UNIVERSITY OF KWAZULU-NATAL

INVESTIGATION OF GRAVITATIONAL COLLAPSE OF GENERALIZED VAIDYA SPACETIMES

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INVESTIGATION OF GRAVITATIONAL COLLAPSE OF GENERALIZED VAIDYA SPACETIMES

by

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As the candidate's supervisors, we have approved this thesis for submission.

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Abstract

In this thesis, we study the gravitational collapse of generalized Vaidya spacetimes which describe a combination of lightlike and timelike matter fields, commonly known as Type I and Type II fields, respectively, in the context of the cosmic censorship conjecture. This conjecture suggests that singularities forming in gravitational collapse should always be covered by event horizons of gravity. Many studies have been made to establish this conjecture in a rigorous mathematical framework but it still remains an open problem. We develop a general mathematical framework to study the conditions on the mass function of generalized Vaidya spacetimes so that future directed nonspacelike geodesics can terminate at the singularity in the past. Our result generalizes earlier works on gravitational collapse. There exist classes of generalized Vaidya mass functions for which the collapse terminates with a locally naked central singularity. We calculate the strength of these singularities, to show that they are strong curvature singularities, and there can be no extension of spacetime through them. We then extend this analysis to higher dimensions and present sufficient conditions on the generalized Vaidya mass functions that will generate a locally naked singular end state. With specific examples, we show the existence of classes of mass functions that lead to a naked singularity in four dimensions, which gets covered on transition to higher dimensions. Hence for these classes of mass functions, cosmic censorship gets restored in higher dimensions, and the transition to higher dimensions restricts the set of initial data that results in a naked singularity.

Preface

The work described in this thesis was carried out in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Westville campus, from July 2013 to November 2015, under the supervision of Professor Sunil D. Maharaj and Doctor Rituparno Goswami.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

M. D. Mkenyeleye | November 2015

Declaration 1 - Plagiarism

I, Maombi Daud Mkenyeleye declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:

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Declaration 2 - Publications

The following publications are in support of this thesis.
(There were regular meetings between myself and my supervisors to discuss research work for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were written by myself with some input from both my supervisors).

Publication 1

Maombi D. Mkenyeleye, Rituparno Goswami, and Sunil D. Maharaj, *Gravitational collapse of generalised Vaidya spacetime*, Phys. D **90**, 064034 (2014). [Chapter 3]

Publication 2

Maombi D. Mkenyeleye, Rituparno Goswami, and Sunil D. Maharaj, *Is Consimic censorship restored in higher dimensions?*, Phys. Rev. D **92**, 024041 (2015). [Chapter 4]

Publication 3

Maombi D. Mkenyeleye, Rituparno Goswami, and Sunil D. Maharaj, *Covariant description of generalized Vaidya spacetime*, Phys. Rev. D (to be submitted) (2015). [Chapter 5]

Signed:

To

my lovely wife Neema L. Gwehela, our son Brighton J. Mkenyeleye and my mother Esther E. Balosha

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Contents

1 Introduction **1**

2 Spacetime manifold and Einstein field equations **8**

2.1 Introduction 8

2.2 The manifold model 8

2.3 Differentiable manifold 10

2.4 The metric tensor 11

2.5 Connection and covariant derivatives 12

2.6 Geodesics 13

2.7 Riemann and Ricci tensors 13

2.8 The Einstein tensor 15

2.9 Energy momentum tensor 16

2.10 The Einstein field equations 17

3 Gravitational collapse of generalized Vaidya spacetime **19**

3.1 Introduction 19

3.2 Generalized Vaidya spacetimes 20

3.3 Collapsing model 25

3.4 Conditions for locally naked singularity 28

3.4.1 Structure of the central singularity 29

3.4.2 Existence of outgoing nonspacelike geodesics 34

3.4.3 Apparent horizon 35

3.5 Strength of singularity 39

3.6 Some special sub-classes of generalized Vaidya spacetimes 40

4 Cosmic censorship in higher dimensions 45

4.1 Introduction 45

4.2 Higher dimensional generalized Vaidya spacetimes 47

4.3 Higher dimensional collapse model 49

4.4 Conditions for locally naked singularity 51

4.4.1 Existence of outgoing nonspacelike geodesics 51

4.4.2 Apparent horizon 52

4.4.3 Sufficient conditions 53

4.5 A general Laurent expandable mass function 54

4.5.1 Example: Class of naked singularity in 4D being eliminated in higher dimensions 56

4.5.2 Proof of existence of open set of mass functions with the above properties 57

4.5.3 Numerical verification 59

5 Covariant description of generalized Vaidya spacetime 61

5.1 Introduction 61

5.2 Covariant formalisms 62

5.2.1 LRS spacetimes 65

5.2.2 LRS class II spacetimes 65

5.3 Generalized Vaidya spacetime 67

6 Conclusions **70**

References **73**

List of Tables

3.1 Equations of tangents to the singularity curve X_0 and values of $\lim_{k\to 0} k^2\psi$ for some special sub-classes of generalized Vaidya spacetime 44

4.1 Algebraic equations for X_0 and X_{AH} for different values of n and N . . 55

4.2 Values of X_0 and Min$[X_{AH}]$ for different dimensions for $\lambda_1 = 5.0$, $\lambda_2 = 0.01$, $\lambda_3 = 2.3$, $\lambda_4 = 0.05$. 57

4.3 Range for X_0 and Min$[X_{AH}]$ for different dimensions: $\{\lambda_1 = 5.2, 0.009 < \lambda_2 < 0.012, \lambda_3 = 2.3, 0 \leq \lambda_4 < 0.4\}$ 58

4.4 Range for X_0 and Min$[X_{AH}]$ for different dimensions: $\{4.8 < \lambda_1 < 5.25, 0.009 < \lambda_2 < 0.012, 2.25 < \lambda_3 < 2.38, \lambda_4 = 0.05\}$ 59

List of Figures

1.1 Gravitational collapse of the matter cloud 3

3.1 Variation of $X_0 = b+$ (red) and $\left(\frac{dv}{dr}\right)_{AH}$ (green) with m'_0 at a fixed value $\dot{m}_0 = 0.0015$ 37

3.2 Variation of $X_0 = b-$ (blue) and $\left(\frac{dv}{dr}\right)_{AH}$ (green) with m'_0 at a fixed value $\dot{m}_0 = 0.0015$ 38

Chapter 1

Introduction

When a massive star has exhausted its nuclear fuel which provides a balance against the internal pull of gravity, such a star undergoes continual gravitational collapse. In this situation, gravity dominates other forces of nature, particularly the weak and strong nuclear forces in such a way that the entire matter cloud collapses and shrinks under the force of its own gravity. The dynamical evolution of gravitational collapse is governed by the Einstein's field equations (EFE). The study of collapse dynamics of the matter clouds is very important because it is this study that would decide whether the ultra-strong gravity regions are visible or not to the external universe. The visibility of these regions is usually determined by the formation of the event horizon during the collapse evolution. If the event horizons of gravity develop before the spacetime singularity forms, then these regions are hidden from the external observers, and this results in a black hole formation. Contrarily, if such horizons are delayed or fail to develop during the process of gravitational collapse, as governed by the internal dynamics of the collapsing cloud, then the naked singularity forms and these extreme gravity regions are visible to the external universe.

The questions then remains about what is the final fate of this continual gravitational collapse. There have been extensive studies on the gravitational collapse and its final end state. Several collapse models developed provide useful insights into the final fate of a massive star within the framework of Einsteins theory of gravity in the past two decades. Many of these models conclude that the final outcome of the continual collapsing star depends on the initial mass of the star on the ground that the gravitational collapse evolves and develops from regular initial data, defined on an initial surface of the collapsing matter.

Landau (1932) pointed out that there exists a critical mass in the quantum theory above which greater masses must collapse to a point. This conclusion came after the discovery of the Schwarzschild solution to the Einstein's field equations. Then, Chandrashekhar (1934) developed a model in the quantum theory of white dwarfs, and pointed out that the life-history of a star of small mass must be different from that of a large mass in such a way that a large mass star cannot pass into the white-dwarf stage. A further question was about the amount of time that a collapsing star would take to settle down to its final end state. Oppenheimer & Snyder (1939) pointed out that for an observer comoving with the stellar matter, the total time of collapse of a spherically symmetric homogeneous and marginally bound dust cloud is finite, and that an external observer sees the star asymptotically shrinking to its gravitational radius. After this study, it was suggested that the spacetime settles to a vacuum Schwarzschild geometry at the end of the gravitational collapse and the central spacetime singularity is hidden by the event horizon which leads to a black hole.

Studies have also shown that a star, with a mass below two or three solar masses will, stabilize as a white dwarf or neutron star during the collapse as it loses some of its original mass. In these cases, after an initial collapse of the cloud when the star has exhausted its nuclear fuel, the star again stabilizes at a much smaller radius due to internal balancing forces provided by either electron or neutron degeneracy pressures. For heavier stars that are several solar masses, they may again settle to a neutron star final state if the star could throw away the excess mass in the process of its evolution. However, for more massive stars, none of the above internal pressures can achieve the required balance, and a continual gravitational collapse becomes inevitable. The collapse then must proceed towards creating a spacetime singularity, as predicted by general relativity theorems, which may be hidden within a black hole or may be visible to the external universe. In his book, Joshi (1993) defines a spacetime singularity as a region where the physical parameters such as mass, energy density, and the spacetime curvature go to their extreme values and blow up, so that the usual laws of physics break down at such a singularity. This can clearly be seen in Figure 1.1. In this region, the time scales and the length are comparable to the Planck scales and therefore the quantum theory combined with the effects of gravity must be taken into account. Some studies suggest that if the quantum gravity theory is correct, then naked singularities should exist in nature (Goswami *et al.* 2006; Goswami & Joshi 2007; Martin 2005). Raychaudhuri (1955) introduced a famous equation, commonly known as the

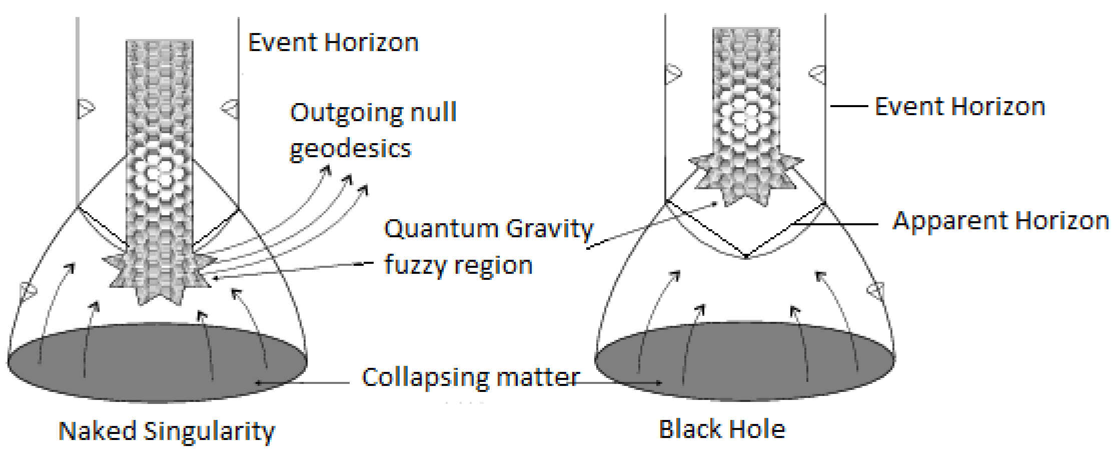

Figure 1.1: Gravitational collapse of the matter cloud

Raychaudhuri equation, to study the effects of gravitational field on timelike and null geodesics under general conditions. Before that, the spacetime singularity was considered to be a result of solving the EFE by assuming exact symmetry conditions. Based on the EFE, the Raychaudhuri equation attempted to study the behaviour of gravitating cloud under general conditions. However, several other extensive studies later showed that spacetime singularities are not a result of the exact symmetries assumed in solving EFE (Hawking & Ellis 1973). This analysis showed that under some general and physical conditions such as energy, the spacetime singularities is inevitable.

Despite all the spacetime singularity theorems developed so far, there was still a fundamental problem to whether the spacetime singularity developed during gravitational collapse would always be hidden below the event horizon or it would be communicated to the external observers. Penrose (1969) proposed a conjecture commonly known as the *Cosmic Censorship Conjecture (CCC)* stating that the singularities forming in gravitational collapse should always be covered by event horizons of gravity and remain invisible to any external observer. *"We are thus presented with what is perhaps the most fundamental unanswered question of general-relativistic collapse theory, namely: does there exist a cosmic censor who forbids the appearance of naked singularities, clothing each one in an absolute event horizon?...it is not known whether singularities observable from outside will ever arise in a generic collapse which starts off from a perfectly reasonable nonsingular initial state"*. The conjecture can be stated in two forms, the strong and weak forms. The strong form proposes that no null rays could emerge from the singularity in a reasonable spacetime, and hence it is invisible for all observers. That is, there occurs no naked singularity for any observer. The weak form of the CCC states that null rays can emerge from the singularity which is however covered by an event horizon and hence they cannot reach the external observer. In this case there is a possibility of forming a locally naked singularity, say for an observer sitting on the collapsing star, but it is globally not because it is hidden behind an event horizon. Despite many studies, this conjecture has not been established in a rigorous mathematical framework to confirm the already widely accepted and applied theory of black holes and dynamics.

