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On Group Factorizations

by

Monde Nasilele

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If you have no reason to study Mathematics why not study it for its beauty!

Abstract

All groups considered in this dissertation are finite. A group G is said to be factorizable if $G = AB$ is the product of two proper subgroups A and B of G, $i.e.$ every element g of G can be expressed in the form $g = ab$ for some $a \in A$ and $b \in B$. If N is a normal subgroup of a finite group G we are guaranteed that the product NH of N and H is a subgroup of the group G for every subgroup H of G. However, normality of one factor is a sufficient and not a necessary condition for the product NH to be a subgroup of G.

In this dissertation, conditions under which a proper subgroup H of a group G has a proper supplement in G have been investigated. We have also investigated conditions under which a finite group is factorizable. A special factorization called an exact factorization is also investigated in this dissertation. This is a factorization of the form $G = HK$ where H and K are subgroups of G such that $H \cap K = \{1\}$; here H is said to be complemented in G by K.

The last chapter briefly reviews the applications and contributions of group factorizations to the study of group theory and abstract Algebra in general.

Preface and Declaration

The work described in this dissertation was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, from February 2015 to March 2016, under the supervision of Professor Bernardo Rodrigues.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Student name: Monde Nasilele

Signature:

As the candidate's Supervisor I agree/do not agree to the submission of this thesis.

Signature: 

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Notation and conventions

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
G	a group
H, K	subgroups
$1, 1_G$	the identity element of G
$H \leq G$	H is a subgroup of G
$N \trianglelefteq G$	N is a normal subgroup of G
$H \cong K$	H is isomorphic to K
HK	the product of H and K
Hg	the right coset of G
x_G	a conjugacy class of x in G
$C_G(x)$	the centralizer of x in G
$Z(G)$	the center of the group G
$N_G(H)$	the normalizer of the subgroup H in G

$\mathrm{Stab}_G(x)$	stabilizer of x in G
$\mathrm{Orb}_G(x)$	orbit of x in G
$[G, G]\,,\ G'$	the commutator subgroup of G
$\mathrm{Ker}f$	kernel of homomorphism f
$\mathrm{Im}f$	image of homomorphism f
$\Phi(G)$	the Frattini subgroup of G
G^S	the residual subgroup of G
$\langle g\rangle$	the subgroup generated by g
h^g	conjugation of h by g
$o(g)$	order of g
$\|G\|$	order of the group G
$p\|m$	p divides m
$(p, m) = 1$	p and m are coprime
D_n	the dihedral group of order $2n$
$\mathbf{V}_4$	the Klein $4-$group
C_n	cyclic group of order n
S_n	the symmetric group on n symbols
A_n	the alternating group on n symbols
Q_8	quaternion group of order 8
F	a formation

Contents

CHAPTER 1

Introduction

In this dissertation we study factorisable groups and subgroups with proper supplements. Let H and K be subgroups of a finite group G and suppose $G = HK$. If any of H or K is equal to G, then $G = HK$ is called a trivial factorization of G, and by a proper factorization (or non-trivial factorization) we mean $G = HK$ with both H and K being proper subgroups of G. If G admits a proper factorization say $G = HK$, then we call G a factorizable group while H and K are called factors.

In our study, two closely related problems shall be distinguished and dealt with in two separate chapters. The first problem is to determine conditions under which a proper subgroup H of a group G has a proper supplement in G. Though only proper normal subgroups were considered, the first problem is also addressed in [12]. The second problem is to investigate conditions under which a finite group G admits a proper factorization. Joseph in [13] tackled the second problem using an approach involving $aS-$groups.

By defining that a group G is called factorizable if it has two proper subgroups H and K such that $G = HK$, it is clear that finite groups of prime order are not factorizable as they do not have a single non-trivial proper subgroup. These are not the only groups that do not admit a proper factorization. Cyclic $p-$groups are not factorizable too. It is an interesting problem to know the groups which admit proper factorizations.

Group factorizations have produced influential results in group theory. For example, Kegel proved that if $G = HK$ with H and K nilpotent, then G is solvable, and Ito showed that if $G = HK$ with H and K abelian, then G is metabelian. Consider the factorization $A_4 = \mathbf{V}_4\langle(123)\rangle$ of the alternating group A_4 into the Klein $4-$group $\mathbf{V}_4$ and $\langle(123)\rangle$. Since the factors $\mathbf{V}_4$ and $\langle(123)\rangle$ are both nilpotent, Kegel's observation can be used to deduce that A_4 is solvable without using the traditional definition of solvability. Similarly, since the factors $\mathbf{V}_4$ and $\langle(123)\rangle$ are both abelian, we could conclude that the alternating group A_4 is metabelian by simply using Ito's observation.

In Chapter 2 we start by presenting basic group theoretical results and definitions that will be required in our study.

In Chapter 3 we revise products of groups and the subgroups useful to our study. This chapter is simply the pivot of the dissertation as a whole. We recall that if H and K are subgroups of a group G, then the product HK of H and K is defined by $HK = \{hk \mid h \in H, k \in K\}$. It is known that even if H and K were subgroups of a group G, the product HK of H and K might not be a subgroup of G. One of the main results for Section 3.1 (Theorem 3.1.9) asserts that the product HK of two subgroups H and K of a group G is a subgroup of G if and only if $HK = KH$.

In the next sections of Chapter 3 we continue to revise important concepts to group factorizations. We shall prove in this dissertation that if G is a non-cyclic group with $G \neq G'$, then G is factorizable. Sylow $p-$subgroups are equally useful to group factorizations as we may observe from Frattini's argument (Lemma 3.5.16).

The Frattini subgroup, defined to be the intersection of all the maximal subgroups of a group, is arguably one of the most important subgroups as far as determining proper factorizations is concerned. Thus, in Section 3.5 we recall the Frattini subgroup and its properties such as the

Frattini subgroup is nilpotent and normal in G. We also show that the Frattini subgroup $\Phi(G)$ of a group G has no proper supplement in G.

If G is a non-trivial group, then G is called an $aS-$group if every non-trivial subgroup of G has a proper supplement in G. The smallest example of a non-abelian $aS-$group is the symmetric group S_3. The residual subgroup of a group G is simply the smallest normal subgroup N of G such that the factor group G/N is an $aS-$group. The importance of this subgroup can be realised in many results of our work. In Theorem $4.1.16$, we observe that if N is a normal subgroup of a group G and if every subgroup of G/N had a proper supplement in G/N, then every subgroup H of G containing N would have a proper supplement in G. Therefore, the search for proper factorizations of a group G is extremely assisted by finding a normal subgroup N of G such that G/N is an $aS-$group.

Chapter 4 is devoted to investigating conditions under which a proper subgroup H of a group G will have a proper supplement in G. Condition 1 states that if H is a normal subgroup of a group G and if H is not contained in the Frattini subgroup of G, then H has a proper supplement in G. Using the residual subgroup G^S of a group G, we obtain a similar but stronger condition in Theorem $4.1.17$. The condition in this theorem asserts that if $\{1\} < H$ is a proper subgroup of G and if H is not contained in the residual subgroup G^S of G, then H has a proper supplement in G. It is a stronger condition than Condition 1 because it does not limit H to being a normal subgroup of G.

In Chapter 5 we study conditions under which a finite group G will be factorizable. The first condition states that if G is a non-cyclic group and if $G \neq G'$, where G' denotes the commutator subgroup of G, then G is factorizable. Nilpotent groups and $p-$groups share several important theoretical properties in group theory. One of the properties is the fact that every maximal subgroup of a nilpotent group or a $p-$group is normal. Furthermore, both nilpotent groups and $p-$groups satisfy the normalizer condition. That is, if G is either a nilpotent group or a $p-$group, then every proper subgroup of G is properly contained in its normalizer. The two properties mentioned above made it possible for us to prove, in Chapter 5, that every non-cyclic $p-$group and every non-cyclic nilpotent group are factorizable.

In Chapter 6 we explore a special type of factorization: a factorization of the form $G = HK$ where H and K are subgroups of G and $H \cap K = \{1\}$. In such a factorization, H is said to be complemented in G by K and if every subgroup of G is complemented, then G is called a complemented group. The notion of subgroup complementation has produced important characterisation results in group theory. Arad and Ward in [2] proved that a group G is soluble if and only if every Sylow $2-$subgroup and every Sylow $3-$subgroup of G are complemented in G.

In Chapter 7, we demonstrate how group factorizations can help us solve problems in group theory. This chapter is simply meant to illustrate that group factorizations is an interesting approach through which to study group theory.

CHAPTER 2

Basic Results and Definitions

2.1 Normal subgroups and Series of groups

A subset of a group G may or may not be a subgroup of the group G. If H is a subgroup of the group G then H is a group under the operation of the group G. Below is the formal definition:

Definition 2.1.1. [19] Let G be a group G with the binary operation $'*'$ and identity element 1. A nonempty subset H of G is called a subgroup if the conditions below are satisfied:

(i) $1 \in H$;

(ii) If $x, y \in H$ then $x * y \in H$;

(iii) If $x \in H$ then $x^{-1} \in H$.

A group G always has at least two subgroups, namely G itself and the subgroup $\{1\}$ consisting of the identity element alone. We call $\{1\}$ the trivial subgroup of G, and we call H a non-trivial subgroup of G if $H \neq \{1\}$. If H is a subgroup of G, we write $H \leq G$; if H is a proper subgroup of G, that is, if $H \neq G$, then we write $H < G$.

Below is the subgroup criterion:

Theorem 2.1.2 (Subgroup criterion)**.** *Let H be a subset of a group G. Then H is a subgroup of G if and only if H is not empty and $xy^{-1} \in H$ whenever $x, y \in H$.*

Proof. See [6]. $\square$

Definition 2.1.3. Let G be a group. Then G is called **abelian** if for all $a, b \in G,\ ab = ba$. If two elements x and y of G are such that $xy = yx$, then the elements x and y are said to **commute**.

Definition 2.1.4. Let G be a group. If G contains an element x such that for every $g \in G$,

$$g = x^k$$

for some $k \in \mathbb{N}$, then G is called a **cyclic** group.

Definition 2.1.5. Let G be a group. Then, the **center** of G denoted by $Z(G)$ is defined by $Z(G) = \{z \in G \mid gz = zg\ \forall g \in G\}$. That is, $Z(G)$ is a set of those elements of G which commute with every other element of G.

Definition 2.1.6. Let x be an element of a group G. Then, the subset

$$\{g \in G \mid gx = xg\}$$

of G denoted by $C_G(x)$ is called the **centralizer** of x in G.

Definition 2.1.7. A subgroup K of a group G is called a normal subgroup of G if $gkg^{-1} \in K$ for every element $k \in K$ and for every element $g \in G$. If K is a normal subgroup of G, we write $K \trianglelefteq G$.

Definition 2.1.8. Let $\{1\} < K \trianglelefteq G$. Then K is said to be a minimal normal subgroup of G if there is no normal subgroup L of G such that $\{1\} < L < K$.

If G is a group it can be easily verified that the trivial subgroup $\{1\}$ and the whole group G are normal subgroups of G. A group $G \neq \{1\}$ is called **simple** if G has no normal subgroups other than $\{1\}$ and G itself. We also recall the following theorem:

Theorem 2.1.9. *Every subgroup of an abelian group G is normal in the group G.*

Proof. See [6]. $\square$

Definition 2.1.10. A subgroup H of a group G is said to be subnormal in G if there exists a chain of subgroups $H_0,\ H_1,\ H_2,\ \ldots,\ H_r$ such that

$$H = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_r = G.$$

This is called a subnormal chain from H to G.

Definition 2.1.11. Let H be a subgroup of a group G. Then the normalizer of H in G denoted by $N_G(H)$ is the subgroup

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}.$$

It can be easily noticed that if H is a subgroup of G, then $H \leq N_G(H)$. If it turns out that every proper subgroup H of a group G is properly contained in its normalizer, then G is said to satisfy the **normalizer condition**.

Maximal subgroups shall be needed in this thesis and so the definition below is in order.

Definition 2.1.12. A proper subgroup M of a group G is called a maximal subgroup if there is no subgroup L of G such that $M < L < G$.

Definition 2.1.13. Let H be a subgroup of a group G and g be any element of G. Then the set $Hg = \{hg \mid g \in G\}$ is called a right coset of H in G generated by g. Putting the element g to the left of H would give the coset gH the name left coset of H.

Cosets of a subgroup can be used to determine whether the subgroup is normal or not as we may see in the next lemma:

Lemma 2.1.14. A subgroup K of a group G is normal if and only if

$$gK = Kg$$

for every $g \in G$.

Proof. See [19]. □

We observe, from Lemma $2.1.14$, that if K is a normal subgroup of a group G, then every right coset of K is also a left coset of K.

Remark 2.1.15. Let G/K denote the family of all the left cosets of a subgroup K of G. It turns out that if K is a normal subgroup of G, then

$$(xK)(yK) = xyK$$

for all $x, y \in G$, and G/K is a group under this operation. The group G/K is called the quotient group G mod K.

Definition 2.1.16. Let H be a subgroup of a finite group G. Then the index of H in G, denoted by $[G : H]$, is the number of left cosets of H in G.

Theorem 2.1.17. *Let H and K be subgroups of the group G such that $H \leq K$. Then $[G : H] = [G : K] \times [K : H]$.*

Proof. See [20]. □

Theorem 2.1.18 (The Correspondence Theorem)**.** *Let $K \trianglelefteq G$. Then every subgroup of G/K is of the form H/K where $K \leq H \leq G$. That is, there is a one to one correspondence between the subgroups of G/K and those subgroups of G which contain K.*

Proof. See [18]. □

Theorem 2.1.19. *Let $K \trianglelefteq G$. Then K is a maximal subgroup of G if and only if*

$$|G/K| = [G : K] = p$$

for some prime p.

Proof. Since K is normal and proper in G, we have that $|G/K| > 1$. Now by the correspondence theorem, K is a maximal subgroup of G if and only if G/K has no non-trivial subgroup; that is if and only if $|G/K| = p$ for some prime p. □

Below are the definitions of a subnormal series and a normal series of a finite group:

Definition 2.1.20. A subnormal series of a group G is a finite sequence of subgroups

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \ \dots \ \leq G_{n-1} \leq G_n = G$$

in which G_i is a normal subgroup of G_{i+1} for all $i = 0,\ 1,\ 2, \dots,\ n-1$.
The collection $G_1/G_0,\ G_2/G_1, \dots,\ G_n/G_{n-1}$ are the factors of the series and n is the length of the series.

Definition 2.1.21. A normal series of a group G is a finite sequence of subgroups

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \ \dots \ \leq G_{n-1} \leq G_n = G$$

in which G_i is a normal subgroup of G_{i+1} and $G_i \trianglelefteq G$ for all $i = 0,\ 1,\ 2, \dots,\ n-1$.

2.2 The Isomorphism Theorems

Definition 2.2.1. [6] Let A and B be non-empty sets. Then a mapping $f : A \to B$ from set A to set B is called

(a) injective (or one-to-one) if, for all $x_1, x_2 \in A,\ x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

(b) surjective (or onto) if, for every $y \in B$, $y = f(x)$ for some $x \in A$.

