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Conformal symmetry and applications to spherically symmetric spacetimes

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This dissertation is submitted to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal, Durban, in fulfilment of the requirements for the degree of Doctor of Philosophy.
As the candidate's supervisors, we have approved this dissertation for submission.

Signed: Prof S. D. Maharaj 12 February 2018.

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Abstract

In this thesis we study static spherically symmetric spacetimes with a spherical conformal symmetry and a nonstatic conformal factor. We analyse the general solution of the conformal Killing vector equation subject to integrability conditions which impose restrictions on the metric functions. The Weyl tensor is used to characterise the conformal geometry. An explicit relationship between the gravitational potentials for both conformally and nonconformally flat cases is obtained. The Einstein equations can then be written in terms of a single gravitational potential. Previous results of conformally invariant static spheres are special cases of our solutions. For isotropic pressure we can find all metrics explicitly and show that the models always admit a barotropic equation of state. We show that this treatment contains well known metrics such Schwarzschild (interior), Tolman, Kuchowicz, Korkina and Orlyanskii, Patwardhan and Vaidya, and Buchdahl and Land. For anisotropic pressures the solution of the fluid equations is found in general. We then consider an astrophysical application of conformal symmetries. We investigate spherical exact models for compact stars with anisotropic pressures and a conformal symmetry. We generate a new anisotropic solution to the Einstein field equations. We demonstrate that this exact solution produces a relativistic model of a compact star. The model generates stellar radii and masses consistent with PSR J1614-2230, Vela X1, PSR J1903+327 and Cen X-3. A detailed physical examination shows that the model is regular, well behaved and stable. The mass-radius limit and the surface red shift are consistent with observational constraints.

Declaration I - Plagiarism

I, Addial Mackingtosh Manjonjo, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

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Declaration II - Publications

Details of contributions that form part of the research presented in this thesis.

- Publication 1

  Manjonjo, A. M., Maharaj, S. D., Moopanar, S., Conformal vectors and stellar models, *European Physical Journal Plus*, **132**, 62 (2017).

- Publication 2

  Mafa Takisa, P., Maharaj, S. D., Manjonjo, A. M., Moopanar, S., Spherical conformal model for compact stars. *European Physical Journal C*, **77**, 713 (2017).

- Publication 3

  Manjonjo, A. M., Maharaj, S. D., Moopanar, S., Static models with conformal symmetry, *Classical and Quantum Gravity*, **35**, 045015 (2018).

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Dedication

To my parents who gave me everything

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Chapter 1

Introduction

Albert Einstein's theory of general relativity is undoubtedly one of the most astounding achievements in science. The recent detection of the long predicted gravitational waves continues to affirm general relativity's position as the most successful theory of gravity. Its applications include the understanding of black holes, formation of galaxies, motions of planets and stars, and the origins and evolution of the universe itself.

At the heart of this theory are the Einstein field equations; a system of ten coupled nonlinear differential equations. Exact solutions to these field equations are of importance in the study of astrophysical objects. They provide the link between the geometry of a spacetime and the matter that reside within and interact with that spacetime. A large portion of general relativity literature is concerned with analysis of exact solutions.

It is difficult to find the exact solutions without making some simplifying assumptions on either the spacetime geometry or the matter content. One of the ways to simplify these equations is to assume the spacetime has some form of symmetry. The concept of symmetry therefore, plays a central role in general relativity. The well-known Noether's theorem reveals that symmetry of a spacetime leads to conserved quantities, which will simplify the solution. Symmetry properties are also used to impose structure on the set of solutions of the Einstein equations by a classification scheme based on groups of motions or other invariant vectors. The type of symmetry we are concerned with in this thesis is spherical symmetry. In a spacetime with spherical symmetry no radial direction is special. Solutions obtained by making such an assumption are often simple and physically reasonable, and they provide a good starting point in examining more complicated spacetimes. The first known such solution is the Schwarzschild solution published in 1916. Spherically symmetric spacetimes are physically significant as they have wide applications to cosmology and relativistic astrophysics. A spacetime symmetry of physical interest is the conformal Killing vector as it preserves the metric up to a conformal factor. Conformal symmetries have been extensively studied in the literature. Conformal symmetries have the geometric property of preserving the structure of the null cone by mapping null geodesics to null geodesics. They are physically significant as they generate constants of the motion along null geodesics for massless particles. There are different types of vector symmetries on a manifold, namely those generated by Killing vectors, conformal Killing vectors and homothetic vectors. In conformally symmetric spacetimes the change in the metric as it is dragged along a congruence of curves is directly proportional to itself. Therefore conformal symmetries are a more general type of symmetry than Killing symmetries. Conformal symmetries also serve as an additional mathematical tool for simplifying gravitational potentials. This is achieved by introducing restrictions on the potentials. It is our hope that a study of conformal symmetries will provide a deeper insight into the behaviour of the gravitational field and yield new classes of exact solutions to field equations. This thesis is primarily on the study of conformal symmetries in general relativity. To give a complete general classification of spacetimes, we make use of the Weyl tensor. The Weyl tensor represents the tracefree part of the Riemann curvature tensor and describes how the shape of a body is distorted whilst the volume is preserved as it experiences tidal forces. This enables us to obtain an explicit relationship connecting the two gravitational potentials in general. Here we relate the two potentials in static spacetimes.

Comprehensive analysis of the conformal geometry, especially in spherically symmetric spacetimes, and their kinematical and dynamical quantities were performed by

Coley and Tupper (1990a, 1990b, 1994) and by Maartens *et al.* (1986) and Mason and Maartens (1987) in anisotropic fluids. The complete conformal group in static spherically symmetric spacetimes was explicitly derived by Maharaj *et al.* (1995) and Maartens *et al.* (1995, 1996), and for the non-static case by Moopanar and Maharaj (2010). Note that the classification of Noether symmetries into distinct classes of algebras was completed by Ali *et al.* (2015). A complete classification of all spherically symmetric spacetimes has been done by Tupper *et al.* (2012) in terms of conformal motions through decomposition into $2+2$ reducible spacetimes and their Petrov types. Various applications of conformal symmetries in relativistic astrophysics arise in geometries which are static and spherically symmetric. Some early models of perfect fluids spheres admitting a group of conformal symmetries were found by Herrera and Ponce de León (1985a,b). Anisotropic spheres consistent with a conformal vector were analysed by Herrera and Ponce de León (1985c), Herrera *et al.* (1984), Rahaman *et al.* (2010a) and Rahaman *et al.* (2010b). A recent investigation by Rahaman *et al.* (2017) with a Tolman-like interior geometry with anisotropy and a conformal symmetry is consistent with a compact star satisfying all energy conditions. The influence of an electric field in the presence of a conformal vector has been considered by Ray *et al.* (2008), Radinschi *et al.* (2010), Maartens and Maharaj (1990) and Krori *et al* (1986a, 1986b). If the physical spaces $\{t = \text{constant}\}$ are spherical then quintessence fields can be generated by a conformal symmetry as shown by Bhar (2015). The gravastar model with charge is consistent with a conformal motion as demonstrated by Usmani *et al.* (2011). Gravastars have also been studied in the braneworld framework with conformal symmetry by Banerjee *et al.* (2016). Esculpi and Aloma (2010) produced conformally invariant charged spheres with a linear equation of state. Herrera *et al.* (2012) studied irreversible dissipative processes and Landau damping in relativistic stellar systems. Mak and Harko (2004) studied charged strange stars with a quark equation of state. The simple geometry of a static spherical symmetric manifold leads to multiple solutions of the Einstein-Maxwell system of equations. These solutions are important as they may be used to model highly dense astronomical objects with strong gravitational interactions. Some recent investigations in this direction include the works of Pandya *et al.* (2015), Kiess (2012), Fatema and Murad (2013), Murad and Fatema (2013) and Murad (2016). It is also possible to find solutions with a barotropic equation of state. Recent examples of papers with linear, quadratic and polytropic equations of state are contained in the analyses of Sunzu *et al.* (2014), Maharaj and Mafa Takisa (2012) and Mafa Takisa and Maharaj (2013) respectively. These models have been shown to provide a basis for an accurate gravitational description of a relativistic star. For a comprehensive analysis establishing the linkage between the stellar model and observations see the treatment of Mafa Takisa *et al.* (2014b). Conformal symmetries have been applied to cosmology in different spacetimes. The conformal geometry has been studied in Robertson-Walker spacetimes by Maartens and Maharaj (1986) and Keane and Barrett (2000). A detailed analysis of conformal vectors has been undertaken by Maartens and Maharaj (1991) and Keane and Tupper (2004) in *pp*-wave spacetimes. Castejon-Amenedo and Coley (1992) and Hansraj *et al.* (2005) have considered the applications of conformal symmetries in conformally related spacetimes. Charged gravitating fluids in dimensions different from four have been studied in the presence of a conformal motion. In the lower dimensions of $2+1$ the model was developed by Mallick *et al.* (2016) and Rahaman *et al.* (2014). In higher dimensions, greater than four, an anisotropic compact star was built by Bhar *et al.* (2015). However note that the stability of conformally invariant compact objects may be affected by dimensions different from four as indicated by Rahaman *et al.* (2015b).

In Chapter 2 the relationship between conformal symmetries and relativistic spheres in astrophysics is studied. We use the nonvanishing components of the Weyl tensor to classify the conformal symmetries in static spherical spacetimes. An explicit connection between the two gravitational potentials for both conformally flat and nonconformally flat cases is obtained. In Chapter 3, we conduct a more general treatment of static spherically symmetric spacetimes with a spherical conformal symmetry and a nonstatic conformal factor. With these assumptions we find an explicit relationship between the gravitational potentials. We use this relation to obtain the general solution of the Einstein field equations with a conformal symmetry. We display the known solutions of the Einstein field equation which are special cases of our general solution. In Chapter 4 we consider spherical exact models for compact stars with anisotropic pressures and a conformal symmetry. The conformal symmetry condition generates an integral relationship between gravitational potentials. We solve this condition to find a new anisotropic solution to the Einstein field equations. We demonstrate that the exact solution produces a relativistic model of a compact star. In Chapter 5 we briefly summarise the work done in this thesis.

Chapter 2

Static conformal classification

2.1 Introduction

Previous work on particular general relativistic models has yielded interesting solutions of the Einstein field equations. This has been achieved by choosing a particular conformal symmetry or restricting the form of the gravitational potentials. We demonstrate that this approach can be generalised more systematically with minimum assumptions. It turns out that we need only assume that the spherically symmetric spacetime is static and a spherically symmetric conformal symmetry exists. Then we can show that the general solution of the Einstein field equations with anisotropic fluid sources is fully determined. Even though the spacetime is static, the conformal factor may be nonstatic. To achieve our result we make use of the Weyl tensor and the integrability condition for the existence of a conformal Killing vector. A nonlinear partial differential equation arises which relates the two metric functions; a detailed study generates different cases which can all be integrated. In §2.3 we investigate the properties of the Weyl tensor in the classification of conformal symmetries. The conformally flat case is analysed in §2.4. The non-conformally flat case is considered in §2.5. For both cases we can integrate the consistency conditions and generate the conformal geometry. The various known cases of exact solutions to the Einstein field equations with conformal geometry are identified in §2.6. Our approach has the advantage of yielding the general functional form of the metric functions for both the isotropic and anisotropic pressures.

2.2 Spacetime geometry

The general form of the line element for static spherically symmetric spacetimes in the absence of shear is given by

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{2\lambda(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{2.1}$$

in comoving coordinates. The quantities $\nu(r)$ and $\lambda(r)$ represent the gravitational potentials. The metric (2.1) admits four linearly independent Killing vectors

$$\mathbf{X} = \frac{\partial}{\partial t}, \tag{2.2a}$$

$$\mathbf{X} = \frac{\partial}{\partial \phi}, \tag{2.2b}$$

$$\mathbf{X} = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}, \tag{2.2c}$$

$$\mathbf{X} = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}. \tag{2.2d}$$

We study groups of conformal motions that preserve the metric up to a factor. The conformal Killing vector field is defined by the following relation

$$\mathcal{L}_{\mathbf{X}}g_{ab} = 2\psi g_{ab}, \tag{2.3}$$

where $\mathcal{L}_{\mathbf{X}}$ is the Lie derivative with respect to the vector $\mathbf{X}$ and $\psi(x^a)$ is the conformal factor. The set of all conformal Killing vectors generate a Lie algebra. The conformal equation (2.3) has been integrated in general by Maharaj *et al.* (1995). They found the conformal vector $\mathbf{X} = (X^0, X^1, X^2, X^3)$ as

$$X^0 = -r^2 e^{-2\nu} A^i{}_t \eta_i + A^0 + a_0, \tag{2.4a}$$

$$X^1 = r^2 e^{-2\lambda} A^i{}_r \eta_i + A^4, \tag{2.4b}$$

$$X^2 = -A^i (\eta_i)_\theta + a_1 \sin\phi + a_2 \cos\phi, \tag{2.4c}$$

$$X^3 = -\csc^2\theta A^i (\eta_i)_\phi + \cot\theta (a_1 \cos\phi - a_2 \sin\phi) + a_3, \tag{2.4d}$$

where the functions $A^0, A^4, A^i = (A^1, A^2, A^3)$, and the constants a_0-a_3 result from the integration process. Also we have set

$$\eta_i = (\eta_1, \eta_2, \eta_3) = (\sin\theta \sin\phi, \sin\theta \cos\phi, \cos\theta). \tag{2.4e}$$

The conformal factor is then given by

$$\psi = r^2 \Big(\nu_r e^{-2\lambda} A^i{}_r - e^{-2\nu} A^i{}_{tt} \Big) \eta_i + A^0_t + \nu_r A^4. \tag{2.4f}$$

The solution above is subject to integrability conditions which have the form

$$(r e^{-\nu} A^i{}_t)_r = 0, \tag{2.5a}$$

$$e^{2(\lambda-\nu)} A^i{}_{tt} + A^i{}_{rr} + (2r^{-1} - \lambda_r - \nu_r) A^i{}_r = 0, \tag{2.5b}$$

$$A^i{}_{tt} + e^{2(\nu-\lambda)} (r^{-1} - \nu_r) A^i{}_r + r^{-2} e^{2\nu} A^i = 0, \tag{2.5c}$$

$$A^4{}_r + (\lambda_r - r^{-1}) A^4 = 0, \tag{2.5d}$$

$$e^{2\nu} A^0{}_r - e^{2\lambda} A^4{}_t = 0, \tag{2.5e}$$

$$A^0{}_t + (\nu_r - r^{-1}) A^4 = 0. \tag{2.5f}$$

2.3 The Weyl tensor

As indicated earlier the conformal vectors in spherically symmetric spacetimes have been investigated and classified by Tupper *et al.* (2012) in terms of Lie algebras and the general structure of admissible metrics. However, this approach does not explicitly give the general functional forms for the metric functions $\nu(r)$ and $\lambda(r)$. These forms are necessary to integrate the Einstein and Einstein-Maxwell equations. In this section we follow the approach of classifying the conformal geometry in terms of the Weyl tensor. The approach has the advantage of yielding the general form of the metric coefficients $\nu(r)$ and $\lambda(r)$. The nonlinear differential equations relating to the gravitational potentials can be obtained in general. The metric functions fall into three classes depending on whether the Weyl tensor is vanishing (one case) or nonzero (two cases). Thus our classification is characterised physically by the presence or absence of tidal forces.

Note that the existence of a conformal Killing vector is connected with the Weyl tensor by the condition

$$\mathcal{L}_{\mathbf{X}} C^{a}{}_{bcd} = 0, \tag{2.6}$$

where $C^{a}{}_{bcd}$ denotes the nonvanishing components of the Weyl tensor. The nonvanishing Weyl tensor components for the spacetime (2.1) are given by

$$C^{0}{}_{101} = -\frac{1}{3}\Gamma, \tag{2.7a}$$

$$C^{0}{}_{202} = \frac{1}{6} r^2 \mathrm{e}^{-2\lambda}\Gamma, \tag{2.7b}$$

$$C^{0}{}_{303} = \frac{1}{6} r^2 \sin^2\theta \mathrm{e}^{-2\lambda}\Gamma, \tag{2.7c}$$

$$C^{1}{}_{212} = C^{0}{}_{202}, \tag{2.7d}$$

$$C^{1}{}_{313} = C^{0}{}_{303} \tag{2.7e}$$

$$C^2{}_{323} = -2C^0{}_{303}, \tag{2.7f}$$

where we have set

$$\Gamma = \nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda}). \tag{2.8}$$

The introduction of the quantity Γ assists in the classification scheme for the conformal symmetry. When $\Gamma = 0$ all components of the Weyl tensor vanish and the spacetime is conformally flat. We apply the general Lie derivative formula (2.6) to obtain the conditions

$$-\frac{1}{3}\left\{r^2 A^i{}_r\Gamma_r + 2\Gamma\left[2r(1 - r\lambda_r)A^i{}_r + r^2 A^i{}_{rr}\right]\right\}\mathrm{e}^{-2\lambda}\eta_i$$

$$-\frac{1}{3}(A^4\Gamma_r + 2A^4{}_r\Gamma) = 0, \tag{2.9a}$$

$$\frac{1}{6}\left\{\mathrm{e}^{-2\lambda}A^i{}_r\left[2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r\right] + 2\Gamma A^i\right\}r^2\mathrm{e}^{-2\lambda}\eta_i$$

$$+\frac{1}{6}\left[2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r\right]\mathrm{e}^{-2\lambda}A^4 = 0. \tag{2.9b}$$

Then using the linear independence of the trigonometric functions in η_i we obtain the following conditions

$$\left[r^2\Gamma_r + 4r(1 - r\lambda_r)\Gamma\right]A^i{}_r + 2r^2 A^i{}_{rr}\Gamma = 0, \tag{2.10a}$$

$$\Gamma_r A^4 + 2\Gamma A^4{}_r = 0, \tag{2.10b}$$

$$\left[2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r\right]\mathrm{e}^{-2\lambda}A^i{}_r + 2\Gamma A^i = 0, \tag{2.10c}$$

$$\left[2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r\right]A^4 = 0. \tag{2.10d}$$

The equations in the system (2.10) are the necessary and sufficient conditions for the integrability condition (2.6) to be satisfied. The structure of the system (2.10) suggests that there is a natural classification system for the conformal symmetries of the static spherically symmetric spacetimes (2.1). We consider the two cases, namely $\Gamma = 0$ and $\Gamma \neq 0$. The classification naturally separates the spacetimes into conformally flat and non-conformally flat categories.

2.4 Conformally flat: $\Gamma = 0$

This case has vanishing Weyl tensor and it is characterised by

$$\nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda}) = 0. \tag{2.11}$$

This second order nonlinear partial differential equation was solved by Herrera *et al.* (2001) in their work on conformally flat anisotropic spheres. Integrating (2.11) leads to

$$\left(\nu_r - \frac{1}{r}\right)^2 \mathrm{e}^{-2\lambda} = \frac{1}{r^2}\left(1 - B^2 r^2 \mathrm{e}^{-2\nu}\right), \tag{2.12}$$

where B is a constant of integration. A second integration then leads to

$$\mathrm{e}^{\nu} = Br \cosh\left(\int \frac{\mathrm{e}^{\lambda}}{r} \mathrm{d}r + l\right), \tag{2.13}$$

where l is constant. The result (2.13) is the general solution of the nonlinear equation (2.11). We can state our result in terms of the following theorem.