Some models with counter examples have been formulated to show that there exist shell focusing naked singularities occurring at the centre of spherically symmetric dust, radiation shells or perfect fluids. Christodoulou (1984) and Newman (1986) established models in which the matter was assumed to be marginally or non-marginally bound dust, and the initial data functions are smooth profiles. In these models it was shown that there exist a naked singularity at the end state of continual gravitational collapse. These singularities can only, in principle, be ruled out by pointing out that perfect fluids or dust are not really fundamental forms of matter. However, since the formulation of these models is based on the assumption that the matter satisfies all the reasonable energy conditions, the ruling out of the above models should be made in terms of clear and simple restrictions on the energy momentum tensor so that the cosmic censorship is well established.

Though the general mathematical proof of this conjecture still remains indefinable, there are a number of important counter-examples that give light to the solution. Investigations of spherically symmetric dynamical collapse models in general relativity for large classes of matter fields in four dimensional spacetimes, indicate that there exist sets of initial data of non-zero measure at the epoch of the commencement of the collapse, that lead to the formation of a locally naked singularity. In these cases the trapped surfaces are delayed during the collapse process, i.e. they do not form early enough to shield the singularity (or the spacetime fireball) from external observers. It is also shown in these studies that there exist families of future outgoing non-spacelike geodesics that emerge from such a naked singularity, providing a non-zero measure set of trajectories escaping away (Baier *et al.* 2015; Joshi 2007; Lemos 1992). Though these examples are mainly presented in the case of spherical symmetry (with a few exceptions of non-spherical models), they can be considered to be relevant; if the censorship is one of the key aspects of gravitation theory, then it should not depend on symmetries of spacetime.

This thesis is organized as follows:

Chapter 1 is the introduction of this thesis where we give the general overview of gravitational collapse and spacetime singularities. With counter-examples, we explain the importance of this study and indicate how some studies in this field, since the introduction of cosmic censorship conjecture, have transformed the general understanding about the final fate of the collapsing star under its own gravity.

Chapter 2 gives a brief introduction to the spacetime manifold and field equations. We define a manifold, differentiable manifold and and some objects (tensors) that characterize a manifold.

In Chapter 3 we study the gravitational collapse of generalized Vaidya spacetimes in the context of the cosmic censorship conjecture. We establish a mathematical framework to study how the tangent vectors behave near the central singularity. We find the *differential equation* that governs the behaviour of these tangent vectors near the central singularity. We analyse the nature of this equation using usual techniques of differential equations, where we give the conditions of the generalized mass function for which the singular points become a node and the outgoing non-spacelike geodesics can come out of the singularity with a definite value of the tangent. We use the contracting mapping to show the existence and uniqueness of the solution to this differential equation. We also calculate the equation of the apparent horizon. With a well chosen and defined generalized Vaidya mass function (i.e. the generalized Vaidya mass function that obeys the energy conditions), we show with a specific example that there exists a class of parameter values for which the apparent horizon is always above the event horizon and that the central singularity is locally naked. We show that this singularity is strong in the sense that no extension of spacetime is possible through them. We also recover some already known solutions of the generalized Vaidya spacetimes such as charged Vaidya, charged Vaidya de-Sitter and Husain solutions using the same differential equation near the central singularity.

In Chapter 4 we extend the analysis of the gravitational collapse of generalized Vaidya spacetimes to arbitrary N-dimensional spacetimes. Though the general mathematical framework remains similar to that in the previous chapter, the conditions on the mass function and it's derivatives for the collapse leading to a locally naked singularity, change as we make a transition to higher dimensional spacetimes. Using explicit examples we show that there exist classes of mass functions, that lead the collapsing star to a naked singularity in four dimensions, will necessarily end in a black hole end state in dimensions greater than four. The reason for this remains the same as in dust models: formation of trapped surfaces is favoured in higher dimensions, and hence the vicinity of the central singularity gets trapped even before the singularity is formed. This gives a definite indication that the dynamics of trapped regions do depend on the spacetime dimensions for a large class of matter fields and the occurrence of trapped surfaces advance in time in higher dimensions.

Chapter 5 is about the covariant description of the generalized Vaidya spacetime, where we calculate quantities that entirely define the spacetime covariantly, with respect to the gravitational collapse.

Chapter 6 is the conclusion of this thesis, where we give a short summary and outlook of the main results of this study.

Chapter 2

Spacetime manifold and Einstein field equations

2.1 Introduction

As stated in Chapter 1, the end state of the collapsing star, whether locally naked or a black hole is usually determined by the initial data of the collapsing matter within Einstein's theory of gravity. In this chapter we introduce and give some preliminaries on the spacetime manifold, the objects characterising it, called tensors, and the Einstein field equations governing the dynamics of matter in spacetime in the context of general relativity.

2.2 The manifold model

A manifold $\mathcal{M}$ of dimension n (or $n-$manifold) is a topological space which is Hausdorff, locally Euclidean and has a countable basis of open sets (Boothby 1986). In general relativity, the universe is usually modelled as a four-dimensional spacetime manifold $\mathcal{M}$ together with a Lorentzian metric tensor $\mathbf{g}$ (defined in section 2.4).

However, recently, several studies have focused on studying the possible existence of spacetimes in more than four dimensions and the observational consequences to both cosmological and black holes contexts. This has mainly resulted from different approaches in studying particle physics to the unification of all forces including gravity. Some theories such as Kaluza-Klein and string theories that address some gravitational issues in higher dimensions have emerged in recent times.

In Kaluza-Klein theories for instance, issues such as dimensional reduction through generalized Kasner solutions, solution to the vacuum field equations of general relativity in $4+1$ spacetime dimensions that leads to a cosmology which at the present epoch has $3+1$ observable dimensions in which the Einstein-Maxwell equations are obeyed (Chodos & Detweiler 1980), presence of entropy flow from the extra dimensions greater than the usual four to the main spacetime (Alvarez & Gavela 1983), the effects of thermal history of the early universe (Sahdev 1984), classification of the $11-$ dimensional classical homogeneous Kaluza-Klein cosmologies (Demianski *et al.* 1987) and so on, were addressed.

On the other hand, string theory emerged in the late 1960s as a result of attempts to understand the strong nuclear force, that is responsible for holding protons and neutrons together inside the nucleus of an atom as well as quarks together inside the protons and neutrons. In this theory, a quantum theory in 11-dimensions, called M-theory, where two of the superstring theories (the type IIA superstring and the $E_8 \times E_8$ heterotic string) exhibit an eleventh dimension at strong coupling, emerged. At low energies the M-theory is approximated by a classical field theory called 11-dimensional supergravity (Berker *et al.* 2007; Green *et al.* 1987). Also in this theory the issue of dealing with the extra dimensions called the *brane-world* scenario was addressed. In this approach the four dimensions are identified with a defect embedded in a higher-dimensional spacetime (Berker *et al.* 2007).

Several other works in dimensions greater than four include the solutions of spherically symmetric spacetimes in higher dimensions such as Schwarzschild and Reisnner-

Nordström black holes (Chodos & Detweiler 1980), thermodynamics and Hawking radiation (Myers and Perry 1986), the generalization of the rotating Kerr black hole (Frolov *et al.* 1987; Mazur and Bombelli 1987; Myers and Perry 1986), higher-dimensional black holes in compactified spacetime (Myers 1987), and the generalization of the Vaidya metric in higher dimensions (Iyer & Vishveshwara 1989). Other works done in the context of higher dimensions, particularly in gravitational collapse, are discussed in Chapter 4.

The manifold model for the universe naturally incorporates the observed continuity of space and time at the classical level, and the basic principle of general relativity where the locally flat regions combine to produce a globally curved continuum in such a way that we can make a transition from one coordinate system to another (Joshi 2007).

2.3 Differentiable manifold

If $\mathbb{R}^n$ denotes the Euclidean space of n dimensions, that is, a set of all $n-$tuples $(x^1, x^2, \cdots, x^n)$ such that $-\infty < x^i < \infty$, $i = 1, 2, \cdots n$, with the usual topology of the open interval, and $\frac{1}{2}\mathbb{R}^n$ denotes the lower half of $\mathbb{R}^n$ with $x^1 \leq 0$ then:

Definition 2.3.1. A map ϕ from an open set $\mathcal{O} \in \mathbb{R}^n$ to an open set $\mathcal{O}' \in \mathbb{R}^m$ is said to be of class C^r if the coordinates (n-tuples) $(x^{1'}, x^{2'}, \cdots, x^{n'})$ of the image point $\phi(p)$ in $\mathcal{O}'$ are $r-$times continuous differentiable functions (r^{th} derivatives exist and are continuous) of the coordinates $(x^1, x^2, \cdots, x^n)$ of p in $\mathcal{O}$. An n-dimensional differentiable manifold is simply a set that is locally similar to an open set of $\mathbb{R}^n$.

Definition 2.3.2. If a map is C^r for all $r \geq 0$, then it said to be C^∞ map.

Definition 2.3.3. A function f from an open set $\mathcal{O}$ of $\mathbb{R}^n$ to $\mathbb{R}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be locally Lipschitz, if for each open set $u \in \mathcal{O}$ with compact closure, there is some constant L, such that for each pair of points $p, q \in u$, $|f(p) - f(q)| \leq L\,|p - q|$.

2.4 The metric tensor

The metric tensor defines a notion of distance between any two infinitesimally separated points of the spacetime manifold. It is a tensor that acts on pairs of vectors to give a number, and is symmetric in its indices. In terms of coordinate basis, the metric tensor is defined as

$$\mathbf{g} = g_{ab} dx^a \otimes dx^b, \tag{2.1}$$

where $g_{ab} = g(\partial/\partial x^a, \partial/\partial x^b)$. For any two vectors $\mathbf{V}$ and $\mathbf{W}$, we can write this as $g(\mathbf{V}, \mathbf{W}) = g_{ab} V^a W^b$ (Joshi 2007), which can be written in the form of a distance between two infinitesimally separated points in spacetime as

$$ds^2 = g_{ab} dx^a dx^b. \tag{2.2}$$

The matrix $[g_{ab}]$ is nonsingular with inverse g^{ab} such that

$$g_{ab} g^{bc} = \delta_a{}^c, \tag{2.3}$$

where $\delta_a{}^c$ is the Kronecker delta. If $[g^{ij}]$ defines an entry of the matrix $\left[g^{ab}\right]$, we can compute each entry using the formula

$$g^{ij} = \frac{\mathcal{G}(ij)}{\det \mathbf{g}}, \tag{2.4}$$

where $\mathcal{G}(ij)$ is the cofactor of $[g_{ij}]$. The tensors g^{ab} and g_{ab} can be used to define the relationships between the covariant and the contravariant vectors as

$$X_a = g_{ab} X^b, \qquad X^a = g^{ab} X_b. \tag{2.5}$$

For a second rank tensor $\mathbf{T}$, the relationships are given as

$$T_{ab} = g_{ac} g_{bd} T^{cd}, \qquad T^{ab} = g^{ac} g^{bd} T_{cd}, \qquad T^a{}_b = g^{ac} T_{cb}. \tag{2.6}$$

The metric is indefinite in the sense that the magnitude of a nonzero vector could be positive, negative or zero. The vector $\mathbf{X} \in T_p$ is called *timelike, null*, or *spacelike* if

$$g(\mathbf{X}, \mathbf{X}) < 0, \qquad g(\mathbf{X}, \mathbf{X}) = 0, \qquad \text{or} \qquad g(\mathbf{X}, \mathbf{X}) > 0 \tag{2.7}$$

respectively (Joshi 2007), where T_p is a tangent space at a point $p \in \mathcal{M}$. The *signature* of the metric g at a point p is the difference between the number of positive eigenvalues and the number of negative eigenvalues. For a nondegenerate and continuous metric tensor, the signature is constant on the entire manifold. A four dimensional manifold has signature $(-, +, +, +)$.

2.5 Connection and covariant derivatives

For a metric tensor $\mathbf{g}$, it is possible to have a unique torsion-free connection ∇ which preserves the metric such that

$$\nabla \mathbf{g} = 0 \qquad \text{or} \qquad g_{ab;c} = 0, \tag{2.8}$$

where ; denotes a covariant derivative. The special connection called the *Levi-Cevita* connection with the coefficients known as *Christoffel symbols* is defined by

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \left(\frac{\partial g_{bd}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d} \right) \equiv \frac{1}{2} g^{cd} (g_{bd,a} + g_{ad,b} - g_{ab,d}) \tag{2.9}$$

(Boothby 1986). This connection is *symmetric* in its lower indices, i.e. $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$. The Christoffel symbols contain all the information about the curvature of the coordinate system and can therefore be transformed to zero when a suitable coordinate transformation is chosen, therefore it is not a tensor. However, it can be considered as any other ordinary tensor in terms of index notation.

For a vector field X^a, the covariant derivative is defined as

$$\nabla_b X^a = \partial_b X^a + \Gamma^a{}_{bc} X^c, \qquad \nabla_b X_a = \partial_b X_a - \Gamma^c{}_{ab} X_c. \tag{2.10}$$

For a tensor T^{ab}, we then have

$$\nabla_c T^{ab} = \partial_c T^{ab} + \Gamma^a{}_{cd} T^{db} + \Gamma^b{}_{cd} T^{ad}, \tag{2.11}$$

and for a mixed tensor $T^a_{\ b}$ we have

$$\nabla_c T^a_{\ b} = \partial_c T^a_{\ b} + \Gamma^a{}_{cd} T^d_{\ b} - \Gamma^d{}_{ab} T^a_{\ d}. \tag{2.12}$$

2.6 Geodesics

In Euclidean space, a *geodesic* is a straight line, defined by two equivalent properties. First, its tangent vector always points in the same direction (along the line) and, second, it is the curve of shortest length between two points (Hobson *et al.* 2006). In the general torsion-free manifold however, the geodesic is considered as a curve $x^a(u)$ described by a parameter u by the fixed direction of its tangent vector $\mathbf{t}(u)$ and satisfies the condition

$$\frac{d\mathbf{t}}{du} = \lambda(u)\mathbf{t}, \tag{2.13}$$

where $\lambda(u)$ is a parametric function of u. Generally, the equations satisfied by both null and non-null geodesics parametrized by a parameter u are given by

$$\frac{d^2x^a}{du^2} + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = \lambda(u)\frac{dx^a}{du}. \tag{2.14}$$

The curve can be parametrized in such a way that $\lambda(u)$ vanishes. Then the geodesics are defined by the equations

$$\frac{d^2x^a}{du^2} + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0, \tag{2.15}$$

where u is called an *affine parameter*. A geodesic in $(\mathcal{M}, g)$ is *timelike*, *spacelike*, or *null* if its tangent vector is timelike, spacelike, or null respectively.