A mapping that is both injective and surjective is said to be bijective.

Definition 2.2.2. Let G and H be groups. A map $\phi : G \to H$ is called a homomorphism if

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in G$. If the homomorphism ϕ is bijective, then we call ϕ an isomorphism. If $\phi : G \to H$ is a surjective homomorphism, then H is called a homomorphic image of G.

Definition 2.2.3. Let G and H be groups and let $f : G \to H$ be a homomorphism from G to H. The kernel of f, denoted by Ker$f$, is defined to be the set

$$\text{Ker}f = \{g \in G \mid f(g) = 1_H\},$$

where 1_H is the identity element in H. From the definition, we note that Ker$f$ is a subset of G. In fact, it can be easily shown that Ker$f$ is a subgroup of G.

Remark 2.2.4. If $f : G \to H$ is a homomorphism from a group G to a group H, then the image of f, denoted by Imf, is a subgroup of H.

Below are the three well-known isomorphism theorems:

Theorem 2.2.5 (The First Isomorphism Theorem)**.** *Let G and H be groups and let $\phi : G \to H$ be a homomorphism. Then*

$$G/\mathit{Ker}\phi \cong \mathit{Im}\phi.$$

Hence, in particular, if ϕ is surjective, then $G/\mathit{Ker}\phi \cong H$.

Proof. Consider the map $\varphi : G/K \to \mathbf{Im}\phi$ given by $\varphi(xK) = \phi(x)$, where $K = \text{Ker}\phi$. Then for all $x, y \in G$, we have that

$$xK = yK \Leftrightarrow y^{-1}x \in K \Leftrightarrow \phi(y^{-1}x) = 1_H \Leftrightarrow \phi(x) = \phi(y) \Leftrightarrow \varphi(xK) = \varphi(yK).$$

Hence, φ is well defined and injective. Furthermore,

$$\varphi((xK)(yK)) = \varphi(xyK) = \phi(xy) = \phi(x)\phi(y) = \varphi(xK)\varphi(yK).$$

Thus, φ is a homomorphism. Since φ is obviously surjective, we conclude that φ is an isomorphism between $G/\text{Ker}\phi$ and Imϕ and hence the proof. □

Theorem 2.2.6 (The Second Isomorphism Theorem)**.** *Let H and N be subgroups of G, and $N \trianglelefteq G$. Then*

$$H/(H \cap N) \cong HN/N.$$

This theorem is also known as the "diamond isomorphism theorem".

Proof. See [6]. □

Theorem 2.2.7 (The Third Isomorphism Theorem)**.** *Let H and K be normal subgroups of a group G such that $K \subset H$. Then*

$$(G/K)/(H/K) \cong G/H.$$

This theorem is also known as the "double quotient isomorphism theorem".

Proof. See [6]. □

2.3 The class equation

Before we end this section, we discuss briefly the class equation of a group.

Definition 2.3.1. Let G be a group. Two elements a and b are said to be **conjugate** if there exists an element g in G such that $gag^{-1} = b$.

Definition 2.3.2. Let x be an element of a group G. Then, the subset

$$\{gxg^{-1} \mid g \in G\}$$

is called the **conjugacy class** of x and is denoted by $C(x)$.

Below are the definitions of the stabilizer and the orbit of an element of a group:

Definition 2.3.3. Let G be a group acting, by left multiplication, on a non-empty set X, and let $x \in X$. Then the set

$$\text{Stab}_G(x) = \{g \in G \mid gx = x\},$$

which can be easily shown to be a subgroup of G, is called the **stabilizer** of x in G.

Definition 2.3.4. Let G be a group acting, by left multiplication, on a non-empty set X, and let $x \in X$. Then the set

$$\text{Orb}_G(x) = \{gx \mid g \in G\},$$

is called the **orbit** of x in G.

Theorem 2.3.5 (Orbit-stabilizer theorem)**.** *Let G be a group acting on a set X, and let $x \in X$. Then,*

$$|Orb_G(x)| = [G : Stab_G(x)]\,.$$

Proof. Fix $x \in X$ and let $S = \text{Stab}_G(x)$. Consider the map

$$\phi : G/S \to \text{Orb}_G(x)$$

given by $\phi(gS) = gx$ for all $g \in G$. Then we note that for all $g, h \in G$,

$$gS = hS \Leftrightarrow h^{-1}gS = S \Leftrightarrow h^{-1}g \in S \Leftrightarrow h^{-1}gx = x \Leftrightarrow gx = hx.$$

Hence, the map ϕ is well defined and injective. Moreover, ϕ is obviously surjective. Therefore, ϕ is a bijection and so $|\text{Orb}_G(x)| = |G/S| = [G : S]$. □

Corollary 2.3.6. Let G be a finite group acting on a set X, and let x be an element of X. Then,

$$|G| = |\text{Stab}_G(x)||\text{Orb}_G(x)|.$$

Proof. The proof follows from the Orbit-stabilizer theorem (above) and Lagrange's theorem. □

If a group G acts on itself by conjugation and if $x \in G$, then the stabilizer of x

$$\{g \in G \mid gxg^{-1} = x\}$$

is simply the centralizer of x in G, while the orbit $\{gxg^{-1} \mid g \in G\}$ of x is precisely the conjugacy class of x in G.

If G is a group, then the number h of distinct conjugacy classes of G is known as the **class number** of G. Suppose that $C_1,\ C_2, \cdots,\ C_h$ are the conjugacy classes of a group G and let n_i be the number of elements in the conjugacy class C_i for all $i = 1,\ 2, \cdots,\ h$. Then, the integers $n_1,\ n_2, \cdots,\ n_h$ satisfy the equation

$$|G| = n_1 + n_2 + \cdots + n_h. \tag{2.1}$$

Equation (2.1) is called the **class equation** of G.

Remark 2.3.7. We observe that an element x of a group G belongs to the center $Z(G)$ of G if and only if the conjugacy class $C(x)$ of x consists of only one element, namely x itself. For this reason, G is the disjoint union of $Z(G)$ and all conjugacy classes containing more than one element. Hence with the help of the Orbit-stabilizer theorem, the class equation can be expressed as

$$|G| = |Z(G)| + \sum_{x \in C} [G : C_G(x)]$$

where C contains exactly one element from each conjugacy class with more than one element.

2.4 The Commutator subgroup

The commutator subgroup is one of the subgroups we cannot do without in this thesis and so the need to remind ourselves of what it is.

Definition 2.4.1. [18] Let G be a group and let $g_1,\ g_2 \in G$. Then the commutator of the elements g_1 and g_2 is the element

$$[g_1,\ g_2] = g_1 g_2 g_1^{-1} g_2^{-1} \in G.$$

Definition 2.4.2. Let G be a group. Then the commutator subgroup (or derived subgroup) of G, denoted by $[G,\ G]$ or G', is the subgroup generated by all the commutators of G. That is

$$[G,\ G] = \left\langle\ [g_1,\ g_2] = g_1 g_2 g_1^{-1} g_2^{-1} \mid g_1,\ g_2 \in G\ \right\rangle.$$

Theorem 2.4.3. *Let G be a group and G' be the commutator subgroup of G. Then $G' \trianglelefteq G$.*

Proof. With the notation $a^t = t^{-1}at$, it suffices to prove that $[x, y]^t \in G'$ for all $x, y, t \in G$. Since conjugation obeys the multiplicative rule, $i.e.$ since $(xy)^t = x^t y^t$, we have that

$$[x, y]^t = (xyx^{-1}y^{-1})^t = x^t y^t (x^{-1})^t (y^{-1})^t = x^t y^t (x^t)^{-1} (y^t)^{-1} = [x^t, y^t] \in G'.$$

And this completes the proof. $\square$

Theorem 2.4.4. *Let G be a group and let $K \trianglelefteq G$. Then the factor group G/K is abelian if and only if $G' \leq K$.*

Proof. Let $K \trianglelefteq G$. Then G/K is abelian if and only if $(xK)(yK) = (yK)(xK)$ for all $x, y \in G$; that is, if and only if $xyK = yxK$, or, equivalently, $x^{-1}y^{-1}xyK = K$; that is, if and only if $[x, y] \in K$ for all $x, y \in G$. Thus G/K is abelian if and only if $G' \leq K$. $\square$

Definition 2.4.5. Let G be a group. Then the higher commutator subgroups of G are defined inductively as:

$$G^0 = G; \qquad G^{(i+1)} = (G^i)';$$

that is, $G^{(i+1)}$ is the commutator subgroup of $G^{(i)}$. The series

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots$$

is called the **derived series** of G.

CHAPTER 3

A Review of Groups

The aim of this chapter is to assemble a selection of basic results from group theory which will be needed to understand the main topic of this thesis. Not every result will be proved as most of these can be found in standard group theory text books such as [20], [17], [11], [18] and [14].

3.1 Products of Groups

We start this chapter by discussing products of groups or subgroups:

Definition 3.1.1. If G is a finite group with proper subgroups H and K, the product of H and K is defined by $HK = \{hk \mid h \in H,\ k \in K\}$.

The question that arises almost naturally is: If H and K are subgroups of a group G, is the product HK always a subgroup of G? It turns out that this is not always the case. Below is an example illustrating this fact:

Example 3.1.2. Take G to be the symmetric group $S_3 = \{(1),\ (12),\ (13),\ (23),\ (123),\ (132)\}$. This group has subgroups, among others, $H = \{(1),\ (12)\}$ and $K = \{(1),\ (13)\}$; yet the product $HK = \{(1),\ (12),\ (13),\ (132)\}$ of H and K is clearly not a subgroup of S_3 since the order of HK does not divide the order of S_3.

Before we go any further, it may be of help to define something closely related to the concept of products of groups: the direct product of two groups.

Definition 3.1.3. [20] Let $(H, *_1)$ and $(K, *_2)$ be two groups. Then the direct product of H and K, denoted by $H \times K$, is the group with elements all ordered pairs (h, k) where $h \in H$ and $k \in K$, and with operation

$$(h, k) \circ (h', k') = (h *_1 h', k *_2 k').$$

The operations $'*_1'$ and $'*_2'$ may or may not be different. It is easy to check that $H \times K$ is indeed a group: the identity element is $(1_H, 1_K)$ while the inverse $(h, k)^{-1}$ of the element (h, k) is (h^{-1}, k^{-1}) in this group. We also notice that even though neither H nor K is a subgroup of $H \times K$, the group $H \times K$ does contain isomorphic replicas of H and K, namely, $H \times \{1_K\} = \{(h, 1_K) \mid h \in H\}$ and $\{1_H\} \times K = \{(1_H, k) \mid k \in K\}$. The example below is an illustration of how this group can be formed from two groups with different structures.

Example 3.1.4. Let H be the quaternion group of order 8 denoted by Q_8 and defined by

$$\begin{aligned} Q_8 &= \big\langle -1\ ,\ i\ ,\ j\ ,\ k \mid (-1)^2 = 1\ ,\ i^2 = j^2 = k^2 = ijk = -1 \big\rangle \\ &= \{1\ ,\ -1\ ,\ i\ ,\ -i\ ,\ j\ ,\ -j\ ,\ k\ ,\ -k\} \end{aligned}$$

and let K be the alternating group $A_3 = \{(1),\ (123),\ (132)\}$. Then examples of elements of $H \times K$ include $(1, (1)),\ (-1, (123)),\ (i, (132))$ and $(j, (123))$. Since the inverse of the element j in the group H is $-j$ and the inverse of (123) in the group K is (132) we have that the inverse of the element $(j, (123)) \in H \times K$ is $(-j, (132)) \in H \times K$. It can be easily verified that $(1, (1))$ is the identity element of the group $H \times K$ since 1 is the identity element of H and (1) is the identity element of K.

Even if H and K are subgroups of a group G, the direct product $H \times K$ of H and K need not be isomorphic to the group G or to any subgroup of G. As an example, consider the symmetric group S_3. Clearly $H = \langle(123)\rangle$ and $K = \langle(12)\rangle$ are subgroups of S_3; yet $H \times K$ is not isomorphic to S_3 since $H \times K$ is an abelian group while S_3 is not abelian. However, under certain conditions, the direct product $H \times K$ of two subgroups of a group G may coincide with the group. Below is the theorem which states clearly when this coincidence happens.

Theorem 3.1.5. *If G is a group containing normal subgroups H and K with $H \cap K = \{1\}$ and $HK = G$, then $G \cong H \times K$.*

Proof. See [19]. $\square$

Theorem 3.1.6. *Let G_1 and G_2 be groups, and $N_1 \trianglelefteq G_1$, $N_2 \trianglelefteq G_2$. Then*

$$(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2).$$

Proof. See [6]. $\square$

Remark 3.1.7. Let H and K be subgroups of a group G. Then, in this thesis, when we say that G is factorised into H and K we mean that $G = HK$ is the product of H and K and not the direct product of H and K.

Definition 3.1.8. [3] Let H and K be subgroups of a group G. Then H and K are said to permute if $HK = KH$.

Theorem 3.1.9. *Let H and K be subgroups of a group G. Then the product HK is a subgroup of G if and only if $HK = KH$ (i.e. if and only if H and K permute).*

Proof. Suppose $HK = KH$. We use the subgroup criterion to show that HK is a subgroup of G. Pick $a, b \in HK$ and let $a = h_1k_1, b = h_2k_2$ with $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then $b^{-1} = k_2^{-1}h_2^{-1}$. Thus, $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1}$. Let $k_3 = k_1k_2^{-1} \in K$ and $h_3 = h_2^{-1} \in H$, then $ab^{-1} = h_1k_3h_3$. Since $HK = KH$, we have that $k_3h_3 = h_4k_4$ for some $h_4 \in H$ and $k_4 \in K$, and so $ab^{-1} = h_1h_4k_4 \in HK$.
Conversely, suppose HK is a subgroup of G. Since $K \leq HK$ and $H \leq HK$, we have that $KH \subseteq HK$ by the closure property of a group. It remains to show that $HK \subseteq KH$. Let $hk \in HK$. Since HK is a subgroup write $hk = b^{-1}$ for some $b \in HK$. If $b = h_1k_1$, then $hk = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH$. Thus, $HK \subseteq KH$ and so $HK = KH$. $\square$

Definition 3.1.10. Let H be a subgroup of G. Then H is said to be permutable if the product HK is a subgroup of G for all subgroups K of G.

Corollary 3.1.11. Let N be a normal subgroup of a group G. Then N is permutable (i.e. NK is a subgroup of G for all subgroups K of G).

Proof. Let N be a normal subgroup of a group G and K be a subgroup of G. Since N is normal we have $Ng = gN$ for all $g \in G$. Thus $Nk = kN$ for all $k \in K$ since $k \in K \Rightarrow k \in G$. Hence, $NK = KN$ for all subgroups K of G and the proof follows by Theorem $3.1.9$. $\square$

Below are the definitions of a supplement and a complement to a subgroup of a group:

Definition 3.1.12. Let H and K be proper subgroups of a group G. Then H is said to be a proper supplement to K (or to be supplemented by K) in G if $G = HK$.