Theorem 2.4.1. *In a static spherically symmetric spacetime which is conformally flat* $(\Gamma = 0)$, *the gravitational potentials* ν *and* λ *are related by the equation*

$$\mathrm{e}^{\nu} = Br \cosh\left(\int \frac{\mathrm{e}^{\lambda}}{r} \mathrm{d}r + l\right),$$

where B *and* l *are constants. The conformal vector* $\mathbf{X}$ *is defined by* (2.4) *and the functions* A^0, A^4 *and* A^i $(i = 1, 2, 3)$ *depend on the spacetime coordinates.*

Further restrictions on ν and λ will arise when the field equations are applied. The quantities A^0, A^4 and A^i are subject to the condition (2.5). These equations may be integrated to give

$$A^0 = F^0(r) + \mathrm{e}^{-\lambda}(1 - r\nu_r)G^0(t), \tag{2.14}$$

$$A^i = F^i(r) + \frac{\mathrm{e}^{\nu}}{r}G^i(t), \tag{2.15}$$

$$A^4 = r\mathrm{e}^{-\lambda}G^0_t(t), \tag{2.16}$$

where G^0, F^0, G^i and F^i arise from integration. For these forms to exist we have the conditions

$$G^0_{tt} + \frac{\mathrm{e}^{2\nu-\lambda}}{r}\left[\mathrm{e}^{-\lambda}(r\nu_r - 1)\right]_r G^0 = \frac{\mathrm{e}^{2\nu-\lambda}}{r}F^0_r, \tag{2.17}$$

$$\frac{\mathrm{e}^{2\nu-\lambda}}{r}F^0_r = c[\epsilon], \tag{2.18}$$

$$\frac{\mathrm{e}^{2\nu-\lambda}}{r}\left[\mathrm{e}^{-\lambda}(r\nu_r - 1)\right]_r = \epsilon w^2, \tag{2.19}$$

$$G^i_{tt} + \frac{\mathrm{e}^{2\nu}}{r^2}\left[1 - \mathrm{e}^{-2\lambda}(1 - r\nu_r)^2\right] G^i = \mathrm{e}^{\nu-2\lambda}(r\nu_r - 1)F^i_r - \frac{\mathrm{e}^{\nu}}{r}F^i, \tag{2.20}$$

$$\mathrm{e}^{\nu-2\lambda}(r\nu_r - 1)F^i_r - \frac{\mathrm{e}^{\nu}}{r}F^i = c_i[\epsilon], \tag{2.21}$$

where $\epsilon = 0,\ \pm 1,\ c[\epsilon]$, and $c_i[\epsilon]$ are constants. We can integrate to obtain

$$G^0(t) = g_0 + \begin{cases} h_0 t + k_0 t^2, & \epsilon = 0, \\ h_0 \cos(wt) + k_0 \sin(wt), & \epsilon = 1, \\ h_0 \cosh(wt) + k_0 \sinh(wt), & \epsilon = -1, \end{cases} \tag{2.22}$$

$$F^0(r) = c[\epsilon]\int r\mathrm{e}^{\lambda-2\nu}\mathrm{d}r, \qquad c[\epsilon] = \begin{cases} 2k_0, & \epsilon = 0, \\ \epsilon g_0 w^2, & \epsilon = \pm 1, \end{cases} \tag{2.23}$$

$$G^i(t) = g_i + \begin{cases} h_i t + k_i t^2, & \epsilon = 0, \\ h_i \cos(wt) + k_i \sin(wt), & \epsilon = 1 \\ h_i \cosh(wt) + k_i \sinh(wt), & \epsilon = -1, \end{cases} \tag{2.24}$$

$$F^i(r) = -g_i \frac{\mathrm{e}^{\lambda}}{r} + \begin{cases} f_i - k_i r^2 \mathrm{e}^{-2\nu}, & \epsilon = 0 \\ \dfrac{\epsilon f_i}{w^2} \mathrm{e}^{\nu-\lambda} \left(\nu_r - \dfrac{1}{r} \right), & \epsilon = \pm 1, \end{cases} \tag{2.25}$$

where $g_0, h_0, k_0, g_i, h_i, k_i$ and f_i are arbitrary constants. In the above we have followed the notation and conventions of Maartens *et al.* (1995).

Therefore for a conformally flat static spherically symmetric spacetime ($\Gamma = 0$), the potentials are related by equation (2.13). The conformal Killing vector $\mathbf{X}$ may be nonstatic, and the functional dependence on the coordinate t is given in terms of elementary functions. The only possibilities for the time dependence are through polynomials, trigonometric and hyperbolic functions.

2.5 Non-conformally flat: $\Gamma \neq 0$

This case is not conformally flat and we can write (2.10a) in the form

$$\left(A^i{}_r \right)_r + \left(\frac{\Gamma_r}{2\Gamma} + \frac{2}{r} - 2\lambda_r \right) A^i{}_r = 0. \tag{2.26}$$

If we consider this as an equation in $A^i{}_r$ then we can integrate to get

$$A^i{}_r = \frac{\mathrm{e}^{2\lambda}}{r^2 \sqrt{\Gamma}} h_1(t), \tag{2.27}$$

where $h_1(t)$ is a function arising from the integration process. Substituting (2.27) into (2.10c) we get

$$A^i = -\frac{1}{\sqrt{\Gamma}} \left[2 \left(\frac{1}{r} - \Gamma_r \right) + \frac{\Gamma_r}{\Gamma} \right] h_1(t), \tag{2.28}$$

which gives us an explicit form for A^i. Integrating (2.10b) we get

$$A^4 = \frac{g_1(t)}{\sqrt{\Gamma}}, \tag{2.29}$$

where $g_1(t)$ is a function of integration. Substituting (2.29) into (2.10d) we have

$$\left[2r(1-r\lambda_r)\Gamma + r^2\Gamma_r\right] g_1(t) = 0. \tag{2.30}$$

This then leads to the following two subcases.

Case a: $g_1(t) = 0$

In this case we get

$$A^4 \;=\; 0, \tag{2.31a}$$

$$A^i \;=\; -\frac{1}{\sqrt{\Gamma}}\left[2\left(\frac{1}{r}-\Gamma_r\right)+\frac{\Gamma_r}{\Gamma}\right] h_1(t). \tag{2.31b}$$

The above forms, together with (2.5e) and (2.5f), imply

$$A^0 = C, \tag{2.31c}$$

where C is a constant. The above equations (2.31) should satisfy the integrability conditions (2.5). If $h_1(t) = 0$ then (2.5) is identically satisfied. Suppose $h_1(t) \neq 0$, then we first we write (2.31b) as

$$A^i = \alpha(r) h_1(t), \tag{2.32}$$

for simplification purposes, where

$$\alpha(r) = -\frac{1}{\sqrt{\Gamma}}\left[2\left(\frac{1}{r}-\Gamma_r\right)+\frac{\Gamma_r}{\Gamma}\right]. \tag{2.33}$$

Substituting (2.32) into (2.5a)-(2.5c) and simplifying we obtain the following equations

$$(r\mathrm{e}^{-\nu}\alpha)_r \;=\; 0, \tag{2.34a}$$

$$\left[\frac{\alpha_{rr}}{\alpha}+(2r^{-1}-\lambda_r-\nu_r)\frac{\alpha_r}{\alpha}\right]\mathrm{e}^{2(\nu-\lambda)} \;=\; \tilde{C}_1, \tag{2.34b}$$

$$\frac{\ddot{h}}{h} \;=\; -\tilde{C}_1, \tag{2.34c}$$

$$\mathrm{e}^{2(\nu-\lambda)}(r^{-1}-\nu_r)\frac{\alpha_r}{\alpha} - r^{-2}\mathrm{e}^{2\nu} \;=\; -\tilde{C}_2. \tag{2.34d}$$

From (2.34a) we get

$$\alpha = \frac{K}{r}\mathrm{e}^{\nu}, \tag{2.35}$$

and (2.34c) gives

$$h(t) = \begin{cases} c_1 + c_2 t, & \tilde{C}_1 = 0, \\ c_1 \cos\left(\sqrt{-\tilde{C}_1} t\right) + c_2 \sin\left(\sqrt{-\tilde{C}_1} t\right), & \tilde{C}_1 < 0, \\ c_1 \cosh\left(\sqrt{-\tilde{C}_1} t\right) + c_2 \sinh\left(\sqrt{-\tilde{C}_1} t\right), & \tilde{C}_1 > 0, \end{cases} \tag{2.36}$$

where K, c_1 and c_2 are constants of integration. We can state our result in terms of the following theorem.

Theorem 2.5.1. *In a static spherical spacetime which is not conformally flat* $(\Gamma \neq 0)$ *if the condition* $[2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r] \neq 0$ *is satisfied, then the conformal Killing vector* $\mathbf{X}$ *exists with*

$$\begin{aligned} A^0 &= C, \\ A^i &= -\frac{1}{\sqrt{\Gamma}}\left[2\left(\frac{1}{r} - \Gamma_r\right) + \frac{\Gamma_r}{\Gamma}\right] h_1(t) = \alpha(r) h_1(t), \\ A^4 &= 0. \end{aligned}$$

The metric potentials are restricted by $(r\mathrm{e}^{-\nu}\alpha)_r = 0$.

Unlike the conformally flat case we considered in §2.4, the metric potentials are not simply related as in equation (2.13). The potentials in this case have to satisfy equations $(r\mathrm{e}^{-\nu}\alpha)_r = 0$ in addition to (2.34b) and (2.34d). Note this class of solution also allows for a nonstatic conformal Killing vector in static spacetimes. The solutions (2.36) of the time-dependent equation are in terms of polynomials, trigonometric and hyperbolic functions. These are the only permissible forms. Effectively, the integrability conditions (2.5) have been reduced to equations (2.34b) and (2.34d).

Case b: $g_1(t) \neq 0$

In this case we have from (2.30) that

$$2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r = 0. \tag{2.37}$$

Integrating (2.37) we get

$$\Gamma = \frac{\mathrm{e}^{2\lambda}}{r^2}k, \tag{2.38}$$

where k is a constant of integration. Then (2.8) and (2.38) imply

$$\nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}\left[1 - (1+k)\mathrm{e}^{2\lambda}\right] = 0. \tag{2.39}$$

Equation (2.39) is highly nonlinear but it can be integrated. We first write it in the following form

$$\left[\frac{\mathrm{e}^{-2\lambda}\nu_r}{r}\right]_r + \mathrm{e}^{-2(\lambda+\nu)}\left[\frac{\mathrm{e}^{-2\nu}\nu_r}{r}\right]_r - \left[\frac{\mathrm{e}^{-2\lambda} - (1+k)}{r^2}\right]_r = 0. \tag{2.40}$$

Now introduce the following variables y and u by letting

$$y = \mathrm{e}^{-2\lambda},\ \frac{u_r}{u} = \nu_r. \tag{2.41}$$

This transforms (2.40) into

$$y_r = \frac{2\left[u_{rr} - \frac{u_r}{r} + \frac{u}{r^2}\right]}{\left[u_r - \frac{u}{r}\right]}y - \frac{2(1+k)u}{r^2\left[u_r - \frac{u}{r}\right]}. \tag{2.42}$$

The integrating factor is $\left[u_r - \frac{u}{r}\right]^2$; this allows us to integrate (2.42) yielding

$$\left[u_r - \frac{u}{r}\right]^2 y = (1+k)\frac{u^2}{r^2} + c_2, \tag{2.43}$$

where c_2 is a constant of integration. Transforming back to the original variables we obtain

$$\left[\nu_r - \frac{1}{r}\right]^2 \mathrm{e}^{-2\lambda} = \frac{(1+k)}{r^2}\left[1 - B^2r^2\mathrm{e}^{-2\nu}\right], \tag{2.44}$$

where we have set $\frac{c_2}{1+k} = -B^2$. Rearranging (2.44) and integrating we get

$$\int \frac{\nu_r - \frac{1}{r}}{\sqrt{1 - B^2r^2\mathrm{e}^{-2\nu}}}\mathrm{d}r = \sqrt{1+k}\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r. \tag{2.45}$$

Completing the integration we finally obtain

$$\mathrm{e}^{\nu} = Br\cosh\left(\sqrt{1+k}\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + l\right), \tag{2.46}$$

where l is a new constant of integration. We conclude from (2.28) and (2.29) that

$$A^i = 0, \tag{2.47}$$

$$A^4 = \frac{1}{\sqrt{k}} r\mathrm{e}^{-\lambda} g_1(t). \tag{2.48}$$

From (2.5f) we have

$$A^0_t = \frac{1}{\sqrt{k}}(1 - r\nu_r)\mathrm{e}^{-\lambda} g_1(t). \tag{2.49}$$

Integrating this we obtain

$$A^0 = \frac{1}{\sqrt{k}}(1 - r\nu_r)\mathrm{e}^{-\lambda} g_2(t) + g_3(r), \tag{2.50}$$

where $g_2(t) = \int g_1(t)\mathrm{d}t$ and $g_3(r)$ is a new function of integration. We substitute (2.48) and (2.50) into (2.5e) and simplify to obtain the following set of equations

$$(g_2)_{tt} + \frac{\mathrm{e}^{2\nu-\lambda}}{r}\left[\mathrm{e}^{-\lambda}(r\nu_r - 1)\right]_r g_2 = \frac{\mathrm{e}^{2\nu-\lambda}}{r}(g_3)_r, \tag{2.51}$$

$$\frac{\mathrm{e}^{2\nu-\lambda}}{r}(g_3)_r = c[\epsilon], \tag{2.52}$$

$$\frac{\mathrm{e}^{2\nu-\lambda}}{r}\left[\mathrm{e}^{-\lambda}(r\nu_r - 1)\right]_r = \epsilon w^2, \tag{2.53}$$

where w and $c[\epsilon]$ are constants. Proceeding in the same way as we did in §2.4, we get

$$g_2(t) = \tilde{g} + \begin{cases} \tilde{g}t + \tilde{k}t^2, & \epsilon = 0, \\ \tilde{h}\cos(wt) + \tilde{k}\sin(wt), & \epsilon = 1, \\ \tilde{h}\cosh(wt) + \tilde{k}\sinh(wt), & \epsilon = -1, \end{cases} \tag{2.54}$$

$$g_3(r) = \frac{\tilde{c}[\epsilon]}{\sqrt{k}} \int r\mathrm{e}^{\lambda-2\nu}\mathrm{d}r, \qquad \tilde{c}[\epsilon] = \begin{cases} 2\tilde{k} & \epsilon = 0 \\ \epsilon\tilde{g}w^2 & \epsilon = \pm 1, \end{cases} \tag{2.55}$$

where $\tilde{g}, \tilde{h}, \tilde{k}$ and $\tilde{c}[\epsilon]$ are arbitrary constants.

We gather all these results in the following theorem.

Theorem 2.5.2. *For static spherically symmetric spacetimes which are non-conformally flat* ($\Gamma \neq 0$) *if the condition* $[2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r] = 0$ *is satisfied, then the conformal Killing vector* $\mathbf{X}$ *exists with*

$$
\begin{aligned}
A^i &= 0, \\
A^4 &= \frac{1}{\sqrt{k}} r\mathrm{e}^{-\lambda} g_1(t), \\
A^0 &= \frac{1}{\sqrt{k}} (1 - r\nu_r)\mathrm{e}^{-\lambda} g_2(t) + g_3(r),
\end{aligned}
$$

where $g_1(t)$ *and* $g_2(t)$ *are related by* $g_2(t) = \int g_1(t)\mathrm{d}t$. *Furthermore, the gravitational potentials are related by the equation*

$$
\mathrm{e}^{\nu} = Br\cosh\left(\sqrt{1+k}\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + l\right),
$$

where $k \neq 0$, B *and* l *are constants.*

Therefore for this third category of conformally flat static spherically symmetric spacetimes (with $\Gamma \neq 0$), the metric potentials are related by (2.46). The conformal Killing vector $\mathbf{X}$ may be nonstatic with time dependence in terms of elementary functions. The only possibilities for the analytic time dependence are through polynomials, trigonometric and hyperbolic functions. Below we present two tables displaying known spacetimes with a conformal symmetry from various authors.

2.6 Exact solutions

We have shown that the classification of the conformal geometry in terms of the Weyl tensor in section §2.4 and §2.5 leads to simple relationships between the gravitational potentials ν and λ given by equations (2.13) and (2.46). This enables us to identify known exact solutions to the Einstein field equations which have been previously found. In Table 2.1 and Table 2.2 we present the known exact solutions with a conformal Killing vector that have been used to model conformally invariant spheres in relativistic physics.

Potentials	CKV	Features
$e^{2\nu} = c^2 r^2 e^{(-2k_1 \int r^{-1} e^{\lambda} dr)}$	$\mathbf{X} = (X^0, X^1, 0, 0)$	Maartens and Maharaj (1990)
$k_1 = \pm 1$	$\psi = \frac{1}{2} k_1 t + r\nu_r e^{-2\lambda}$	charged anisotropic sphere
		conformally flat ($\Gamma = 0$)
		nonstatic vector $\mathbf{X}$
$e^{2\nu} = 1$	$\mathbf{X} = (X^0, X^1, 0, 0)$	Tello-Llanos (1988)
$e^{2\lambda} = (ar^2 + b)^{-1}$	$\psi = F^0(r) + e^{-\lambda}\left[r\nu_r G^0_t(t)\right]$	charged anisotropic sphere
$b = 1$	$+e^{-\lambda}\left[(1 - r\nu_r)\, G^0(t)\right]$	conformally flat ($\Gamma = 0$)
		nonstatic vector $\mathbf{X}$
$e^{2\nu} = c^2 r^2 e^{(-2k_1 \int r^{-1})}$	$\mathbf{X} = (X^0, X^1, 0, 0)$	Rahaman *et al.* (2017)
$k_1 = \pm 1$	$\psi = F^0(r) + e^{-\lambda}\left[r\nu_r G^0_t(t)\right]$	charged anisotropic sphere
	$+e^{-\lambda}\left[(1 - r\nu_r)\, G^0(t)\right]$	conformally flat ($\Gamma = 0$)
		nonstatic vector $\mathbf{X}$
$e^{2\nu} = 1$	$\mathbf{X} = (X^0, X^1, 0, 0)$	Tello-Llanos (1988)
$e^{2\lambda} = e^{2c} = 1$	$\psi = F^0(r) + e^{-\lambda}\left[r\nu_r G^0_t(t)\right]$	charged anisotropic sphere
	$+e^{-\lambda}\left[(1 - r\nu_r)\, G^0(t)\right]$	conformally flat ($\Gamma = 0$)
		nonstatic vector $\mathbf{X}$

Table 2.1: Known spherically symmetric spacetimes admitting a conformally flat geometry.

Exact solutions to the field equations with a conformal symmetry, for a specified form of the potentials, have been investigated by various authors. In their models particular forms of the potentials ν and λ are made. It is not necessary to make this specification as solutions with a conformal symmetry can be shown to exist in general. We can apply the conformal symmetries found in this chapter to generate exact solutions of the Einstein field equations for an uncharged anisotropic matter distribution.