2.7 Riemann and Ricci tensors

The *Riemann* or the *Curvature* tensor, $R^a{}_{bcd}$, is a tensor of the fourth rank defined as

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}. \tag{2.16}$$

where $\Gamma^a{}_{bd,c} = \partial_c\Gamma^a{}_{bd}$. The curvature tensor has symmetry properties which can be observed by changing from mixed components $R^a{}_{bcd}$ to covariant components $R_{abcd} = g_{ae}R^e{}_{bcd}$. By simple transformation, it can easily be seen that

$$R_{abcd} = -R_{bacd} = -R_{abdc}, \tag{2.17}$$

$$R_{abcd} = R_{cdab}. \tag{2.18}$$

Thus, the tensor is antisymmetric in each of the index pairs a, b, c and d, and is symmetric under the interchange of any two pairs with one another. The cyclic sum of the components of R_{abcd}, obtained by permutation of any three indices, is equal to zero; for example,

$$R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{2.19}$$

The curvature tensor can also be used to prove the following *Bianchi identity*

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0. \tag{2.20}$$

The *Ricci* tensor is a tensor of the second rank formed by contracting the curvature tensor. The Ricci tensor can therefore be defined as

$$R_{ab} = g^{dc} R_{dacb} = R^c{}_{acb}. \tag{2.21}$$

According to (2.16), we have

$$R_{ab} = R^c{}_{acb} = \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{ab}\Gamma^e{}_{ce} - \Gamma^c{}_{be}\Gamma^e{}_{ac}. \tag{2.22}$$

The Ricci tensor is symmetric, i.e.

$$R_{ab} = R_{ba}. \tag{2.23}$$

If we contract the Ricci tensor, R_{ab}, we obtain the invariant

$$R = g^{ab} R_{ab} = g^{ac} g^{bd} R_{abcd}, \tag{2.24}$$

which is called the *scalar curvature* or *Ricci scalar* of space. The *Gaussian curvature* K, also known as the total curvature (Kreyszig 1991), of a two-dimensional surface, is an intrinsic property of the space independent of the coordinate system used to describe it. The curvature of a two-dimensional surface is defined by (Hobson *et al.* 2006)

$$K = \frac{R_{1212}}{\det \mathbf{g}}. \tag{2.25}$$

For a 2−surface sphere metric defined by $ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$, the Gaussian curvature is given by $K = r^{-2}$.

2.8 The Einstein tensor

The Einstein tensor, named after Albert Einstein, is used to express the curvature of a manifold. In general relativity, the Einstein tensor occurs in the Einstein field equations for gravitation describing spacetime curvature in a manner consistent with energy considerations. It is defined in terms of the Ricci and metric tensors as

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \tag{2.26}$$

The Einstein tensor can easily be derived from the Bianchi identity (2.20), which can also be written as

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{2.27}$$

Raising a and contracting with d gives

$$\nabla_e R_{bc} + \nabla_c R^a{}_{bae} + \nabla_a R^a{}_{bec} = 0. \tag{2.28}$$

On using the antisymmetry property (2.17) in the second term we get

$$\nabla_e R_{bc} - \nabla_c R_{be} + \nabla_a R^a{}_{bec} = 0. \tag{2.29}$$

If we now raise b and contract with e, we find that

$$\nabla_b R^b{}_c - \nabla_c R + \nabla_a R^{ab}{}_{bc} = 0. \tag{2.30}$$

Using the antisymmetric properties (2.17), the third term may be written as

$$\nabla_a R^{ab}{}_{bc} = \nabla_a R^{ba}{}_{cb} = \nabla_a R^a{}_c = \nabla_b R^b{}_c. \tag{2.31}$$

It can be seen that the first and last terms in (2.30) are identical, so that

$$2\nabla_b R^b{}_c - \nabla_c R = \nabla_b (2R^b{}_c - \delta^b{}_c R) = 0. \tag{2.32}$$

Finally, raising the index c, we obtain the result

$$\nabla_b (R^{bc} - \frac{1}{2} g^{bc} R) = 0. \tag{2.33}$$

The term in brackets is called the *Einstein tensor* and is denoted as

$$G^{ab} \equiv R^{ab} - \frac{1}{2} g^{ab} R. \tag{2.34}$$

Similarly,

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R. \tag{2.35}$$

2.9 Energy momentum tensor

The energy momentum tensor (also known as stress energy tensor), T_{ab}, is an attribute of matter, radiation, and non-gravitational force fields in the spacetime. It is the source of the gravitational field in the Einstein field equations. The energy momentum tensor, T^{ab} can be described as the flux of the a-th component of four-momentum across a surface of constant x^b, such that:

- T^{00} is the flux of $0-$th component of four-momentum (energy) across the time surface (x^0), called the *energy density.*
- $T^{0i} = T^{i0}$ is the energy flux across surface of constant x^i, called the *heat conduction.*
- T^{ij} is the flux of $i-$momentum across the j-surface, called the *stress.*
- T^{ii} is the *pressure* in the i-th direction (no sum over i).

For example: A *perfect fluid* (a fluid with no heat conduction and viscosity and moves through spacetime with constant four-velocity u^a with respect to any inertial frame), when considered in the 'instantaneous rest frame', is uniquely characterised by its rest energy density ρ and rest isotropic pressure p, *i.e*, $T^{00} = \rho$, $T^{0i} = T^{i0} = 0$ and $T^{ij} = p\delta^{ij}$

(for T^{ij} to be diagonal for any orientation of axes). Thus,

$$T^{ab} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{2.36}$$

where c is the speed of light. This can generally be written as

$$T^{ab} = \left(\rho + \frac{p}{c^2}\right) u^a u^b + p g^{ab}, \tag{2.37}$$

which can be considered as the definition of a perfect fluid in general relativity (Landau & Lifshitz 1971). The conservation of energy momentum tensor throughout the manifold implies that $\nabla_a T^{ab} = 0$.

2.10 The Einstein field equations

The *Einstein field equations*, also called the gravitational field equations, were derived by Albert Einstein (1916). These equations describe how matter and energy (described by the energy momentum tensor T_{ab}) curve the geometry of the manifold. As we have already seen, the conservation of the energy tensor gives

$$\nabla_a G^{ab} = 0 = \nabla_a T^{ab}. \tag{2.38}$$

Hence, it is natural to equate these two equations to obtain

$$R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab}, \tag{2.39}$$

where κ is a constant given by $\kappa = \frac{8\pi G}{c^4}$, G is a gravitational constant and c is the speed of light. Equations (2.39) are known as *Einstein field equations* (EFE). An alternative form of EFE can be obtained by writing equation (2.39) in terms of mixed components,

$$R^a{}_b - \frac{1}{2} \delta^a{}_b R = \kappa T^a{}_b, \tag{2.40}$$

and contracting by setting $a = b$. This gives $R = -\kappa T$ (in four dimensions), where $T \equiv T^a{}_a$. Therefore, equations (2.39) can be written as

$$R_{ab} = \kappa(T_{ab} - \frac{1}{2}Tg_{ab}). \tag{2.41}$$

Since T_{ab} in general, contains all forms of energy and momentum in the matter fields (including electromagnetic radiation if present), then for an *empty region* (a region of spacetime in which $T_{ab} = 0$), the gravitational field equations are given by

$$R_{ab} = 0. \tag{2.42}$$

In four dimensions, g_{ab} has ten independent components. Hence, the gravitational field equations give a set of ten nonlinear differential equations and twenty independent components of the curvature tensor R_{abcd} that describe the fundamental interaction of gravitation of matter and energy. This shows that the field equations can be satisfied in empty space with a nonvanishing curvature tensor, and thus gives a conclusion that gravitational fields can exist in empty space in four or more dimensions only.

The Einstein field equations as derived here are however not unique because if we add Λg_{ab} (where Λ is a universal constant of nature, commonly known as the *cosmological constant*) to either G_{ab} or T_{ab}, they will continue to be divergence-free. Therefore, these equations can generally be written as

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = \kappa T_{ab}. \tag{2.43}$$

Chapter 3

Gravitational collapse of generalized Vaidya spacetime

3.1 Introduction

The Vaidya spacetime, also known as the radiating Schwarzschild spacetime, describes the geometry outside a radiating spherically symmetric star (Vaidya 1951). The radiation effects are important in the later stages of gravitational collapse of a star, when a considerable amount of energy in the form of photons or neutrinos is ejected from the star. This makes the collapsing star to be surrounded by an ever expanding zone of radiation. If we treat the complete nonstatic configuration of the radiating star and the zone of radiation as an isolated system within an asymptotically flat universe, then beyond the expanding zone of radiation the spacetime may be described by the Schwarszchild solution. The Vaidya solution is of Petrov type D and possesses a normal shear-free null congruence with nonzero expansion. In terms of exploding (imploding) null coordinates the metric is given as

$$ds^2 = -\left[1 - \frac{2m(v)}{r}\right] dv^2 + 2\epsilon dvdr + r^2 d\Omega^2, \tag{3.1}$$

where $\epsilon = \pm 1$ describes incoming (outgoing) radiation shells respectively, the function '$m(v)$' is the mass function and $d\Omega^2$ describes the line element on the 2-sphere.

One of the earliest counterexamples of the Cosmic Censorship Conjecture (CCC), with a reasonable matter field satisfying physically reasonable energy conditions, was found in the shell focussing singularity formed by imploding shells of radiation in the Vaidya-Papapetrou model (Dwivedi & Joshi 1989; Papapetrou 1985). In this case, radially injected radiation flows into an initially flat and empty region, and is focussed into a central singularity of growing mass. It was shown that the central singularity at ($v = 0$, $r = 0$) becomes a node with definite tangent for families of nonspacelike geodesics, for a non-zero measure of parameters in the model. Hence the singularity at ($v = 0$, $r = 0$) is naked in the sense that families of future directed nonspacelike geodesics going to future null infinity terminate at the singularity in the past. The existence of naked singularities is important because it would be possible for external observers to observe the gravitational collapse of a star to infinite density. This would help to address some foundational problems in general relativity since it cannot make predictions about the future evolution of spacetime near a singularity. This is not the problem in generic black holes as an outside observer cannot observe the spacetime within the event horizon. For a detailed discussion on the censorship violation in radiation collapse we refer to Joshi (1993).

3.2 Generalized Vaidya spacetimes

The generalization of the Vaidya solution, also known as the generalized Vaidya spacetime, that includes all the known solutions of Einstein field equations with combination of Type I and Type II matter fields, was given by Wang and Wu (1999). This generalization comes from the observation that the energy momentum tensor for these matter fields are linear in terms of the mass function. As a result, the linear superposition of particular solutions is also a solution to the field equations. Hence, by superposition we can explicitly construct solutions such as the monopole-de Sitter-charged Vaidya solution and the Husain solution. Generalized Vaidya spacetimes are also widely used in describing the formation of regular black holes (Sean 2006), dynamical black holes (Dawood & Ghosh 2004) and black holes with closed trapped regions (Frolov 2014). Recently, it was shown that the generalized Vaidya model can be matched to a heat conducting interior of a radiating star (Maharaj *et al.* 2004). Also, the generalized Vaidya spacetime emerges naturally while solving many other astrophysical and cosmological scenarios (Alishahiha *et al.* 2014; Sungwook *et al.* 2010).

Unless otherwise specified, we use natural units ($c = 8\pi G = 1$) throughout this work, Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $(-,+,+,+)$ signature and the Riemann tensor defined by equation (2.16).

The Hilbert–Einstein action in the presence of matter is given by

$$\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{-g}\left[R - 2\Lambda - 2\mathcal{L}_m\right] \,, \tag{3.2}$$

variation of which also gives the Einstein field equations (2.39).

We know that the most general spherically symmetric line element for an arbitrary combination of Type I matter fields (whose energy momentum tensor has one timelike and three spacelike eigenvectors) and Type II matter fields (whose energy momentum tensor has double null eigenvectors) is given by (Barrabes & Israel 1991)

$$\begin{aligned} ds^2 &= -e^{2\psi(v,r)}\left[1 - \frac{2m(v,r)}{r}\right]dv^2 + 2\epsilon e^{\psi(v,r)}dvdr \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\phi^2),\ (\epsilon = \pm 1). \end{aligned} \tag{3.3}$$

Here '$m(v,r)$' is the mass function related to the gravitational energy within a given radius r (Lake & Zannias 1991). When $\epsilon = +1$, the null coordinate v represents the Eddington advanced time, where r is decreasing towards the future along a ray $v =$ constant and depicts ingoing null congruence while $\epsilon = -1$ depicts an outgoing null congruence.