Definition 3.1.13. Let H and K be proper subgroups of a group G. Then H is said to be a complement to K (or to be complemented by K) in G if $G = HK$ and $H \cap K = \{1\}$.

Definition 3.1.14. Let H and K be proper subgroups of a group G. Then H is called a minimal supplement to K if H is a supplement to K in G and there does not exist a proper subgroup L of H such that $G = KL$.

We observe from Corollary $3.1.11$ that if N is a normal subgroup of a group G and H is any subgroup of G, then we are guaranteed that the product NH is a subgroup of G. However, it should be noted that N being a normal subgroup of a group G is a sufficient condition, and not a necessary condition, for the product NH to be a subgroup of G for any other subgroup H of G. What this means is that the product HK of two subgroups H and K may be a subgroup of G even when none of the two subgroups is normal in G. The example below verifies this fact:

Example 3.1.15. The dihedral group

$$\begin{aligned} D_4 &= \left\langle r, s \mid r^4 = s^2 = 1,\ srs^{-1} = r^{-1} \right\rangle \\ &= \{1,\ r,\ r^2,\ r^3,\ s,\ sr,\ sr^2,\ sr^3\} \end{aligned}$$

has subgroups $H = \{1,\ s\}$ and $K = \{1,\ sr^2\}$. Both H and K are not normal subgroups of D_4. H is not normal since $r^{-1}(s)r = srs^{-1}(s)r = srr = sr^2 \notin H$. Also K is not normal since $r^{-1}(sr^2)r = srs^{-1}(sr^2)r = srr^2r = s \notin K$. However the product $HK = \{1,\ s,\ sr^2,\ r^2\}$ is a subgroup of the group D_4.

A product HK is a way of combining two subgroups of a group G to get a new subset which may or may not be a subgroup of G. In that sense it is worthy discussing intersections and unions of subgroups briefly. We note the two theorems below:

Theorem 3.1.16. *Let H and K be two subgroups of a group G. Then the intersection $H \cap K$ is also a subgroup of G.*

Proof. Since $1_G \in H$ and $1_G \in K$, we have that $1_G \in H \cap K$ and so $H \cap K$ is non-empty. Now let $a, b \in H \cap K$. We need to show that $ab^{-1} \in H \cap K$. But $a, b \in H \cap K \Rightarrow a, b \in H$ and $a, b \in K$. Since H and K are groups themselves, $b \in H$ and $b \in K$ implies that $b^{-1} \in H$ and $b^{-1} \in K$. Hence, $ab^{-1} \in H$ and $ab^{-1} \in K$ by closure. Thus, $ab^{-1} \in H \cap K$ and the proof follows by the subgroup criterion. $\square$

Theorem 3.1.17. *Let H and K be subgroups of a group G. Then the union $H \cup K$ is a subgroup of the group G if and only if $H \leq K$ or $K \leq H$.*

Proof. Suppose $H \subseteq K$ or $K \subseteq H$. Then $H \cup K = K$ or $H \cup K = H$ correspondingly. In either case $H \cup K$ is a subgroup of G since both H and K are subgroups of G.
Conversely, suppose that $H \cup K$ is a subgroup of G. To show that $H \subseteq K$ or $K \subseteq H$ we need only show $H \nsubseteq K$ implies $K \subseteq H$. Suppose $H \nsubseteq K$. Then there is an $h \in H$ such that $h \notin K$. Let $k \in K$. Then $h, k \in H \cup K$ which means that $hk \in H \cup K$ since $H \cup K$ is a group in its own right. If $hk \in K$, then $(hk)k^{-1} = h(kk^{-1}) = he = h \in K$, a contradiction. Thus, $hk \in H$ which means $h^{-1}(hk) = (h^{-1}h)k = ek = k \in H$. Since k was picked arbitrarily from the subgroup K we have shown that $K \subseteq H$ and this completes the proof. $\square$

Lemma 3.1.18 (Dedekind's Lemma)**.** [18] Let $H,\ K$ and L be subgroups of a group G such that $K \subseteq L$. Then $(HK) \cap L = (H \cap L)K$.

Proof. Since $H \cap L \subseteq H$, then $(H \cap L)K \subseteq HK$. Similarly, since $H \cap L \subseteq L$, we have that $(H \cap L)K \subseteq LK = L$. Thus,

$$(H \cap L)K \subseteq HK \cap L. \tag{3.1}$$

Choose an element $x \in (HK) \cap L$. Then $x = hk$ for some $h \in H$ and $k \in K$ and $x \in L$. Thus, $h = xk^{-1} \in LK = L$. Now since $H \cap L \subseteq L$, it follows that $h \in H \cap L$, and so $x = hk \in (H \cap L)K$. Therefore

$$(HK) \cap L \subseteq (H \cap L)K. \tag{3.2}$$

Now combining (3.1) and (3.2) gives that $(HK) \cap L = (H \cap L)K$. $\square$

3.2 Nilpotent groups

If G is a group we recall that the center of G, denoted by $Z(G)$, is the set of all elements of G which commute with every element of G. That is $Z(G) = \{z \in G \mid zg = gz \ \forall g \in G\}$. We also remember that $Z(G)$ is a subgroup of the group G, a normal subgroup for that matter.

Definition 3.2.1. Let G be a group. We define inductively the n-th center of G, denoted by $Z_n(G)$, as follows:

$$Z_n(G) = \{x \in G \mid xyx^{-1}y^{-1} \in Z_{n-1}(G) \ \forall y \in G\}.$$

For $n = 0$ we have $Z_0(G) = \{1\}$ and so we observe that

$$Z_1(G) = \{x \in G \mid xyx^{-1}y^{-1} \in \{1\} \ \forall y \in G\} = \{x \in G \mid xy = yx \ \forall y \in G\} = Z(G).$$

The ascending series

$$\{1\} = Z_0(G) \subset Z_1(G) \subset Z_2(G) \subset \cdots \subset Z_n(G) \subset \dots$$

of subgroups of the group G is called the upper central series of G.

A nilpotent group can be defined using the upper central series as below:

Definition 3.2.2. A group G is said to be nilpotent if $Z_m(G) = G$ for some positive integer m. The smallest m such that $Z_m(G) = G$ is called the class of nilpotency of G.

It is well known from elementary group theory that if G is an abelian group, then $Z(G) = G$. Thus, trivially, every abelian group is nilpotent since $Z_1(G) = Z(G)$.

Theorem 3.2.3. *If G is a nilpotent group and H is a proper subgroup of G then H is also a proper subgroup of the normaliser $N_G(H)$ of H in G.*

Proof. Let G be a nilpotent group of class r. It is trivial that $\{1\} = Z_0(G) \leq H$. It is also obvious that $G = Z_r(G) \nleq H$. Hence there exits a unique integer k with $0 \leq k \leq r-1$ such that

$$Z_k(G) \leq H,\ Z_{k+1}(G) \nleq H.$$

Thus there is an element u such that $u \in Z_{k+1}(G)$ and $u \notin H$. It suffices to show that $u \in N_G(H)$. Let $h_1 \in H$. Then $[u, h_1] \in [Z_{k+1}(G), G] \leq Z_k(G) \leq H$. This means that $u^{-1}h_1^{-1}uh_1 = h_2$ for some $h_2 \in H$. Hence $u^{-1}h_1^{-1}u \in H$. Since h_1^{-1} runs through H together with h_1, we have shown that $u^{-1}Hu \subset H$. Using the same argument with u replaced by u^{-1} we have that $uHu^{-1} \subset H$; that is $H \subset u^{-1}Hu$ which then imply that $u^{-1}Hu = H$. Thus, $u \in N_G(H)$. $\square$

Corollary 3.2.4. Every maximal subgroup of a nilpotent group G is normal in the group G.

Proof. Let M be a maximal subgroup of the group G. Since M is properly contained in G we have that $M < N_G(M) \leq G$ by Theorem 3.2.3. Now the maximality of M implies that $N_G(M) = G$. Thus $M \trianglelefteq G$. $\square$

Theorem 3.2.5. *Let G be a group. Then G is nilpotent if and only if $G' \leq \Phi(G)$, where $\Phi(G)$ denotes the Frattini subgroup of G.*

Proof. Suppose G is nilpotent and let M be a maximal subgroup of G. Then by Corollary 3.2.4, we have that $M \trianglelefteq G$. Also by Theorem 2.1.19 of Chapter 2, we have that G/M is cyclic of prime order and so G/M is an abelian group. Therefore $G' \leq M$. This is true for every maximal subgroup M of G and so, by definition of the Frattini subgroup we deduce that $G' \leq \Phi(G)$. Conversely, suppose $G' \leq \Phi(G)$. We need to show that G is nilpotent. But $G' \leq \Phi(G)$ implies that $G' \leq M$ for every maximal subgroup M of G. Thus, M/G' is a subgroup of an abelian group G/G' and so $M/G' \trianglelefteq G/G'$ as every subgroup of an abelian group is normal. Therefore, $M \trianglelefteq G$ and so we have that every maximal subgroup of G is normal. Hence G is nilpotent. $\square$

3.3 Solvable groups

Definition 3.3.1. A group G is said to be solvable (or soluble) if it has a solvable series, by which we mean a series

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

in which each factor G_{i+1}/G_i is abelian. The length of a shortest solvable series of the group G is called the derived length of G.

Remark 3.3.2. Trivially every abelian group G is solvable (with solvable length 1) since the series

$$\{1\} \trianglelefteq G$$

is already a solvable series of G.

Theorem 3.3.3. *Every subgroup H of a solvable group G is itself solvable.*

Proof. Since G is solvable, let $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ be a solvable series of G. Consider the series

$$\{1\} = (H \cap G_0) \leq (H \cap G_1) \leq (H \cap G_2) \leq \cdots \leq (H \cap G_n) = H. \tag{3.3}$$

This is a subnormal series of H since $H \cap G_i = (H \cap G_{i+1}) \cap G_i \trianglelefteq H \cap G_{i+1}$. By the second isomorphism theorem, we have that

$$(H \cap G_{i+1})/(H \cap G_i) \cong G_i(H \cap G_{i+1})/G_i \leq G_{i+1}/G_i.$$

Now since the factor groups G_{i+1}/G_i are abelian for all i, the subgroups $(H \cap G_{i+1})/(H \cap G_i)$ are also abelian for all $1 \leq i \leq n-1$. Therefore, the series given in (3.3) is a solvable series of H and so H is solvable. $\square$

Theorem 3.3.4. *Let G be a group and let $H \trianglelefteq G$. If both H and G/H are solvable, then G is solvable.*

Proof. See [20]. $\square$

3.4 Sylow $p-$subgroups

We recall Lagrange's theorem which states that if H is a subgroup of a finite group G, then the order of H divides the order of G. We also know that the converse of Lagrange's theorem is false: that is, if G is a finite group and d divides $|G|$, it is not always true that G contains a subgroup of order d. An example of this is the alternating group A_5 whose order is 60; and yet it has no subgroup of order 30.

However, the Norwegian mathematician Peter Ludwig Sylow discovered that the converse of Lagrange's theorem is partially true as we shall see later in this section. Sylow subgroups will be needed in this dissertation and so the need to revisit them. Since Sylow subgroups are $p-$groups we first define a $p-$group.

Definition 3.4.1. Let p be a prime. A group G is called a $p-$group if every element g of G has order p^k for some $k \in \mathbb{N}$.

Theorem 3.4.2 (Cauchy's Theorem)**.** *Let p be a prime. If G is a finite group such that p divides $|G|$, then G contains an element of order p.*

Proof. See [19]. $\square$

The corollary below suggests that the converse of Lagrange's theorem is true if the divisor d is a prime.

Corollary 3.4.3. Let p be a prime. If G is a finite group such that p divides $|G|$, then G contains a subgroup of order p.

Proof. Let G be a finite group and let p be a prime such that p divides the order of G. Then by Cauchy's Theorem G contains an element x of order p. Thus, the cyclic subgroup $H = \langle x \rangle$ generated by the element x has order p. $\square$

Theorem 3.4.4. *A finite group G is a $p-$group if and only if $|G| = p^m$, where p is a prime and m is a non-negative integer.*

Proof. Let G be a group and suppose $|G| = p^m$. Then if $g \in G$ we have that $o(g)|\ p^m$ by Lagrange's theorem. That is, every element of G has order p^k for some positive integer $k \leq m$. Hence, G is a $p-$group.
Conversely, suppose that G is a $p-$group. Assume that there is a prime $q \neq p$ which divides the order of G. By Cauchy's theorem, G contains an element of order q, which contradicts the fact that G is a $p-$group. Thus, there does not exist such a prime as q and so $|G| = p^m$. $\square$

Lemma 3.4.5. Let G be a finite $p-$group. If H is a proper subgroup of G, then $H < N_G(H)$.

Proof. See [20]. $\square$

Theorem 3.4.6. *Let G be a finite $p-$group. Then every maximal subgroup of G is normal and has index p.*

Proof. Let M be a maximal subgroup of the $p-$group G. Then by Lemma 3.4.5, we have that $M < N_G(M) \leq G$. Consequently, the maximality of M gives that $N_G(M) = G$, that is, $M \trianglelefteq G$. Now that K is both a maximal and normal subgroup of G, the index $[G : K]$ is prime by Theorem 2.1.19. $\square$

Definition 3.4.7. Let G be a finite group and p be a prime. If $|G| = p^k q$, where k is a non-negative integer and q is a positive integer such that p does not divide q, then a subgroup of G of order p^k is called a Sylow $p-$subgroup of G or Sylow subgroup for short.

From the definition of a Sylow $p-$subgroup of a group, we note that an equivalent definition of a Sylow $p-$subgroup would be the definition below:

Definition 3.4.8. Let p be a prime. A Sylow $p-$subgroup of a finite group G is a maximal $p-$subgroup P.

The fact that Sylow $p-$subgroups are maximal subgroups helps us prove the following theorem:

Theorem 3.4.9. *Let P be a Sylow $p-$subgroup of a finite group G, then every conjugate of P is also a Sylow $p-$subgroup of G.*

Proof. Let $g \in G$, then gPg^{-1} is a $p-$subgroup of G. If gPg^{-1} is not maximal then there exists a maximal $p-$subgroup Q with $gPg^{-1} < Q$. Hence, $P < g^{-1}Qg$, which is a contradiction since P is a maximal subgroup. Thus, gPg^{-1} is a maximal $p-$subgroup and the proof follows by Definition 3.4.8. □

Corollary 3.4.10. Let p be a prime. Then a finite group G has a unique Sylow $p-$subgroup P if and only if $P \trianglelefteq G$.