The matter distribution is taken to be an anisotropic fluid with a comoving fluid

Potentials	CKV	Features
$e^{2\nu} = c^2 r^2 e^{(-2k_1 \int r^{-1} e^{\lambda} dr)}$	$X^0 = k_1 g_2(t) + g_3(r) + a_0$	Maartens and Maharaj (1990)
$k_1 = \pm\sqrt{1+k}$	$X^1 = \frac{1}{\sqrt{k}} r e^{-\lambda} g_1(t)$	charged anisotropic sphere
	$\psi = k_1 g_2(t) + (e^{-\lambda} - k_1) g_1(t)$	nonconformally flat ($\Gamma \neq 0$)
	$+g_3(r)$	
$e^{2\lambda} = 1 + \frac{r^2}{R^2}$	$X^0 = k_1 g_2(t) + g_3(r) + a_0$	Rahaman *et al.* (2017)
$e^{2\nu} = C^2 r^2 e^{(-2k_1 \int r^{-1} e^{\lambda} dr)}$	$X^1 = \frac{1}{\sqrt{k}} \frac{rR}{\sqrt{R^2+r^2}} g_1(t)$	charged anisotropic sphere
$k_1 = \pm\sqrt{1+k}$	$\psi = k_1 g_2(t) + (\frac{R}{R^2+r^2} - k_1) g_1(t)$	nonconformally flat ($\Gamma \neq 0$)
	$+g_3(r)$	
$e^{2\nu} = 1$	$X^0 = \frac{1}{\sqrt{k}}(ar^2+b)^{\frac{1}{2}} g_2(t)$	Tello-Llanos (1988)
$e^{2\lambda} = (ar^2+b)^{-1}$	$+g_3(r) + a_0$	charged anisotropic sphere
	$X^1 = r(ar^2+b)^{\frac{1}{2}} g_1(t)$	nonconformally flat ($\Gamma \neq 0$)
$b = 1+k$	$\psi = (ar^2+b)^{\frac{1}{2}} g_2(t) + g_3(r)$	
$e^{2\nu} = 1$	$X^0 = e^{-c} g_2(t) + g_3(r) + a_0$	Tello-Llanos (1988)
$e^{2\lambda} = e^{2c} = \frac{1}{1+k}$	$X^1 = \frac{1}{\sqrt{k}} r e^{-c} g_1(t)$	charged anisotropic sphere
	$\psi = e^{-c} g_2(t) + g_3(r)$	nonconformally flat ($\Gamma \neq 0$)
$e^{2\nu} = C_1^2 r^2$	$X^0 = g_3(r) + a_0$	Herrera *et al.* (1984)
$e^{2\lambda} = \frac{4}{cr^2+2}$	$X^1 = \frac{1}{2\sqrt{k}} r(cr^2+2)^{\frac{1}{2}} g_1(t)$	charged anisotropic sphere
	$\psi = \frac{1}{2}(cr^2+2)^{\frac{1}{2}} g_1(t) + g_3(r)$	nonconformally flat ($\Gamma \neq 0$)
$e^{2\nu} = C_1^2 r^2$	$X^0 = g_3(r) + a_0$	Mak and Harko (2004)
$e^{2\lambda} = \frac{3}{1-Br^2}$	$X^1 = \frac{1}{\sqrt{k}} r \left(\frac{1-Br^2}{3}\right)^{\frac{1}{2}} g_1(t)$	charged anisotropic sphere
	$\psi = \left(\frac{1-Br^2}{3}\right)^{\frac{1}{2}} g_1(t) + g_3(r)$	nonconformally flat ($\Gamma \neq 0$)
		B is the bag constant
$e^{2\nu} = C_1^2 r^2$	$X^0 = g_3(r) + a_0$	Usmani *et al.* (2011)
$e^{2\lambda} = \frac{1}{C_1^2 r^2}$	$X^1 = C_1 \frac{1}{\sqrt{k}} r^2 g_1(t)$	charged anisotropic sphere
	$\psi = C_1 r g_1(t) + g_3(r)$	nonconformally flat ($\Gamma \neq 0$)
$e^{2\nu} = C_1^2 r^2$	$X^0 = g_3(r) + a_0$	Esculpi and Aloma (2010)
	$X^1 = \frac{1}{\sqrt{k}} r e^{-\lambda} g_1(t)$	charged anisotropic sphere
	$\psi = e^{-\lambda} g_1(t) + g_3(r)$	nonconformally flat ($\Gamma \neq 0$)

Table 2.2: Known spherically symmetric spacetimes admitting a nonconformally flat geometry and a nonstatic vector $\mathbf{X}$.

4-vector $u^a = \mathrm{e}^{-\nu}\delta^a_0$. The energy momentum tensor has the form

$$T_{ab} = \mathrm{diag}\left(-\rho, p_{\|}, p_{\perp}, p_{\perp}\right), \tag{2.56}$$

where $\rho, p_{\|}$ and $p_{\perp}$ are the energy density, radial pressure and tangential pressure respectively. When the anisotropy vanishes $p_{\|} = p_{\perp}$ and the pressure is isotropic. The Einstein field equations for charged matter which is self-gravitating are

$$8\pi\rho = \frac{1}{r^2}\left[r(1-\mathrm{e}^{2\lambda})\right]_r, \tag{2.57a}$$

$$8\pi p_{\|} = \mathrm{e}^{-2\lambda}\left(\frac{2\nu_r}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{2.57b}$$

$$8\pi p_{\perp} = \mathrm{e}^{-2\lambda}\left(\nu_{rr} + {\nu_r}^2 + \frac{\nu_r}{r} - \frac{\lambda_r}{r} - \nu_r\lambda_r\right). \tag{2.57c}$$

The conformal vector $\mathbf{X}$ provides the connection between the potentials ν and λ via equation (2.46). It is possible to write the system (2.57) in the form

$$8\pi\rho = \frac{1}{r^2}\left[r(1-\mathrm{e}^{2\lambda})\right]_r, \tag{2.58a}$$

$$8\pi p_{\|} = \frac{1}{r^2}\left[3\mathrm{e}^{-2\lambda} - 1 + 2\sqrt{1+k}\,\mathrm{e}^{-\lambda}\tanh\left(\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + l\right)\right], \tag{2.58b}$$

$$8\pi p_{\perp} = \frac{1}{r^2}\left[(1-2r\lambda_r)\mathrm{e}^{-2\lambda} + 2\sqrt{1+k}\,\mathrm{e}^{-\lambda}\tanh\left(\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + l\right)\right] + \frac{(1+k)}{r^2}. \tag{2.58c}$$

Thus we have demonstrated that the matter variables $\rho, p_{\|}$ and $p_{\perp}$ can be written in terms of the potential λ. Any choice of λ will give an exact solution to the system (2.58). Physical conditions on the matter distribution will further restrict the form of the potential λ.

2.7 Discussion

In some recent investigations invariant relativistic static spheres have been found which contain a conformal Killing vector. Particular spacetimes have been identified satisfying the field equations for a gravitating sphere. We have attempted to study this problem in general without specifying the static spacetimes. Our objective was to find general conditions for the existence of a conformal symmetry which are consistent with the field equations. We achieved this by following the work done by Maartens *et al.* (1995). We have used the non-vanishing Weyl tensor components to calculate the conformal Killing vector solution. We can summarise our results in terms of the quantity Γ related to the Weyl tensor. For the conformal Killing vector in static spherically symmetric spacetimes to exist there are three classes of metrics possible: conformally flat ($\Gamma = 0$) and potentials ν and λ related by (2.13); non-conformally flat case ($\Gamma \neq 0$, $[2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r] \neq 0$) and no simple relation between ν and λ; non-conformally flat ($\Gamma \neq 0$, $[2r(1 - r\lambda_r)\Gamma + r^2\Gamma_r] = 0$) and potentials ν and λ related by (2.46). Even though the spacetimes are static it is possible to find conformal vectors which are nonstatic. In all the three classes the time dependence of the conformal Killing vector $\mathbf{X}$ can be found explicitly. These are given in terms of polynomials (of degree at most two), trigonometric and hyperbolic functions. Particular static relativistic stellar models with time dependence in the conformal symmetries have been found recently. An example is the model of Shee *et al.* (2016). Our detailed analysis has shown that time dependence in the conformal symmetry, in static metrics, is restricted to a small family of elementary functions. The metric potentials ν and λ are related by equations (2.13) and (2.46). This enables us to write the Einstein equations in terms of a single gravitational potential. Thus any choice of this potential generates a neutral anisotropic star. Lastly we constructed a table to detail the work that has been done by various authors on the subject of conformal symmetries in astrophysics. We observe that their solutions are indeed special cases of the general solutions discussed in this chapter.

Chapter 3

Spherical conformal models

3.1 Introduction

Note that the conformal symmetry equation (2.3) is a purely geometrical condition and it allows for a general form for the conformal vector **X**. The gravitational dynamics will further restrict the form of **X** through the field equations. To make further progress and obtain specific metrics it is necessary to restrict the form of the conformal Killing vector **X**. On the grounds of spherical symmetry we would expect that the angular dependence in θ and ϕ should be absent, and gravity would restrict the vector **X** to lie in the t-r plane. This observation helps to make progress. In §3.2 we introduce the static spherically symmetric line element and the conformal symmetry. We derive a system of four coupled differential equations connecting the components of the conformal Killing vector, the conformal factor and the gravitational potentials. We generate the conformal integrability conditions using the Weyl tensor in §3.3 for both conformally flat and non-conformally flat spacetimes. In both cases we obtain a relation between the two gravitational potentials. In §3.4 we give Einstein's field equations for a general static spherically symmetric line element with an anisotropic fluid with the Durgapal and Bannerji (1983) spacetime coordinates. In §3.5 we find all exact solutions of the Einstein field equations with a conformal symmetry for isotropic and anisotropic pressures. Known solutions are regained from our general result.

Concluding remarks are made in §3.6.

3.2 Conformal symmetry

All known solutions are related to the special case of the conformal vector

$$\mathbf{X} = \alpha(t,r)\frac{\partial}{\partial t} + \beta(t,r)\frac{\partial}{\partial r}, \tag{3.1a}$$

$$\psi = \psi(t,r), \tag{3.1b}$$

which is spherically symmetric. Solutions to the Einstein-Maxwell system with conformal vector (3.1a)-(3.1b) were first considered by Herrera and Ponce de León (1985a, 1985c) and Herrera *et al.* (1984), with a static vector **X**, and by Maartens and Maharaj (1990), with a nonstatic vector **X**. In Maartens and Maharaj (1990) the conformal factor was taken to be static. In this treatment we assume the existence of the conformal vector (3.1a) and a nonstatic $\psi(t,r)$. This will generalise the analysis of Maartens and Maharaj (1990) and also possibly give new solutions to the field equations.

Performing the Lie derivative in (2.3) for (3.1a)-(3.1b) yields the following conditions

$$\nu_r\beta + \alpha_t = \psi, \tag{3.2a}$$

$$\mathrm{e}^{2\lambda}\beta_t - \mathrm{e}^{2\nu}\alpha_r = 0, \tag{3.2b}$$

$$\lambda_r\beta + \beta_r = \psi, \tag{3.2c}$$

$$\beta = r\psi. \tag{3.2d}$$

Equations (3.2a)-(3.2d) comprise an underdetermined system of nonlinear partial differential equations in α, β, ψ, ν and λ.

3.3 The Weyl tensor

The Weyl tensor plays an important role in this thesis. It has the same symmetries as the Riemann tensor with the extra condition that it is trace-free, i.e. it is a Riemann tensor with the Ricci terms subtracted out. The Weyl tensor is associated with a phenomenon known as tidal deformation, which distorts the shape of an object but preserves its volume. An important property of the Weyl tensor is that it remains invariant under conformal transformations. Hence, it is sometimes known as the conformal tensor. It is difficult to integrate the system (3.2a)-(3.2d) in general. However, we find that there is simplification if we use the integrability condition for (2.3) in terms of the Weyl tensor. The existence of a conformal Killing vector is related to the Weyl tensor by (2.6). As show previously this leads to the expression (2.8):

$$\Gamma = \nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda}).$$

The introduction of the quantity Γ assists in analysing the conformal symmetry. Observe that if $\Gamma = 0$ then all components of the Weyl tensor vanish; the spacetime is then conformally flat.

We apply the general Lie derivative formula (2.6) to obtain, after some calculations, the conditions

$$\Gamma_r \beta + 2\Gamma\beta_r = 0, \tag{3.3a}$$

$$\left(r^2 \mathrm{e}^{-2\lambda}\Gamma\right)_r \beta = 0. \tag{3.3b}$$

The equations in (3.3a)-(3.3b) are the conditions arising from the integrability condition (2.6). The equations (3.3a)-(3.3b) have to be solved in conjunction with the system (3.2a)-(3.2d). Observe that the conformal function α is absent in (3.3a)-(3.3b), and only the conformal function β is present. The two cases $\Gamma = 0$ and $\Gamma \neq 0$ are considered separately.

3.3.1 Conformally flat: $\Gamma = 0$

In this case the Weyl tensor vanishes and

$$\nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda}) = 0. \tag{3.4}$$

This second order nonlinear partial differential equation was solved by Grøn and Johannesen (2013) in their detailed study on conformally flat spherical spacetimes. Equation (3.4) can be rewritten in the following form

$$\left\{ \ln \left[\mathrm{e}^{\nu - \lambda} \left(\nu_r - \frac{1}{r} \right) + \frac{\mathrm{e}^{\nu}}{r} \right] \right\}_r = \frac{\mathrm{e}^{\lambda}}{r}. \tag{3.5}$$

Integration of (3.5) leads to

$$\mathrm{e}^{\nu} = Ar \exp\left(\int \frac{\mathrm{e}^{\lambda}}{r} \mathrm{d}r \right) + Br \exp\left(- \int \frac{\mathrm{e}^{\lambda}}{r} \mathrm{d}r \right), \tag{3.6}$$

where A and B are constants. The result (3.6) is the general solution of the nonlinear equation (2.11). The general solution (3.6) can also be written as a sum of hyperbolic cosh and sinh functions. The solution in terms of the cosh functions was given by Herrera *et al.* (2001) and Manjonjo *et al.* (2017).

3.3.2 Non-conformally flat: $\Gamma \neq 0$

This case is not conformally flat and (3.3b) leads to the two subcases: $\beta = 0$ or $\beta \neq 0$. If $\beta = 0$ then $\psi = 0$ from (3.2d) and we have a Killing vector. Hence we take $\beta \neq 0$ for a proper conformal symmetry. We have from (3.3b) that

$$\left(r^2 \mathrm{e}^{-2\lambda} \Gamma \right)_r = 0. \tag{3.7}$$

Integrating (3.7) we obtain

$$\Gamma = \frac{\mathrm{e}^{2\lambda}}{r^2} k, \tag{3.8}$$

where k is a constant of integration. Then (2.8) and (3.8) imply

$$\nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + \frac{1}{r^2} = (1 + k)\frac{\mathrm{e}^{2\lambda}}{r^2}. \tag{3.9}$$

Equation (3.9) is highly nonlinear but it can be integrated. To integrate (3.9) completely we need to consider the separate cases: $1 + k > 0, 1 + k = 0, 1 + k < 0$.

3.3.2.1 $1+k>0$:

To achieve this we apply the approach of Grøn and Johannesen (2013) for conformally flat spacetimes. We modify (3.9) by adding two new terms as follows

$$\nu_{rr}+{\nu_r}^2-\lambda_r\nu_r+r^{-1}(\lambda_r-\nu_r)+\frac{1}{r^2}+\sqrt{1+k}\left(\frac{\nu_r}{r}-\frac{1}{r^2}\right)\mathrm{e}^{\lambda}$$
$$=(1+k)\frac{\mathrm{e}^{2\lambda}}{r^2}+\sqrt{1+k}\left(\frac{\nu_r}{r}-\frac{1}{r^2}\right)\mathrm{e}^{\lambda}. \tag{3.10}$$

We multiply both sides by $r\mathrm{e}^{\nu-2\lambda}$ so that (3.10) becomes

$$r\mathrm{e}^{-\lambda}\left[\left(\nu_r-\frac{1}{r}\right)\mathrm{e}^{\nu-\lambda}+\sqrt{1+k}\left(\frac{\mathrm{e}^{\nu}}{r}\right)\right]_r$$
$$=\sqrt{1+k}\left[\left(\nu_r-\frac{1}{r}\right)\mathrm{e}^{\nu-\lambda}+\sqrt{1+k}\left(\frac{\mathrm{e}^{\nu}}{r}\right)\right]. \tag{3.11}$$

Simplifying (3.11) gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\ln\left[\left(\nu_r-\frac{1}{r}\right)\mathrm{e}^{\nu-\lambda}+\sqrt{1+k}\left(\frac{\mathrm{e}^{\nu}}{r}\right)\right]\right\}=\sqrt{1+k}\frac{\mathrm{e}^{\lambda}}{r}. \tag{3.12}$$

Equation (3.12) is integrated to yield

$$\left(\nu_r-\frac{1}{r}\right)\mathrm{e}^{\nu-\lambda}+\sqrt{1+k}\left(\frac{\mathrm{e}^{\nu}}{r}\right)=2\tilde{B}\exp\left(\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right), \tag{3.13}$$

where $\tilde{B}$ is a constant. Multiplying both sides of (3.13) by $\frac{\mathrm{e}^{\lambda}}{r}\exp\left(\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right)$ and simplifying we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{\nu}}{r}\exp\left(\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right)\right]=2\tilde{B}\frac{\mathrm{e}^{\lambda}}{r}\exp\left(2\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right). \tag{3.14}$$

We integrate (3.14) and simplify to obtain

$$\mathrm{e}^{\nu}=Ar\exp\left(\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right)+Br\exp\left(-\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right), \tag{3.15}$$

where $A=\frac{\tilde{B}}{\sqrt{1+k}}$ and B are constants.

3.3.2.2 $1 + k = 0$:

In this case the equation (3.9) becomes

$$\nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + \frac{1}{r^2} = 0. \tag{3.16}$$

We multiply (3.16) by $e^{\nu-\lambda}$ and simplify to obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\nu_r - \frac{1}{r}\right) e^{\nu-\lambda}\right] = 0. \tag{3.17}$$

Equation (3.17) is integrated to obtain

$$\frac{e^{\nu}}{r} = A \int \frac{e^{\lambda}}{r} \mathrm{d}r + B, \tag{3.18}$$

which relates λ to ν.

3.3.2.3 $1 + k < 0$:

For this case we modify (3.9) by adding $\sqrt{-(1+k)}\left(\frac{\nu_r}{r} - \frac{1}{r^2}\right) e^{\lambda}$ to both sides of the equation. We proceed in the same manner as we did in (3.10)-(3.14) to obtain

$$e^{\nu} = Ar \exp\left(\sqrt{-(1+k)} \int \frac{e^{\lambda}}{r} \mathrm{d}r\right) + Br \exp\left(-\sqrt{-(1+k)} \int \frac{e^{\lambda}}{r} \mathrm{d}r\right), \tag{3.19}$$

which is similar to (3.15).

This completes the solution of (3.9) for all values of k. It is remarkable to note that the nonlinear equation (3.9) admits a general solution given for all three cases. The existence of a conformal symmetry implies an explicit relationship between the gravitational potentials λ and ν. Note that the case $k = 0$ reduces (3.15) to the conformally flat solution (3.6). We can now express our result as the theorem:

Theorem 3.3.1. *For a static metric with a spherically symmetric conformal symmetry* $\mathbf{X} = \alpha(t,r)\frac{\partial}{\partial t} + \beta(t,r)\frac{\partial}{\partial r}$ *and a nonstatic conformal factor* $\psi(t,r)$ *the gravitational* *potentials are related by*

$$\mathrm{e}^{\nu} = \begin{cases} Ar\exp\left(\sqrt{1+k}\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right) & \\ \quad +Br\exp\left(-\sqrt{1+k}\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right) & \text{, where } 1+k>0 \\ Ar\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + Br & \text{, where } 1+k=0 \\ Ar\exp\left(\sqrt{-(1+k)}\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right) & \\ \quad +Br\exp\left(-\sqrt{-(1+k)}\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right) & \text{, where } 1+k<0, \end{cases}$$

for both conformally flat ($k=0$) *and non-conformally flat* ($k\neq 0$) *spacetimes.*

3.4 Field equations

The Einstein field equations for an uncharged anisotropic fluid are given by

$$\rho = \frac{1}{r^2}\left[r(1-\mathrm{e}^{-2\lambda})\right]_r, \tag{3.20a}$$

$$p_{\parallel} = \mathrm{e}^{-2\lambda}\left[\frac{2\nu_r}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2}, \tag{3.20b}$$

$$p_{\perp} = \mathrm{e}^{-2\lambda}\left[\nu_{rr}+{\nu_r}^2+\frac{\nu_r}{r}-\frac{\lambda_r}{r}-\nu_r\lambda_r\right], \tag{3.20c}$$

for the potentials $\nu(r)$ and $\lambda(r)$.