The specific combination of matter fields that makes $\psi(v,r) = 0$ gives the generalized Vaidya geometry. In this thesis, as we are considering a collapse scenario, we take $\epsilon = +1$. Particularly, we consider a line element of the form

$$ds^2 = -\left(1 - \frac{2m(v,r)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{3.4}$$

We use the following definitions

$$\dot{m}(v,r) \equiv \frac{\partial m(v,r)}{\partial v}, \quad m'(v,r) \equiv \frac{\partial m(v,r)}{\partial r}. \tag{3.5}$$

We find the connections (Christoffel symbols) using equation (2.9). This gives the following nonvanishing components

$$\Gamma^v{}_{vv} = \frac{m(v,r)}{r^2} - \frac{m'(v,r)}{r}, \tag{3.6a}$$

$$\Gamma^v{}_{\theta\theta} = -r, \tag{3.6b}$$

$$\Gamma^v{}_{\phi\phi} = -r\sin^2\theta, \tag{3.6c}$$

$$\Gamma^r{}_{vv} = \frac{\dot{m}(v,r)}{r} + \frac{m(v,r)}{r^2} - \frac{2m(v,r)^2}{r^3} - \frac{m'(v,r)}{r} + \frac{2m(v,r)m'(v,r)}{r^2}, \tag{3.6d}$$

$$\Gamma^r{}_{vr} = \Gamma^r{}_{rv} = \frac{m'(v,r)}{r} - \frac{m(v,r)}{r^2}, \tag{3.6e}$$

$$\Gamma^r{}_{\theta\theta} = 2m(v,r) - r, \tag{3.6f}$$

$$\Gamma^r{}_{\phi\phi} = 2m(v,r)\sin^2\theta, \tag{3.6g}$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \tag{3.6h}$$

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \tag{3.6i}$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}, \tag{3.6j}$$

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta. \tag{3.6k}$$

The Ricci tensor is calculated using equation (2.22), which gives the following nonvanishing components

$$R_{vv} = \frac{2m(v,r)m''(v,r)}{r^2} - \frac{m''(v,r)}{r} + \frac{2\dot{m}(v,r)}{r^2}, \tag{3.7a}$$

$$R_{vr} = R_{rv} = \frac{m''(v,r)}{r}, \tag{3.7b}$$

$$R_{\theta\theta} = 2m'(v,r), \tag{3.7c}$$

$$R_{\phi\phi} = 2m'(v,r)\sin^2\theta. \tag{3.7d}$$

If we express the above components of the Ricci tensor in the form of mixed tensors $R^a{}_b$, we obtain

$$R^v{}_v = R^r{}_r = \frac{m''(v,r)}{r}, \quad (3.8a)$$
$$R^\theta{}_\theta = R^\phi{}_\phi = \frac{2m'(v,r)}{r^2}. \quad (3.8b)$$

The Ricci scalar is calculated using equation (2.24). This gives

$$R = \frac{2m''(v,r)}{r} + \frac{4m'(v,r)}{r^2}. \quad (3.9)$$

The Einstein tensor is defined by the formula in equation (2.26), from which we get the following nonvanishing components

$$G_{vv} = \frac{2\dot{m}(v,r)}{r^2} + \frac{2m'(v,r)}{r^2} - \frac{4m(v,r)m'(v,r)}{r^3}, \quad (3.10a)$$
$$G_{vr} = G_{rv} = -\frac{2m'(v,r)}{r^2}, \quad (3.10b)$$
$$G_{\theta\theta} = -rm''(v,r), \quad (3.10c)$$
$$G_{\phi\phi} = -rm''(v,r)\sin^2\theta. \quad (3.10d)$$

If we express the above components in the form of $G^a{}_b = g^{ac}G_{bc}$, we get

$$G^v{}_v = G^r{}_r = -\frac{2m'(v,r)}{r^2}, \quad (3.11a)$$
$$G^r{}_v = \frac{2\dot{m}(v,r)}{r^2}, \quad (3.11b)$$
$$G^\theta{}_\theta = G^\phi{}_\phi = -\frac{m''(v,r)}{r}. \quad (3.11c)$$

Using the Einstein field equations (2.39) ($\Lambda = 0$), the corresponding energy momentum tensor can be written in the form

$$T_{ab} = T^{(n)}_{ab} + T^{(m)}_{ab}, \quad (3.12)$$

where

$$T^{(n)}_{ab} = \vartheta l_a l_b, \quad (3.13a)$$
$$T^{(m)}_{ab} = (\rho + \varrho)(l_a k_b + l_b k_a) + \varrho g_{ab}, \quad (3.13b)$$

with l_a and k_a being two null vectors defined by,

$$l_a = \delta^0{}_a, \quad k_a = \frac{1}{2}\left[1 - \frac{2m(v,r)}{r}\right]\delta^0{}_a - \delta^1{}_a, \tag{3.14}$$

where $l_a l^a = k_a k^a = 0$ and $l_a k^a = -1$ (Husain 1996).

Equation (3.12) can be considered as a generalized energy momentum tensor of the Vaidya solution, with the component $T^{(n)}_{ab}$ being considered as the matter field that moves along the null hypersurfaces $v =$ constant while $T^{(m)}_{ab}$ describes the matter moving out along timelike trajectories. When $\rho = \varrho = 0$, the solutions reduce to the Vaidya solution with $m = m(v)$.

If the EMT of equation (3.12) is projected to the orthonormal basis, defined by the four vectors,

$$\begin{aligned} E^{(0)}_a &= \frac{l_a + k_a}{\sqrt{2}}, & E^{(1)}_a &= \frac{l_a - k_a}{\sqrt{2}}, \\ E^{(2)}_a &= \frac{1}{r}\delta^a_2, & E^{(3)}_a &= \frac{1}{r\sin\theta}\delta^a_3, \end{aligned} \tag{3.15}$$

it can be found that

$$\left[T_{(a)(b)}\right] = \begin{bmatrix} \frac{\vartheta}{2} + \rho & \frac{\vartheta}{2} & 0 & 0 \\ \frac{\vartheta}{2} & \frac{\vartheta}{2} - \rho & 0 & 0 \\ 0 & 0 & \varrho & 0 \\ 0 & 0 & 0 & \varrho \end{bmatrix}, \tag{3.16}$$

where

$$\vartheta = \frac{2\dot{m}(v,r)}{r^2}, \quad \rho = \frac{2m'(v,r)}{r^2}, \quad \varrho = -\frac{m''(v,r)}{r}, \tag{3.17}$$

(Wang & Wu 1999). This form of the energy momentum is a combination of Type I and Type II fluids (Hawking & Ellis 1973), with the following energy conditions

a) *The weak and strong energy conditions*:

$$\vartheta \geq 0, \quad \rho \geq 0, \quad \varrho \geq 0, \quad (\vartheta \neq 0). \tag{3.18}$$

b) *The dominant energy conditions*:

$$\vartheta \geq 0, \quad \rho \geq \varrho \geq 0, \quad (\vartheta \neq 0). \tag{3.19}$$

These energy conditions can be satisfied by suitably choosing the mass function $m(v, r)$. In particular, when $m = m(v)$, all the energy conditions (weak, strong, and dominant) reduce to $\vartheta \geq 0$, while when $m = m(r)$ we have $\vartheta = 0$, and the matter field degenerates to a Type I fluid with the usual energy conditions (Hawking & Ellis 1973).

3.3 Collapsing model

In this section, we examine the gravitational collapse of imploding radiation and matter described by the generalized Vaidya spacetime. For this situation, a thick shell of radiation and Type I matter collapses at the centre of symmetry (Joshi 1993).

If K^a is the tangent to nonspacelike geodesics with $K^a = \frac{dx^a}{dk}$, where k is the affine parameter, then $K^a{}_{;b}K^b = 0$ and

$$g_{ab}K^aK^b = B, \tag{3.20}$$

where $B = 0$ for null vectors and $B = \mp 1$ for timelike and spacelike vectors respectively.

The equations for dK^v/dk and dK^r/dk are calculated from the Lagrangian

$$L = \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b, \tag{3.21}$$

where $\dot{x}^a = \frac{dx^a}{dk}$, and Lagrange-Euler equations

$$\frac{\partial L}{\partial x} - \frac{d}{dk}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0. \tag{3.22}$$

Using (3.4), the Lagrangian is given by

$$L = -\frac{1}{2}\left(1 - \frac{2m(v,r)}{r}\right)\dot{v}^2 + \dot{v}\dot{r} + \frac{1}{2}r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2). \tag{3.23}$$

- v-component

$$\frac{\partial L}{\partial v} - \frac{d}{dk}\left(\frac{\partial L}{\partial \dot{v}}\right) = 0, \tag{3.24}$$

$$\begin{aligned} \frac{\partial L}{\partial v} &= \frac{1}{r}\dot{m}(v,r)\dot{v}^2, && (3.25) \\ \frac{\partial L}{\partial \dot{v}} &= -\left(1 - \frac{2m(v,r)}{r}\right)\dot{v} + \dot{r}, && (3.26) \end{aligned}$$

$$\begin{aligned} &\frac{dK^r}{dk} + \frac{1}{r}\dot{m}(v,r)(K^v)^2 - \left(1 - \frac{2m(v,r)}{r}\right)\frac{dK^v}{dk} \\ &- \frac{2}{r^2}\left(m(v,r) - rm'(v,r)\right)K^vK^r = 0. \end{aligned} \tag{3.27}$$

- r-component

$$\frac{\partial L}{\partial r} - \frac{d}{dk}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0, \tag{3.28}$$

$$\begin{aligned} \frac{\partial L}{\partial r} &= \left(\frac{1}{r}m'(v,r) - \frac{m(v,r)}{r^2}\right)\dot{v}^2 + r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2), && (3.29) \\ \frac{\partial L}{\partial \dot{r}} &= \dot{v}, && (3.30) \end{aligned}$$

$$\frac{dK^v}{dk} - \left(\frac{m'(v,r)}{r} - \frac{m(v,r)}{r^2}\right)(K^v)^2 - r\left((K^\theta)^2 + \sin^2\theta(K^\phi)^2\right) = 0. \tag{3.31}$$

- θ-component

$$\frac{\partial L}{\partial \theta} - \frac{d}{dk}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0, \tag{3.32}$$

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= r^2\sin\theta\cos\theta(K^\phi)^2, && (3.33) \\ \frac{\partial L}{\partial \dot{\theta}} &= r^2\dot{\theta}, && (3.34) \end{aligned}$$

$$\frac{dK^\theta}{dk} + \frac{2}{r}K^rK^\theta - \sin\theta\cos\theta(K^\phi)^2 = 0. \tag{3.35}$$

- ϕ-component

$$\frac{\partial L}{\partial \phi} - \frac{d}{dk}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0, \tag{3.36}$$

$$\begin{aligned} \frac{\partial L}{\partial \phi} &= 0, && (3.37) \\ \frac{\partial L}{\partial \dot{\phi}} &= r^2\sin^2\theta\dot{\phi}, && (3.38) \end{aligned}$$

$$\frac{dK^\phi}{dk} + \frac{2}{r}K^r K^\phi + 2\cot\theta K^\theta K^\phi = 0. \tag{3.39}$$

We can write the above Lagrange-Euler equations as

$$\frac{d}{dk}(r^2 \sin^2\theta K^\phi) = 0, \tag{3.40a}$$

$$\frac{d}{dk}(K^\theta) + \frac{2}{r}K^r K^\theta - \sin\theta\cos\theta(K^\phi)^2 = 0, \tag{3.40b}$$

$$\frac{dK^v}{dk} - \left(\frac{m'(v,r)}{r} - \frac{m(v,r)}{r^2}\right)(K^v)^2 - r\left((K^\theta)^2 + \sin^2\theta(K^\phi)^2\right) = 0, \tag{3.40c}$$

$$\begin{aligned} \frac{dK^r}{dk} &+ \frac{1}{r}\dot{m}(v,r)(K^v)^2 - \left(1 - \frac{2m(v,r)}{r}\right)\left(\frac{m'(v,r)}{r} - \frac{m(v,r)}{r^2}\right)(K^v)^2 \\ &- \frac{2}{r^2}\big(m(v,r) - rm'(v,r)\big)K^v K^r \\ &- r\left(1 - \frac{2m(v,r)}{r}\right)\left((K^\theta)^2 + \sin^2\theta(K^\phi)^2\right) = 0. \end{aligned} \tag{3.40d}$$

Using the condition in equation (3.20), we can write equations (3.40c) and (3.40d) as

$$\frac{dK^v}{dk} + \left(\frac{m(v,r)}{r^2} - \frac{m'(v,r)}{r}\right)(K^v)^2 - \frac{\ell^2}{r^3} = 0, \tag{3.41a}$$

$$\frac{dK^r}{dk} + \frac{\dot{m}(v,r)}{r}(K^v)^2 - \frac{\ell^2}{r^3}\left(1 - \frac{2m(v,r)}{r}\right) - B\left(\frac{m(v,r)}{r^2} - \frac{m'(v,r)}{r}\right) = 0, \tag{3.41b}$$

where B is a constant defined by equation (3.20).

