cdots, \partial_{2n}, \xi\}$ in M one has $(\phi_i^j \circ \pi^2)(\phi_j^k \circ \pi^2) = g(\phi^2 \partial_i, \partial_i) = -g(\partial_i, \partial_i) = -g_{ij}$, which implies $A^2(\delta_{2i}) = -\delta_{2i}$ then $A^3 + A = 0$.

Therefore, in view of Proposition 6.1.1 the $(1,1)$-tensor A is an almost n-contact structure and it depends only on the almost structure ϕ on M and compatible with the metric

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} dy^{(1)i} \otimes dy^{(1)j} + g_{ij} dy^{(2)i} \otimes dy^{(2)j}. \tag{6.20}$$

Theorem 6.2.1. *Let (M, ϕ, ξ, η, g) be an almost contact manifold, the manifolds $Osc^2(M)$ endowed with the structure $(A, \underset{s}{\xi}, \overset{s}{\eta}, \mathbb{G})$ where $\underset{s}{\xi} = \delta_{1s}$ and $\overset{s}{\eta} = \delta y^{(1)s}$ is an almost n-contact manifold. Moreover, if Φ is the fundamental 2-forms of M then the fundamental 2-form $\Phi^{\mathbb{G}}$ satisfies $\Phi^{\mathbb{G}}(X,Y) = \mathbb{G}(X, AY)$, $X, Y \in TOsc^2(M)$ of $Osc^2(M)$ is given by*

$$\Phi^{\mathbb{G}} = \Phi_{ij} dx^i \wedge \delta y^{(2)j}. \tag{6.21}$$

Proof. The proof of the first part of the theorem follows from the proof of Proposition 6.1.1 by replacing the $(1,1)$-tensor $\mathbb{F}$ by the $(1,1)$-tensor A given in (6.19). Using the adapted vector fields δ_i and δ_{2j} one gets

$$\Phi^{\mathbb{G}}(\delta_i, \delta_{2j}) = \mathbb{G}(\delta_i, A\delta_{2j}) = \mathbb{G}(\delta_i, a_j^k \delta_k) = a_j^k \mathbb{G}(\delta_i, \delta_k) = a_j^k g_{ik}. \tag{6.22}$$

Since $\Phi_{ij} = \Phi(\partial_i, \partial_j) = g(\partial_i, \phi\partial_j) = g(\partial_i, \phi_j^k \partial_k) = (\phi_j^k \circ \pi^2) g_{ik} = a_j^k g_{ik}$. Hence, we have

$$\Phi^{\mathbb{G}} = \Phi_{ij} dx^i \wedge \delta y^{(2)j},$$

which completes the proof. □

Then the proof is completed.

Proposition 6.2.1. *If (M, ϕ, ξ, η, g) is an locally conformal almost cosymplectic manifold then the fundamental 2-form of $(Osc^2(M), A, \underset{s}{\xi}, \overset{s}{\eta}, \mathbb{G})$ satisfies*

$$d\Phi^{\mathbb{G}} \wedge dx^j = 2\omega \wedge \Phi^{\mathbb{G}} \wedge dx^j + d(\delta y^{(2)j}) \wedge \Phi. \tag{6.23}$$

Proof. By $\wedge$-producting the equation (6.21) with dx^j and taking into account the equation (3.5), one gets

$$\Phi^{\mathbb{G}} \wedge dx^j = -\Phi_{ij} dx^i \wedge dx^j \wedge \delta y^{(2)j} = -\Phi \wedge \delta y^{(2)j}. \tag{6.24}$$

Then differentiating (6.24), one obtains (6.23). □

For a given co-basis

$$dx^i,\ \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M_j^i} dx^j,\ \delta y^{(2)i} = dy^{(2)i} + \underset{(1)}{M_j^i} dy^{(1)j} + \underset{(2)}{M_j^i} dx^j, \tag{6.25}$$

we recall that

Lemma 6.2.1. *[40, page 104] The exterior differential of the 1-forms (5.35) are given by the following formula*

$$d(dx^i) = 0,\ d(\delta y^{(1)j}) = \frac{1}{2} \underset{(01)}{R_{im}^j} dx^m \wedge dx^i + \underset{(11)}{B_{im}^j} \delta y^{(1)m} \wedge dx^i + \underset{(21)}{B_{im}^j} \delta y^{(2)m} \wedge dx^i,$$

$$\begin{aligned} d(\delta y^{(2)j}) = {} & \frac{1}{2} \underset{(02)}{R_{im}^j} dx^m \wedge dx^i + \underset{(12)}{B_{im}^j} \delta y^{(1)m} \wedge dx^i + \underset{(22)}{B_{im}^j} \delta y^{(2)m} \wedge dx^i + \underset{(12)}{R_{im}^j} \delta y^{(1)m} \wedge \delta y^{(1)i} \\ & + \underset{(21)}{B_{im}^j} \delta y^{(2)m} \wedge \delta y^{(1)i}. \end{aligned} \tag{6.26}$$