Proof. Assume that P, a Sylow $p-$subgroup of G, is unique. By Theorem 3.4.9, for each $g \in G$, the conjugate gPg^{-1} is also a Sylow $p-$subgroup of G. Now by uniqueness of P, we have that $gPg^{-1} = P$ for all $g \in G$, and so $P \trianglelefteq G$.
Conversely, assume that $P \trianglelefteq G$. If Q is any Sylow $p-$subgroup of G, then $Q = bPb^{-1}$ for some $b \in G$ as we shall see in Theorem 3.4.12 that any two Sylow $p-$subgroups are conjugate. But since P is normal, we deduce that $bPb^{-1} = P$, and so $Q = P$. □

Below are the three major Sylow's theorems:

Theorem 3.4.11. *(Sylow I). Let G be a finite group and p be a prime. If $|G| = p^k q$ with p and q coprime, then G contains a subgroup of order p^k, that is, Sylow $p-$subgroups always exist.*

Proof. See [19]. □

Theorem 3.4.12. *(Sylow II). Let G be a finite group of order $p^k q$ where p is a prime. Then all Sylow $p-$subgroups are conjugate to each other.*

Proof. See [19]. □

Theorem 3.4.13. *(Sylow III). For each prime p let n_p denote the number of Sylow $p-$subgroups. If $|G| = p^k m$ where $(p, m) = 1$, then $n_p \equiv 1 \bmod p$ and $n_p | m$.*

Proof. See [19]. □

There are various applications of Sylow subgroups in group theory. Sylow subgroups can be used to characterise certain types of groups as we may see in the following lemma and in the subsequent theorem:

Lemma 3.4.14. Let G be a group. The following statements are equivalent:

(i) G is nilpotent.

(ii) Every Sylow $p-$subgroup of G is normal in G.

Proof. See [18]. □

Theorem 3.4.15. *A finite group G is nilpotent if and only if it is the direct product of its Sylow subgroups.*

Proof. See [20]. □

3.5 The Frattini Subgroup

Definition 3.5.1. Let G be a group. The intersection of all the maximal subgroups of G is called the **Frattini subgroup** of G, and denoted by $\Phi(G)$.

Remark 3.5.2. Let G be a group. If $G \neq \{1\}$ and G is finite, then G certainly has at least one maximal subgroup. After all every proper subgroup of a group G is either a maximal subgroup of G or is contained in a maximal subgroup of G. However, if G is infinite it may have no maximal subgroups. If an infinite group G does not have a single maximal subgroup, then we define $\Phi(G) = G$. Also, if $G = \{1\}$, we define $\Phi(G) = \{1\} = G$.

Definition 3.5.3. If G is a group, an automorphism of G is an isomorphism from G to G. The set of all automorphisms of G, denoted by $Aut(G)$, forms a group under functional composition.

Remark 3.5.4. A subgroup K of a group G is called **characteristic** if $\phi(K) = K$ for every automorphism ϕ of G. Since any automorphism of G sends a maximal subgroup into another maximal subgroup, the set of all maximal subgroups of G is invariant under any automorphism of G, and so is the Frattini subgroup $\Phi(G)$ of G. Thus, $\Phi(G)$ is a characteristic subgroup and since characteristic subgroups are normal, we deduce that $\Phi(G) \trianglelefteq G$.

Definition 3.5.5. An element x of a group G is called a non-generator of G if for every subset S of G such that $\langle S, x\rangle = G$, then $\langle S\rangle = G$.

Theorem 3.5.6. *For every finite group G the Frattini subgroup of G is the set of all non-generators of G.*

Proof. Let x be a non-generator of the group G and suppose that $x \notin \Phi(G)$. Then by definition of $\Phi(G)$ there must exist a maximal subgroup M of G such that $x \notin M$. Hence $M \neq \langle x, M\rangle$ and so $G = \langle x, M\rangle$ as M is maximal in G. But since x is a non-generator of G we have that $G = \langle M\rangle = M$, which is a contradiction. Thus, our supposition is defeated and so $x \in \Phi(G)$. □

Corollary 3.5.7. Let G be a group and K be a subgroup of G. If $G = \Phi(G)K$, then $G = K$.

Proof. If $G = \Phi(G)K$, then we have that $G = \langle \Phi(G), K\rangle$. Now by Theorem 3.5.6 we obtain that $G = \langle K\rangle = K$. □

Corollary 3.5.8. Let G be a group. If $G/\Phi(G)$ is cyclic, then G is cyclic.

Proof. Let $x \in G$ such that the coset $x\Phi(G)$ generates the quotient group $G/\Phi(G)$. Then we have that $G = \langle x, \Phi(G)\rangle$ and so by Theorem 3.5.6, we deduce that $G = \langle x\rangle$. Therefore G is cyclic. □

Lemma 3.5.9. Let G be a group and $N \trianglelefteq G$. Then $\Phi(N) \leq \Phi(G)$.

Proof. Since $\Phi(N)$ is a characteristic subgroup of N and $N \trianglelefteq G$, we have that $\Phi(N)$ is normal in G. Suppose $\Phi(N) \nleq M$, where M is a maximal subgroup of G. Since $M \leq \Phi(N)M \leq G$, maximality of M gives that $\Phi(N)M = G$. Now since $\Phi(N) \leq N$, Dedekind's Lemma gives that $N = (\Phi(N)M) \cap N = (M \cap N)\Phi(N)$. By Corollary 3.5.7 we obtain $M \cap N = N$, so that

$\Phi(N) \leq N \leq M$. But this contradicts our assumption that $\Phi(N) \nleq M$. Hence $\Phi(N) \leq M$ for any maximal subgroup M of G, and so $\Phi(N) \leq \Phi(G)$. $\square$

Remark 3.5.10. It is not true in general that if H is a subgroup of G then $\Phi(H) \leq \Phi(G)$. As a counter example, consider the symmetric group S_4. It can be easily checked that $\Phi(S_4) = \{(1)\}$. However, consider the Sylow $2-$subgroup

$$H = \{(1),\ (1234),\ (13)(24),\ (1432), (13),\ (12)(34),\ (24),\ (14)(23)\}$$

of S_4 isomorphic to the dihedral group D_4. The maximal subgroups of H are
$\langle(1234)\rangle$, $\langle(13)(24),\ (13)\rangle$ and $\langle(13)(24),\ (12)(34)\rangle$. Thus, $\Phi(H) = \{(1),\ (13)(24)\}$, and so $\Phi(H) \nleq \Phi(S_4)$.

Lemma 3.5.11. Let G be a group and let $K \trianglelefteq G$. If $K \nleq \Phi(G)$, then K has a proper supplement in G.

Proof. K not contained in $\Phi(G)$ implies that K is not contained in M for some maximal subgroup M of G. Since $K \trianglelefteq G$, we deduce that the product KM is a subgroup of G. Thus, $M < KM \leq G$. Now, since M is maximal in G and since KM contains M properly, we have that $KM = G$. Hence, K has a proper supplement in G. $\square$

Theorem 3.5.12. *Let G be a group such that $H \leq G$ and $K \trianglelefteq G$. If $K \leq \Phi(H)$, then $K \leq \Phi(G)$.*

Proof. We shall prove the contra-positive of this statement: that is, if $K \nleq \Phi(G)$ then $K \nleq \Phi(H)$. Suppose $K \nleq \Phi(G)$. Then, by Lemma 3.5.11, there is a proper subgroup J of G such that $G = JK$.
Assume that $K \leq \Phi(H)$. Then

$$K \leq H \leq G = JK.$$

Now, by Dedekind's Lemma we have that

$$H = H \cap G = H \cap (JK) = (H \cap J)K.$$

By Lemma $3.5.11$ again, the assumption that $K \leq \Phi(H)$ implies that $H \cap J = H$ which would further imply that

$$K \leq H \leq J.$$

Therefore, $G = JK = J < G$, a contradiction. Hence, if $K \nleq \Phi(G)$, it follows that $K \nleq \Phi(H)$. □

Theorem 3.5.13. *Let G be a group and let $K \trianglelefteq G$. If H is a minimal supplement to K in G, then $H \cap K \leq \Phi(H)$.*

Proof. Since H is a supplement to K in G, we have that $G = HK$. Suppose, to the contrary, that $H \cap K \nleq \Phi(H)$. Then, by Lemma $3.5.11$, there exists a proper subgroup L of H such that $H = (H \cap K)L$. Now, Dedekind's Lemma gives that

$$H = (H \cap K)L = H \cap (KL),$$

which then implies that $G = KL$. But this contradicts the fact that H is a minimal supplement to K. □

Lemma 3.5.14. Let $N \trianglelefteq G$. Then the following are true:

(i) $\Phi(G)N/N \leq \Phi(G/N)$.

(ii) If $N \leq \Phi(G)$, then $\Phi(G/N) = \Phi(G)/N$.

Proof. (i) Let K be a maximal subgroup of G/N, then by the correspondence theorem there exists a maximal subgroup M of G such that $N \leq M < G$ and $K = M/N$. Now $\Phi(G/N) = \bigcap(M/N)$, where M is maximal in G. Since $\Phi(G) \leq M$, we have that $\Phi(G)N \leq MN = M$. Thus, $\Phi(G)N/N \leq M/N$ for all M maximal in G. Therefore,

$$\Phi(G)N/N \leq \bigcap(M/N) = \Phi(G/N).$$

(ii) By the correspondence theorem there is a one-to-one correspondence between the subgroups of G/N and those subgroups of G that contain N. Now $N \leq \Phi(G)$ implies that N

is contained in every maximal subgroup M of G. Hence,

$$\begin{aligned}\Phi(G/N) &= \bigcap (M/N), \text{ where } M \text{ runs through all maximal subgroups of } G \\ &= \left(\bigcap M\right)/N = \Phi(G)/N.\end{aligned}$$

□

Corollary 3.5.15. $\Phi\left(G/\Phi(G)\right) = \{1\}$.

Proof. Since $\Phi(G)$ is a normal subgroup of G, by Lemma 3.5.14 part (ii) we have that

$$\Phi\left(G/\Phi(G)\right) = \Phi(G)/\Phi(G) = \{1\}.$$

□

The lemma below is usually referred to as Frattini's argument.

Lemma 3.5.16. Let H be a normal subgroup of a group G and let P be a Sylow $p-$subgroup of H. Then $G = HN_G(P)$, where $N_G(P)$ is the normalizer of P in G.

Proof. We know that $HN_G(P) \leq G$ since H is a normal subgroup of G. Now, to show that $G = HN_G(P)$ it remains to show that $G \leq\ HN_G(P)$. Since $P \leq H$ we have that $gPg^{-1} \leq gHg^{-1} = H$. The equality $gHg^{-1} = H$ holds because $H \trianglelefteq G$. Since all Sylow $p-$subgroups of H are conjugate in H there exists $h \in H$ such that

$$P = h^{-1}gPg^{-1}h = h^{-1}gP(h^{-1}g)^{-1}.$$

Now, $P = h^{-1}gP(h^{-1}g)^{-1}$ implies that $h^{-1}g \in N_G(P)$ which further mean that $hh^{-1}g = g \in HN_G(P)$. Thus, $G \leq\ HN_G(P)$ and so $G = HN_G(P)$. □

Lemma 3.5.17. If G is a finite group, then the Frattini subgroup $\Phi(G)$ of G is nilpotent.

Proof. Let P be a Sylow subgroup of $\Phi(G)$. Since $\Phi(G) \trianglelefteq G$, Frattini's argument (See Lemma 3.5.16) attest that

$$G = N_G(P)\Phi(G).$$

Hence, by Corollary $3.5.7$, $N_G(P) = G$. Thus $P \trianglelefteq G$, and so $P \trianglelefteq \Phi(G)$. Now we have shown that every Sylow subgroup of $\Phi(G)$ is normal in $\Phi(G)$ and it follows from Lemma $3.4.14$ that $\Phi(G)$ is nilpotent. $\square$

Theorem 3.5.18. *The Frattini factor group $G/\Phi(G)$ of a finite $p-$group G is an elementary abelian $p-$group.*

Proof. Let $\mathcal{M}$ denote the set of all maximal subgroups of G. Since G is a $p-$group, we have that $M \trianglelefteq G$ and G/M is a cyclic $p-$group of order p for all $M \in \mathcal{M}$. Since G/M is a cyclic group of order p for all $M \in \mathcal{M}$, it follows that $(gM)^p = g^p M = M$ and so $g^p \in M$ for all $M \in \mathcal{M}$. Thus we deduce that $g^p \in \Phi(G)$ for every $g \in G$. Hence for every $g \in G$ we have that

$$(g\Phi(G))^p = g^p\Phi(G) = \Phi(G).$$

Therefore, the factor group $G/\Phi(G)$ is an elementary abelian $p-$group. $\square$

Lemma 3.5.19. Let $G_1,\ G_2, \ldots, G_r$ be finite groups. Then

$$\Phi(G_1 \times G_2 \times \cdots \times G_r) = \Phi(G_1) \times \Phi(G_2) \times \cdots \times \Phi(G_r).$$

Proof. See [7]. $\square$

3.6 The Residual subgroup

Before we end this chapter, we define a vital subgroup to group factorizations, the residual subgroup. Here is the motivation for this concept: Let A and N be subgroups of a group G such that $N < A$ and $N \trianglelefteq G$. We shall prove, in Chapter 4 (See Theorem $4.1.4$), that if $A/N < G/N$ has a proper supplement in G/N, then A has a proper supplement in G. Consequently, if every subgroup of G/N had a proper supplement in G/N, then every subgroup H of G which contains the subgroup N would have a proper supplement in G.

A non-trivial group G is called an $aS-$group if every non-trivial subgroup of G has a proper supplement in G. The search for a proper factorization of a group G is therefore highly assisted by the existence of a proper normal subgroup N of G such that G/N is an $aS-$group. The residual subgroup of a group G is, in simplest terms, the smallest normal subgroup N such that G/N is an $aS-$group, which is why we want to study it. A few definitions will be given before we finally define the said concept.

Below is the definition of a formation:

Definition 3.6.1. A class of finite groups F is said to be a formation if every homomorphic image of an $F-$group is an $F-$group and if $G/(N \cap M)$ belongs to F whenever G/N and G/M belong to F.

Below are some examples of formations:

- The class of finite solvable groups;
- The class of finite nilpotent groups;
- The class of finite abelian groups.

Definition 3.6.2. A formation F is called a saturated formation if $G \in F$ whenever $G/\Phi(G) \in F$, where $\Phi(G)$ denotes the Frattini subgroup of G.

Below we give an example of a saturated formation and an example of a non-saturated formation:

Example 3.6.3. The class $\mathbf{S}$ of finite solvable groups is a saturated formation.

Proof. Let G be a finite group and assume that $G/\Phi(G) \in \mathbf{S}$. With this assumption, we should show that $G \in \mathbf{S}$, that is, we should show that G is solvable. By Lemma 3.5.17, we deduce that $\Phi(G)$ is solvable since every nilpotent group is solvable. Thus, so far we have that both $G/\Phi(G)$ and $\Phi(G)$ are solvable and the proof follows by Theorem 3.3.4. $\square$

Example 3.6.4. The formation of finite abelian groups denoted by $\mathbf{A}b$ is a non-saturated formation. To justify this claim consider the group

$$\begin{aligned} G = D_4 &= \left\langle r, s \mid r^4 = s^2 = 1,\ srs^{-1} = r^{-1} \right\rangle \\ &= \{1,\ r,\ r^2,\ r^3,\ s,\ sr,\ sr^2,\ sr^3\} \end{aligned}$$

The maximal subgroups of G are $M_1 = \{1,\ r,\ r^2,\ r^3\}$, $M_2 = \{1,\ r^2,\ s,\ sr^2\}$ and $M_3 = \{1,\ r^2,\ sr,\ sr^3\}$. Thus $\Phi(G) = M_1 \cap M_2 \cap M_3 = \{1,\ r^2\}$. With this example we observe that $G/\Phi(G) \in \mathbf{A}b$ (since every group of order 4 is abelian); yet G does not belong to $\mathbf{A}b$ as it is not abelian.