If we write

$$K^v = \frac{P(v,r)}{r}, \tag{3.42}$$

then $K_a K^a = g_{ab}K^a K^b = B$ gives

$$K^v = \frac{dv}{dk} = \frac{P}{r}, \qquad P = P(v,r), \tag{3.43}$$

$$K^r = \frac{dr}{dk} = \frac{P}{2r}\left(1 - \frac{2m(v,r)}{r}\right) - \frac{\ell^2}{2rP} + \frac{Br}{2P}. \tag{3.44}$$

From (3.40a), it can be observed that $K^\phi = \text{const.}/r^2\sin^2\theta$. Substituting this in (3.40b), we can integrate (3.40a) and (3.40b) to get

$$K^\theta = \frac{\ell\cos\beta}{r^2\sin^2\theta}, \tag{3.45}$$

$$K^{\phi} = \frac{\ell \sin\beta \cos\phi}{r^2}, \tag{3.46}$$

where ℓ and β are constants of integration (Joshi 1993). ℓ is the impact parameter and β is the isotropy parameter given by the relation $\sin\phi \tan\beta = \cot\theta$. Using equation (3.42), we get

$$\frac{dK^v}{dk} = \frac{d}{dk}\left(\frac{P}{r}\right) = \frac{1}{r}\frac{dP}{dk} - \frac{P}{r^2}\frac{dr}{dk}. \tag{3.47}$$

Thus

$$\frac{dP}{dk} = \frac{1}{r}\left(r^2 \frac{dK^v}{dk} + P\frac{dr}{dk}\right). \tag{3.48}$$

Substituting equations (3.41a) and (3.44) into equation (3.48) gives

$$\frac{dP}{dk} = \frac{P^2}{2r^2}\left(1 - \frac{4m(v,r)}{r} + 2m'(v,r)\right) + \frac{\ell^2}{2r^2} + \frac{B}{2}, \tag{3.49}$$

which is the equation satisfied by the function P. The value of P can be obtained by integrating this equation when the form and conditions for the mass function $m(v,r)$ are specified.

3.4 Conditions for locally naked singularity

In this section we examine, given the generalized Vaidya mass function, how the final fate of collapse is determined in terms of either a black hole or a naked singularity. If there are families of future directed non-spacelike trajectories reaching faraway observers in spacetime, which terminate in the past at the singularity, then we have a naked singularity forming as the collapse final state. Otherwise when no such families exist and event horizon forms sufficiently early to cover the singularity, we have a black hole. The equation for the radial null geodesics ($\ell = 0, \beta = 0$) for the line element (3.4) can be easily found, using equations (3.42) and (3.44), which is given by

$$\frac{dv}{dr} = \frac{2r}{r - 2m(v,r)}. \tag{3.50}$$

The above differential equation has a singularity at $r = 0, \quad v = 0$. The nature of this singularity can be analysed by the usual techniques of the theory of ODE's (Tricomi

1961; Perko 1991). Whereas the procedures used below are standard, we shall describe the case treated here in some detail so as to give the exact picture of the nature of the central singularity at $r = 0, \quad v = 0$.

3.4.1 Structure of the central singularity

We can generally write equation (3.50) in the form

$$\frac{dv}{dr} = \frac{M(v,r)}{N(v,r)}, \tag{3.51}$$

with the singular point at $r = v = 0$, where both the functions $M(v,r)$ and $N(v,r)$ vanish. Hence we should carefully analyze the existence and uniqueness of the solution of the above differential equation in the vicinity of this singularity. At this point it is useful to introduce a new independent variable t with differential dt such that

$$\frac{dv}{M(v,r)} = \frac{dr}{N(v,r)} = dt, \tag{3.52}$$

so that the differential equation (3.51) can be replaced by the system

$$\begin{aligned} \frac{dv(t)}{dt} &= M(v,r), \\ \frac{dr(t)}{dt} &= N(v,r). \end{aligned} \tag{3.53}$$

We would like to emphasise here that all the solutions of equation (3.51) are solutions of the system (3.53) and hence we study the behaviour of this system of equations near the singular point $r = v = 0$ in the (r, v) plane. We can easily see that the singular point of (3.51) is a fixed point of the system (3.53). To find the necessary and sufficient conditions for existence of the solutions of this system in the vicinity of the fixed point $r = v = 0$, let us write (3.53) as a differential equation of the vector $\boldsymbol{x}(t) = [v(t), r(t)]^T$ on $\mathbb{R}^2$ as

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t)) \tag{3.54}$$

Now to show the existence and uniqueness of the solution with respect to the initial conditions arbitrarily near the fixed point of the above system (since the initial conditions on the fixed point will imply the system stays on the fixed point) we give the following definitions:

Definition 3.4.1. The function $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at $\boldsymbol{x} = \boldsymbol{x}_0$, if the partial derivatives of the functions M and N with respect to r and v exist at that point. The derivative of the function, $\boldsymbol{Df}$, is given by the 2×2 Jacobian matrix

$$\begin{bmatrix} M_{,v} & M_{,r} \\ N_{,v} & N_{,r} \end{bmatrix}$$

Definition 3.4.2. Suppose U is an open subset of $\mathbb{R}^2$, then $\boldsymbol{f} : U \to \mathbb{R}^2$ is of class C^1 iff the partial derivatives $M_{,v}, M_{,r}, N_{,v}, N_{,r}$ exist and are continuous on U.

Henceforth we will consider the function $\boldsymbol{f}$ to be of class C^1 throughout the space-time. Let us now show that there exists a unique solution to the system (3.54) subject to the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, where $\boldsymbol{x}_0$ is arbitrarily near the fixed point of the equation. Let us define an operator T in the following way:

Definition 3.4.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an operator acting on all continuous and differentiable vectors $\boldsymbol{y}(t)$ on $\mathbb{R}^2$ and takes them to the image $T\boldsymbol{y}(t)$ defined as

$$T\boldsymbol{y}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{y}(s))ds \ .$$

We now prove an important property of this operator T, subject to the function $\boldsymbol{f}$ being class C^1,

Lemma 3.4.4. *Let $U \ni \boldsymbol{x}_0$ be an open subset of $\mathbb{R}^2$ and $\boldsymbol{f} : U \to \mathbb{R}^2$ is of class C^1 and $\boldsymbol{y}(t)$, $\boldsymbol{z}(t)$ are continuous and differentiable vectors on U. Then there always exists an ϵ-neighbourhood $B_\epsilon(\boldsymbol{x}_0)$ of $\boldsymbol{x}_0$ in which $|T\boldsymbol{y}(t) - T\boldsymbol{z}(t)| \leq \kappa|\boldsymbol{y}(t) - \boldsymbol{z}(t)|$ where $0 \leq \kappa \leq 1$. In other words T is an contraction mapping on $B_\epsilon(\boldsymbol{x}_0)$.*

Proof. Let $K_0 = \max_{|\boldsymbol{x}-\boldsymbol{x}_0| \leq \epsilon} ||\boldsymbol{Df}(\boldsymbol{x})||$. Then we have

$$|T\boldsymbol{y}(t) - T\boldsymbol{z}(t)| = \left| \int_{t_0}^{t} \left(\boldsymbol{f}(\boldsymbol{y}(s)) - \boldsymbol{f}(\boldsymbol{z}(s))\right) ds \right| . \tag{3.55}$$

The above equation can be written as

$$|T\boldsymbol{y}(t) - T\boldsymbol{z}(t)| = \left| \int_{t_0}^{t} \left(\int_{\boldsymbol{z}(s)}^{\boldsymbol{y}(s)} \boldsymbol{D}\boldsymbol{f}(\boldsymbol{r}) dr \right) ds \right| , \tag{3.56}$$

and therefore we get the inequality

$$|T\boldsymbol{y}(t) - T\boldsymbol{z}(t)| \leq K_0|(t - t_0)|\, |\boldsymbol{y}(t) - \boldsymbol{z}(t)|. \tag{3.57}$$

Hence there always exists an open interval $(t_0 - h, t_0 + h)$ (that corresponds to a neighbourhood around $\boldsymbol{x}_0$) where $K_0|(t - t_0)| \leq 1$ and T is a contraction mapping. □

Having established the existence of a contraction mapping in a neighbourhood of the point $\boldsymbol{x}_0$ and recalling that $\mathbb{R}^2$ is a complete metric space, we now use the following theorem to establish the existence and uniqueness of the solution of the system (3.54) subject to the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$.

Theorem 3.4.5. *If $T : \mathbb{X} \to \mathbb{X}$ is a contraction mapping on a complete metric space $\mathbb{X}$, then there is exactly one solution of the equation $T\boldsymbol{x} = \boldsymbol{x}$.*

The above theorem establishes a unique solution of the system (3.54) with the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ in an ϵ-neighbourhood of the point $\boldsymbol{x}_0$ which is given by

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{x}(s)) ds\ . \tag{3.58}$$

The assumption that $\boldsymbol{f} : U \to \mathbb{R}^2$ is of class C^1 assures the solution to be continuous and differentiable in this neighbourhood. Let us now find the nature of the fixed point $r = v = 0$ of the system (3.54). As the partial derivatives of the functions M and N exist and are continuous in the neighbourhood of the fixed point, we can linearise the system near the fixed point and hence the general behaviour of this system near the singular point is similar to the characteristic equations (Tricomi 1961).

$$\begin{aligned} \frac{dv}{dt} &= Av + Br \\ \frac{dr}{dt} &= Cv + Dr, \end{aligned} \tag{3.59}$$

where $A = \dot{M}(0,0)$, $B = M'(0,0)$, $C = \dot{N}(0,0)$, $N'(0,0) = D$, with the dot denoting partial differentiation with respect to the variable v while the dash denotes partial differentiation with respect to the coordinate r and $AD - BC \neq 0$. By using a linear substitution of the type

$$\begin{aligned} \xi &= \alpha v + \omega r \\ \eta &= \gamma v + \delta r, \end{aligned} \tag{3.60}$$

where $\alpha\delta - \omega\gamma \neq 0$, and the equation

$$\frac{d\eta}{d\xi} = \frac{\chi_2 \eta}{\chi_1 \xi}, \tag{3.61}$$

the system (3.59) can be reduced into the form

$$\begin{aligned} \frac{d\xi}{dt} &= \chi_1 \xi, \\ \frac{d\eta}{dt} &= \chi_2 \eta. \end{aligned} \tag{3.62}$$

Using equations (3.59), (3.60) and (3.62), it can be found that

$$\begin{aligned} \alpha(Av + Br) + \omega(Cv + Dr) &= \chi_1(\alpha v + \omega r), \\ \gamma(Av + Br) + \delta(Cv + Dr) &= \chi_2(\gamma v + \delta r). \end{aligned}$$

By equating the coefficients of v and r in the above equations, we obtain

$$\begin{aligned} (A - \chi_1)\alpha + C\omega &= 0, \\ B\alpha + (D - \chi_1)\omega &= 0, \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} (A - \chi_2)\gamma + C\delta &= 0, \\ B\gamma + (D - \chi_2)\delta &= 0. \end{aligned} \tag{3.64}$$

The above equations in α, ω and γ, δ may be satisfied by the values of α, ω, γ, δ not all zero if the determinant of the coefficients is zero. That is

$$\begin{vmatrix} A - \chi & C \\ B & D - \chi \end{vmatrix} = 0, \tag{3.65}$$

or

$$\chi^2 - (A + D)\chi + AD - BC = 0. \tag{3.66}$$

This is the characteristic equation with roots (eigenvalues) χ_1 and χ_2 given by

$$\chi = \frac{1}{2}\left((A + D) \pm \sqrt{(A - D)^2 + 4BC}\right). \tag{3.67}$$

The singularity of equation (3.59) is classified as a node if $(A - D)^2 + 4BC \geq 0$ and $BC > 0$. Otherwise, it may be a centre or focus.

Now, for the equation (3.50) we have $M(v, r) = 2r$, $N(v, r) = r - 2m(v, r)$. If at the central singularity, $v = 0$, $r = 0$, we define the following limits

$$m_0 = \lim_{v\to 0, r\to 0} m(v, r), \tag{3.68a}$$

$$\dot{m}_0 = \lim_{v\to 0, r\to 0} \frac{\partial}{\partial v} m(v, r), \tag{3.68b}$$

$$m_0' = \lim_{v\to 0, r\to 0} \frac{\partial}{\partial r} m(v, r), \tag{3.68c}$$

then the null geodesic equation can be linearized near the central singularity as

$$\frac{dv}{dr} = \frac{2r}{(1 - 2m_0')r - 2\dot{m}_0 v}. \tag{3.69}$$

Clearly, this equation has a singularity ar $v = 0, \quad r = 0$. We can determine the nature of this singularity by observing the value of the discriminant of the characteristic equation. Using equation (3.67), the roots of the characteristic equation are given by

$$\chi = \frac{1}{2}\left((1 - 2m_0') \pm \sqrt{(1 - 2m_0')^2 - 16\dot{m}_0}\right). \tag{3.70}$$

For the singular point at $r = 0, v = 0$ to be a node, it is required that

$$(1 - 2m_0')^2 - 16\dot{m}_0 \geq 0 \quad \text{and} \quad \dot{m}_0 > 0. \tag{3.71}$$

Thus, if the mass function $m(v, r)$ is chosen such that the condition in equation (3.71) is satisfied, then the singularity at the origin ($v = 0$, $r = 0$) will be a node and outgoing nonspacelike geodesics can come out of the singularity with a definite value of the tangent.

3.4.2 Existence of outgoing nonspacelike geodesics

Let us now return to the physical problem of the collapsing generalized Vaidya spacetime, and choose the mass function that has the following properties

1. The mass function $m(v, r)$ obeys all the physically reasonable energy conditions throughout the spacetime.

2. The partial derivatives of the mass function exist and are continuous on the entire spacetime.

3. The limits of the partial derivatives of the mass function $m(v, r)$ at the central singularity obey the conditions: $(1 - 2m_0')^2 - 16\dot{m}_0 \geq 0$ and $\dot{m}_0 > 0$.

The choice of the mass function with the above properties would ensure the existence and uniqueness of the solutions of the null geodesic equation in the vicinity of the central singularity, and will also make the central singularity a node of C^1 solutions with definite tangents.