Equivalent forms of the field equations are obtained if we introduce new coordinate transformations. Here we utilise the transformation due to Durgapal and Bannerji (1983). We introduce a new variable x, and functions y and Z as follows

$$x = r^2, \tag{3.21a}$$

$$Z(x) = \mathrm{e}^{-2\lambda(r)}, \tag{3.21b}$$

$$y^2(x) = \mathrm{e}^{2\nu(r)}. \tag{3.21c}$$

Then we obtain a different but equivalent form of the field equations given by

$$\frac{1-Z}{x} - 2\dot{Z} = \rho, \tag{3.22a}$$

$$4Z\left(\frac{\dot{y}}{y}\right) + \frac{Z-1}{x} = p_{\parallel}, \tag{3.22b}$$

$$4xZ\left(\frac{\ddot{y}}{y}\right) + (4Z + 2x\dot{Z})\left(\frac{\dot{y}}{y}\right) + \dot{Z} = p_{\perp}, \tag{3.22c}$$

where $(\cdot)$ represents differentiation with respect to x. From (3.22b) and (3.22c) we obtain

$$4xZ\left(\frac{\ddot{y}}{y}\right) + 2x\dot{Z}\left(\frac{\dot{y}}{y}\right) + \dot{Z} - \frac{Z-1}{x} = \Delta, \tag{3.23}$$

where $\Delta = p_{\perp} - p_{\parallel}$ is the degree of pressure anisotropy. Pressure isotropy implies that $\Delta = 0$. For isotropic pressures (3.23) becomes

$$4xZ\left(\frac{\ddot{y}}{y}\right) + 2x\dot{Z}\left(\frac{\dot{y}}{y}\right) + \dot{Z} - \frac{Z-1}{x} = 0, \tag{3.24}$$

which is a consistency condition on the potentials.

3.5 Exact solutions

We seek to investigate the types of exact solutions admitted by the field equations (3.22a)-(3.22c) in the presence of the conformal Killing vector $\mathbf{X}$. The conformal conditions (3.15), (3.18) and (3.19) can be expressed in terms of the new variables (3.21a)-(3.21c). We obtain equivalent conditions

$$y = \begin{cases} Ax^{\frac{1}{2}}\exp\left(\frac{1}{2}\sqrt{1+k}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}}\right) \\ \quad +Bx^{\frac{1}{2}}\exp\left(-\frac{1}{2}\sqrt{1+k}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}}\right) & \text{, where } 1+k>0 \\ \frac{A}{2}x^{\frac{1}{2}}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}} + Bx^{\frac{1}{2}} & \text{, where } 1+k=0 \\ Ax^{\frac{1}{2}}\exp\left(\frac{1}{2}\sqrt{-(1+k)}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}}\right) \\ \quad +Bx^{\frac{1}{2}}\exp\left(-\frac{1}{2}\sqrt{-(1+k)}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}}\right) & \text{, where } 1+k<0 \end{cases} \tag{3.25}$$

for the potential y.

Taking the first and second derivatives of each case of (3.25), substituting into (3.23) and simplifying we obtain the same equation

$$\dot{Z} - \frac{1}{x}Z + \frac{k+2}{2x} = \frac{\Delta}{2}, \qquad \text{for } 1+k \gtreqless 0, \tag{3.26}$$

which is a simple linear equation in Z. We can integrate (3.26) to obtain

$$Z = x\int \frac{1}{2x}\left(\Delta - \frac{k+2}{x}\right)\mathrm{d}x + mx, \tag{3.27}$$

where m is a new constant. Hence the second potential Z is a known quantity. We can state our general result in the following theorem:

Theorem 3.5.1. *For a static anisotropic fluid distribution with a spherically symmetric conformal symmetry* $\mathbf{X} = \alpha(t,r)\frac{\partial}{\partial t} + \beta(t,r)\frac{\partial}{\partial r}$ *with nonstatic conformal factor* $\psi(t,r)$ *the general solution to the Einstein field equations is fully determined. The metric functions* $y(=\mathrm{e}^{\nu})$ *and* $Z(=\mathrm{e}^{-2\lambda})$ *are given by* (3.25) *and* (3.27) *respectively. Isotropic gravitating fluids are regained when* $\Delta = 0$.

A particular choice of the anisotropy Δ gives a functional form for Z in (3.27) and y then follows from (3.25). Several known solutions will fall in our category of exact models with a conformal Killing vector. We demonstrate this by considering the cases $\Delta = 0$ and $\Delta \neq 0$ separately.

3.5.1 Isotropic pressure: $\Delta = 0$

For isotropic pressures $\Delta = 0$ and $p_{\|} = p_{\perp} = p$. In this case (3.26) becomes

$$\dot{Z} - \frac{1}{x}Z + \frac{k+2}{2x} = 0. \tag{3.28}$$

We can integrate (3.28) to obtain

$$Z = \begin{cases} mx & , k=-2, m\neq 0 \\ mx + \dfrac{k+2}{2} & , k \neq -2, \end{cases} \tag{3.29}$$

where m is an integration constant. Hence the gravitational potential $Z(x)$ is a linear function. The remaining metric function $y(x)$ in (3.25) follows by direct integration. For notational convenience we set $n = \frac{k+2}{2}$. Then the general solution may be categorized in terms of the critical values

$$n = 0 \quad (k = -2), \quad n = \frac{1}{2} \quad (k = -1), \quad n = 1 \quad (k = 0, \quad \text{conformally flat case})$$

To complete our analysis we integrate

$$\int \frac{\mathrm{d}x}{xZ^{\frac{1}{2}}} = \int \frac{\mathrm{d}x}{x\sqrt{mx+n}}, \tag{3.30}$$

for different values of m and n. The solutions of (3.30) are substituted into (3.25) to obtain the metric functions with conformal geometry. The matter variables ρ and p then follow from (3.22a)-(3.22c).

It is possible to present all possible exact solutions after performing the integration in (3.30). This gives the general solution of all isotropic spherically symmetric spacetimes admitting conformal symmetry. The exact solutions are presented in Table 3.1 and Table 3.2. In Table 3.1 the gravitational potentials y and Z are listed. In Table 3.2 we present the matter variables ρ and p. It is interesting to note that for all the exact solutions in Table 3.2, the variable x can be written as a function of the energy density ρ. This means that all solutions are of the form

$$p = p(\rho), \tag{3.31}$$

and a barotropic equation of state exists. Note that in certain cases, for particular values of the constants A and B, the equation of state (3.31) becomes linear. We present these cases in Table 3.3. The gravitational potentials in Table 3.1 contain several known models found previously. These can be identified for particular parameter values m, n, A and B. Table 3.4 gives the known exact solutions with conformal symmetry with isotropic pressure and the references to the literature where they were reported. All known exact solutions of the Einstein field equations with isotropic pressure are listed in the comprehensive treatment of static spherically symmetric solutions of Delgaty and Lake (1998). In Table 3.4 we have given the solutions with conformal symmetry in terms of the radial coordinate so that comparison with the treatment of Delgaty and Lake (1998) is simplified. These include the metrics of (interior) Schwarzschild (1916), Tolman (1939), Kuchowicz (1967), Korkina and Orlyanskii (1991), Patwardhan and Vaidya (1943), and Buchdahl and Land (1968). We have also added the metric of Saslaw *et al.* (1996) which may be interpreted as the isothermal sphere in a general relativistic setting. The corresponding line elements have been expressed in a form which is consistent with the notation of Delgaty and Lake (1998).

3.5.2 Anisotropic pressure: $\Delta \neq 0$

Spherically symmetric conformal vectors $\mathbf{X}$ lead to the relationship (3.25) between the gravitational potentials ν and λ. For anisotropic pressures, $\Delta \neq 0$ and $p_{\parallel} \neq p_{\perp}$. The Einstein field equations (3.22a)-(3.22c) with an anisotropic matter distribution can then be written as

$$\rho = \frac{1-Z}{x} - 2\dot{Z}, \tag{3.32a}$$

$$p_{\parallel} = \left(\frac{2\sqrt{(1+k)Z}}{x}\right)\left[\frac{A\exp\left(\sqrt{1+k}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}}\right) - B}{A\exp\left(\sqrt{1+k}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}}\right) + B}\right] + \frac{3Z-1}{x}, \tag{3.32b}$$

$$p_{\perp} = \left(\frac{2\sqrt{(1+k)Z}}{x}\right)\left[\frac{A\exp\left(\sqrt{1+k}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}}\right) - B}{A\exp\left(\sqrt{1+k}\int\frac{\mathrm{d}x}{xZ^{\frac{1}{2}}}\right) + B}\right] + \frac{1+k+Z}{x} + 2\dot{Z}. \tag{3.32c}$$

In this representation we note that all the matter variables depend only on the potential Z (see Theorem 3.5.1). Therefore any choice of Z will lead to an exact solution after integration in (3.32a)-(3.32c). Clearly only those solutions with desirable physical features will lead to an acceptable model for a gravitating relativistic sphere.

Several solutions of the field equations are known with anisotropic pressures with a conformal Killing vector. All such solutions will be contained in equations (3.32a)-(3.32c). A recent example is the anisotropic stellar model with nonstatic conformal symmetry and highly dense interior of Shee *et al.* (2016). We show that their model is a special case of our treatment. Take $A = 0$ with $\tilde{k} = \frac{\sqrt{1+k}}{2}$ in (3.32a)-(3.32c) and let

$$\rho = \frac{a}{x} + 3b, \tag{3.33}$$

for the energy density. This profile of the energy density was also studied by Gleiser and Dev (2004) and Dev and Gleiser (2002) yielding relativistic stars with desirable

Case	Parameters	Gravitational potentials
1	$m \neq 0, n < 0$	$y = Ax^{\frac{1}{2}} \exp\left[\sqrt{2-\frac{1}{n}}\tan^{-1}\left(\frac{\sqrt{mx+n}}{\sqrt{-n}}\right)\right]$ $+Bx^{\frac{1}{2}} \exp\left[-\sqrt{2-\frac{1}{n}}\tan^{-1}\left(\frac{\sqrt{mx+n}}{\sqrt{-n}}\right)\right]$ $Z = mx + n$
2	$m \neq 0, n = 0$	$y = Ax^{\frac{1}{2}} \exp\left(-\frac{1}{\sqrt{mx}}\right) + Bx^{\frac{1}{2}} \exp\left(\frac{1}{\sqrt{mx}}\right)$ $Z = mx$
3	$m \neq 0, 0 < n < \frac{1}{2}$	$y = Ax^{\frac{1}{2}} \left[\frac{\sqrt{mx+n}-\sqrt{n}}{\sqrt{mx+n}+\sqrt{n}}\right]^{\frac{1}{2}\sqrt{\frac{1}{n}-2}} + Bx^{\frac{1}{2}} \left[\frac{\sqrt{mx+n}+\sqrt{n}}{\sqrt{mx+n}-\sqrt{n}}\right]^{\frac{1}{2}\sqrt{\frac{1}{n}-2}}$ $Z = mx + n$
4	$m \neq 0, n = \frac{1}{2}$	$y = \frac{A\sqrt{2}}{2} x^{\frac{1}{2}} \ln\left[\frac{\sqrt{mx+\frac{1}{2}}-\sqrt{\frac{1}{2}}}{\sqrt{mx+\frac{1}{2}}+\sqrt{\frac{1}{2}}}\right] + Bx^{\frac{1}{2}}$ $Z = mx + \frac{1}{2}$
5	$m \neq 0, n > \frac{1}{2}$	$y = Ax^{\frac{1}{2}} \left[\frac{\sqrt{mx+n}-\sqrt{n}}{\sqrt{mx+n}+\sqrt{n}}\right]^{\frac{1}{2}\sqrt{2-\frac{1}{n}}} + Bx^{\frac{1}{2}} \left[\frac{\sqrt{mx+n}+\sqrt{n}}{\sqrt{mx+n}-\sqrt{n}}\right]^{\frac{1}{2}\sqrt{2-\frac{1}{n}}}$ $Z = mx + n$
6	$m = 0, 0 < n < \frac{1}{2}$	$y = Ax^{\frac{1}{2}\left(1+\sqrt{\frac{1}{n}-2}\right)} + Bx^{\frac{1}{2}\left(1-\sqrt{\frac{1}{n}-2}\right)}$ $Z = n$
7	$m = 0, n = \frac{1}{2}$	$y = A\frac{\sqrt{2}}{2} x^{\frac{1}{2}} \ln x + Bx^{\frac{1}{2}}$ $Z = \frac{1}{2}$
8	$m = 0, n > \frac{1}{2}$	$y = Ax^{\frac{1}{2}\left(1+\sqrt{2-\frac{1}{n}}\right)} + Bx^{\frac{1}{2}\left(1-\sqrt{2-\frac{1}{n}}\right)}$ $Z = n$

Table 3.1: Gravitational potentials y and Z of the Einstein field equations with conformal symmetry and isotropic pressure for various parameter values of m and n.

Case	Parameters	Matter variables
1	$m \neq 0, n < 0$	$\rho = \frac{1-n}{x} - 3m$ $p = \frac{2\sqrt{(1-2n)(mx+n)}}{x} \left\{ \frac{A\exp\left[2\sqrt{2-\frac{1}{n}}\tan^{-1}\left(\frac{\sqrt{mx+n}}{\sqrt{-n}}\right)\right]-B}{A\exp\left[2\sqrt{2-\frac{1}{n}}\tan^{-1}\left(\frac{\sqrt{mx+n}}{\sqrt{-n}}\right)\right]+B} \right\} + \frac{3(mx+n)-1}{x}$
2	$m \neq 0, n = 0$	$\rho = \frac{1}{x} - 3m$ $p = \frac{2\sqrt{mx}}{x} \left[\frac{A\exp\left(-\frac{2}{\sqrt{mx}}\right)-B}{A\exp\left(-\frac{2}{\sqrt{mx}}\right)+B} \right] + 3m - \frac{1}{x}$
3	$m \neq 0, 0 < n < \frac{1}{2}$	$\rho = \frac{1-n}{x} - 3m$ $p = \left(\frac{2\sqrt{(1-2n)(mx+n)}}{x}\right) \left\{ \frac{A\left(\frac{\sqrt{mx+n}-\sqrt{n}}{\sqrt{mx+n}+\sqrt{n}}\right)^{\sqrt{\frac{1}{n}-2}}-B}{A\left(\frac{\sqrt{mx+n}-\sqrt{n}}{\sqrt{mx+n}+\sqrt{n}}\right)^{\sqrt{\frac{1}{n}-2}}+B} \right\} + \frac{3(mx+n)-1}{x}$
4	$m \neq 0, n = \frac{1}{2}$	$\rho = \frac{1}{2x} - 3m$ $p = \frac{4A\sqrt{mx+\frac{1}{2}}}{x\left[A\sqrt{2}\ln\left(\frac{\sqrt{mx+\frac{1}{2}}-\sqrt{\frac{1}{2}}}{\sqrt{mx+\frac{1}{2}}+\sqrt{\frac{1}{2}}}\right)+2B\right]} + \frac{3(mx+\frac{1}{2})-1}{x}$
5	$m \neq 0, n > \frac{1}{2}$	$\rho = \frac{1-n}{x} - 3m$ $p = \left(\frac{2\sqrt{(2n-1)(mx+n)}}{x}\right) \left\{ \frac{A\left(\frac{\sqrt{mx+n}-\sqrt{n}}{\sqrt{mx+n}+\sqrt{n}}\right)^{\sqrt{2-\frac{1}{n}}}-B}{A\left(\frac{\sqrt{mx+n}-\sqrt{n}}{\sqrt{mx+n}+\sqrt{n}}\right)^{\sqrt{2-\frac{1}{n}}}+B} \right\} + \frac{3(mx+n)-1}{x}$
6	$m = 0, 0 < n < \frac{1}{2}$	$\rho = \frac{1-n}{x}$ $p = \frac{A\left(3n+2n\sqrt{\frac{1}{n}-2}-1\right)x^{\sqrt{\frac{1}{n}-2}}+B\left(3n-2n\sqrt{\frac{1}{n}-2}-1\right)}{x\left(Ax^{\sqrt{\frac{1}{n}-2}}+B\right)}$
7	$m = 0, n = \frac{1}{2}$	$\rho = \frac{1}{2x}$ $p = \frac{1}{2x} + \frac{A\sqrt{2}}{x\left(A\sqrt{2}\ln x+2B\right)}$
8	$m = 0, n > \frac{1}{2}$	$\rho = \frac{1-n}{x}$ $p = \frac{A\left(3n+2n\sqrt{2-\frac{1}{n}}-1\right)x^{\sqrt{2-\frac{1}{n}}}+B\left(3n-2n\sqrt{2-\frac{1}{n}}-1\right)}{x\left(Ax^{\sqrt{2-\frac{1}{n}}}+B\right)}$

Table 3.2: Matter variables ρ and p of the Einstein field equations with conformal symmetry and isotropic pressure for various parameter values of m and n for the same cases as in Table 3.1.

Case	Parameters	Equation of state
4	$A = 0$	$p = -\rho$
6	$A = 0$	$p = \left(\frac{3n-2n\sqrt{\frac{1}{n}-2}-1}{1-n}\right)\rho$
6	$B = 0$	$p = \left(\frac{3n+2n\sqrt{\frac{1}{n}-2}-1}{1-n}\right)\rho$
7	$A = 0$	$p = \rho$
8	$A = 0$	$p = \left(\frac{3n-2n\sqrt{2-\frac{1}{n}}-1}{1-n}\right)\rho$
8	$B = 0$	$p = \left(\frac{3n+2n\sqrt{2-\frac{1}{n}}-1}{1-n}\right)\rho$

Table 3.3: Linear equation of state that arises in Table 3.2 with conformal symmetry and isotropic pressure.