Theorem 6.2.2. *Let (M, ϕ, ξ, η, g) be an l.c. almost cosymplectic manifold. If the horizontal and the vertical distribution N_0 and N_1 in (2.28) are integrable and the coefficients $\underset{(21)}{B}{}^j_{im}$ vanish then the fundamental 2-form $\Phi^{\mathbb{G}}$ satisfies*

$$d\Phi^{\mathbb{G}} = 2\omega \wedge \Phi^{\mathbb{G}} \tag{6.27}$$

and there exists 1-forms α_j such that one has

$$d \overset{i}{\eta} = \alpha_j \wedge \overset{j}{\eta} . \tag{6.28}$$

Proof. According to the Remark 5.3.1, the distributions N_0 and N_1 in (2.28) are integrable if and only if $\underset{(01)}{R}{}^j_{im} = \underset{(02)}{R}{}^j_{im} = \underset{(12)}{R}{}^j_{im} = 0$, then the second and the third equation in (6.26) become

$$\begin{aligned} d(\delta y^{(1)j}) &= \underset{(11)}{B}{}^j_{im} \delta y^{(1)m} \wedge dx^i + \underset{(21)}{B}{}^j_{im} \delta y^{(2)m} \wedge dx^i, \\ d(\delta y^{(2)j}) &= \underset{(12)}{B}{}^j_{im} \delta y^{(1)m} \wedge dx^i + \underset{(22)}{B}{}^j_{im} \delta y^{(2)m} \wedge dx^i + \underset{(21)}{B}{}^j_{im} \delta y^{(2)m} \wedge \delta y^{(1)i}, \end{aligned}$$

and using the assumption $\underset{(21)}{B}{}^j_{im} = 0$ together with the observation that $\Phi = \Phi_{ij} dx^i \wedge dx^j$ one obtains

$$\begin{aligned} d(\delta y^{(1)j}) &= \underset{(11)}{B}{}^j_{im} \delta y^{(1)m} \wedge dx^i = -\underset{(11)}{B}{}^j_{im} dx^i \wedge \delta y^{(1)m}, \\ d(\delta y^{(2)j}) \wedge \Phi &= \underset{(12)}{B}{}^j_{im} \delta y^{(1)m} \wedge dx^i \wedge \Phi_{ij} dx^i \wedge dx^j + \underset{(22)}{B}{}^j_{im} \delta y^{(2)m} \wedge dx^i \wedge \Phi_{ij} dx^i \wedge dx^j = 0, \end{aligned}$$

Therefore, we obtain $d(\delta y^{(2)j}) \wedge \Phi = 0$, then in virtue of (6.23) and posing $\alpha = -\underset{(11)}{B}{}^j_{im} dx^i$ we complete the proof. □

Corollary 6.2.1. *Under the assumptions of Theorem 6.2.2, if the Lee form ω of the l.c. almost cosymplectic manifold M satisfies $\omega = -\underset{(11)}{B}{}^j_{im} dx^i$ then the bundle $Osc^2(M)$ is an l.c. almost cosymplectic manifold.*

Note that the coefficients $\underset{(21)}{B}{}^j_{im}$ identically vanish if the coefficients $\underset{(1)}{N}{}^i_j$ are constant in $y^{(2)}$-direction.

Conclusion and Perspectives

7.1 Conclusion

This thesis contains recent author's investigations on differential geometry of smooth manifolds equipped with almost complex and almost contact structures. One of the main goals was the relationships between geometric objects of the almost contact structures (ϕ, ξ, η, g) and $(\phi, \xi_U, \eta_U, g_U)$. We recalled the Chern-Hamilton tensor and used its parallelism to prove that, under some geometric conditions, the class of l.c. almost cosymplectic manifolds contains in the class of cosymplectic manifolds. The Lee form ω and its dual, Lee vector field B, played a primary role in this study. The Lee vector field B was used by G. Olszack in [43, theorem 3.3] to define a subclass of l.c. almost cosymplectic manifolds based on its proportionality with the structure vector field ξ. We enrich the Olszack's study by proving that there are many classes of this kind in which the proportionality condition introduced by Olszack is also satisfied. Furthermore, we used the distributions $\ker \eta$ and its orthogonal to study foliations on l.c. almost cosymplectic manifolds.

The indefinite case of locally conformal almost cosymplectic manifolds was also studied. Here we paid attention to canonical foliations $\mathcal{F}$ whose leaves are the maximal connected integral manifolds of the Pfaffian equation $\omega = 0$. The casual character of the Lee vector B played a very important role in this study. Leaves of $\mathcal{F}$, seen as submanifolds of the ambient space under consideration, led to the study of intrinsic geometric objects. The formulas of change of screen distributions were derived. Moreover, the non-tangential leaves, the screen conformal leaves and the higher order geodesibility of leaves of $\mathcal{F}$ were studied. The latter yielded to some integral formulas.

The Conformal-type Lagrangian functions were defined on 2-osculator bundle $Osc^2(M)$. For a given conformal-type regular Lagrangian functions L and $\widetilde{L}$, some examples in Riemannian and Finsler cases were given. In the latter case, the 0-homogeneity was required on the conformal change in order to get a conformal fundamental tensor which is still satisfying Finsler assumptions. We also showed how geometric objects, such as Lie brackets, d-tensors of Riemannian curvatures, Ricci tensors, and scalar curvatures, are related.

By the means of Vaisman characterization of l.c. almost cosymplectic structures on a smooth manifold M, we characterized an l.c. almost cosymplectic structure on the 2-osculator bundle $Osc^2(M)$. Finally, under some conditions, we prolonged the l.c. almost cosymplectic structure from M to $Osc^2(M)$.

7.2 Future research directions

As perspectives, we would like to focus on the l.c. almost cosymplectic submersions, that is, submersions from l.c. almost cosymplectic manifolds.

As known there are 4,096 classes of almost contact metric structures and here we have examined only a few number of them. It shall be of interest to pursue the study with new manifolds. What shall be needed is the defining relations of some new almost contact metric manifolds. Other manifolds can be obtained by the use of warped product following the formalism of Kenmotsu [25]. An investigation shall also be oriented towards the study of the geometry of singular 2-Lagrange spaces and the conformal 2-Hamilton space using the Legendre mapping.

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