To find the condition for the existence of outgoing radial nonspacelike geodesics from the nodal singularity, we consider the tangent of these curves at the singularity. Suppose X denotes the tangent to the radial null geodesic. If the limiting value of X at the singular point is positive and finite then we can see that outgoing future directed null geodesics do terminate in the past at the central singularity. The existence of these radial null geodesics characterises the nature (a naked singularity or a black hole) of the collapsing solutions. In order to determine the nature of the limiting value of X at $r = 0, v = 0$ we define

$$X_0 = \lim_{v \to 0, r \to 0} X = \lim_{v \to 0, r \to 0} \frac{v}{r}. \tag{3.72}$$

Using equation (3.69) and L'Hospital's rule (for the C^1 null geodesics) we get

$$X_0 = \lim_{v \to 0, r \to 0} \frac{v}{r} = \frac{dv}{dr} = \frac{2}{(1 - 2m_0') - 2\dot{m}_0(\frac{v}{r})}, \tag{3.73}$$

which simplifies to

$$X_0 = \frac{2}{(1-2m_0') - 2\dot{m}_0 X_0}. \tag{3.74}$$

Solving for X_0 gives

$$X_0 = b_\pm = \frac{(1-2m_0') \pm \sqrt{(1-2m_0')^2 - 16\dot{m}_0}}{4\dot{m}_0}. \tag{3.75}$$

If we can get one or more positive real roots by solving equation (3.74), then the singularity may be locally naked if the null geodesic lies outside the trapped region. In the next subsection we will calculate the dynamics of the trapped region to find the conditions for the existence of such geodesics.

3.4.3 Apparent horizon

The occurrence of a naked singularity or a black hole is usually decided by causal behaviour of the trapped surfaces developing in the spacetime during the collapse evolution. The apparent horizon is the boundary of the trapped surface region in the spacetime. In spherically symmetric spacetime, the equation of the apparent horizon is generally given as,

$$g^{ab} R_{,a} R_{,b} = 0. \tag{3.76}$$

For the generalized Vaidya spacetime the equation of the apparent horizon is given as

$$\frac{2m(v,r)}{r} = 1. \tag{3.77}$$

Thus, the slope of the apparent horizon can be calculated in the following way: we know

$$\frac{2dm(v,r)}{dr} = 1, \tag{3.78a}$$

$$2\left(\frac{\partial m}{\partial v}\right)\left(\frac{dv}{dr}\right)_{AH} + \frac{2\partial m}{\partial r} = 1, \tag{3.78b}$$

which finally gives the slope of the apparent horizon at the central singularity ($v \to 0$, $r \to 0$) as

$$\left(\frac{dv}{dr}\right)_{AH} = \frac{1-2m_0'}{2\dot{m}_0}. \tag{3.79}$$

Thus now we have sufficient conditions for the existence of a locally naked central singularity for a collapsing generalized Vaidya spacetime, which we state in the following proposition:

Proposition 3.4.6. *Consider a collapsing generalized Vaidya spacetime from a regular epoch, with a mass function $m(v,r)$ that obeys all the physically reasonable energy conditions and is differentiable in the entire spacetime. If the following conditions are satisfied:*

1. *The limits of the partial derivatives of the mass function $m(v,r)$ at the central singularity obey the conditions: $(1-2m'_0)^2 - 16\dot{m}_0 \geq 0$ and $\dot{m}_0 > 0$,*
2. *There exist one or more positive real roots X_0 of equation (3.75),*
3. *At least one of the positive real roots is less than $\left(\frac{dv}{dr}\right)_{AH}$ at the central singularity,*

then the central singularity is locally naked with outgoing C^1 radial null geodesics escaping to the future.

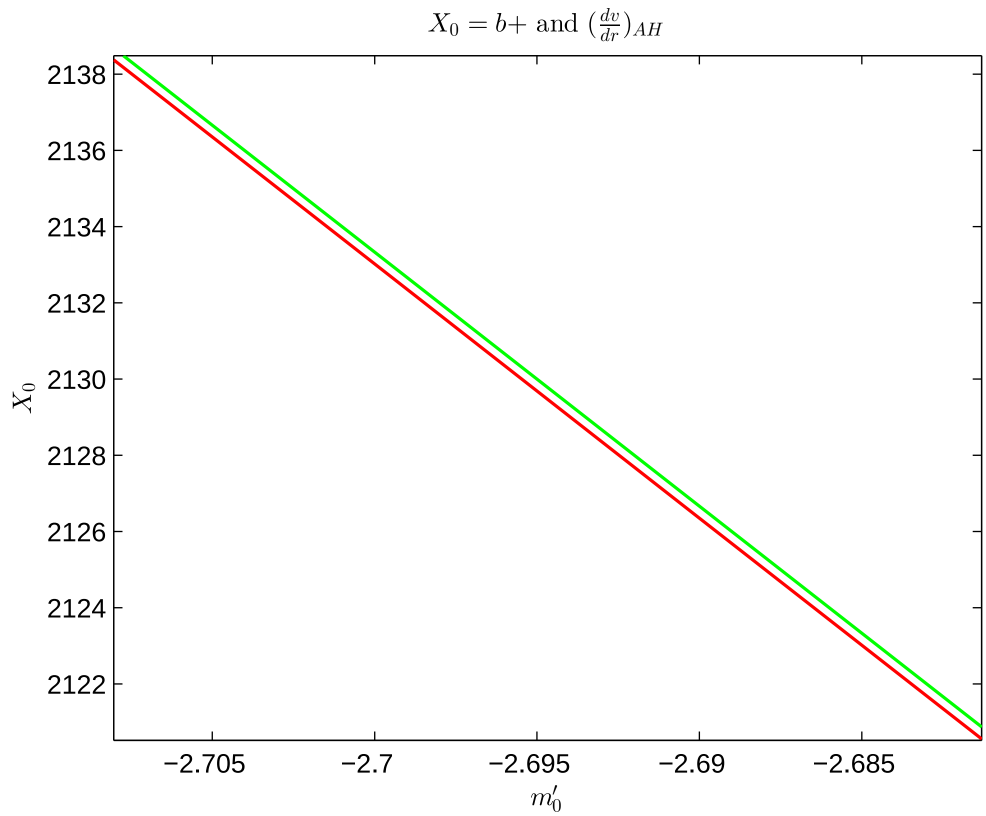

Figure 3.1: Variation of $X_0 = b+$ (red) and $\left(\frac{dv}{dr}\right)_{AH}$ (green) with m_0' at a fixed value $\dot{m}_0 = 0.0015$

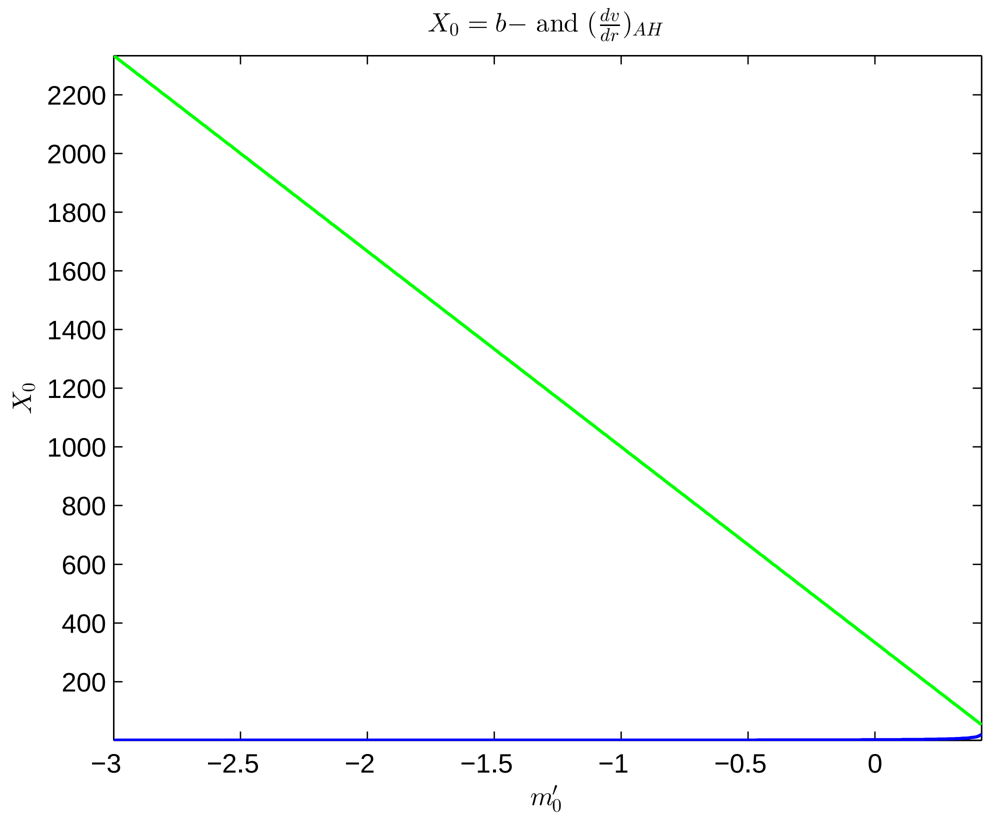

Figure 3.2: Variation of $X_0 = b-$ (blue) and $\left(\frac{dv}{dr}\right)_{AH}$ (green) with m_0' at a fixed value $\dot{m}_0 = 0.0015$

Figures 3.1 and 3.2 show the values of X_0 and $\left(\frac{dv}{dr}\right)_{AH}$ when m_0' is varied in the interval $-3.0 \leq m_0' \leq 0.42$ for a fixed value of $\dot{m}_0$. It can be observed from the figures that the value of $X_0 = b_\pm$ is always below the value of $\left(\frac{dv}{dr}\right)_{AH}$, and thus there exist open sets of parameter values for which the singularity is locally naked.

3.5 Strength of singularity

To compute the strength of singularity according to Tipler (1961), which is the measure of its destructive capacity in the sense that whether extension of spacetime is possible through them or not (Ghosh & Dadhich 2001), we consider the null geodesics parameterized by the affine parameter k and terminating at the shell focusing singularity $r = v = k = 0$. Following Clarke and Krolack (1985), a singularity would be strong if the condition

$$\lim_{k \to 0} k^2 \psi = \lim_{k \to 0} k^2 R_{ab} K^a K^b > 0, \tag{3.80}$$

as defined by Tipler (1977), (which is the sufficient condition for the singularity to be Tipler strong) and R_{ab} is the Ricci tensor, is satisfied. We find the scalar $\psi = R_{ab} K^a K^b$ using equations (3.8a) (3.42) and (3.44) as

$$\psi = (2\dot{m}_0) \left(\frac{P}{r^2}\right)^2, \tag{3.81}$$

and therefore,

$$k^2 \psi = (2\dot{m}_0) \left(\frac{Pk}{r^2}\right)^2. \tag{3.82}$$

Using equations (3.42), (3.44) and L'Hospital's rule, we can evaluate the limit along nonspacelike geodesics as $k \to 0$. This limit is found to be

$$\lim_{k \to 0} k^2 \psi = (2\dot{m}_0) \lim_{k \to 0} \left(\frac{Pk}{r^2}\right)^2. \tag{3.83}$$

If we assume that $P \neq 0, \infty$, then by using L'Hospital's rule we have

$$\lim_{k \to 0} \left(\frac{Pk}{r^2}\right) = \lim_{k \to 0} \left(\frac{P}{2r} \frac{dk}{dr}\right). \tag{3.84}$$

From equation (3.42), $\frac{P}{r} = \frac{dv}{dk}$. Therefore

$$\lim_{k \to 0} \left(\frac{Pk}{r^2} \right) = \frac{1}{2} \frac{dv}{dk} \frac{dk}{dr} = \frac{1}{2} \frac{dv}{dr} = \frac{1}{2} X_0. \tag{3.85}$$

Thus, we finally get

$$\lim_{k \to 0} k^2 \psi = \frac{1}{4} X_0^2 (2\dot{m}_0). \tag{3.86}$$

We observe that the strength of the central singularity depends only on the limit of the derivative of mass function with respect to v and the limiting value X_0.

With the suitable choice of the mass function (see Table 3.1 for some special cases), it can be shown that

$$\lim_{k \to 0} k^2 \psi = \frac{1}{4} X_0^2 (2\dot{m}_0) > 0. \tag{3.87}$$

If this condition is satisfied for some real and positive root X_0, then we conclude that the observed naked singularity is strong. It is interesting to note that when the energy conditions are satisfied, then if a naked singularity is developed as a end state of the collapse, then that naked singularity is always strong.