Model	Case	Parameters	Metric
Schwarzschild	**5**	$m=-\frac{1}{R^2}, n=1$	$-\left(\tilde{A}-\tilde{B}\sqrt{1-\frac{r^2}{R^2}}\right)^2 \mathrm{d}t^2$
(interior)		$A=-\frac{\sqrt{m}}{2}\left(\tilde{A}+\tilde{B}\right)$	$+\left(1-\frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2$
(1916)		$B=\frac{\sqrt{m}}{2}\left(\tilde{A}-\tilde{B}\right)$	
Tolman VI	**8**	$n=\frac{1}{2-n_1^2}$	$-\left(-\tilde{A}r^{1+n_1}+\tilde{B}r^{1-n_1}\right)^2\mathrm{d}t^2$
(1939)		$A=-\tilde{A}, B=\tilde{B}$	$+\left(2-n_1^2\right)\mathrm{d}r^2+r^2\mathrm{d}\Omega^2$
Kuchowicz Ia	**5**	$A=R^{a-1}\left(\frac{m}{n}\right)^{\frac{1}{2}a}\tilde{A}$	$-z^{2(1-a)}\Big[\tilde{B}\left(1+\sqrt{1+bz^2}\right)^a$
(1967)		$B=R^{a-1}\left(\frac{m}{n}\right)^{\frac{1}{2}a}\tilde{B}$	$+\tilde{A}\left(1-\sqrt{1+bz^2}\right)^a\Big]^2\mathrm{d}t^2$
			$+\left(n+mr^2\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2,$
			where $a=\sqrt{2-\frac{1}{n}}$, $b=\frac{m}{n}R^2$,
			$z=\frac{r}{R}$
Kuchowicz Ib	**4**	$A=-\frac{\sqrt{2}}{2R}\tilde{A}$	$-z^2\left[\tilde{A}\ln\left\|\frac{1+\sqrt{1+bz^2}}{1-\sqrt{1+bz^2}}\right\|+\tilde{B}\right]^2\mathrm{d}t^2$
(1967)		$B=\frac{1}{R}\tilde{B}$	$+\frac{2}{1+bz^2}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2,$
			where $b=2mR^2$, $z=\frac{r}{R}$
Kuchowicz Ib	**7**	$A=\frac{\sqrt{2}}{2}\tilde{A}$	$-\left(\tilde{A}r\ln r+\tilde{B}r\right)^2\mathrm{d}t^2+\frac{1}{n}\mathrm{d}r^2$
(1967)		$B=\tilde{B}$	$+r^2\mathrm{d}\Omega^2$
Korkina and	**8**	$n=1$	$-\tilde{B}^2\left(1+ar^2\right)^2\mathrm{d}t^2+\mathrm{d}r^2+r^2\mathrm{d}\Omega^2$
Orlyanskii III (1991)		$A=a\tilde{B}, B=\tilde{B}$	
Patwardhan and	**5**	$m=\frac{4\sqrt{7}}{7}, n=\frac{4}{7}$	$-\left[\tilde{A}\xi^{\frac{1}{4}}\left(\xi-1\right)^{\frac{3}{4}}+\tilde{B}\xi^{\frac{3}{4}}\left(\xi-1\right)^{\frac{1}{4}}\right]^2\mathrm{d}t^2$
Vaidya V (1943)		$A=\frac{7^{\frac{1}{4}}\tilde{A}}{2}$	$+\frac{7}{4\left(\sqrt{7}r^2+1\right)}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2,$
		$B=\frac{7^{\frac{1}{4}}\tilde{B}}{2}$	where $\xi=\frac{1}{2}\left(1+\sqrt{1+\sqrt{7}r^2}\right)$
Buchdahl and	**4**	$m=-\frac{\rho_b}{6}, n=\frac{1}{2}$	$-\frac{\rho_b}{3}r^2\mathrm{d}t^2+\left(\frac{1}{2}-\frac{\rho_b}{6}r^2\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2$
Land (1968)		$A=0, B=\sqrt{\frac{\rho_b}{3}}$	
Saslaw *et al.*	**8**	$n=\frac{(1+\alpha)^2}{(1+\alpha)^2+4\alpha}$	$-\tilde{B}^2r^{\left(\frac{2\alpha}{1+\alpha}\right)}\mathrm{d}t^2+\left(1+\frac{4\alpha}{(1+\alpha)^2}\right)\mathrm{d}r^2$
(1996)		$A=0, B=\tilde{B}$	$+r^2\mathrm{d}\Omega^2$

Table 3.4: Previously discovered line elements $\mathrm{d}s^2$ for exact solutions of the Einstein field equations admitting a conformal symmetry with isotropic pressure for cases that arise in Table 3.1. The metrics are labelled according to Delgaty and Lake (1998).

physical features. Then the gravitational potentials are given by

$$y = \int \frac{\mathrm{d}x}{x\sqrt{1-a-bx+\frac{d}{x^{\frac{1}{2}}}}}, \tag{3.34}$$

$$Z = 1-a-bx+\frac{d}{x^{\frac{1}{2}}}, \tag{3.35}$$

with anisotropy

$$\Delta = \frac{1}{x}\left(\tilde{k}^2 + 2a - 1 + \frac{d}{x^{\frac{1}{2}}}\right). \tag{3.36}$$

The conformal symmetry associated with the Shee *et al.* (2016) model is given by

$$\mathbf{X} = \frac{1}{2}\tilde{k}t\frac{\partial}{\partial t} + x\left(1-a-bx+\frac{d}{x^{\frac{1}{2}}}\right)^{\frac{1}{2}}\frac{\partial}{\partial x}, \tag{3.37a}$$

$$\psi = \left(1-a-bx+\frac{d}{x^{\frac{1}{2}}}\right)^{\frac{1}{2}}. \tag{3.37b}$$

Clearly many other exact solutions to the field equations can be found with different forms of $\mathbf{X}$ and ψ. It would be interesting to identify those cases which are physically reasonable.

3.6 Discussion

In this chapter we have performed a systematic study of static spherically symmetric spacetimes with a conformal symmetry. The conformal Killing vector is spherically symmetric with a nonstatic conformal factor. We have shown that under these assumptions a second order nonlinear differential equation arises which dictates the behaviour of the gravitational field. Integrating this nonlinear equation leads to an explicit relationship between the gravitational potentials $\nu(r)$ and $\lambda(r)$. This relationship applies to both conformally flat and non-conformally flat spacetimes. We have expressed this result as Theorem 3.3.1. The Einstein field equations then allow us to generate explicit forms for the potentials in terms of the anisotropy Δ: the solution of the field equations has been found in general. This result is given in Theorem 3.5.1.

When $\Delta = 0$, the pressure is isotropic, we are in a position to find all metrics and matter variables explicitly, and these are presented in Table 3.1 and Table 3.2 respectively. We note that models listed in Table 3.2 with isotropic pressures admit a barotropic equation of state. The special cases of linear equations of state are identified and tabulated in Table 3.3. For particular parameter values we are able to regain well-known solutions such as the Schwarzschild interior metric (1916), the Tolman IV (1939) metric, and several other line elements; we have also included the isothermal metric of Saslaw *et al.* (1996) which admits a conformal vector. We list the known metrics with isotropic pressures and conformal symmetry in Table 3.4. We show that the known isotropic metrics are contained in the comprehensive catalogue of Delgaty and Lake (1998).

In the presence of anisotropy when $\Delta \neq 0$, we show that the matter variables depend only on the choice of the potential $\lambda(r)(= Z(x))$. Therefore the choice of $\lambda(r)$ only will lead to an exact solution with conformal symmetry, and will regain known solutions. We demonstrate that this is the case by regaining the model of Shee *et al.* (2016) and identify the associated conformal symmetry.

Chapter 4

Astrophysical model

4.1 Introduction

In this chapter we look at an astrophysical application of conformal symmetries. We investigate how these symmetries are useful in modelling dense relativistic stars. As stated earlier in this thesis, Herrera *et al.* (1984) were the first to model a conformally invariant gravitating sphere. Herrera and Ponce de León (1985a, 1985b, 1985c) presented other stellar models with conformal symmetry. However, many of these solutions were not regular at the centre. Maartens and Maharaj (1990) generated models of conformally invariant spheres with an anisotropic energy momentum tensor which is regular at the centre of the star. We will apply the relationship between the gravitational potentials established in Chapter 2 as the basis of our study. We choose a form for one of the gravitational potentials that enables the conformal condition in (2.39) to be integrated. In this work we present a new anisotropic solution for a compact star with the use of a conformal Killing vector in the spherical spacetime geometry. We demonstrate that the exact solution produces a relativistic model of a compact star. The model generates stellar radii and masses consistent with the pulsars PSR J1614-2230, Vela X1, PSR J1903+327 and Cen X-3.

In §4.2, we present the relationship between gravitational potentials and the Einstein field equations. A new exact anisotropic solution is generated in §4.3. In §4.4, physical requirement conditions for acceptability of the stellar model and the model parameter constraints are discussed. In §4.5, we generate masses and radii for selected pulsars PSR J1614-2230, Vela X-1, PSR J1903+327 and Cen X-3. The results are presented in Table I and Table II, and graphical plots of matter variables for PSR J1614-2230 are displayed. A detailed analysis of the physical features is presented. A brief conclusion is made in §4.6.

4.2 The model

To establish the exact solution required for an astrophysical application we utilize earlier results of this thesis. In Chapter 2 we found that in a static spherically symmetric spacetime the conformal symmetry yields the condition

$$e^{\nu} = Br\cosh\left(\sqrt{1+k}\int\frac{e^{\lambda}}{r}\mathrm{d}r + l\right), \tag{4.1}$$

where B, k and l are constants. When $k = 0$ then the spacetime is conformally flat, and $k \neq 0$ leads to a non-conformally flat model. In §2.6 we found that the Einstein field equation for a spacetime admitting a conformal symmetry with relationship (4.1) can be expressed as follows

$$8\pi\rho = \frac{1-e^{-2\lambda}}{r^2} + \frac{2\lambda' e^{-2\lambda}}{r}, \tag{4.2a}$$

$$8\pi p_r = \frac{3e^{-2\lambda}-1}{r^2} + \frac{2e^{-\lambda}\sqrt{1+k}\tanh\left(\sqrt{1+k}\int\frac{e^{\lambda}}{r}\mathrm{d}r + l\right)}{r^2}, \tag{4.2b}$$

$$8\pi p_t = \frac{(1-2\lambda' r)e^{-2\lambda}}{r^2} + \frac{2\sqrt{1+k}e^{-\lambda}\tanh\left(\sqrt{1+k}\int\frac{e^{\lambda}}{r}\mathrm{d}r + l\right)}{r^2} + \frac{1+k}{r^2}, \tag{4.2c}$$

where primes denote differentiation with respect to the coordinate r. The quantities ρ, p_r and p_t are the energy density, radial pressure and tangential pressure respectively. In the above we are using units where $G = c = 1$.

We find that all matter variables depend only on one metric function, namely $e^{2\lambda}$. A particular choice of $\lambda(r)$ will lead to an exact solution of the field equations after integration. We demonstrate the existence of an exact solution in the next section.

The mass of an uncharged compact object contained within a radius r of the relativistic sphere is

$$M(r) = 4\pi \int_0^r \rho(\omega)\omega^2 \mathrm{d}\omega. \tag{4.3}$$

4.3 Exact solution

We need to choose the function e^{λ} and complete the integration in (4.1) to generate a solution to the Einstein system. We perform this choice so that both metric functions remain regular at the centre $r = 0$. We take e^{λ} in the form

$$e^{\lambda} = \frac{1}{\sqrt{k+1} - br^2}, \tag{4.4}$$

were b is a constant. Then (4.1) gives the second potential

$$e^{\nu} = \frac{B}{2\sqrt{b}e^{l}} \left[\frac{\left(e^{2l} - 1\right) br^2 + \sqrt{1+k}}{\sqrt{\sqrt{1+k} - br^2}} \right]. \tag{4.5}$$

At $r = 0$ the potentials are regular at the stellar centre.

Then using the potentials (4.4) and (4.5), an exact solution to the Einstein system (2.58) is given by

$$\rho = \frac{6b\sqrt{1+k} - 5b^2r^2}{8\pi} - \frac{k}{8\pi r^2}, \tag{4.6a}$$

$$\begin{aligned} p_r = & \frac{3b^3(n-1)r^4 + b^2\sqrt{1+k}(7-8n)r^2 + (5k+4)(n-1)b}{8\pi\left[(n-1)\,br^2 + \sqrt{1+k}\right]} \\ & + \frac{k\sqrt{1+k}}{8\pi r^2\left[(n-1)\,br^2 + \sqrt{1+k}\right]}, \end{aligned} \tag{4.6b}$$

$$p_t = \frac{5b^3(n-1)r^4 + b^2\sqrt{1+k}(9-8n)r^2 + 4(k+1)(n-1)b}{8\pi\left[(n-1)\,br^2 + \sqrt{1+k}\right]}. \tag{4.6c}$$

In (4.6) we have set $n = e^{2l}$ for notational convenience. Equations (4.6) represent an anisotropic star which is gravitating in the presence of the conformal Killing vector $\mathbf{X}$. Note that the mass function (4.3) becomes

$$M(r) = b(\sqrt{1+k})r^3 - \frac{b^2}{2}r^5 - \frac{k}{2}r, \tag{4.7}$$

for the above density.

4.4 Physical features of the stellar model

4.4.1 Regularity conditions inside and at the boundary $r = \mathcal{R}$

For physical acceptability, the model should comply with several requirements throughout the star. These include:

1. Regularity:

    (a) The gravitational potentials $e^{2\nu}$ and $e^{2\lambda}$ and the matter variables ρ, p_r, p_t should be positive at the centre and regular throughout the star

    (b) At the centre ρ, p_r and p_t should be finite: $\rho(r=0) = \rho_c$, $p_r(r=0) = p_{rc}$ and $p_t(r=0) = p_{tc}$

    (c) the gradients $\frac{d\rho}{dr} \leq 0$, $\frac{dp_r}{dr} \leq 0$ and $\frac{dp_t}{dr} \leq 0$ within the star

    (d) The anisotropy at the centre should vanish: $\Delta(r=0) = p_t - p_r = 0$.

2. Stable configuration:

    (a) We require that the speed of sound must be less than the speed of light. This implies that

    $$0 \leq v_r^2 = \frac{dp_r}{d\rho} \leq 1,$$

    and

    $$0 \leq v_t^2 = \frac{dp_t}{d\rho} \leq 1,$$

    inside the stellar body.

(b) To prevent cracking or overturning of the star we must have

$$-1 < v_t^2 - v_r^2 < 0$$

and

$$0 < v_r^2 - v_t^2 < 1$$

.

(c) the adiabatic index Γ should obey.

$$\Gamma = \frac{\rho + p_r}{p_r} \frac{\mathrm{d}\rho}{\mathrm{d}p_r} > \frac{4}{3}$$

.

3. The equilibrium condition for stability is related to the Tolman-Oppenheimer-Volkoff (TOV) equation. The TOV equation is given by

$$\frac{\mathrm{d}p_r}{\mathrm{d}r} = -\nu'(\rho + p_r) + \frac{2}{r}(p_t - p_r) \tag{4.8}$$

if we let

$$F_g = -\nu'(\rho + p_r),$$

$$F_h = -\frac{dp_r}{dr},$$

$$F_a = \frac{2}{r}\left(p_t - p_r\right),$$

where F_g, F_h and F_a are known as gravitational, hydrostatic and anisotropic forces respectively. Then (4.8) becomes

$$F_g + F_h + F_a = 0, \tag{4.9}$$

so that the anisotropic gravitating sphere is in equilibrium.

4. Within the star, the dominant energy conditions require that:

(a) $\rho - p_r \geq 0$,

(b) $\rho - p_t \geq 0$,

(c) $\rho - p_r - 2p_t \geq 0$.

5. Matching conditions:

(a) The metric functions $e^{2\lambda}$ and $e^{2\nu}$ at the boundary $r = \mathcal{R}$ should match smoothly to the Schwarzschild exterior metric i.e.

$$e^{2\nu(\mathcal{R})} = 1 - \frac{2M}{\mathcal{R}}, \tag{4.10}$$

$$e^{2\lambda(\mathcal{R})} = \left(1 - \frac{2M}{\mathcal{R}}\right)^{-1}. \tag{4.11}$$

(b) The radial pressure should vanish at the surface

$$p_r(\mathcal{R}) = 0. \tag{4.12}$$

The mass-radius ratio and the surface redshift are also important physical quantities. The maximum limit of the mass-radius ratio for an uncharged compact star is given by the inequality

$$\frac{2M}{\mathcal{R}} < \frac{8}{9},$$

proposed by Buchdahl (1959). For a realistic compact object, the upper bound on the surface redshift is given by

$$Z_s(\mathcal{R}) = \frac{1}{\sqrt{1 - \frac{2M}{\mathcal{R}}}} - 1 \leq 2. \tag{4.13}$$

4.4.2 Model parameter constraints

The regularity of the model is based on the constraints choice of the parameters in the physical quantities arising in §4.4.1. The parameters are chosen so that the model is well behaved at the centre and within the stellar structure. Therefore we use the value $k = 0$ in the energy density (4.6a) for regularity at the centre. When $(r = 0)$ we have the values:

$$\rho_c = \frac{3b}{4\pi}, \tag{4.14a}$$

$$p_{rc} = \frac{(n-1)b}{2\pi}, \tag{4.14b}$$

$$p_{tc} = \frac{(n-1)b}{2\pi}, \tag{4.14c}$$

$$v_{rc}^2 = \frac{1}{5}(4n^2 - 3), \tag{4.14d}$$

$$v_{tc}^2 = \frac{1}{5}(4n^2 - 5). \tag{4.14e}$$

The central values (4.14) restrict the parameter n to the range of $\frac{\sqrt{5}}{2} < n < \sqrt{2}$.

We name a new constant $\sqrt{H} = \frac{B}{2\sqrt{b e^l}}$ and use the three boundary conditions (4.10)-(4.12) with five unknowns $M, \mathcal{R}, n, b$ and H. We can express particular parameters in terms of others. The physically relevant model quantities are the mass M and the radius $\mathcal{R}$, and the model parameters n and b of a compact object.

1. Firstly we can write $\mathcal{R}$, M in terms of n, b. We obtain

$$\mathcal{R} = 0.408248\sqrt{\frac{8n}{b(n-1)} - \frac{\sqrt{16(n-1)n+1}}{b(n-1)} - \frac{7}{b(n-1)}}, \tag{4.15}$$

   which is the radius of star. In addition we find the quantity

$$M = b\mathcal{R}^3\left(1 - \frac{b\mathcal{R}^2}{2}\right), \tag{4.16}$$

   which is the total mass of the star. Equation (4.16) restricts the parameter b in the range $0 < b < \frac{2}{\mathcal{R}^2}$. The parameter H can be written in terms of n, b as

$$H = \frac{(b\mathcal{R}^2 - 1)\left(2(b\mathcal{R}^3 - 0.5b^2\mathcal{R}^5) - \mathcal{R}\right)}{\mathcal{R}\left(nb\mathcal{R}^2 - b\mathcal{R}^2 + 1\right)^2}, \tag{4.17}$$

where H is a constant scaling in our static stellar model. Then the recipe for physical analysis of our model can be outlined as follows:

(a) Select the central density $\rho_c = \frac{3b}{4\pi}$ and central pressure $p_{rc} = \frac{(n-1)b}{2\pi}$ with n in the range $\frac{\sqrt{5}}{2} < n < \sqrt{2}$ and $0 < b < \frac{2}{\mathcal{R}^2}$.

(b) Calculate $\mathcal{R}$ using (4.15).

(c) Use (4.16) to calculate the mass M.

(d) The parameter H can be found from (4.17).

The rest of the matter variables then follow.

2. Secondly, we can write b and n in terms of $\mathcal{R}$ and M. We obtain

$$b = \frac{\mathcal{R}^3 - \sqrt{\mathcal{R}^5(\mathcal{R} - 2M)}}{\mathcal{R}^5}, \tag{4.18}$$

which is the model parameter linked to the central density. The parameter n is given by

$$\begin{aligned} n &= \frac{18M^2 - 13M\mathcal{R} + 2\mathcal{R}^2}{M(18M - 8\mathcal{R})} - \frac{2\sqrt{\mathcal{R}^5(\mathcal{R} - 2M)}}{M\mathcal{R}(18M - 8\mathcal{R})} \\ &\quad + \frac{3\sqrt{\mathcal{R}^5(\mathcal{R} - 2M)}}{\mathcal{R}^2(18M - 8\mathcal{R})}, \end{aligned} \tag{4.19}$$

which is the parameter related to the matter variables except the central density. The parameter H for this case is in the form

$$\begin{aligned} H &= \frac{M^2\left(-1.125\sqrt{\mathcal{R}^5(\mathcal{R} - 2M)} - 1.5\mathcal{R}^3\right) - 0.25\mathcal{R}^2\sqrt{\mathcal{R}^5(\mathcal{R} - 2M)} - 0.25\mathcal{R}^5}{\mathcal{R}^4(M - 0.5\mathcal{R})} \\ &\quad + \frac{M\left(\mathcal{R}\sqrt{\mathcal{R}^5(\mathcal{R} - 2M)} + 1.25\mathcal{R}^4\right)}{\mathcal{R}^4(M - 0.5\mathcal{R})}. \end{aligned} \tag{4.20}$$

In equations (4.18)-(4.20) we note the mass $M < \frac{\mathcal{R}}{2}$ and $M \neq \frac{4\mathcal{R}}{9}$. Now the recipe for the physical analysis of our model can be outlined as follows:

(a) Select the radius $\mathcal{R}$ and mass M of a given star such that $M < \frac{\mathcal{R}}{2}$ and $M \neq \frac{4\mathcal{R}}{9}$.