Definition 3.6.5. A group G is an $aS-$group if it has order 1 or if every non-trivial subgroup H of G has a supplement in G.

It turns out that the collection of all $aS-$groups, denoted by $a\mathfrak{S}$ forms a formation.

Theorem 3.6.6. *Every subgroup of an $aS-$group is an $aS-$group.*

Proof. Let G be an $aS-$group and $H \leq G$. If $H = \{1\}$ or G, then the result follows trivially. Suppose that H is non-trivial and proper in G. Let K be a non-trivial subgroup of H. Since K is a non-trivial subgroup of G, there exists a proper subgroup L of G such that $G = KL$. By the Dedekind's Lemma,

$$H = G \cap H = (KL) \cap H = K(L \cap H).$$

If $L \cap H = H$, then $H \leq L$. This would then imply that $K \leq L$ and $G = L$, a contradiction. Thus, $L \cap H$ is a proper subgroup of H, and H is an $aS-$group. $\square$

Theorem 3.6.7. *If G is an $aS-$group, then the Frattini subgroup of G is trivial.*

Proof. Suppose that $\Phi(G) \neq \{1\}$. Let $x \in \Phi(G)$, such that $x \neq 1$. Since G is an $aS-$group, there is a proper subgroup H of G such that $G = \langle x \rangle H$. Consequently, $G = \langle x, H \rangle = \langle H \rangle = H$, a contradiction. Thus, $\Phi(G) = \{1\}$. $\square$

Definition 3.6.8. Let G be a group. Then G is supersolvable (or supersoluble) if there exists a normal series

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_k = G$$

such that each quotient group G_{i+1}/G_i is cyclic for all $0 \leq i \leq k-1$.

Theorem 3.6.9. *Suppose G is a finite group with the property that all its maximal subgroups are of index a prime. Then G is supersolvable.*

Proof. See [9]. □

We recall that an elementary abelian group G is an abelian group in which every non-trivial element has order p, where p is a prime. The elementary groups have the property that the Frattini subgroup, $\Phi(G)$, of the group G is the identity and each element $x \in G$ is not only a generator of G, but also of each subgroup of G containing it. Bechtell in [5] has defined an elementary group as below:

Definition 3.6.10. A group G is an elementary group if the Frattini subgroup of each subgroup H of G, $\Phi(H)$, is the identity.

Below is the characterisation theorem of $aS-$groups.

Theorem 3.6.11. *A group G is an $aS-$group if and only if G is supersolvable with elementary abelian Sylow subgroups.*

Proof. First suppose G is an $aS-$group, and let $x \in G, x \neq 1$. If $G = \langle x \rangle$, then G is supersolvable. Suppose $G \neq \langle x \rangle$. Then there exists a proper normal subgroup N of G such that $G = \langle x \rangle N$ for G is an $aS-$group. Let M be a normal subgroup of G maximal with respect to $x \notin M$ and $N \subseteq M$. We claim that G/M has prime order:
Since $G = \langle x \rangle M$, we deduce that $G/M \cong \langle x \rangle M/M \cong \langle x \rangle / \langle x \rangle \cap M$ by the second isomorphism theorem. Thus, G/M is cyclic and so abelian. To prove our claim, it now remains to show that G/M is simple. Suppose, to the contrary, that G/M is not simple, then there is a non-trivial proper subgroup K/M such that $K/M \trianglelefteq G/M$. Consequently, K is a proper normal subgroup of G with $M \subseteq K$. If $x \in K$, then $\langle x \rangle M = G \subseteq K$, a contradiction. Thus, $x \notin K$, which again contradicts the maximality of M. This implies that G/M is simple and the claim is proved. Hence, M has prime index. Since M was arbitrary, we have that every maximal subgroup of G has prime index. Therefore, G is supersolvable by Theorem 3.6.9.
Conversely, suppose G is supersolvable and every Sylow subgroup of G is elementary abelian.

By Theorem 1.1 of [12], the condition that all the Sylow subgroups of G are elementary abelian is equivalent to G being elementary itself. Since all elementary groups have trivial Frattini, we have that $\Phi(G) = \{1\}$. If H is a non-trivial normal subgroup of G, then H has a proper supplement in G by Theorem 3.5.11. Otherwise, suppose H is a non-trivial permutable subgroup of G. Then since $\Phi(G) = \{1\}$, Theorem 2.1 of [12] asserts that H has a proper supplement in G and so G is an $aS-$group. $\square$

Corollary 3.6.12. Let H and K be $aS-$groups. Then, the direct product $H \times K$ of H and K is also an $aS-$group.

Proof. Since H and K are $aS-$groups, by Theorem 3.6.11 we have that both H and K are supersolvable groups. Hence, $H \times K$ is supersolvable. It remains to show that all the Sylow $p-$subgroups of $H \times K$ are elementary abelian. By Theorem 1.1 of [12], we have that H and K are elementary groups. Now since H and K are elementary groups, we have that $\Phi(H) = \{1_H\}$ and $\Phi(K) = \{1_K\}$. Furthermore, by Theorem 3.5.19 it follows that $\Phi(H \times K) = \Phi(H) \times \Phi(K) = \{1\}$ and so $H \times K$ is an elementary group. Hence, the proof follows by Theorem 3.6.11. $\square$

Example 3.6.13. Here we give an example of an $aS-$group. It is easy to check that all the subgroups of the symmetric group S_3 of order 2 are supplemented by the alternating subgroup $A_3 = \langle(123)\rangle$. Thus, every non-trivial subgroup of S_3 has a proper supplement in S_3 and so the symmetric group S_3 is an $aS-$group by definition. Also using the characterisation theorem it is easy to see that S_3 is an $aS-$group as it is supersoluble and all its Sylow subgroups are elementary abelian.

Example 3.6.14. A non-example of an $aS-$group is the alternating group A_4. Two reasons will be given to justify this claim:

- The series $\{1\} \trianglelefteq \mathbf{V}_4 \trianglelefteq A_4$ is the longest normal series of the alternating group A_4. Now since the quotient factor $\mathbf{V}_4/\{1\} \cong \mathbf{V}_4$ is not cyclic we have that A_4 is not supersoluble. Hence, by the characterisation theorem, A_4 cannot be an $aS-$group as it is not supersoluble in the first place.

- By using the definition we also observe that A_4 cannot be an $aS-$group. This is because the three subgroups $A = \{(1),\ (12)(34)\}$, $B = \{(1),\ (13)(24)\}$ and $C = \{(1),\ (14)(23)\}$ of A_4 do not have proper supplements in the group A_4.

Definition 3.6.15. Let G be a group. Then the residual subgroup of the group G denoted by G^S is the intersection of all the normal subgroups N of G such that G/N is an $aS-$group.

The example below is an illustration of how to find the residual subgroup of a group.

Example 3.6.16. Suppose we want to find the residual subgroup of the alternating group A_4. The first thing we do is to find all the normal subgroups N of A_4 such that A_4/N is an $aS-$group. Clearly A_4 will be one of such normal subgroups since A_4/A_4 is a group of order 1 and we know that all groups of order 1 are $aS-$groups. The next one will be the Klein $4-$group $\mathbf{V}_4$ since $A_4/\mathbf{V}_4$ is an $aS-$group. Though normal, the trivial subgroup $\{1\}$ does not qualify in our list because $A_4/\{1\} \cong A_4$ is not an $aS-$group as we saw from the previous example. Thus, the residual subgroup of A_4 is $A_4^S = A_4 \cap \mathbf{V}_4 = \mathbf{V}_4$.

From the small example above we observe that the process of calculating the residual subgroup of a group can be tedious especially when the order of the group large enough. Lucky enough there are some properties that one can use to find it without going through the process of finding all the normal subgroups N such that G/N is an $aS-$group and taking the intersection. Below are some important properties of the residual subgroup G^S of a non-trivial group G.

Lemma 3.6.17. Let G be a group and G^S be the residual subgroup of G. Then the following properties hold:

(i) $\Phi(G) \leq G^S$.

(ii) G' is a proper subgroup of G if and only if G^S is a proper subgroup of G.

(iii) If $N \trianglelefteq G$, then $N^S \leq G^S$ and $G^S N/N = (G/N)^S$.

(iv) For a group L with residual L^S, $(G \times L)^S = G^S \times L^S$.

(v) $G^S \cap Z(G) \leq \Phi(G)$.

(vi) For any subgroup H of G, $\Phi(H) \leq G^S$.

Proof. (i) Since G^S is a normal subgroup of G, by part (i) of Lemma $3.5.14$ we have that $\Phi(G)G^S/G^S \leq \Phi(G/G^S)$. Since G/G^S is an $aS-$group, Theorem $3.6.7$ asserts that $\Phi(G/G^S) = \{1\}$. Hence, $\Phi(G)G^S/G^S = \{1\}$ and so $\Phi(G)G^S = G^S$. Therefore, $\Phi(G) \leq G^S$.

(ii) Suppose G' is a proper subgroup of G. Since G/G' is abelain and all abelian groups are supersolvable, we have that G/G' is supersolvable. By Theorem $3.6.11$, it follows that G/G' is an $aS-$group. Hence, $G^S \leq G'$.

The converse follows from the fact that G/G^S is supersolvable.

(iii) Note that the result follows if $N \leq G^S$. If N is not contained in G^S, then $G^S N/G^S$ is a non-trivial subgroup of the $aS-$group G/G^S. By Theorem $3.6.6$, we have that $G^S N/G^S$ is also an $aS-$group. The second isomorphism theorem gives that $G^S N/G^S \cong N/(G^S \cap N)$ and so $N/(G^S \cap N)$ is an $aS-$group. By definition of the residual subgroup, $N^S \leq G^S \cap N \leq G^S$. Thus, $N^S \leq G^S$.

To prove the second part: Let $(G/N)^S = S/N$. Given that $(G/N)/(S/N) \cong G/S$, which is an $aS-$group, $G^S \leq S$. Thus,

$$G^S N/N \leq S/N = (G/N)^S.$$

To obtain equality, it remains to show that $(G/N)^S \leq G^S N/N$. If $G^S N = G$, then $(G/N)^S \leq G/N = G^S N/N$ and $G^S N/N = (G/N)^S$.

Now suppose that $G^S N < G$. Then it follows, by the third isomorphism theorem, that $(G/N)/(G^S N/N) \cong G/G^S N$. Since $G^S \leq G^S N < G$ and since G/G^S is an $aS-$group, we deduce that $G/G^S N$ is an $aS-$group by Theorem $3.6.6$, that is $(G/N)/(G^S N/N)$ is an $aS-$group. Therefore, $(G/N)^S \leq G^S N/N$ and so $G^S N/N = (G/N)^S$.

(iv) Since $G^S \trianglelefteq G$ and $L^S \trianglelefteq L$, Theorem $3.1.6$ gives that $(G \times L)/(G^S \times L^S) \cong G/G^S \times L/L^S$. Furthermore, since G/G^S and L/L^S are $aS-$groups, the direct product $G/G^S \times L/L^S$ is also an $aS-$group by Corollary $3.6.12$. Hence, $(G \times L)^S \leq G^S \times L^S$ by definition of the residual subgroup of a group.

Secondly, by part (iii) both G^S and L^S are contained in $(G \times L)^S$. Thus, $G^S \times L^S \leq (G \times L)^S$ and $G^S \times L^S = (G \times L)^S$.

(v) Let $G^S \cap Z(G) = K \neq \{1\}$ and assume that K is not contained in $\Phi(G)$. Thus, there is a maximal subgroup M of G such that $K \nleq M$. Since both G^S and $Z(G)$ are normal subgroups of G, we have that $K \trianglelefteq G$. Now the maximality of M gives that $KM = G$. Let $g \in G$. Then $g = km$, where $k \in K$ and $m \in M$, and $M^g = M^{km}$. Since $k \in Z(G)$,

$$M^g = M^{km} = M^m = M$$

and $M \trianglelefteq G$. Since M is maximal in G, Theorem 2.1.19 asserts that $[G : M] = p$ for some prime p. Given that G/M is an $aS-$group, we have that $G^S \leq M$, which implies that $K = G^S \cap Z(G) \leq G^S \leq M$, a contradiction. Thus, the assumption that K is not contained in $\Phi(G)$ is false. Hence, $K = G^S \cap Z(G) \leq \Phi(G)$.

(vi) If $H \leq G^S$, then the proof follows immediately. Consider the case that H is not contained in G^S. Then HG^S/G^S is a non-trivial subgroup of G/G^S. Since G/G^S is an $aS-$group and since $HG^S/G^S \cong H/(H \cap G^S)$ is a subgroup of G/G^S, Theorem 3.6.6 asserts that $H/(H \cap G^S)$ is an $aS-$group too. Thus $H^S \leq H \cap G^S \leq G^S$. By part (i) of this lemma, we deduce that $\Phi(H) \leq G^S$.

□

If G is a group, then the Frattini subgroup, $\Phi(G)$ of G is contained in the residual subgroup G^S of G by part (i) of Lemma 3.6.17. However, the Frattini subgroup and the residual subgroup may coincide as the theorem below indicates.

Theorem 3.6.18. *[13] If G is nilpotent group then $\Phi(G) = G^S$.*

Proof. By part (i) of Lemma 3.6.17, all that needs to be shown is that $G^S \leq \Phi(G)$. Since G is nilpotent, it is the direct product of its Sylow subgroups. That is, $G = S_1 \times S_2 \times \cdots \times S_t$, where for each i, $1 \leq i \leq t$, S_i is a Sylow p_i-subgroup of G. Now by Lemma 3.5.19 we have that

$\Phi(G) = \Phi(S_1 \times S_2 \times \cdots \times S_t) = \Phi(S_1) \times \Phi(S_2) \times \cdots \times \Phi(S_t)$ and so

$$G/\Phi(G) \cong S_1/\Phi(S_1) \times S_2/\Phi(S_2) \times \cdots \times S_t/\Phi(S_t).$$

Since Sylow subgroups are $p-$groups, by Theorem $3.5.18$ we obtain that $S_i/\Phi(S_i)$ is elementary abelian for all i, where $1 \leq i \leq t$. So far we have shown that the Sylow subgroups of $G/\Phi(G)$ are elementary abelian. Theorem $3.5.17$ asserts that $\Phi(G)$ is nilpotent. Since all nilpotent groups are supersolvable, we deduce that both G and $\Phi(G)$ are supersolvable. Consequently, $G/\Phi(G)$ is supersolvable and so is an $aS-$group by Theorem $3.6.11$. Therefore, $G^S \leq \Phi(G)$ by definition of residual subgroup. $\square$

CHAPTER 4

Subgroup supplementation

4.1 Subgroups with proper supplements

This chapter is dedicated to investigating conditions under which a subgroup H of a group G has a proper supplement in G. Here is the motivation behind this topic:
The dihedral group

$$\begin{aligned} D_4 &= \left\langle r, s \mid r^4 = s^2 = 1,\ srs^{-1} = r^{-1} \right\rangle \\ &= \{1,\ r,\ r^2,\ r^3,\ s,\ sr,\ sr^2,\ sr^3\} \end{aligned}$$

has eight non-trivial subgroups:

(i) Three of order 4 : $H_1 = \{1,\ r,\ r^2,\ r^3\},\ H_2 = \{1,\ r^2,\ s,\ sr^2\}$ and $H_3 = \{1,\ r^2,\ sr,\ sr^3\}$.