3.6 Some special sub-classes of generalized Vaidya spacetimes

Using equation (3.74) we calculate the equations of tangents to the null geodesics at the central singularity for some special sub-classes of the generalized Vaidya spacetimes with the specific mass function, $m(v, r)$. In all these mass functions, we can see that it is possible to obtain at least one or more real and positive value of X_0.

i. *The self-similar Vaidya spacetime*

In this case we consider the situation of a radial influx of null fluid in an initially empty region of Minkowski spacetime (Dadhich & Ghosh 2001; Joshi 1993). The first shell arrives at $r = 0$ at time $v = 0$ and the final shell at $v = T$. A central singularity of the collapsing mass is developed at $r = 0$. For $v < 0$ we have $m(v,r) = 0$ and for $v > T$ we have $m(v,r) = M_0$ where M_0 is the constant Schwarzschild mass. For the weak energy conditions to be satisfied, it is required that $\dot{m}(v,r)$ to be a nonnegative. We define the mass function as

$$m(v,r) = m(v), \tag{3.88}$$

where

$$m(v) = \begin{cases} 0, & v < 0, \\ \frac{1}{2}\lambda v, & 0 \leq v \leq T, \\ M_0, & v > T. \end{cases} \tag{3.89}$$

The mass function is a nonnegative increasing function of v for imploding radiation. For $0 \leq v \leq T$, the solution is the self-similar Vaidya spacetime. For this choice of mass function, using equation (3.74), we get

$$X_0 = \frac{2}{1 - \lambda X_0} \quad \text{or} \quad X_0 = \frac{1 \pm \sqrt{1 - 8\lambda}}{2\lambda}. \tag{3.90}$$

This is similar to the solution obtained by Joshi (1993). This equation gives positive values of X_0 for all values of λ in the range $0 < \lambda \leq \frac{1}{8}$. It can also be observed that $\lim\limits_{k \to 0} k^2 \psi = \frac{1}{4}\lambda X_0^2 > 0$ for all positive values of X_0; hence the singularity is strong.

ii. *The charged Vaidya spacetime*

This subclass of the generalized Vaidya spacetime has been studied in great detail (Israel 1967; Lindquist *et al.* 1965; Patil *et al.* 1967). We consider here the form of the mass function

$$m(v,r) = f(v) - \frac{e^2(v)}{2r}, \tag{3.91a}$$

where $f(v)$ and $e(v)$ are arbitrary functions representing the mass and electric charge respectively (limited only by the energy conditions), at the advanced time v (Dadhich & Ghosh 2001; Wang & Wu 1999). Particularly, we define these functions as

$$f(v) = \begin{cases} 0, & v < 0, \\ \lambda v (\lambda > 0) & 0 \leq v \leq T, \\ f_0 (> 0), & v > T, \end{cases} \tag{3.91b}$$

and

$$e^2(v) = \begin{cases} 0, & v < 0, \\ c^2 v^2 (\mu^2 > 0), & 0 \leq v \leq T, \\ e_0^2 (> 0), & v > T \end{cases} \tag{3.91c}$$

(Beesham & Ghosh 2003). For this choice of mass function, using equation (3.74) we obtain

$$c^2 X_0^3 - 2\lambda X_0^2 + X_0 - 2 = 0. \tag{3.92}$$

This equation is a polynomial of degree three with the negative last term and positive first coefficient. By the theory of polynomial functions, every equation of this nature must have at least one root which is positive. The existence of these roots signifies that the singularity is naked. In particular, when $c^2 = 0.001,\ \lambda = 0.01$, then one of the roots of equation (3.92) is 2.077 and $\lim_{k \to 0} k^2 \psi = \frac{1}{2} X_0^2 \left(\lambda - c^2 X_0 \right) = 0.0171 > 0$. Therefore the condition for a strong naked singularity is satisfied.

iii. *The charged Vaidya-deSitter spacetime*

The charged Vaidya-deSitter solution is a generalized Vaidya solution of a charged null fluid in an expanding de-Sitter background (Beesham & Ghosh 2003). We define the mass mass function as

$$m(v, r) = m(v) - \frac{e^2(v)}{2r} + \frac{\Lambda r^3}{6}, \tag{3.93}$$

where $f(v)$ and $e(v)$ are arbitrary functions representing the mass and electric charge respectively, and $\Lambda \neq 0$ is the cosmological constant. For the weak energy condition to be satisfied, it is required that $r\dot{m}(v) - e(v)\dot{e}(v)$ to be nonnegative (Beesham & Ghosh 2003; Wang & Wu 1999). We specifically define the functions similar to that of charged Vaidya and the algebraic equation that governs the behaviour of the tangent vectors near the central singularity comes out to be the same.

iv. *The Husain solution*

This is a solution of the Einstein field equations for the null fluid with the equation of state $\varrho = k\rho$ where $\rho = \frac{g(v)}{4\pi r^{2k+2}}$, $k \neq \frac{1}{2}$ (Husain 1996; Wang & Wu 1999). This solution is a subclass to the generalized Vaidya solutions with the mass function given by

$$m(v,r) = \begin{cases} q(v) - \frac{g(v)}{(2k-1)r^{2k-1}}, & k \neq \frac{1}{2}, \\ q(v) + g(v)\ln r, & k = \frac{1}{2}, \end{cases} \tag{3.94a}$$

where $q(v)$ and $g(v)$ are arbitrary functions which are restricted only by the energy conditions. For the dominant energy conditions to be satisfied, it is required that $g(v) \geq 0$ and either $\dot{g}(v) > 0$ for $k < \frac{1}{2}$ or $\dot{g}(v) < 0$ for $k > \frac{1}{2}$. The weak or strong energy conditions are satisfied when $\rho \geq 0$, $\varrho \geq 0$. We consider the case when $k \neq \frac{1}{2}$ and define the mass function as

$$q(v) = \begin{cases} 0, & v < 0, \\ \frac{1}{2}\lambda v(\lambda > 0), & 0 \leq v \leq T, \\ q_0(> 0), & v > T, \end{cases} \tag{3.94b}$$

and

$$g(v) = \begin{cases} 0, & v < 0, \\ c^2 v^{2k}, & 0 \leq v \leq T, \\ g_0(> 0), & v > T. \end{cases} \tag{3.94c}$$

For this mass function using equation (3.74), we get

$$2c^2\left(1 - \frac{2k}{2k-1}\right)X_0^{2k+1} + \lambda X_0^2 - 2X_0 + 4 = 0. \tag{3.95}$$

This equation can be solved to get some positive roots X_0 for some particular values of c^2, k and λ. In particular, when $c^2 = 0.001$, $k = \lambda = 0.01$, then one of the roots of equation (3.95) is 2.00408 and $\lim_{k\to 0} k^2\psi = \frac{1}{4}X_0^2\left(\lambda - \frac{4kc^2}{2k-1}X_0^{2k-1}\right) = 0.506 > 0$. This shows that the singularity is naked and strong.

Table 3.1 gives a summary of the equations of tangent to the singularity curve X_0 and the value of $\lim_{k\to 0} k^2\psi$ for chosen mass functions in some sub-classes of the generalized Vaidya spacetime.

Table 3.1: Equations of tangents to the singularity curve X_0 and values of $\lim_{k\to 0} k^2\psi$ for some special sub-classes of generalized Vaidya spacetime

Spacetime	Equation for tangent to the singularity curve	$\lim_{k\to 0} k^2\psi$
Vaidya	$X_0 = \frac{2}{1-\lambda X_0}$ or	$\frac{1}{4}\lambda X_0^2$
	$X_0 = \frac{1\pm\sqrt{1-8\lambda}}{2\lambda}, \quad 0 < \lambda \leq \frac{1}{8}$	
Charged Vaidya	$c^2X_0^3 - 2\lambda X_0^2 + X_0 - 2 = 0$	$\frac{1}{2}X_0^2\left(\lambda - c^2X_0\right)$
Charged Vaidya-de Sitter	$c^2X_0^3 - 2\lambda X_0^2 + X_0 - 2 = 0$	$\frac{1}{2}X_0^2\left(\lambda - c^2X_0\right)$
Husain solution	$2c^2\left(1 - \frac{2k}{2k-1}\right)X_0^{2k+1} + \lambda X_0^2$	
	$-2X_0 + 4 = 0$	$\frac{1}{4}X_0^2(\lambda - \frac{4kc^2}{2k-1}X_0^{2k-1})$

Chapter 4

Cosmic censorship in higher dimensions

4.1 Introduction

In this chapter, we extend the analysis of gravitational collapse of generalized Vaidya spacetimes to higher dimensions. One obvious question that arises (influenced by higher dimensional and emergent theories of gravity - e.g string theory or braneworld models), is as follows:

Does the transition to higher dimensional spacetimes (with compact or non-compact extra dimensions) restrict the set of initial data that leads to a naked singularity?

In other words, how does the number of spacetime dimensions dictate the dynamics of trapped regions in the spacetime? This question is important as most of the proofs of the key theorems of black hole dynamics and thermodynamics demand the spacetimes to be future asymptotically simple, which is not possible if the censorship is violated (Hawking & Ellis 1973). If the locally naked singularities in 4-dimensional spacetime are naturally absent in higher dimensions, then that will be an argument in favour of higher dimensional (or emergent theories) of gravity, as in those cases the important results of black hole dynamics and thermodynamics would be more relevant.

To answer the above question, at least partially, Goswami and Joshi (2004a, 2004b) established the following important result: The naked singularities occurring in dust collapse from smooth initial data (which include those discovered by Eardley and Smarr (1979), Christodoulou (1984), and Newman (1986)) are eliminated when we make transition to higher dimensional spacetimes. The cosmic censorship is then restored for dust collapse, which will always produce a black hole as the collapse end state for dimensions $D \geq 6$, under conditions such as the smoothness of initial data from which the collapse develops, which follow from physical grounds.

The physical reason behind the above result is that higher dimensional spacetimes favour trapped surface formation and the formation of horizons advance in time. Hence for dimensions greater than five, the vicinity of the singularity always gets trapped even before the singularity is formed, and hence the singularity is causally cut-off from any external observer.

Several other works on higher dimensional radiation collapse and perfect fluid collapse have been done (Beesham & Ghosh 2003; Ghosh & Dadhich 2001; Ghosh & Dawood 2008; Ghosh & Deshkar 2007; Ghosh & Saraykar 2000; Dadhich *et al.* 2005; Patil 2003), where the matter field is taken to be of a specific form (for example, perfect fluids with linear equation of state, pure radiation, charged radiation etc.). All of these studies give an indication that higher dimensions do favour trapping and hence the epoch of trapped surface formation advances as we go to higher dimensions.

The main criticism of the dustlike models or pure perfect fluid models is that they are far too idealised. For any realistic massive astrophysical body, which is undergoing gravitational collapse, the pressure and the radiative processes must play an important role together. One of the known spacetimes that can closely mimic such a collapse scenario is the generalized Vaidya spacetime, where the matter field is a specific combination of Type I matter (whose energy momentum tensor has one timelike and three spacelike eigenvectors) that moves along timelike trajectories and Type II matter (whose energy momentum tensor has double null eigenvectors) that moves along null trajectories. Thus, a collapsing generalized Vaidya spacetime depicts the collapse of a usual perfect fluid combined with radiation. Therefore the collapse scenario here is much closer to what is expected for the collapse of a realistic astrophysical star. In our earlier work (Mkenyeleye *et al.* 2014), we investigated the gravitational collapse of generalized Vaidya spacetime in four dimensions and developed a general mathematical framework to study the conditions on the mass function such that future directed nonspacelike geodesics can terminate at the singularity in the past.

4.2 Higher dimensional generalized Vaidya spacetimes

The spherically symmetric line element for an N-dimensional generalized Vaidya spacetime is given as

$$ds^2 = -\left(1 - \frac{2m(v,r)}{r^{(N-3)}}\right) dv^2 + 2dvdr + r^2 d\Omega^2_{(N-2)}, \tag{4.1}$$

where

$$d\Omega^2_{(N-2)} = \sum_{i=1}^{N-2} \left[\prod_{j=1}^{i-1} \sin^2(\theta^j)\right] (d\theta^i)^2, \tag{4.2}$$

is the metric on the $(N-2)$ sphere in polar coordinates with θ^i being spherical coordinates. '$m(v,r)$' is the generalized mass function related to the gravitational energy within a given radius r (Lake & Zannias 1991), which can be carefully defined so that the energy conditions are satisfied. The coordinate v represents the Eddington advanced time where r is decreasing towards the future along a ray $v =$ constant (ingoing). When $N = 4$, the line element reduces to the generalized Vaidya solution in 4-dimensions (Wang & Wu 1999).

The nonvanishing components of the Ricci tensor are given as

$$R^{v}{}_{v} = R^{r}{}_{r} = \frac{m''(v,r)}{r^{(N-3)}} - \frac{(N-4)m'(v,r)}{r^{(N-2)}}, \tag{4.3a}$$

$$R^{\theta^1}{}_{\theta^1} = R^{\theta^2}{}_{\theta^2} = \cdots = R^{\theta^{(N-2)}}{}_{\theta^{(N-2)}} = \frac{2m'(v,r)}{r^{(N-2)}}. \tag{4.3b}$$

The Ricci scalar is given by

$$R = \frac{2m''(v,r)}{r^{(N-3)}} + \frac{4m'(v,r)}{r^{(N-2)}}, \tag{4.4}$$

while the nonvanishing components of the Einstein tensor are given by

$$G^{v}{}_{v} = G^{r}{}_{r} = -\frac{(N-2)m'(v,r)}{r^{(N-2)}}, \tag{4.5a}$$

$$G^{r}{}_{v} = \frac{(N-2)\dot{m}(v,r)}{r^{(N-2)}}, \tag{4.5b}$$

$$G^{\theta^1}{}_{\theta^1} = G^{\theta^2}{}_{\theta^2} = \cdots = G^{\theta^{(N-2)}}{}_{\theta^{(N-2)}} = -\frac{m''(v,r)}{r^{(N-3)}}. \tag{4.5c}$$

The energy momentum tensor (EMT) can be written in the form

$$T_{ab} = T^{(n)}_{ab} + T^{(m)}_{ab}, \tag{4.6}$$

where

$$T^{(n)}_{ab} = \vartheta l_a l_b, \tag{4.7a}$$

$$T^{(m)}_{ab} = (\rho + \varrho)(l_a k_b + l_b k_a) + \varrho g_{ab} \tag{4.7b}$$

(Husian 1996). In the above,

$$\begin{aligned} \vartheta &= \frac{(N-2)\dot{m}(v,r)}{r^{(N-2)}}, \quad \rho = \frac{(N-2)m'(v,r)}{r^{(N-2)}}, \\ \varrho &= -\frac{m''(v,r)}{r^{(N-3)}}, \end{aligned} \tag{4.8}$$

with l_a and k_a being two null vectors,

$$l_a = \delta^0_a, \quad k_a = \frac{1}{2}\left[1 - \frac{2m(v,r)}{r^{(N-3)}}\right]\delta^0_a - \delta^1_a, \tag{4.9}$$

where $l_a l^a = k_a k^a = 0$ and $l_a k^a = -1$.