(b) Calculate b using (4.18).

(c) Use (4.19) to calculate n checking that $\frac{\sqrt{5}}{2} < n < \sqrt{2}$ and $0 < b < \frac{2}{\mathcal{R}^2}$ holds.

(d) The parameter H can be calculated from (4.20).

The remaining matter variables then follow.

4.5 Physical Analysis

For physical acceptability of our model, it is interesting to investigate whether it can be applied to observed stars for some range of parameters presented above. To study the features of the model, we select four pulsars PSR J1614-2230, Vela X-1, PSR J1903+327 and Cen X-3. We use the equations (4.18)-(4.20) with the mass $M = 2.91 M_{\odot}$ and the corresponding radius $\mathcal{R} = 10.30$km for PSR J1614-2230. These values arise in the works of Mafa Takisa *et al.* (2014b) with a linear equation of state and Mafa Takisa *et al.* (2014a) with a quadratic equation of state as input in this investigation. The values of the constants calculated are $b = 0.0032063$ km^{-2}, $n = 1.20921$ and $H = 0.250386$. Plugging these values in relevant equations, we obtain the central density $\rho_c = 1.031 \times 10^{15}$ gcm^{-3}, the surface density $\rho_{\mathcal{R}} = 9.70 \times 10^{14}$ gcm^{-3} and central radial and tangential pressures $p_{r_c} = p_{t_c} = 1.205 \times 10^{35}$ dyne cm^{-2}.

Firstly, we allow simultaneously the parameters b and n in (4.15)-(4.17) to vary so that we can generate the masses and radii of the remaining stars Vela X-1, PSR J1903+327 and Cen X-3. The resulting values are presented in Table I. We note that the central density decreases with the decrease of the mass; this feature was also reported in the works of Mafa Takisa *et al.* (2014a) and Mafa Takisa *et al.* (2014b). Secondly, the parameter n has a fixed value but the parameter b is allowed to vary. We obtain different masses, radii and central densities for Vela X-1, PSR J1903+327 and Cen X-3. These results are given in Table II. We note that the central density increases with the decrease of the mass; this feature is similar to the investigations of Sharma and Ratanpal (2013), Singh *et al.* (2016) and Kileba Matondo *et al.* (2017). For both scenarios, the central density is approximately in the order of 10^{15}gcm^{-3} which is relevant for an anisotropic compact relativistic star as pointed out by Ruderman (1972). The surface density is in the order of 10^{14} gcm^{-3} and the redshift is in the range $0.364-0.515$. This range is close to the values found by Böhmer *et al.* (2007), Rahaman *et al.* (2015a, 2012) and Kileba Matondo *et al.* (2017). Also, we see that the required upper bound of Buchdahl which is equivalent to $Z_s \leq 2$ for a realistic star has been fulfilled. The value of the stellar radius $\mathcal{R}$ is the range of $7.80-10.30$ km, and the mass in the range of $1.49 - 1.97 M_{\odot}$. Similar mass values were obtained by Gangopadhyay *et al.* (2013) and Mafa Takisa *et al.* (2014a) and Mafa Takisa *et al.* (2014b). The compactification factor $\frac{M}{\mathcal{R}}$ is in the range of neutron stars and ultracompact stars.

To illustrate the behaviour of the matter variables inside the stellar structure, we have plotted several profiles in different Figures 4.1-4.12 . All the matter variables are well behaved throughout the stellar structure. The matter density in Fig. 4.1 is decreasing and remains finite. In Fig. 4.2 we display the radial and tangential pressures and the anisotropy. We note that both tangential and radial pressures are monotonic decreasing outwards although the radial pressure vanishes at the boundary; the tangential pressure is still positive inside the star. The anisotropy is finite at the centre and remains positive within the star, and consequently the anisotropic force is repulsive in nature as stated by Gokhroo and Mehra (1994). The mass versus radius displayed in Fig. 4.3 is well behaved and increasing function throughout the compact object. In Fig. 4.4, the gradients of density, radial pressure and tangential pressure are plotted. All the gradient profiles remain negative and regular as required for a realistic star. The square of radial and tangential speed of sound v_r^2, v_t^2 are shown in Fig. 2(a) and they satisfy the causality condition. The radial speed of sound is greater than the tangential speed of sound within the stellar structure. In Fig. 4.6, the quantities $v_t^2 - v_r^2$ and $v_r^2 - v_t^2$ are plotted, showing that $-1 < v_t^2 - v_r^2 < 0$ and $0 < v_r^2 - v_t^2 < 1$ which comply with the cracking stability requirement; it is also observed that these two quantities do not change sign inside the fluid sphere, as required by Herrera (1992). The mass versus central density is provided in Fig. 4.7. For the lower bound radius value $\mathcal{R} = 7.80$ km, the mass increases with central density and reaches the turning point at the central density value of $\rho_c = 5.29 \times 10^{15}\mathrm{gcm}^{-3}$ with corresponding value of mass $M = 2.642 M_{\odot}$. For the upper bound radius value $\mathcal{R} = 10.30$ km, the mass increases with central density and reaches the turning point at the central density value of $\rho_c = 3.03 \times 10^{15}\mathrm{gcm}^{-3}$ with corresponding value of mass $M = 3.489 M_{\odot}$. The stellar masses located on the left of these turning points are in the stable region and the unstable region corresponds to the right side of the turning point. Note that for values greater than $\mathcal{R} = 10.30$ km, the maximum mass for a stable star is at the lower central density compared to the case for $\mathcal{R} = 7.80$ km.

The adiabatic indices Γ_r and Γ_t profiles are shown in Fig. 4.8; indicating that $\Gamma_r > \frac{4}{3}$ and $\Gamma_t > \frac{4}{3}$ throughout the stellar structure. The profiles of gravitational, hydrostatic and anisotropic forces are plotted in Fig. 4.9 showing that anisotropic and hydrostatic forces are positive and balanced by the negative gravitational force. Interestingly the sum of all the forces is zero as required for the equilibrium of a static system. The energy conditions $\rho - p_r$, $\rho - p_t$ and $\rho - p_r - 2p_t$ are plotted in Fig. 4.10 which remain positive thereby implying that the energy conditions are not violated in our model. The matching conditions are presented in Fig. 4.11, the metric potentials $e^{2\nu}$ and $e^{2\lambda}$ are regular within the stellar object, with smooth matching to the Schwarzschild exterior at the radius $\mathcal{R} = 10.30$ km. Moreover at the centre the fluid sphere satisfies $e^{2\nu(r=0)}$= constant and $e^{2\nu(r=0)} = 1$. The gravitational redshift is plotted in Fig. 4.12, with a monotonically decreasing profile.

Star	$b(\text{km}^{-2})$	n	F	$\frac{M}{M_\odot}$	$\mathcal{R}(\text{km})$	$\frac{M}{\mathcal{R}}$	$\rho_c(\text{x}10^{15}\text{gcm}^{-3})$	Z_s
PSR J1614-2230	0.0032063	1.20921	0.250386	1.97	10.30	0.28230	1.031	0.515
Vela X-1	0.00309987	1.17072	0.297192	1.77	9.99	0.26151	0.997	0.448
PSR J1903+327	0.0030455	1.15320	0.322566	1.667	9.82	0.25056	0.980	0.416
Cen X-3	0.00295072	1.12838	0.308379	1.49	9.51	0.23126	0.949	0.364

Table 4.1: Variation of mass, radius and central density in terms of b and n. The parameters b and n are variable.

Star	$b(\text{km}^{-2})$	n	F	$\frac{M}{M_\odot}$	$\mathcal{R}(\text{km})$	$\frac{M}{\mathcal{R}}$	$\rho_c(\text{x}10^{15}\text{gcm}^{-3})$	Z_s
PSR J1614-2230	0.0032063	1.20921	0.250386	1.97	10.30	0.28230	1.031	0.5155
Vela X-1	0.0039718	1.20921	0.250386	1.77	9.25	0.28243	1.28	0.5160
PSR J1903+327	0.0043699	1.20921	0.250386	1.667	8.82	0.27898	1.41	0.5040
Cen X-3	0.0055950	1.20921	0.250386	1.49	7.80	0.28232	1.80	0.5143

Table 4.2: Variation of mass, radius and central density in term of b. The parameter b is variable and n is fixed.

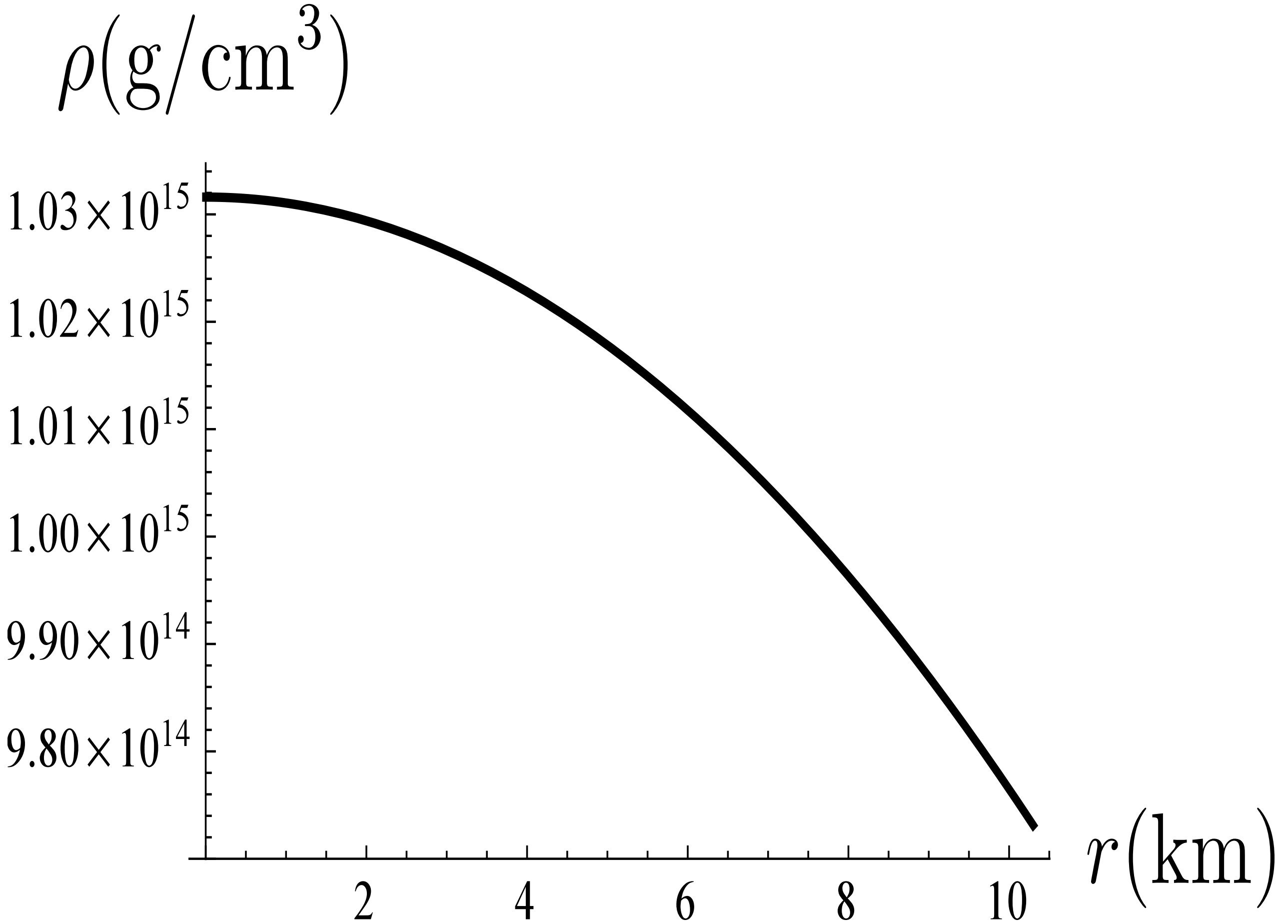

Figure 4.1: Energy density for PSR J1614-2230.

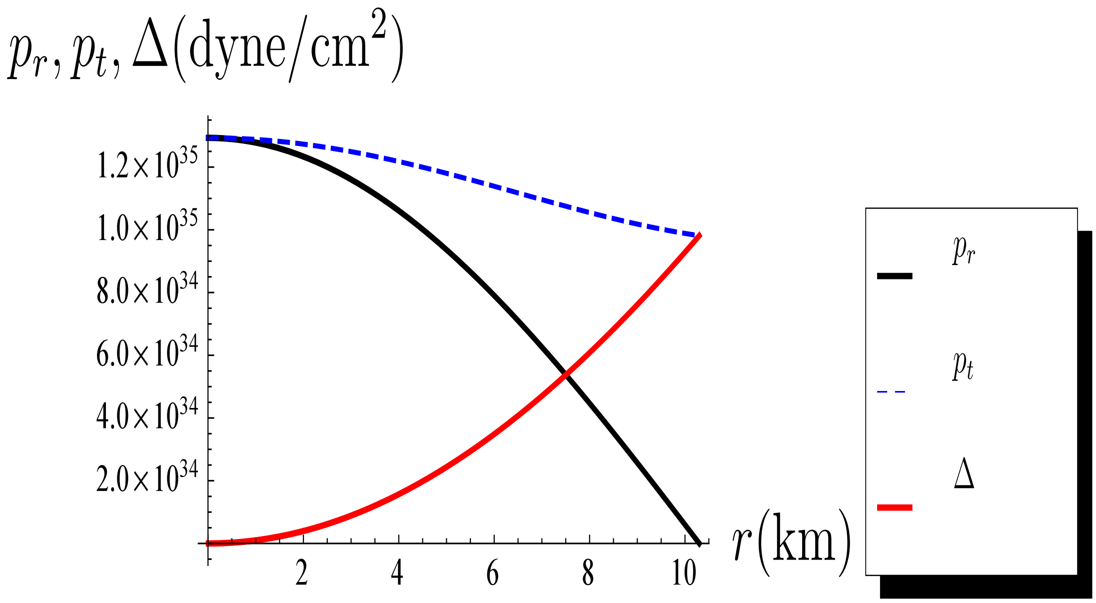

Figure 4.2: Tangential, radial pressure and anisotropy for PSR J1614-2230.

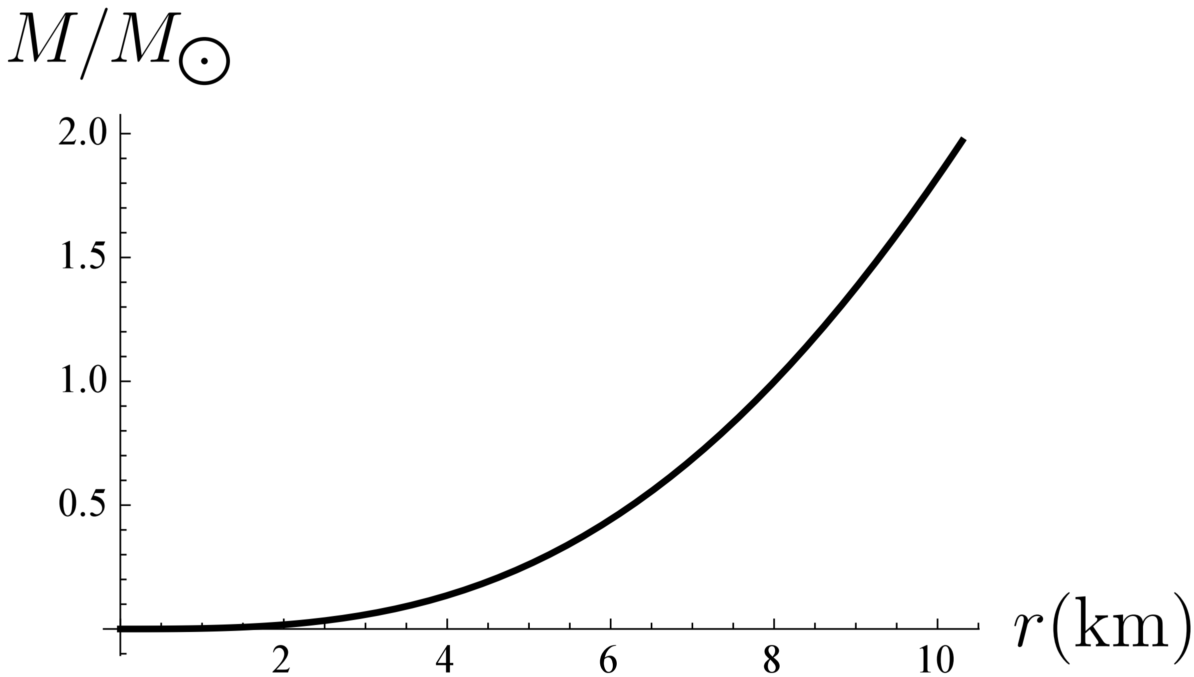

Figure 4.3: Mass vs radius for PSR J1614-2230.