(ii) Five of order 2: $K_1 = \{1,\ s\},\ K_2 = \{1,\ sr\},\ K_3 = \{1,\ sr^2\},\ K_4 = \{1,\ sr^3\}$ and $N = \{1,\ r^2\}$.

We know clearly that when a subgroup H of a finite group G is normal, we are guaranteed by Corollary $3.1.11$ that the product HK is a subgroup of the group G for all subgroups K of G. The subgroup $N = \{1,\ r^2\}$ is normal in D_4 which implies that the product NK is a subgroup of D_4 for all subgroups K of D_4. The trouble here is that out of the eight subgroups listed above there does not exist a single subgroup K of D_4 such that $D_4 = NK$. Something should be wrong with this particular subgroup N. This is why we have chosen to investigate conditions under which a proper subgroup has a proper supplement in the group.

Below is the first condition under which a normal subgroup will have a proper supplement in a finite group.

Theorem 4.1.1. *(Condition 1)· A normal subgroup N of G will have a proper supplement in G if and only if N is not contained in the Frattini subgroup $\Phi(G)$ of G.*

Proof. Suppose N has a supplement H in G. Then H is contained in a maximal subgroup M of G. Now since $G = NH$ and $H \leq M$, we have that $G = NM$. Suppose to the contrary that $N \leq \Phi(G)$, then $N \leq M$ since $\Phi(G) \leq M$ for all maximal subgroups M of G. But $N \leq M$ implies $NM = M \neq G$, a contradiction. Hence, N is not contained in $\Phi(G)$.
Conversely, let N be a normal subgroup of a group G and suppose that N is not contained in $\Phi(G)$. Then N is not contained in M for some maximal subgroup M of G. Since $N \lhd G$, we are guaranteed that the product NM is a subgroup of G. Thus, $M < NM \leq G$. Since there is no proper subgroup between M and G containing M, we have that $NM = G$; and so N has a supplement in G. □

Remark 4.1.2. Condition 1 gives an explanation to the problem presented in the motivation behind this topic: The subgroups $H_1,\ H_2$ and H_3 are the maximal subgroups of the dihedral group D_4. Since the Frattini subgroup of a group is calculated by taking the intersection of all the maximal subgroups of the group, we observe that $\Phi(D_4) = H_1 \cap H_2 \cap H_3 = \{1, r^2\} = N$; which implies that N is contained in $\Phi(D_4)$, no wonder N has no proper supplement in D_4.

Below is the second condition:

Theorem 4.1.3. *(Condition 2)· Let $N \trianglelefteq G$. If N is non-nilpotent then N has a proper supplement in G.*

Proof. Since N is non-nilpotent, we have that $|N|$ is divisible by at least two primes. Furthermore, by Lemma $3.4.14$, there exists a Sylow $p-$subgroup P of N which is not normal in N. By the Frattini's argument, $G = NN_G(P)$ where $N_G(P)$ is the normalizer of P in G. If $N_G(P) = G$, then P would be normal in G which would eventually mean that $P \trianglelefteq N$, a contradiction. Thus, $N_G(P) < G$ and so N has a proper supplement in G. □

Theorem 4.1.4. *(Condition 3)· Let N and A be subgroups of a group G with $N \trianglelefteq G$ and $N < A$. If A/N has a proper supplement in G/N then A has a proper supplement in G.*

Proof. Suppose A/N has a proper supplement B/N in G/N. Then, $G/N = (A/N)(B/N) = AB/N$ and so $G = AB$. □

Theorem 4.1.5. *(Condition 4)· If A is an abelian normal subgroup of a group G such that $A \cap \Phi(G) = \{1\}$, then A has a supplement in G. In fact, this supplement is a complement.*

Proof. Suppose that A is a non-trivial subgroup of G, for otherwise the proof would follow immediately. Now, $A \cap \Phi(G) = \{1\}$ implies that $A \nleq \Phi(G)$. By definition of $\Phi(G)$, $A \nleq \Phi(G)$ implies that there is some maximal subgroup M of G such that $A \nleq M$. Since M does not contain A and since $A \trianglelefteq G$, we have that $M < AM \leq G$, and so the maximality of M gives that $G = AM$. So far, we have shown that M is a supplement to A. To show that M is a complement to A it remains to show that $A \cap M = \{1\}$. If M is a minimal supplement to A, then by Theorem $3.5.12$ we have $A \cap M \leq \Phi(M)$ and by Theorem $3.5.13$ it follows that

$$A \cap M \leq \Phi(G). \tag{4.1}$$

But from elementary set theory we also have

$$A \cap M \leq A. \tag{4.2}$$

Thus, combining (4.1) and (4.2) we deduce that

$$A \cap M \leq A \cap \Phi(G) = \{1\},$$

and so $A \cap M = \{1\}$. If M is not a minimal supplement to A, then we choose a proper subgroup K of M such that K is a minimal supplement to A so that the same argument we used to show that $A \cap M = \{1\}$ would be applied to K to obtain $A \cap K = \{1\}$. $\square$

The example below illustrates and verifies Theorem 4.1.5.

Example 4.1.6. Consider the alternating group

$$G = A_4 = \{(1),\ (123),\ (124),\ (134),\ (234),\ (132),\ (142),\ (143),\ (243),\ (12)(34),\\ (13)(24),\ (14)(23)\}.$$

Apart from the Klein $4-$group $\mathbf{V}_4 = \{(1),\ (12)(34),\ (13)(24),\ (14)(23)\}$, all the four subgroups of A_4 of order 3 are maximal subgroups of the group A_4. Hence, the Frattini subgroup $\Phi(A_4)$ of A_4 is $\{(1)\}$ which implies that $\mathbf{V}_4 \cap \Phi(A_4) = \{1\}$. Now since $\mathbf{V}_4$ is both a normal subgroup of A_4 and abelian it satisfies the role of the subgroup A in the hypothesis of Theorem 4.1.5. Now, to verify the theorem it remains to check if there is such a subgroup B of A_4 with $\mathbf{V}_4 B = A_4$ and $\mathbf{V}_4 \cap B = \{1\}$. Choosing $B = \{(1),\ (123),\ (132)\}$ verifies the theorem.

Before we give the next condition under which a proper subgroup will have a proper supplement in a group G we need to define a normal Hall subgroup.

Definition 4.1.7. Let d and n be positive integers, then d is said to be a Hall divisor of n if d is a factor of n and $\frac{n}{d}$ is coprime to d. As an example, 2 is a Hall divisor of 6 since 2 and $\frac{6}{2} = 3$ are co-prime.

Definition 4.1.8. Let H be a subgroup of a group G, then H is called a Hall subgroup of G if the order $|H|$ of H is a Hall divisor of the order $|G|$ of the group G.

Below is an equivalent definition of a Hall subgroup.

Definition 4.1.9. A Hall subgroup of a finite group G is a subgroup whose order is coprime to its index.

Example 4.1.10. As a quick example: the Klein $4-$group $\mathbf{V}_4 = \{(1),\ (12)(34),\ (13)(24),\ (14)(23)\}$ is a Hall subgroup of the alternating group A_4 since the order $|\mathbf{V}_4| = 4$ of $\mathbf{V}_4$ is a Hall divisor of the order $|A_4| = 12$ of A_4.

Now we can define a normal Hall subgroup of a group.

Definition 4.1.11. A subgroup H of a group G is called a normal Hall subgroup if $H \trianglelefteq G$ and H is a Hall subgroup of G.

Below is the fifth condition under which a subgroup H of a group G will have a proper supplement in G. It is known as Schur-Zassenhaus Theorem.

Theorem 4.1.12. *(Condition 5)· If K is a normal Hall subgroup of a group G, then K has a proper supplement in G. In fact, this proper supplement is a complement to K.*

Proof. Let $|G/K| = n$ and $|K| = m$. Then it suffices to show that G has a subgroup H of order n: for then, since $(m, n) = 1$, $H \cap K = \{1\}$, by the second isomorphism theorem, $|HK| = mn$ and so $G = HK$.
We prove by induction on m that G has a subgroup of order n. Assume $m > 1$, for the result follows trivially if $m = 1$. Let p be a prime divisor of m and P be a Sylow $p-$subgroup of K. Then, by Frattini's argument, $G = NK$, where $N = N_G(P)$. By the second isomorphism theorem, $N \cap K \trianglelefteq N$ and $N/(N \cap K) \cong G/K$, of order n. If $N < G$ then $N \cap K < K$, and so $|N \cap K|$ is a proper divisor of m. Then, by the induction assumption, N has a subgroup H of order n. Thus, also H is a subgroup of G of order n. $\square$

Example 4.1.13. This example is meant to verify Schur-Zassenhaus Theorem. Let G be the special linear group $SL(2, 3)$, $i.e.$, the group of invertible 2×2 matrices having determinant 1 over the field $F_3 = \{0, 1, 2\}$. The subgroup

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

is the commutator subgroup of G and so it is normal in G. We also note that $|K| = 8$ is a Hall divisor of $|G| = 24$, thus H is a normal Hall subgroup of G. To verify the Schur-Zassenhaus Theorem we should find a complement to K. The element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has order 3 and so the subgroup $H = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$ has order 3. It can be easily checked that $G = HK$ and $H \cap K = \{1\}$. Hence, H is a complement to K and so the theorem is verified.

Remark 4.1.14. If the subgroup K is not normal, Schur-Zassenhaus Theorem is not true in general: That is, even if K was a proper subgroup of G such that $|K|$ is relatively prime to its index $[G : K]$, K might not have a proper supplement in G. Consider the sporadic simple group M_{22}. It has a maximal subgroup K of order $5760 = 2^7 3^2 5$, which is relatively prime to its index $[M_{22} : K] = 77$. However, K has no proper supplement in M_{22} as M_{22} is non-factorizable.

By the previous remark, it is clear that Schur-Zassenhaus Theorem is limited to normal subgroups. The next condition is an attempt to generalise Schur-Zassenhaus Theorem:

Theorem 4.1.15. *(Condition 6)· Let G be a group with $G^S \neq G$, and let $\{1\} < H < G$ such that $(|H|, [G : H]) = 1$. If H is not properly contained in G', then H has a proper supplement in G.*

Proof. Given that $G^S < G$, part (ii) of Lemma 3.6.17 implies that $G' < G$. If $G = G'H$, then H has a proper supplement in G. Assume that $G \neq G'H$. If $H = G'$, then $H \trianglelefteq G$ and H has a proper supplement in G by Schur-Zassenhaus Theorem. Suppose $H \neq G'$. Since $H \nleq G'$, $G'H/G'$ is a non-trivial subgroup of G/G'. Furthermore, $G'H/G' \trianglelefteq G/G'$ for G/G' is an abelian group. Let $|G'H/G'| = |H/(H \cap G')| = d$. By the third isomorphism theorem,

$$|(G/G')/(G'H/G')| = |G/G'H| = \frac{|G|}{|G'H|}.$$

Since $|G'H| = \frac{|G'||H|}{|H \cap G'|} > |H|$, let $\frac{|G|}{|G'H|} = m$. Then, $(m, |H|) = 1$. Since d divides $|H|$, we have that $(m, d) = 1$ and by Schur-Zassenhaus Theorem, $G'H/G'$ has a proper supplement K/G' in G/G'. As a result, $G = HK$ and H has a proper supplement in G. □

Theorem 4.1.16. *[13] (Condition 7)· Let A and H be subgroups of a group G such that A is proper in G and $A < H$. If A has a proper supplement in G, then A has a proper supplement in H.*

Proof. The proof follows immediately if $H = G$. Suppose H is a proper subgroup of G. Since A has a proper supplement in G, it follows that $G = AB$ for some subgroup B of G. By Dedekind's Lemma, we have that

$$H = H \cap G = H \cap (AB) = (H \cap B)A.$$

Now we consider the two cases below:

(i) Case 1: If $H \cap B = \{1\}$, then $A = H$; which is a contradiction by hypothesis.

(ii) Case 2: If $H \cap B = H$, then $A < H \leq B$, another contradiction (since $A \leq B$ would imply that $G = AB = B$).

Thus, $H \cap B$ is a proper supplement to A in H. $\square$

The converse of this Condition is not true as the group $G = \langle x, y \mid x^9 = y^9 = 1,\ xy = yx\rangle$ indicates. Consider the subgroups $H = \Phi(G) = \langle x^3,\ y^3\rangle$ and $A = \langle x^3\rangle = \{1,\ x^3,\ x^6\}$. The subgroup A has a proper supplement $B = \langle y^3\rangle = \{1,\ y^3,\ y^6\}$ in H since $H = AB$; yet A has no proper supplement in G as A is contained in the Frattini subgroup of G.

Theorem 4.1.17. *(Condition 8)· Let G be a group and let* $\{1\} < H < G$. *If* $G^S \cap H \neq H$, *then* H *has a proper supplement in G.*

Proof. Since $G^S \cap H \neq H$, it must be that $G^S \neq G$. If $HG^S = G$, then the proof follows immediately. Suppose $HG^S \neq G$, then HG^S/G^S is a non-trivial proper subgroup of the group G/G^S, which is an $aS-$group. Since in an $aS-$group every non-trivial subgroup has a proper supplement, there is a proper subgroup K/G^S of G/G^S such that

$$G/G^S = (HG^S/G^S)(K/G^S) = (HG^SK)/G^S = HK/G^S.$$

Consequently, $G = HK$ where K is a proper subgroup of G, and hence the proof. $\square$

Theorem 4.1.18. *(Condition 9) · Let G be a non-cyclic group with* $G^S \neq G$. *Then every maximal subgroup* M *of G has a proper supplement in G.*

Proof. Let M be a maximal subgroup of G. If $G = MG^S$, then M has a proper supplement right away. We therefore suppose $G \neq MG^S$. If $M < G^S$, then $G^S = G$ which would contradict the hypotheses. If $G^S < M$, then $G^S \cap M \neq M$ and M would have a proper supplement in G by Theorem 4.1.17. Finally, if $M = G^S$, then $M \trianglelefteq G$. Suppose $M = \Phi(G)$, then $G/\Phi(G)$ has prime order and so is a cyclic group. But $G/\Phi(G)$ being cyclic implies, by Corollary 3.5.8, that

G is also a cyclic group, which contradicts the hypotheses of the theorem. Thus, $M \neq \Phi(G)$. Also since M is maximal in G, we have that $M \nsubseteq \Phi(G)$, and by Theorem 4.1.1, M has a proper supplement in G. □

Theorem 4.1.19 (Condition 10)**.** *Let G be a finite group, and let p be a prime. Then a normal Sylow $p-$subgroup of G has a proper supplement in G.*

Proof. Let G be a finite group and let P be a Sylow $p-$subgroup of G. By definition of a Sylow $p-$subgroup, we observe that $|P|$ is relatively prime to $[G : P]$. Thus, P is a Hall subgroup of G. Now since P is normal in G, we have that P is a normal Hall subgroup of G and by Theorem 4.1.12, the proof follows. □

Suppose N is a normal subgroup of a group G. Then Theorem 4.1.1 asserts that N has a proper supplement provided it is not contained in the Frattini subgroup of G. Thus, if the Frattini subgroup $\Phi(G)$ of G is trivial then we obtain the following result.

Theorem 4.1.20 (Condition 11)**.** *Let G be a finite group. If $\Phi(G) = \{1\}$, then every non-trivial normal subgroup of G has a proper supplement in G.*

Proof. Since $\Phi(G) = \{1\}$, G admits maximal subgroups. Suppose that there is a non-trivial normal subgroup N of G that has no proper supplement in G. Then, for every maximal subgroup M of G, $NM \neq G$. Now since $NM \neq G$ and M is maximal in G, we have that $NM = M$. Thus $N \subseteq M$, for every maximal subgroup M of G. Consequently, $N \subseteq \Phi(G)$, a contradiction, since $\Phi(G) = \{1\}$ and N is non-trivial. Thus, N has a proper supplement in G. □

4.2 Subgroups without proper supplements

In as much as this chapter is dedicated to investigating conditions under which a subgroup has a proper supplement; it is equally useful to understand conditions under which a subgroup of a group has no proper supplement. Below are a few such conditions:

Theorem 4.2.1 (Condition 1)**.** *Let $N \trianglelefteq G$. If $N \leq \Phi(G)$ (i.e if N is contained in the Frattini subgroup of G), then N has no proper supplement in G.*

Proof. Already proved in Theorem 4.1.1. $\square$

Theorem 4.2.2. *(Condition 2)· Let H and K be subgroups of the group G such that $H < K$. If K is a cyclic $p-$group then H has no proper supplement in G.*

Proof. In Chapter 5 we have explained why all cyclic $p-$groups are not factorizable. Thus, K is not factorizable, that is, there does not exist a proper subgroup L of K such that $K = HL$. So far, we have established that H has no proper supplement in K. Hence, by considering the contra-positive of Theorem 4.1.16, the proof follows. $\square$

The example below illustrates and verifies Theorem 4.2.2.

Example 4.2.3. The subgroup $H = \{1,\ -1\}$ of the Quaternion group

$$\begin{aligned} Q_8 &= \big\langle -1\ ,\ i\ ,\ j\ ,\ k \mid (-1)^2 = 1\ ,\ i^2 = j^2 = k^2 = ijk = -1 \big\rangle \\ &= \{1\ ,\ -1\ ,\ i\ ,\ -i\ ,\ j\ ,\ -j\ ,\ k\ ,\ -k\}. \end{aligned}$$

has no proper supplement in the group Q_8 because it is contained in the cyclic $p-$group $K = \langle i \rangle = \{1,\ -1,\ i,\ -i\}$.

Theorem 4.2.4 (Condition 3)**.** *Let $N,\ B \leq G$ such that $N \trianglelefteq G$ and $N \leq B$. If N has no proper supplement in B then N has no proper supplement in G.*

Proof. Since $N \trianglelefteq G$ we have that $N \trianglelefteq B$. Thus, by Theorem 4.1.1 N not having a proper supplement in B implies that $N \leq \Phi(B)$. Hence, by Theorem 3.5.12, N is contained in the Frattini subgroup of G. Therefore, the proof follows by Theorem 4.1.1. $\square$

Theorem 4.2.5 (Condition 4)**.** *Let G be a finite group. Then, the Frattini subgroup $\Phi(G)$ of G has no proper supplement in G.*

Proof. See Corollary 3.5.7. $\square$

CHAPTER 5

Factorizable Groups

In this chapter, we investigate conditions under which a finite group G admits a proper factorization. Before we start looking at conditions under which a group is factorizable, we give some examples of non-factorizable groups.

5.1 Groups without proper factorizations

Definition 5.1.1. A group G is called non-factorizable if $|G| \neq 1$ and for all proper subgroups H of G, there does not exist a proper subgroup K of G such that $G = HK$.

With the demand that H and K be proper subgroups in the definition above it is clear that abelian simple groups are non-factorizable as they have only one proper subgroup, the trivial subgroup consisting of the identity element alone. Another family of finite groups that is non-factorizable is the family of cyclic $p-$groups. This is because all proper subgroups of a finite $p-$group G are contained in the Frattini subgroup, $\Phi(G)$, of G.

The example below helps us visualize this fact.

Example 5.1.2. Consider the cyclic $p-$group below:

$$G = \left\langle x \mid x^{16} = 1 \right\rangle = \{1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6,\ x^7,\ x^8,\ x^9,\ x^{10},\ x^{11},\ x^{12},\ x^{13},\ x^{14}, x^{15}\}$$

The group G has three non-trivial subgroups:

$$\begin{aligned} H_2 &= \left\langle x^2 \right\rangle = \{1,\ x^2,\ x^4,\ x^6,\ x^8,\ x^{10},\ x^{12},\ x^{14}\} \\ H_4 &= \left\langle x^4 \right\rangle = \{1,\ x^4,\ x^8,\ x^{12}\} \\ H_8 &= \left\langle x^8 \right\rangle = \{1,\ x^8\}. \end{aligned}$$

We observe that H_2 is the only maximal subgroup of the $p-$group G and it contains all the other proper subgroups of G. Hence $\Phi(G) = H_2$, and so the proper subgroups $H_2,\ H_4$ and H_8 are all contained in the Frattini subgroup of G. Thus as we proved, in the previous chapter, that a normal subgroup has a proper supplement if and only if it is not contained in the Frattini subgroup of the group, none of the subgroups $H_2,\ H_4$ and H_8 has a proper supplement in G. This therefore implies that the group G is non-factorizable. The explanation in this particular example is not the proof but it is the reason all finite cyclic $p-$groups are not factorizable.

From Table 4.1 of [13], we note that among the simple groups of Lie type the unitary groups $U_n(q)$ with n odd are non-factorizable except for $U_3(3),\ U_3(5)$ and $U_9(2)$. The same table also confirms that the sporadic simple groups

$$M_{22},\ Mc,\ C_{O_3},\ C_{O_2},\ Fi_{23},\ Fi'_{24},\ HN,\ Th,\ B,\ M,\ J_1,\ O'N,\ L_y,\ J_3$$

and J_4 are all non-factorizable.

5.2 Groups with proper factorizations

A group G is said to be factorizable (or to admit a proper factorization) if $G = HK$ is a product of its proper subgroups H and K. Below is the first condition under which a finite group G is factorizable:

Theorem 5.2.1 (Condition 1)**.** *If G is a non-cyclic group and $G' \neq G$ then G admits a proper factorization.*

Proof. Consider the commutator subgroup G' of G. If G' is a proper subgroup of G, then two cases arise:

(i) Case 1: If $G' \nsubseteq \Phi(G)$. Since G' is a normal subgroup of G, the result follows by Theorem $4.1.1$ which states that a normal subgroup N of a group G has a proper supplement in G if and only if $N \nsubseteq \Phi(G)$.

(ii) Case 2: If $G' \subseteq \Phi(G)$. Then G is nilpotent by Theorem $3.2.5$. Let M be a maximal subgroup of G. Then $M \trianglelefteq G$ since by Corollary $3.2.4$ every maximal subgroup of a nilpotent group is normal. Now let $x \in G$ such that $x \notin M$ and let $H = \langle x \rangle$, then MH is a subgroup of G as $M \trianglelefteq G$ and $M < MH \leq G$. Since M is maximal the chain $M < MH \leq G$ implies that $MH = G$ and so G is factorizable.

□

Example 5.2.2. Condition 1 above gives an explanation why all symmetric groups are factorizable. Consider the symmetric group S_n. We know that the group S_n is non-cyclic and that $[S_n,\ S_n] = A_n \neq S_n$. Hence, the symmetric group satisfies the hypotheses of the theorem (Condition 1) above and so we expect it to be factorizable. We observe that $S_n = A_n \langle x \rangle$ is a product of the alternating group A_n and any subgroup generated by some element $x \in S_n$ where $x \notin A_n$, and so it is factorizable verifying the theorem.

The condition below is called Frattini's argument.

Lemma 5.2.3 (Condition 2)**.** If a group G has a normal subgroup H and if H has a Sylow $p-$subgroup P then G admits a proper factorization. In fact, $G = HN_G(P)$, where $N_G(P)$ is the normalizer of P in G.

Proof. This is Frattini's argument already proved in Lemma $3.5.16$. □

Example 5.2.4. This example is meant to illustrate and verify the Frattini's argument. Let the group G be the symmetric group S_4 and let the subgroup H be the Alternating subgroup A_4. Then $H \trianglelefteq G$ and since the order $|A_4|$ of A_4 is $12 = 2^2.3$, we have that $P = \{(1),\ (123),\ (132)\}$

is a Sylow $3-$subgroup of H. So the subgroups H and P satisfy the hypothesis of the theorem. It remains to check whether $S_4 = A_4 N_{S_4}(P)$. Now since

$$N_{S_4}(P) = \{(1),\ (12),\ (13),\ (23),\ (123),\ (132)\} = S_3,$$

it can be easily seen that $S_4 = A_4 S_3 = A_4 N_{S_4}(P)$, and so the Frattini's argument is verified.

Theorem 5.2.5 (Condition 3)**.** *A non-cyclic $p-$group is factorizable.*

Proof. Let G be a non-cyclic $p-$group and let M be a maximal subgroup of G. Since G is a $p-$group, by Theorem 3.4.6, we deduce that $M \trianglelefteq G$. Since M is normal in G, the product $M\langle g\rangle$ is a subgroup of G for every $g \in G$. Pick x in G such that $x \notin M$. Since G is non-cyclic, $\langle x\rangle$ is a proper subgroup of G. Furthermore,

$$M < M\langle x\rangle \leq G.$$

Hence, the maximality of M gives that $G = M\langle x\rangle$, and so G is factorisable. □

Theorem 5.2.6 (Condition 4)**.** *Let G be a non-cyclic group containing a normal subgroup M of prime index. Then G admits a proper factorization. In fact, $G = M\langle x\rangle$ for every element $x \notin M$.*

Proof. We consider two cases:

(i) Case 1: Suppose M is a maximal subgroup of G. Then $M < M\langle x\rangle \leq G$ for all x in G and $x \notin M$. Since M is maximal in G and since $M\langle x\rangle$ contains M properly, we have that $M\langle x\rangle = G$.

(ii) Case 2: Suppose M is not maximal in G. Then the series $M < M\langle x\rangle \leq G$ of subgroups of G still holds only that, this time, the subgroup $M\langle x\rangle$ does not need to be equal to G since M is not maximal. Let $[G : M] = p$ and $[G : M\langle x\rangle] = q$. Then $q < p$ since $M < M\langle x\rangle$. In fact q divides p by Theorem 2.1.17. Now, since p is prime there are two possibilities: either $q = p$ or $q = 1$. But $q = p$ would imply that $M = M\langle x\rangle$ which is not possible since $x \notin M$. Thus, we have that $q = [G : M\langle x\rangle] = 1$ and so $M\langle x\rangle = G$.

$\square$

Theorem 5.2.7 (Condition 5)**.** *Let G be a non-cyclic group with $G^S \neq G$. Then G admits a proper factorization where one of the proper subgroups is cyclic.*

Proof. Since $G \neq G^S$ it follows that there exists an element $g \neq 1$ such that $g \in G$ and $g \notin G^S$. We also note that since G is not cyclic the subgroup $\langle g \rangle$ is a proper subgroup of G. Now since $g \notin G^S$, we have that $G^S \cap \langle g \rangle \neq \langle g \rangle$ and by Theorem 4.1.17 the subgroup $\langle g \rangle$ has a proper supplement in G. $\square$

Theorem 5.2.8 (Condition 6)**.** *A non-cyclic nilpotent group is factorizable.*

Proof. Let G be a non-cyclic nilpotent group. Since G is nilpotent, by Corollary 3.2.4 , $M \trianglelefteq G$, for M maximal in G. Now M normal in G implies that the product $M \langle g \rangle$ is a subgroup of G for every $g \in G$. Choose an element b in G such that $b \notin M$, then $M < M \langle b \rangle \leq G$. The maximality of M gives that $G = M \langle b \rangle$. Since G is non-cyclic, we are guaranteed that $\langle b \rangle \neq G$, and so G is factorizable. $\square$

Theorem 5.2.9 (Condition 7)**.** *Let p and q be distinct primes. Then every group of order pq is factorizable.*

Proof. Let G be a group such that $|G| = pq$ and suppose, without loss of generality, that $p < q$. By Cauchy's Theorem, G has an element a of order p and an element b of order q. Let $P = \langle a \rangle$ and $Q = \langle b \rangle$. These subgroups have size p and q respectively. By Theorem 3.4.13, $n_q \equiv 1 \bmod q$ and $n_q |\ p$. Thus since p is prime, the only choices for n_q are $n_q = 1$ or p. Since $1 < p < q$, the congruence condition on n_q implies that $n_q = 1$. Therefore Q is the only Sylow $q-$subgroup of G and by Corollary 3.4.10, we deduce that Q is normal in G. Since P and Q are Sylow subgroups, they are maximal in G. Hence, the normality of Q in G gives that $G = PQ$. $\square$

Theorem 5.2.10 (Condition 8)**.** *Let p and q be distinct primes such that $q < p$. Then every group of order $p^n q$ is factorizable for any positive integer n.*

Proof. Let G be a finite group such that $|G| = p^n q$. Since Theorem 3.4.11 asserts that Sylow $p-$subgroups always exist, let H be a subgroup of G of order p^n. By Cauchy's Theorem, G has a subgroup $K = \langle b \rangle$, where b is an element of G of order q. Now, by Theorem $3.4.13$, we have that $n_p \equiv 1 \mod p$ and $n_p|\ q$. Since q is a prime, n_p is either 1 or q. Besides, the congruence condition on n_p together with the condition that $q < p$ eliminate the possibility of n_p being q. Thus, $n_p = 1$ and by Corollary $3.4.10$, it follows that $H \trianglelefteq G$. Since q does not divide p^n, we have that K is a Sylow $q-$subgroup of G. Hence, both H and K are maximal subgroups of G and by the normality of H we deduce that $G = HK$. $\square$

CHAPTER 6

On Complemented groups

If G is a group and H and K are subgroups of G with $G = HK$, then H is said to be supplemented in G by K. In this chapter we explore a special type of factorization: a factorization which insists that H and K meet trivially. Let G be a finite group and H a subgroup of G. If there exist a subgroup K of G such that $G = HK$ with $H \cap K = \{1\}$, then H is said to be complemented by K in G. If H is a non-trivial normal subgroup of G and is complemented in G by K, then G is said to split over H and is written $G = [H]K$.