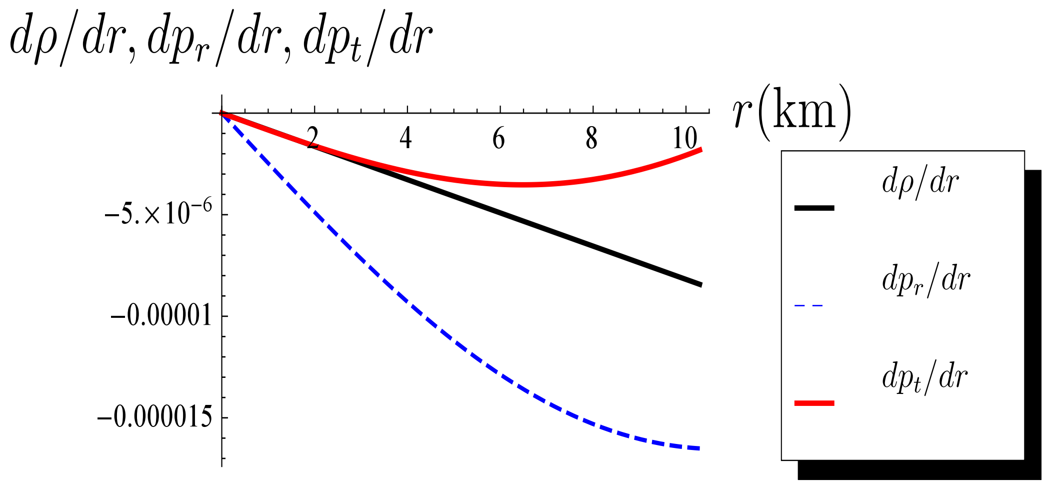

Figure 4.4: Pressure and energy gradients for PSR J1614-2230

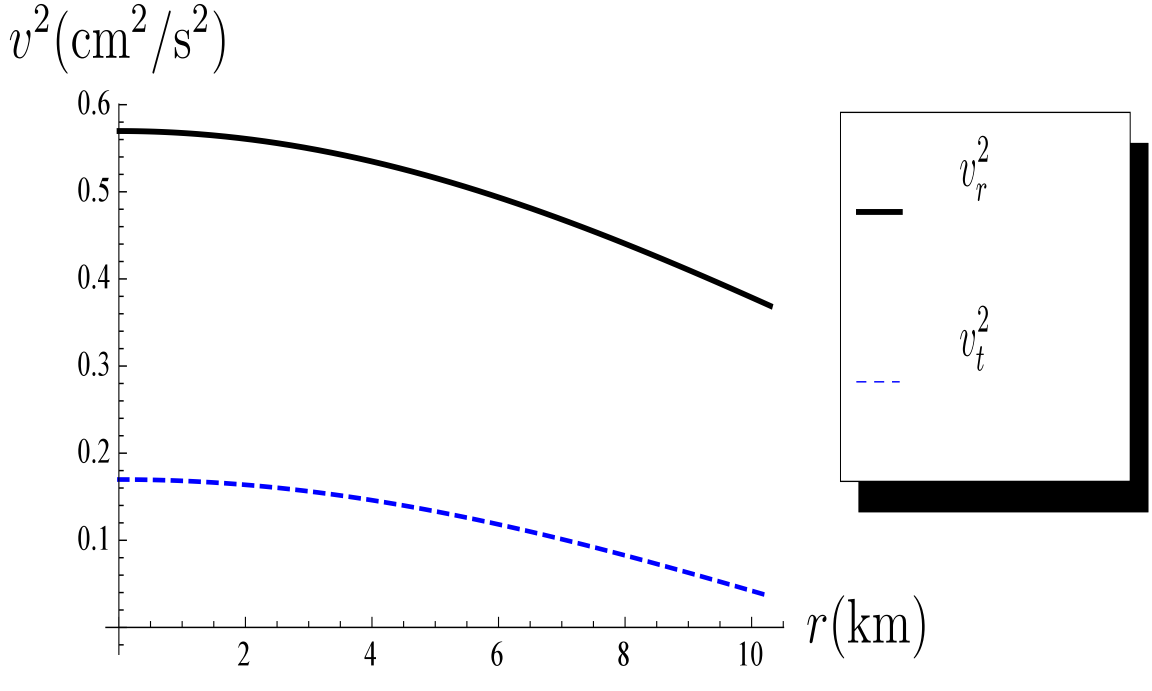

Figure 4.5: Speeds of sound for PSR J1614-2230

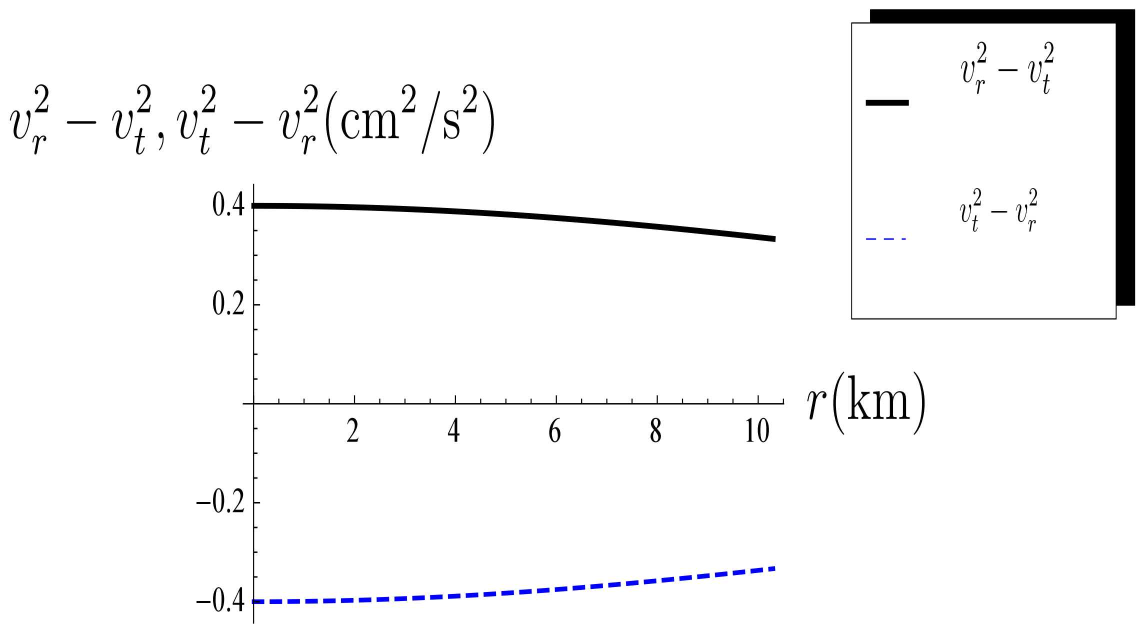

Figure 4.6: The quantities $v_r^2 - v_t^2$ and $v_t^2 - v_r^2$ for PSR J1614-2230

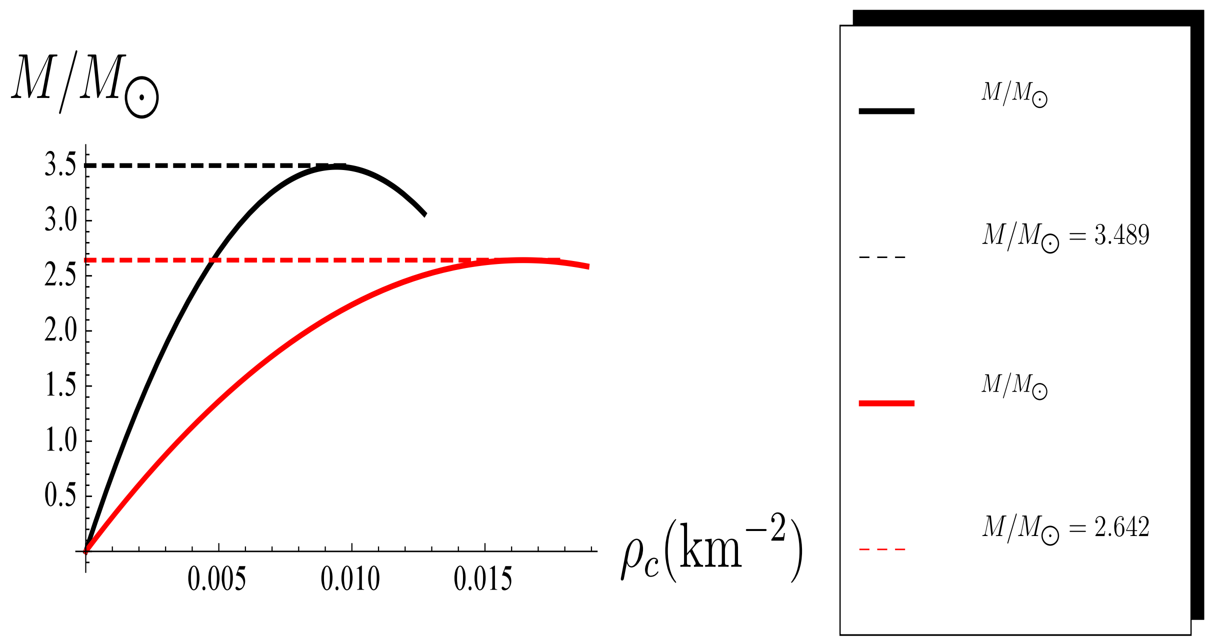

Figure 4.7: Stellar mass versus central density for PSR J1614-2230

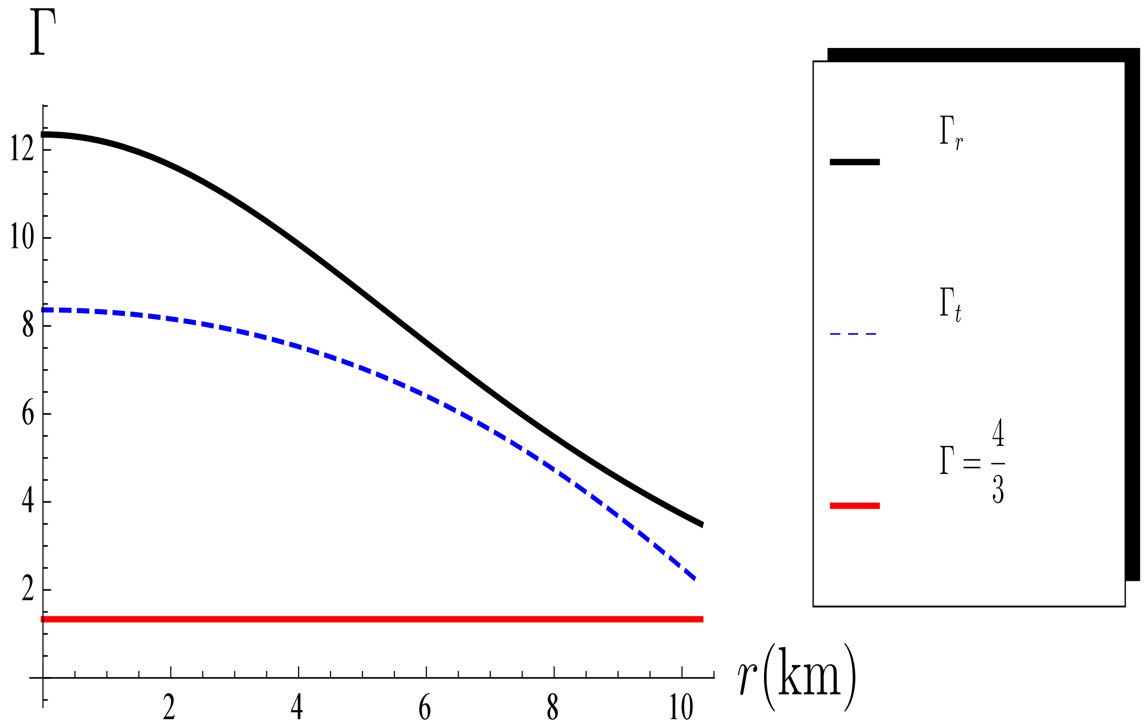

Figure 4.8: Adiabatic index for PSR J1614-2230

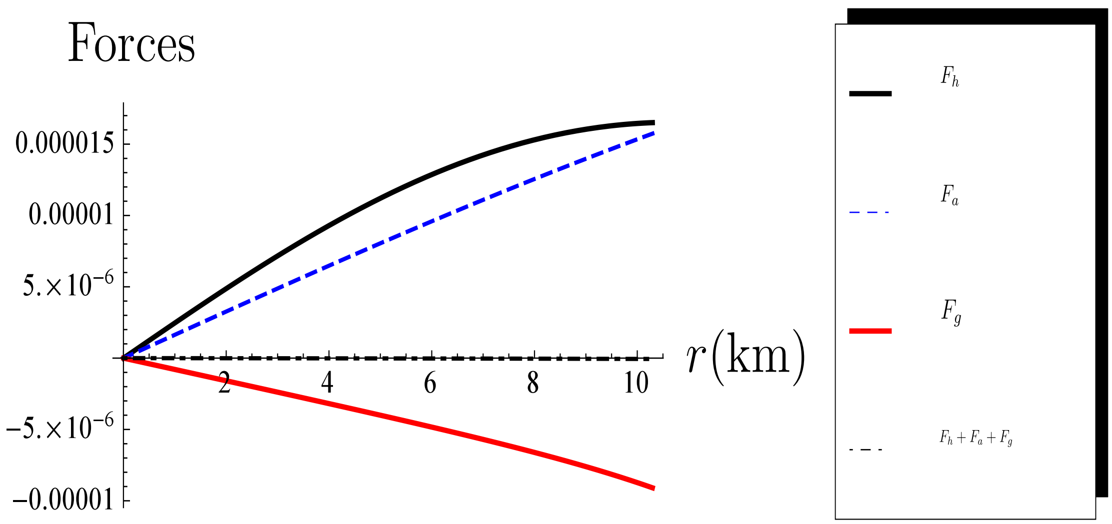

Figure 4.9: Forces F_h, F_g and F_a for PSR J1614-2230

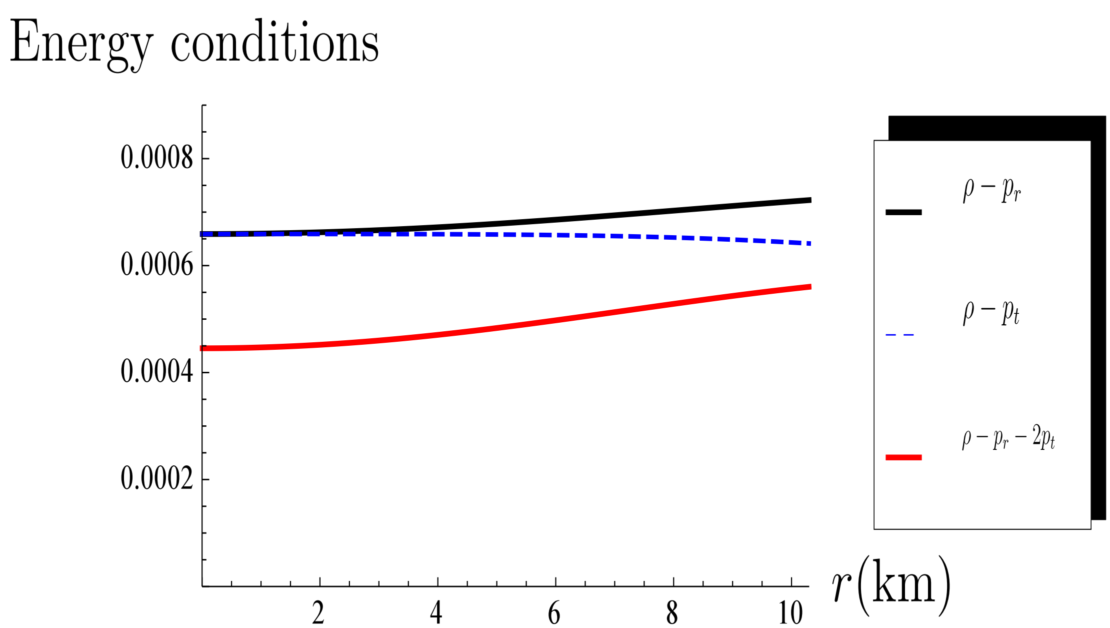

Figure 4.10: Energy conditions for PSR J1614-2230

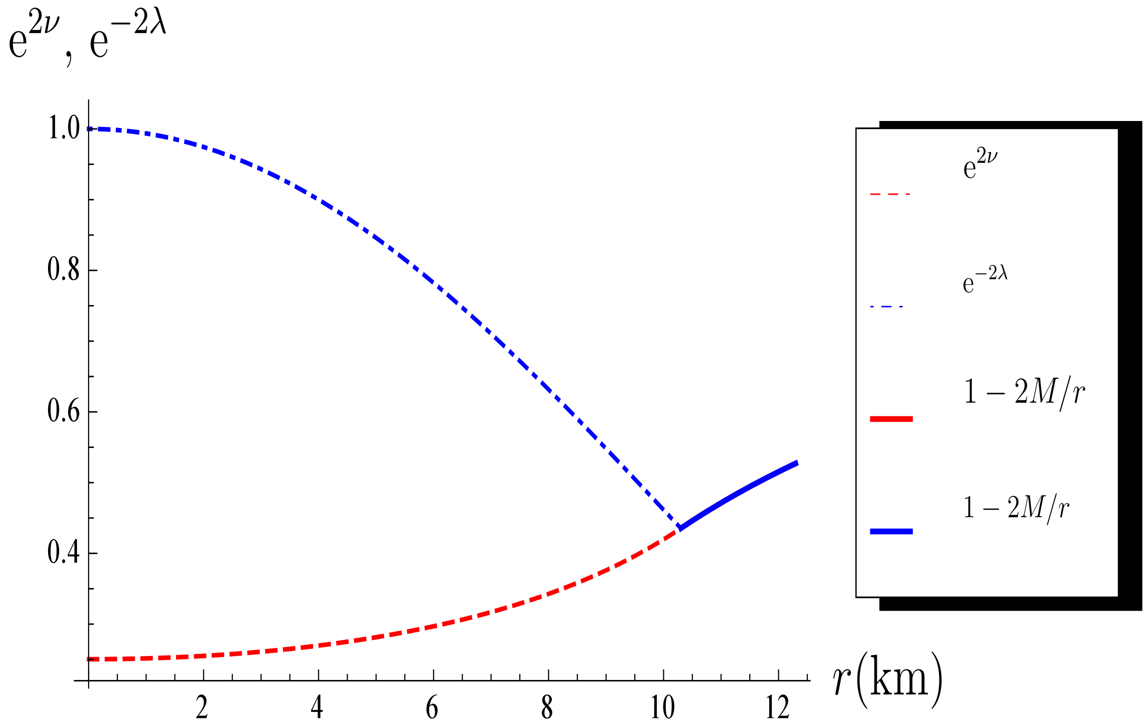

Figure 4.11: Matching at boundary for PSR J1614-2230

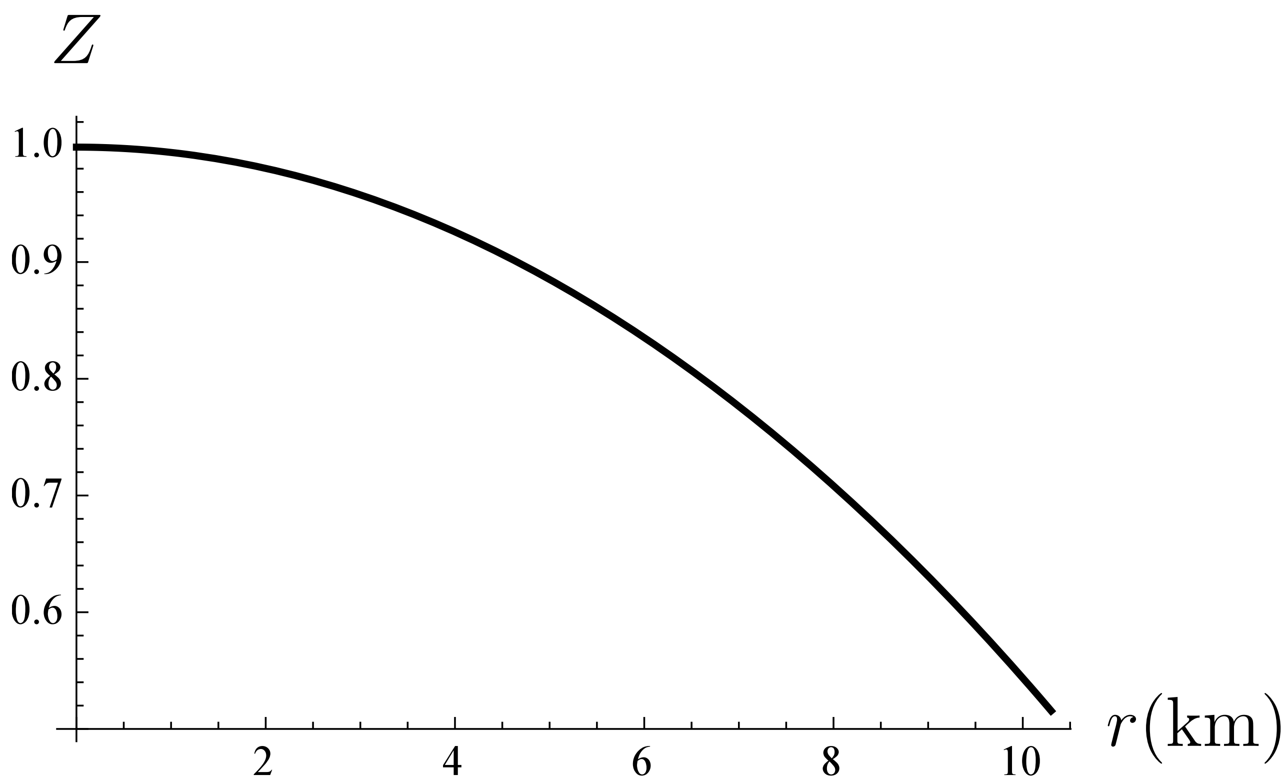

Figure 4.12: The gravitational redshift for PSR J1614-2230

4.6 Discussion

In this chapter we have used a conformal symmetry to model a compact star with anisotropic pressure. The presence of a conformal symmetry leads to an integral relationship between the potentials; only one choice for the potentials has to be made. The exact solution exists for all values of the parameter k. When $k = 0$ the model is conformally flat which ensures regularity at the centre. Furthermore, various parameter constraints are imposed so that the model is well behaved at the centre and within the star. We obtained different masses and radii of four stars, namely PSR J1614-2230, Vela X-1, PSR J1903+0327 and Cen X-3. We achieved this for two different scenarios. Firstly, we varied both parameters b and n simultaneously. We observe that the central density decreases with the decrease of the mass. Secondly, the parameter n was fixed and b was allowed to vary. In this situation the central density increases with a decrease in mass. This behaviour of decreasing central density is present in the investigations of Mafa Takisa *et al.* (2014a) and Mafa Takisa *et al.* (2014b) and Murad (2016). The behaviour of increasing central density is present in the works of Sharma and Ratanpal (2013), Singh *et al.* (2016) and Bhar *et al.* (2015). It is interesting to observe that our model allows for both scenarios. These two different scenarios arise in the works of other researchers and we made reference to their works. We then plotted the matter variables for the compact object PSR J1614-2230. Our analysis reveals that the matter variables and the metric potentials are well behaved within the star. In particular the model is stable and causality is maintained in the interior of the star. The values for the mass-radius ratio and surface redshift are consistent with observed values. In summary, the assumption of a conformal symmetry generates a realistic star in general relativity.