6.1 Complemented subgroups

Definition 6.1.1. Let G be a group and H a subgroup of G. If H is complemented in G by a subgroup K, then the factorization $G = HK$ is called an exact factorization.

It can be easily observed that if $G = HK$ and $H \cap K = \{1\}$, then every element $g \in G$ has a unique expression $g = hk$ where $h \in H$ and $k \in K$. This justifies why the factorization $G = HK$ with $H \cap K = \{1\}$ is termed exact.

Definition 6.1.2. If every subgroup H of a finite group G is complemented in G, then G is called a complemented group.

A quick example of a complemented group would be the symmetric group S_3. In this group, we note that all the subgroups of order 2 are complemented by A_3.

Lemma 6.1.3. Let G be a group and H, K and N subgroups of G with $N \trianglelefteq G$.

(i) If $H < K$ and H is complemented in G, then H is complemented in K.

(ii) If $N < H$ and H is complemented in G, then H/N is complemented in G/N.

Proof. (i) Let J be a subgroup of G such that $G = HJ$ and $H \cap J = \{1\}$. Then by Dedekind's Lemma, we have that

$$K = G \cap K = (HJ) \cap K = (J \cap K)H.$$

If $J \cap K = K$, then $K \leq J$ which implies that $H \leq J$ which further implies that $G = HJ = J$, a contradiction. Hence, $J \cap K$ is a proper subgroup of K. To show that H is complemented in K by $J \cap K$, it remains to show that $H \cap (J \cap K) = \{1\}$. But

$$H \cap (J \cap K) = (H \cap J) \cap K = \{1\} \cap K = \{1\}.$$

(ii) Let L be a subgroup of G and suppose H is complemented in G by L. Then $G = HL$ and $H \cap L = \{1\}$. Now, we note that $(H/N)(L/N) = (HL)/N = G/N$. Since $H \cap L = \{1\}$, we also have that $H/N \cap L/N = \{1_{G/N}\}$. Thus H/N is complemented in G/N.

$\square$

Theorem 6.1.4. *If G is an $aS-$group which satisfies the descending chain condition on subgroups, then G is a complemented group.*

Proof. Let H be a subgroup of G and let K be minimal among subgroups which supplement H in G. Let $H_1 = H \cap K$, and suppose that $H_1 \neq \{1\}$. By Theorem 4.1.16 H_1 has a proper supplement K_1 in K. Thus

$$G = HK = H(H_1K_1) = (HH_1)K_1 = HK_1.$$

This contradicts the minimality of K. Therefore, $H_1 = \{1\}$ and H is complemented in G. $\square$

Below is the Schur-Zassenhaus Theorem restated in the light of subgroup complementation:

Theorem 6.1.5. *Let H be a normal subgroup of G such that $(|H|, [G : H]) = 1$. Then H is complemented in G.*

Definition 6.1.6. Let $H \leq G$. Then H is called a normal $p-$complement of G for a prime p if H is a normal subgroup of G of order coprime to p and index a power of p.

Definition 6.1.7. Let G be a finite group, p a prime dividing the order of G and P a Sylow $p-$subgroup of G. Then G is called a $p-$nilpotent group if P is complemented in G.

Theorem 6.1.8. *If H is a normal $p-$complement of a group G, then H is complemented by any Sylow $p-$subgroup of G.*

Proof. Let H be a normal $p-$complement of a finite group G and suppose $|H| = m$. Then $(m, p) = 1$ by definition of a normal $p-$complement. Let $[G : H] = p^k$, then by Lagrange's theorem we have that $|G| = mp^k$. If P is a Sylow $p-$subgroup of G, then $|P| = p^k$. Since $H \trianglelefteq G$ and P is maximal in G, we deduce that $HP = G$. To complete our proof, it remains to show that $P \cap H = \{1\}$. Let $x \in P \cap H$ and let $o(x)$ denote the order of x. Then $o(x)$ divides $|P|$ and $o(x)$ divides $|H|$. Since $(|H|, |P|) = 1$, we have that $o(x) = 1$. Therefore, $P \cap H = \{1\}$. $\square$

Theorem 6.1.9. *Let G be a finite group such that $|G| = pq$ where p and q are distinct primes with $q < p$. Then G is $p-$nilpotent.*

Proof. Let G be a finite group with $|G| = pq$, where p and q are primes. Then by Cauchy's theorem G contains an element h of order p and an element k of order q. Let $P = \langle h \rangle$ and $K = \langle k \rangle$. Then P is a Sylow $p-$subgroup of G and by Theorem 3.4.13, we have that $n_p \equiv 1$ *mod* p and $n_p | q$. Since $q < p$, it follows that $n_p = 1$. Consequently, $P \trianglelefteq G$. Furthermore, $P \cap K = \{1\}$ since it is a group of order dividing both p and q. Since $P \trianglelefteq G$, we deduce that PK is a subgroup of G. The product formula asserts that

$$|PK| = \frac{|P||K|}{|P \cap K|} = |P||K| = pq = |G|.$$

Thus $G = PK$, and P is complemented. $\square$

CHAPTER 7

Applications of Group Factorizations

In this chapter we discuss, briefly, some contributions of group factorizations to the understanding of group theory. The main purpose of this chapter is to demonstrate how group factorizations can be used to tackle group theory problems.

7.1 Group factorization approach

Group factorizations can be used to characterise some groups. The theorem below was proved by Bertram Huppert:

Theorem 7.1.1. *[1] If $G = AB$ with A and B cyclic, then G is supersolvable.*

Consider the dihedral group

$$D_n = \langle r, s \mid s^2 = 1,\ r^n = 1,\ s^{-1}r^m s = r^{-m} \rangle.$$

If we wanted to determine whether the group D_n is supersolvable or not, Huppert's theorem could be of help. All we need to observe is that the subgroups $\langle s \rangle$ and $\langle r \rangle$ are cyclic. In fact, the latter is normal in D_n as it is a subgroup of index 2. Hence, D_n can be factorized into $D_n = \langle r\rangle\langle s\rangle$. Therefore, by Huppert's theorem (Theorem 7.1.1) we have that D_n is supersolvable. If we use the traditional definition of a supersolvable group to solve the same problem, we note that the series $\{1\} \trianglelefteq \langle r\rangle \trianglelefteq D_n$ is a normal series of D_n in which all the quotient groups are cyclic, confirming that D_n is indeed supersolvable.

Remark 7.1.2. If $G = AB$ with A and B cyclic, that does not mean that G is cyclic too. A counterexample would be obtained from the factorization $S_3 = A_3\langle(12)\rangle$ of the symmetric group S_3. Certainly, A_3 and $\langle(12)\rangle$ are cyclic subgroups of S_3; yet S_3 is not cyclic.

The theorem below is called Ito's Theorem. It is a highly celebrated result in group theory.

Theorem 7.1.3 (Ito's Theorem)**.** *If $G = AB$ is a product of abelian subgroups A and B then G is metabelian, $i.e.$ it is solvable with derived length at most* 2.

Suppose we wanted to determine whether the quaternion group

$$Q_8 = \{1,\ -1,\ i,\ -i,\ j,\ -j,\ k,\ -k\}$$

is metabelian or not: First we note that the subgroups $A = \langle i\rangle = \{1,\ -1,\ i,\ -i\}$ and $B = \langle j\rangle = \{1,\ -1,\ j,\ -j\}$ are cyclic subgroups of Q_8 and that $Q_8 = AB$. Since cyclic groups are abelian, we deduce that A and B are abelian and so Q_8 is metabelian by Ito's theorem. Furthermore, since Q_8 is not abelian we know that its derived length is not 1. Hence, by Ito's theorem we conclude that Q_8 has derived length 2.

Remark 7.1.4. Even if a group $G = AB$ was a product of two abelian subgroups A and B, it would not imply that G is also abelian. Consider the factorization $Q_8 = AB$ of the quaternion group Q_8, where $A = \langle i\rangle$ and $B = \langle j\rangle$. We note that the subgroups A and B are both abelian; yet the product $AB = Q_8$ is not abelian.

Kegel in [3] made the following observation:

Theorem 7.1.5. *If $G = AB$ is a product of nilpotent subgroups A and B then G is solvable.*

Consider the general linear group $GL(2,3)$ of degree two: the group of 2×2 invertible matrices over the field $\mathbb{F}_3 = \{0,\ 1,\ 2\}$ with three elements. Suppose we wish to determine whether $GL(2,3)$ is solvable or not. Let A be the subgroup of $GL(2,3)$ of order 8, isomorphic to the quaternion group and let $B = \left\langle \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \right\rangle$ be the subgroup of order 6, isomorphic to the cyclic group $\mathbb{Z}_6$. Then $A \trianglelefteq GL(2,3)$ and we obtain the factorization $GL(2,3) = AB$. Now since A is a $p-$group, it is nilpotent; B is also nilpotent as it is cyclic. Hence, by Kegel's theorem (Theorem $7.1.5$) we have that $GL(2,3)$ is solvable.

Remark 7.1.6. Let G be a finite group. Even if $G = AB$ was a product of two nilpotent subgroups A and B, it would not guarantee that G is also nilpotent. Here is a counterexample: Consider the factorization $A_4 = \mathbf{V}_4\langle(123)\rangle$ of the alternating group A_4 into the Klein $4-$group $\mathbf{V}_4$ and $\langle(123)\rangle$. Definitely, $\mathbf{V}_4$ and $\langle(123)\rangle$ are nilpotent. Nonetheless, the product $\mathbf{V}_4\langle(123)\rangle = A_4$ is not nilpotent.

From Remark 7.1.6, we observe that a group $G = AB$ which is a product of nilpotent subgroups A and B does not always inherit the property of nilpotency from the factors. Troubled by the same observation, Kegel introduced, in 1961, the notion of a triple factorization. This is a factorization of a group involving three subgroups A, B and C of the type

$$G = AB = BC = CA.$$

The evidence is that the existence of a triple factorization can have greater consequences for the group structure than does a single factorization. For example, Kegel in [1] proved the theorem below:

Theorem 7.1.7. *A finite group G which has a triple factorization $G = AB = BC = CA$ with A, B and C all nilpotent is nilpotent.*

Motivated by Theorem $7.1.7$, we tried (in this dissertation) to check whether a group which is a triple factorization by abelian subgroups would be abelian itself. The following counterexample shuttered our hopes: Consider the quaternion group Q_8 which has the triple factorization

$$Q_8 = AB = BC = CA,$$

where $A = \langle i \rangle$, $B = \langle j \rangle$ and $C = \langle k \rangle$. The subgroups A, B and C are all abelian as they are cyclic; yet the group Q_8 is not abelian.

The methods presented in this chapter are given to show that group factorizations is equally an effective approach through which to study group theory.

CHAPTER 8

Conclusion

The study was set out to explore the concept of group factorizations. Throughout our study we have tried to answer the two major questions stated below:

(i) Question 1: under what conditions does a proper subgroup H of a group G have a proper supplement in G?

(ii) Question 2: under what conditions does a finite group G admit a proper factorization?

The Frattini subgroup and the residual subgroup, among others, are crucial in answering these questions.

Answers to Question 1 are given in Chapter 4 and the first answer states that if N is a normal subgroup of a group G, then N has a proper supplement in G provided that N is not contained in the Frattini subgroup of G. In fact, the Frattini subgroup $\Phi(G)$, itself, of a group G has no proper supplement in G. The Schur-Zassenhaus Theorem which states that if K is a normal subgroup of a finite group G and if $|K|$ is relatively prime to its index, then K has a proper supplement in G is also an adequate answer to Question 1.

Chapter 5 is devoted to answering Question 2. We proved in this chapter that both non-cyclic nilpotent groups and non-cyclic $p-$groups are factorizable. One of the most general answers to Question 2 was obtained with the help of the commutator subgroup G' of a group G. It states that if G is a non-cyclic group and if $G \neq G'$, then G admits a proper factorization. We also attempted to address this question by simply using the order of a group. We managed to show that if p and q are distinct primes and if G is a finite group such that $|G| = pq$, then G is factorizable.

Most properties of a group G carry on to the subgroups of G. For example if G is nilpotent and if H is a subgroup of G, then H is also nilpotent. If $G = HK$ is a factorizable group, a natural path of inquiry opens up when one asks how the structure of the factors H and K affects the structure of G. Obviously, if H and K are finite subgroups of G, then G is finite and its order is given by

$$|G| = \frac{|H||K|}{|H \cap K|}.$$

Thus, a group which is the product of two finite $p-$groups is itself a finite $p-$group. Another property which carries over from the factors of a factorizable group to the group itself is the property of being perfect, $i.e.$, coinciding with the commutator subgroup.

Nonetheless, this occurrence seems to be quite uncommon. Surely if one experiments with properties such as solubility, finite exponent, being abelian, or nilpotency, one soon realizes the difficulty of using the factorization to obtain information about the structure of the group. There are in fact some quite evident counterexamples:

(i) The subgroups A_3 and $\langle(12)\rangle$ of the symmetric group S_3 are clearly nilpotent, yet the product $A_3\langle(12)\rangle = S_3$ is not nilpotent.

(ii) The subgroups A_3 and the Klein $4-$group $\mathbf{V}_4$ of the alternating group A_4 are undoubtedly abelian, yet the product $A_3\mathbf{V}_4 = A_4$ is not abelian.

(iii) The factors A_4 and $\mathbb{Z}_5$ of the factorization $A_5 = A_4\mathbb{Z}_5$ where $\mathbb{Z}_5 = \langle(12345)\rangle$ are undoubtedly soluble, yet the product A_5 is not soluble.

The concept of group factorizations therefore needs to be revisited in order to understand the extent to which we may know the structure of a factorizable group $G = HK$ by simply using properties of the factors H and K.

To show how worthy of studying group factorizations is, consider the following problem: If we wanted to determine whether the special linear group $SL(2,3)$ is soluble or not, we would look for a normal series

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = SL(2,3)$$

of $SL(2,3)$, by using the traditional definition, in which the quotient groups G_i/G_{i-1} are abelian for all i. By using group factorizations to solve the same problem, we would simply consider the fact that $SL(2,3)$ can be factorized into the dihedral group D_4 and the cyclic group $\mathbb{Z}_3$ as we saw in Example $4.1.13$. Since the factors D_4 and $\mathbb{Z}_3$ are nilpotent, we deduce right away (by using Theorem $7.1.5$) that the group $SL(2,3)$ is soluble. Hence, group factorizations has not only produced influential results to group theory but also is undisputedly one of the best approaches to studying group theory.

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