Chapter 5

Conclusion

Our goal in this thesis was to investigate the application of conformal symmetries in the study of general relativity. Some recent work has yielded models admitting symmetry. We set out to investigate this problem without imposing any conditions on the spacetime symmetry other than static spherical symmetry.

We outline the work done in this thesis below:

- In Chapter 2, we performed Weyl tensor analysis to classify static spherically symmetric spacetimes. We found a general relationship between the metric potentials
$$\mathrm{e}^{\nu(r)} = \cosh\left[\sqrt{1+k}\int \frac{\mathrm{e}^{\lambda(r)}}{r}\mathrm{d}r + l\right],$$
for conformally flat and nonconformally spacetimes. We find that our general treatment regains the results of Maartens and Maharaj (1990), Tello-Llanos (1988), Mak and Harko (2004) and Usmani *et al.* (2011), amongst others. These results were tabulated in Tables 2.1 and 2.2. In the Einstein field equations with anisotropic pressures we can write $\lambda(r)$ in term of $\nu(r)$.

- In Chapter 3 we again use the Weyl tensor to find a more general form for relationship between the metric potentials.

$$
\mathrm{e}^{\nu} = \begin{cases} Ar\exp\left(\sqrt{1+k}\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right) & \\ \quad +Br\exp\left(-\sqrt{1+k}\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right) & \text{, where } 1+k>0 \\ Ar\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + Br & \text{, where } 1+k=0 \\ Ar\exp\left(\sqrt{-(1+k)}\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right) & \\ \quad +Br\exp\left(-\sqrt{-(1+k)}\displaystyle\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r\right) & \text{, where } 1+k<0, \end{cases}
$$

for both conformally flat ($k = 0$) and non-conformally flat ($k \neq 0$) solutions. This relationship is applicable for all values of k and therefore encompasses more solutions. We then found a general solution of the Einstein field equations for spacetimes with a conformal symmetry. Theorem 3.3.1 shows that the gravitational potentials can be expressed in terms of Δ, the degree of anisotropy. Any choice of Δ will lead to an exact solution. For isotropic pressure $\Delta = 0$, we found that particular parameter values lead to well-known solutions such as the Schwarzschild interior metric (1916), the Tolman IV (1939) metric, and several other line elements. We listed the known metrics with isotropic pressures and conformal symmetry in Table 3.4. When the pressure is isotropic we found that the Einstein system will always have the barotropic equation of state

$$p = p(\rho).$$

We picked out certain cases of the solution which yield a linear equation of state and tabulated these results in Tables 3.3. For anisotropic pressure we examined the work of Shee *et al.* (2016) and demonstrated that their choice of Δ is consistent with our general analysis.

- In Chapter 4, we analysed an astrophysical application of our work on conformal symmetries. We looked at conformally flat spacetimes as this guarantees regularity at the centre for the matter variables. We have shown that, subject to constraints on certain parameter values, the metric potentials and matter variables are well behaved. We generated stellar radii and masses of four pulsars, namely PSR J1614-2230, Vela X-1, PSR J1903+0327 and Cen X-3 by assuming a spacetime with static spherical symmetry. We proceeded as follows,

- Firstly, we varied both parameters b and n simultaneously. We found that as the mass decreases, the central density decreases. This behaviour is present in the investigations of Mafa Takisa *et al.* (2014a) and Mafa Takisa *et al.* (2014b) and Murad (2016).
- Secondly, the parameter n was fixed and b was allowed to vary. In this scenario the central density increases with a decrease in mass. The behaviour has been analysed by Sharma and Ratanpal (2013), Singh *et al.* (2016) and Bhar *et al.* (2015).

We have demonstrated that

- The adiabatic indices Γ_r and Γ_t profiles shown in Figure 4.8, indicate that $\Gamma_r > \frac{4}{3}$ and $\Gamma_t > \frac{4}{3}$ throughout the stellar structure for stability.
- The profiles of gravitational, hydrostatic and anisotropic forces plotted in Fig. 4.9, show that anisotropic and hydrostatic forces are positive and balanced by the negative gravitational force.
- The sum of all the forces is zero as required for the equilibrium of a static system.
- The gravitational redshift, plotted in Figure 4.12, has a monotonically decreasing profile.
- The upper bound of Buchdahl which is equivalent to $Z_s \leq 2$ for a realistic star is obeyed.

Our analysis is consistent with observation.

In conclusion this thesis has demonstrated the importance of conformal symmetries as a mathematical tool for simplifying the complex nonlinear Einstein field equations.

We then found exact solutions and showed that they yield physically reasonable models. With this general approach we hope to extend our results to a broader class of astrophysical objects in future work.

References

1. Ali, F., Feroze, T. and Ali, S. Complete classification of spherically symmetric static spacetimes via Noether symmetries. *Theoretical and Mathematical Physics* **184,** 973 (2015).

2. Banerjee, A., Rahaman, F., Islam, S. and Govender, M. Braneworld gravastars admitting conformal motion. *The European Physical Journal C* **76,** 34 (2016).

3. Bhar, P. Vaidya–Tikekar type superdense star admitting conformal motion in presence of quintessence field. *The European Physical Journal C* **75,** 123 (2015).

4. Bhar, P., Rahaman, F., Ray, S. and Chatterjee, V. Possibility of higher-dimensional anisotropic compact star. *The European Physical Journal C* **75,** 190 (2015).

5. Böhmer, C. G., Harko, T. and Lobo, F. S. N. Conformally symmetric traversable wormholes. *Physical Review D* **76,** 1 (2007).

6. Buchdahl, H. A. and Land, W. J. The relativistic incompressible sphere. *Journal of the Australian Mathematical Society* **8,** 6 (1968).

7. Buchdahl, H. A. General relativistic fluid spheres. *Physical Review* **116,** 1027 (1959).

8. Castejon-Amenedo, J. and Coley, A. A. Exact solutions with conformal Killing vector fields. *Classical and Quantum Gravity* **9,** 2203 (1992).

9. Coley, A. A. and Tupper, B. O. J. Spacetimes admitting inheriting conformal Killing vector fields. *Classical and Quantum Gravity* **7,** 1961 (1990).

10. Coley, A. A. and Tupper, B. O. J. Spherically symmetric spacetimes admitting inheriting conformal Killing vector fields. *Classical and Quantum Gravity* **7,** 2195 (1990).

11. Coley, A. A. and Tupper, B. O. J. Spherically symmetric anisotropic fluid ICKV spacetimes. *Classical and Quantum Gravity* **11,** 2553 (1994).

12. Delgaty, M. S. R. and Lake, K. Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein's equations. *Computer Physics Communications* **115,** 395 (1998).

13. Dev, K. and Gleiser, M. Anisotropic stars: exact solutions. *General Relativity and Gravitation* **34,** 1793 (2002).

14. Durgapal, M. C. and Bannerji, R. New analytical stellar model in general relativity. *Physical Review D* **27,** 328 (1983).

15. Esculpi, M. and Aloma, E. Conformal anisotropic relativistic charged fluid spheres with a linear equation of state. *The European Physical Journal C* **67,** 521 (2010).

16. Fatema, S. and Murad, M. H. An exact family of Einstein–Maxwell Wyman–Adler solution in general relativity. *International Journal of Theoretical Physics* **52,** 2508 (2013).

17. Gangopadhyay, T., Ray, S., Li, X.-D., Dey, J. and Dey, M. Strange star equation of state fits the refined mass measurement of 12 pulsars and predicts their radii. *Monthly Notices of the Royal Astronomical Society* **431,** 3216 (2013).

18. Gleiser, M. and Dev, K. Anistropic Stars: Exact Solutions and Stability. *International Journal of Modern Physics D* **13,** 1389 (2004).

19. Gokhroo, M. K. and Mehra, A. L. Anisotropic spheres with variable energy density in general relativity. *General Relativity and Gravitation* **26,** 75 (1994).

20. Grøn, Ø. and Johannesen, S. Conformally flat spherically symmetric spacetimes. *The European Physical Journal Plus* **128,** 92 (2013).

21. Hansraj, S., Maharaj, S. D., Msomi, A. M. and Govinder, K. S. Lie symmetries for equations in conformal geometries. *Journal of Physics A: Mathematical and General* **38,** 4419 (2005).

22. Herrera, L. Cracking of self-gravitating compact objects. *Physics Letters A* **165,** 206 (1992).

23. Herrera, L., Di Prisco, A. and Ibanez, J. Reversible dissipative processes, conformal motions and Landau damping. *Physics Letters A* **376,** 899 (2012).

24. Herrera, L., Jimenez, J., Leal, L., Ponce de León, J., Esculpi, M. and Galina, V. Anisotropic fluids and conformal motions in general relativity. *Journal of Mathematical Physics* **25,** 3274 (1984).

25. Herrera, L. and Ponce de León, J. Isotropic and anisotropic charged spheres admitting a one-parameter group of conformal motions. *Journal of Mathematical Physics* **26,** 2302 (1985).

26. Herrera, L. and Ponce de León, J. Anisotropic spheres admitting a one-parameter group of conformal motions. *Journal of Mathematical Physics* **26,** 2018 (1985).

27. Herrera, L. and Ponce de León, J. Perfect fluid spheres admitting a one-parameter group of conformal motions. *Journal of Mathematical Physics* **26,** 778 (1985).

28. Herrera, L., Prisco, A. D., Ospino, J. and Fuenmayor, E. Conformally flat anisotropic spheres in general relativity. *Journal of Mathematical Physics* **42,** 2129 (2001).

29. Keane, A. J. and Barrett, R. K. The conformal group SO (4, 2) and Robertson-Walker spacetimes. *Classical and Quantum Gravity* **17,** 201 (2000).

30. Keane, A. J. and Tupper, B. O. J. Conformal symmetry classes for pp-wave spacetimes. *Classical and Quantum Gravity* **21,** 2037 (2004).

31. Kiess, T. E. Exact physical Maxwell-Einstein Tolman-VII solution and its use in stellar models. *Astrophysics and Space Science* **339,** 329 (2012).

32. Kileba Matondo, D., Mafa Takisa, P., Maharaj, S. D. and Ray, S. Prediction of stellar masses with Finch and Skea geometry. *Astrophysics and Space Science* **362,** 186 (2017).

33. Korkina, M. P. and Orlyanskii, O. Y. A method for generating spherically symmetric static solutions of general relativity theory equations. *The Ukrainian Journal of Physics* **36,** 885 (1991).

34. Krori, K. D., Borgohain, P., Das, K. and Sarma, A. Charged matter and a one-parameter group of conformal motions: I. *Canadian Journal of Physics* **64,** 882 (1986).

35. Krori, K. D., Borgohain, P., Das, K. and Sarma, A. Charged matter and a one-parameter group of conformal motions: II. *Canadian Journal of Physics* **64,** 887 (1986).

36. Kuchowicz, B. A generalization of Tolman's solution VI of Einstein's field equations for a fluid sphere. *Physics Letters A* **24,** 221 (1967).

37. Maartens, R. and Maharaj, M. S. Conformally symmetric static fluid spheres. *Journal of Mathematical Physics* **31,** 151 (1990).

38. Maartens, R. and Maharaj, S. D. Conformal Killing vectors in Robertson-Walker spacetimes. *Classical and Quantum Gravity* **3,** 1005 (1986).

39. Maartens, R. and Maharaj, S. D. Conformal symmetries of pp-waves. *Classical and Quantum Gravity* **8,** 503 (1991).

40. Maartens, R., Mason, D. P. and Tsamparlis, M. Kinematic and dynamic properties of conformal Killing vectors in anisotropic fluids. *Journal of Mathematical Physics* **27,** 2987 (1986).

41. Maartens, R., Maharaj, S. D. and Tupper, B. O. J. General solution and classification of conformal motions in static spherical spacetimes. *Classical and Quantum Gravity* **12,** 2577 (1995).

42. Maartens, R., Maharaj, S. D. and Tupper, B. O. J. Conformal motions in static spherical spacetimes. *Classical and Quantum Gravity* **13,** 317 (1996).

43. Mafa Takisa, P., Maharaj, S. D. and Ray, S. Stellar objects in the quadratic regime. *Astrophysics and Space Science* **354,** 2120 (2014).

44. Mafa Takisa, P. and Maharaj, S. D. Some charged polytropic models. *General Relativity and Gravitation* **45,** 1951 (2013).

45. Mafa Takisa, P., Ray, S. and Maharaj, S. D. Charged compact objects in the linear regime. *Astrophysics and Space Science* **350,** 733 (2014).

46. Maharaj, S. D., Maartens, R. and Maharaj, M. S. Conformal symmetries in static spherically symmetric spacetimes. *International Journal of Theoretical Physics* **34,** 2285 (1995).

47. Maharaj, S. D. and Mafa Takisa, P. Regular models with quadratic equation of state. *General Relativity and Gravitation* **44,** 1419 (2012).

48. Mak, M. K. and Harko, T. Quark stars admitting a one-parameter group of conformal motions. *International Journal of Modern Physics D* **13,** 149 (2004).

49. Mallick, A., Karar, I., Rahaman, F. and Biswas, R. Three dimensional charged interior solutions admitting conformal Killing vectors. *International Journal of Theoretical Physics* **55,** 505 (2016).

50. Manjonjo, A. M., Maharaj, S. D. and Moopanar, S. Conformal vectors and stellar models. *The European Physical Journal Plus* **132,** 62 (2017).

51. Mason, D. P. and Maartens, R. Kinematics and dynamics of conformal collineations in relativity. *Journal of Mathematical Physics* **28,** 2182 (1987).

52. Moopanar, S. and Maharaj, S. D. Conformal symmetries of spherical spacetimes. *International Journal of Theoretical Physics* **49,** 1878 (2010).

53. Murad, M. H. Some analytical models of anisotropic strange stars. *Astrophysics and Space Science* **361,** 20 (2016).

54. Murad, M. H. and Fatema, S. Some exact relativistic models of electrically charged self-bound stars. *International Journal of Theoretical Physics* **52,** 4342 (2013).

55. Pandya, D. M., Thomas, V. O. and Sharma, R. Modified Finch and Skea stellar model compatible with observational data. *Astrophysics and Space Science* **356,** 285 (2015).

56. Patwardhan, G. K. and Vaidya, P. C. Relativistic distributions of matter of radial symmetry. *Journal of the University of Bombay* **12,** 23 (1943).

57. Radinschi, I., Rahaman, F., Kalam, M. and Chakraborty, K. Classical electron model with non static conformal symmetry. *Fizika B* **19,** 125 (2010).

58. Rahaman, F., Jamil, M., Kalam, M., Chakraborty, K. and Ghosh, A. On role of pressure anisotropy for relativistic stars admitting conformal motion. *Astrophysics and Space Science* **325,** 137 (2010).

59. Rahaman, F., Jamil, M., Sharma, R. and Chakraborty, K. A class of solutions for anisotropic stars admitting conformal motion. *Astrophysics and Space Science* **330,** 249–256 (2010).

60. Rahaman, F., Maharaj, S. D., Sardar, I. H. and Chakraborty, K. Conformally symmetric relativistic star. *Modern Physics Letters A* **32,** 1750053 (2017).

61. Rahaman, F., Paul, N., De, S. S., Ray, S. and Jafry, M. A. K. The Finslerian compact star model. *The European Physical Journal C* **75,** 564 (2015).

62. Rahaman, F., Pradhan, A., Ahmed, N., Ray, S., Saha, B. and Rahaman, M. Fluid sphere: Stability problem and dimensional constraint. *International Journal of Modern Physics D* **24,** 1550049 (2015).

63. Rahaman, F., Ray, S., Karar, I., Fatima, H. I., Bhowmick, S. and Ghosh, G. K. Static charged fluid in (2+ 1)-dimensions admitting conformal Killing vectors. *International Journal of Modern Physics D* **23,** 1450042 (2014).

64. Rahaman, F., Sharma, R., Ray, S., Maulick, R. and Karar, I. Strange stars in Krori–Barua space-time. *The European Physical Journal C* **72,** 2071 (2012).

65. Ray, S., Usmani, A. A., Rahaman, F., Kalam, M. and Chakraborty, K. Electromagnetic mass model admitting conformal motion. *Indian Journal of Physics* **82,** 1191 (2008).

66. Ruderman, M. Pulsars: structure and dynamics. *Annual Review of Astronomy and Astrophysics* **10,** 427 (1972).

67. Saslaw, W. C., Maharaj, S. D. and Dadhich, N. An isothermal universe. *The Astrophysical Journal* **471,** 571 (1996).

68. Schwarzschild, K. *Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie* in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, Phys.-Math. Klasse, 424* (1916).

69. Sharma, R. and Ratanpal, B. S. Relativistic stellar model admitting a quadratic equation of state. *International Journal of Modern Physics D* **22,** 1350074 (2013).

70. Shee, D., Rahaman, F., Guha, B. K. and Ray, S. Anisotropic stars with non-static conformal symmetry. *Astrophysics and Space Science* **361,** 167 (2016).

71. Singh, K. N., Pant, N. and Govender, M. Some analytic models of relativistic compact stars. *Indian Journal of Physics* **90,** 1215 (2016).

72. Sunzu, J. M., Maharaj, S. D. and Ray, S. Quark star model with charged anisotropic matter. *Astrophysics and Space Science* **354,** 2131 (2014).

73. Tello-Llanos, R. A. Curvature and conformal collineations in presence of matter. *General Relativity and Gravitation* **20,** 765 (1988).

74. Tolman, R. C. Static solutions of Einstein's field equations for spheres of fluid. *Physical Review* **55,** 364 (1939).

75. Tupper, B. O. J., Keane, A. J. and Carot, J. A classification of spherically symmetric spacetimes. *Classical and Quantum Gravity* **29,** 145016 (2012).

76. Usmani, A. A., Rahaman, F., Ray, S., Nandi, K. K., Kuhfittig, P. K. F., Rakib, S. A. and Hasan, Z. Charged gravastars admitting conformal motion. *Physics Letters B* **701,** 388 (2011).