Equation (4.6) is taken as a generalized energy momentum tensor for the generalized Vaidya spacetime, with the component $T_{ab}^{(n)}$ being considered as the matter field that moves along the null hypersurfaces v = constant, while $T_{ab}^{(m)}$ describes the matter moving along timelike trajectories. If the EMT of equation (4.6) is projected to the orthonormal basis, defined by the vectors,

$$\begin{aligned} E_a^{(0)} &= \frac{l_a + k_a}{\sqrt{2}}, \quad E_a^{(1)} = \frac{l_a - k_a}{\sqrt{2}}, \quad E_a^{(2)} = \frac{1}{r}\delta_2^a, \cdots, \\ E_a^{(N)} &= \frac{1}{r\sin\theta^1 \sin\theta^2 \sin\theta^3 \cdots \sin\theta^{(N-2)}}\delta_N^a, \end{aligned} \tag{4.10}$$

it can be found that the symmetric EMT can be given as the $N \times N$ matrix,

$$\left[T_{(a)(b)}\right] = \begin{bmatrix} \frac{\vartheta}{2} + \rho & \frac{\vartheta}{2} & 0 & \cdots & 0 \\ \frac{\vartheta}{2} & \frac{\vartheta}{2} - \rho & 0 & 0 & 0 \\ 0 & 0 & \varrho & 0 & 0 \\ \vdots & \cdots & 0 & \varrho & \vdots \\ 0 & 0 & 0 & \cdots & \varrho \end{bmatrix}. \tag{4.11}$$

For this fluid the energy conditions are given as (Hawking & Ellis 1973)

1. *The weak and strong energy conditions*:

$$\vartheta \geq 0, \quad \rho \geq 0, \quad \varrho \geq 0, \quad (\vartheta \neq 0). \tag{4.12}$$

2. *The dominant energy condition*:

$$\vartheta \geq 0, \quad \rho \geq \varrho \geq 0, \quad (\vartheta \neq 0). \tag{4.13}$$

These energy conditions can be satisfied by suitable choices of the mass function $m(v, r)$.

4.3 Higher dimensional collapse model

In this section, we examine the gravitational collapse of a collapsing matter field in the generalized Vaidya spacetime when a spherically symmetric configuration of Type I and Type II matter collapse at the centre of symmetry in an otherwise empty universe which is asymptotically flat far away (Joshi 1993).

If K^a is the tangent to nonspacelike geodesics with $K^a = \frac{dx^a}{dk}$, where k is the affine parameter, then $K^a{}_{;b}K^b = 0$ and

$$g_{ab}K^aK^b = \beta, \tag{4.14}$$

where β is a constant that characterizes different classes of geodesics with $\beta = 0$ for null geodesic vectors, $\beta < 0$ for timelike geodesics and $\beta > 0$ for spacelike geodesics (Joshi 1993). Here we consider the case of null geodesics, that is, $\beta = 0$.

We calculate the equations dK^v/dk and dK^r/dk using the Lagrangian given by $L = \frac{1}{2}g_{ab}\frac{dx^a}{dk}\frac{dx^b}{dk}$ and the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^a} - \frac{d}{dk}\left(\frac{\partial L}{\partial x^a_{,k}}\right) = 0, \tag{4.15}$$

In the case of the higher dimensional generalized Vaidya spacetime, these equations are given by

$$\begin{aligned} \frac{dK^v}{dk} + \left(\frac{(N-3)m(v,r)}{r^{(N-2)}} - \frac{m'(v,r)}{r^{(N-3)}}\right)(K^v)^2 &= 0, && (4.16a)\\ \frac{dK^r}{dk} + \frac{\dot{m}(v,r)}{r^{(N-3)}}(K^v)^2 &= 0. && (4.16b) \end{aligned}$$

All other components are considered to be 0. If we follow Dwivedi and Joshi (1989) and write K^v as

$$K^v = \frac{P(v,r)}{r}, \tag{4.17}$$

then using $K_aK^b = 0$ we get

$$\begin{aligned} K^v &= \frac{dv}{dk} = \frac{P(v,r)}{r}, && (4.18a)\\ K^r &= \frac{dr}{dk} = \frac{P}{2r}\left(1 - \frac{2m(v,r)}{r^{(N-3)}}\right). && (4.18b) \end{aligned}$$

4.4 Conditions for locally naked singularity

The nature (for a locally naked singularity or a black hole) of the collapsing solutions can be characterized by the existence of radial null geodesics coming out of the singularity (Ghosh & Dadhich 2001; Joshi 1993).

The radial null geodesics of the line element (4.1) can be calculated using equations (4.18a) and (4.18b). These geodesics are given by the equation

$$\frac{dv}{dr} = \frac{2r^{(N-3)}}{r^{(N-3)} - 2m(v,r)}. \tag{4.19}$$

This differential equation has a singularity at $r = 0$, $v = 0$. Using the same techniques (Mkenyeleye *et al.* 2014; Perko 1991; Tricomi 1961), equation (4.19) can be re-written near the singular point as

$$\frac{dv}{dr} = \frac{2(N-3)r^{(N-3)}}{(N-3)r^{(N-3)} - 2m_0'r - 2\dot{m}_0 v}, \tag{4.20}$$

where

$$\begin{aligned} m_0 &= \lim_{v\to 0, r\to 0} m(v,r), & (4.21a) \\ \dot{m}_0 &= \lim_{v\to 0, r\to 0} \frac{\partial}{\partial v} m(v,r), & (4.21b) \\ m_0' &= \lim_{v\to 0, r\to 0} \frac{\partial}{\partial r} m(v,r). & (4.21c) \end{aligned}$$

4.4.1 Existence of outgoing nonspacelike geodesics

We can clearly see that equation (4.20) has a singularity at $v = 0, \quad r = 0$. The classification of the tangents of both radial and nonradial outgoing nonspacelike geodesics terminating at the singularity in the past can be given by the limiting values at $v = 0$, $r = 0$. The conditions for the existence for such geodesics have been described in detail (Mkenyeleye *et al.* 2014) using the concept of contraction mappings. The existence of these radial null geodesics also characterizes the nature (a naked singularity or a black hole) of the collapsing solutions. If we let X to be the limiting value at $r = 0$, $v = 0$, we can determine the nature of this limiting value on a singular geodesic as

$$X_0 = \lim_{v\to 0, r\to 0} X = \lim_{v\to 0, r\to 0} \frac{v}{r}. \tag{4.22}$$

Using a suitably chosen mass function, equation (4.20) and l'Hopital's rule, we can explicitly find the expression for the tangent values X_0 which governs the behaviour of the null geodesics near the singular point. Thus, the nature of the singularity can then be determined by studying the solution of this algebraic equation. This expression can be calculated as

$$\begin{aligned} X_0 &= \lim_{v\to 0, r\to 0} \frac{dv}{dr} \\ &= \lim_{v\to 0, r\to 0} \frac{2(N-3)r^{(N-4)}}{(N-3)r^{(N-4)} - 2m_0' - 2\dot{m}_0 X_0}. \end{aligned} \tag{4.23}$$

4.4.2 Apparent horizon

The existence of the apparent horizon, which is the boundary of the trapped surface region in the spacetime also determines the nature of the singularity. If at least one value of the limiting positive values X_0 is less than the slope of the apparent horizon at the central singularity, then the central singularity is locally naked with the outgoing radial null geodesics escaping from the past to the future.

For the generalized higher dimensional Vaidya spacetime, the apparent horizon is defined by

$$2m(v, r) = r^{(N-3)}. \tag{4.24}$$

The slope of the apparent horizon can be calculated as follows:

$$2\frac{dm}{dr} = (N-3)r^{(N-4)}, \tag{4.25a}$$

$$2\left(\frac{\partial m}{\partial v}\right)\left(\frac{dv}{dr}\right)_{AH} + 2\frac{\partial m}{\partial r} = (N-3)r^{(N-4)}. \tag{4.25b}$$

Thus, the slope of the apparent horizon at the central singularity is given by

$$X_{AH} = \left(\frac{dv}{dr}\right)_{AH} = \lim_{v\to 0, r\to 0} \frac{(N-3)r^{(N-4)} - 2m_0'}{2\dot{m}_0}. \tag{4.26}$$

4.4.3 Sufficient conditions

We can now write the sufficient conditions for the existence of a locally naked central singularity for a collapsing generalized Vaidya spacetime in arbitrary dimensions N, which we state in the following proposition:

Proposition 4.4.1. *Consider a collapsing N-dimensional generalized Vaidya spacetime from a regular epoch, with a mass function $m(v,r)$, that obeys all physically reasonable energy conditions and is differentiable in the entire spacetime. If the following conditions are satisfied :*

1. *The limits of the partial derivatives of the mass function $m(v,r)$ exist at the central singularity,*
2. *There exist one or more positive real roots X_0 of the equation (4.23),*
3. *At least one of the positive real roots of X_0 is less than the smallest root of equation (4.26),*

then the central singularity is locally naked with outgoing C^1 radial null geodesics escaping to the future.

We emphasise here, that all the previous works of higher dimensional generalized Vaidya collapse (Beesham & Ghosh 2003; Ghosh & Dadhich 2001; Patil 2003), are special cases of the general analysis presented above. In the next section, we give a specific example to transparently demonstrate the effect of transition to higher dimensions on the nature of the central singularity.

4.5 A general Laurent expandable mass function

We consider here a Laurent expandable mass function of the generalized Vaidya space-time in higher dimensions in the general form as

$$2m(v,r) = \lambda_1 m_1(v) - \lambda_2 \frac{m_2(v)}{r^{(N-3)}} - \lambda_3 \frac{m_3(v)}{r^{(N-2)}} + \cdots \,, \tag{4.27}$$

where

$$m_n(v) = v^{(2N+n-8)},\ n = 1, 2, \cdots \quad \text{and } \lambda_n\text{'s are constants.}$$

Using equation (4.23) and (4.26), we get the expression of the tangent to the null geodesics X_0 and tangent to the apparent horizon X_{AH} in higher dimensions as

$$X_0 = \frac{2(N-3)}{(N-3) - (2N-7)\lambda_1 X_0^{(2N-7)} + (N-3)\left(\lambda_2 X_0^{2N-6} + \lambda_3 X_0^{(2N-5)} + \cdots\right)}, \tag{4.28}$$

and

$$X_{AH} = \frac{(N-3) - (N-3)\lambda_2 X_{AH}^{(2N-6)} - (N-2)\lambda_3 X_{AH}^{(2N-5)} - \cdots}{\lambda_1 (2N-7) X_{AH}^{(2N-8)} - (2N-6)\lambda_2 X_{AH}^{(2N-7)} - (2N-5)\lambda_3 X_{AH}^{(2N-6)} - \cdots}, \tag{4.29}$$

respectively. These expressions can be written in the general form as

$$\sum_{n=1}^{\infty} \left(f_n(N, \lambda_i) X_0^{(2N+n-7)} \right) + (N-3)X_0 - 2(N-3) = 0, \tag{4.30}$$

and

$$\sum_{n=1}^{\infty} g_n(N, \lambda_i) X_{AH}^{(2N+n-8)} - (N-3) = 0, \tag{4.31}$$

where $f_n(N, \lambda_i)$ and $g_n(N, \lambda_i)$ are some functions of N and the λ_i's.

These expressions can explicitly be solved for X_0 and X_{AH} using some specific values of n, N and λ_i's (see Table 4.1) and then we can make conclusions about the nature of the singularity by using the following conditions:

(i) If there is no positive real solution for X_0, then there are no outgoing null geodesics from the singularity and the singularity is causally cut off from the external observer.

(ii) If there is no real solution for X_{AH}, then there are no trapped surfaces and the singularity is globally naked, provided there is at least one positive real root of X_0.

(iii) If there are one or multiple real solutions for X_{AH} with the smallest solution less than X_0, then it can be concluded that the collapse results in a black hole end state.

(iv) If the smallest solution Min$[X_{AH}]$ is greater than any one of the positive solutions of X_0, then there will be future directed null geodesics from the singularity and hence the singularity is locally naked.

We can easily see from Table 4.1, that the general expression obtained here contains the expressions for X_0 and X_{AH} corresponding to Vaidya collapse in 4-D ($n = 1, N = 4$) (Dwivedi & Joshi 1989; Joshi 1993), charged Vaidya-de Sitter in 4-D ($n = 2, N = 4$) (Beesham & Ghosh 2003) and charged Vaidya in 5-D ($n = 2, N = 5$) (Patil 2003).

Table 4.1: Algebraic equations for X_0 and X_{AH} for different values of n and N

n and N	Expression for X_0	Expression for X_{AH}
$n = 1,\ N = 4$	$\lambda_1 X_0^2 - X_0 + 2 = 0$	$X_{AH} = \frac{1}{\lambda_1}$
$n = 2, N = 4$	$\lambda_2 X_0^3 - \lambda_1 X_0^2 + X_0 - 2 = 0$	$\lambda_2 X_{AH}^2 - \lambda_1 X_{AH} + 1 = 0$
$n = 2,\ N = 5$	$2\lambda_2 X_0^5 - 3\lambda_1 X_0^4 + 2X_0 - 4 = 0$	$2\lambda_2 X_{AH}^4 - 3\lambda_1 X_{AH}^3 + 2 = 0$

4.5.1 Example: Class of naked singularity in 4D being eliminated in higher dimensions

In this section we will consider a specific example, that can be easily generalized to an open set, to show explicitly how a naked singularity in four dimensions gets covered in higher dimensions. Let us consider a scenario where $n = 4$. In this case the expression for X_0 and X_{AH} become

$$\begin{aligned}(2N-7)\lambda_1 X_0^{2N-6} - (N-3)\lambda_2 X_0^{2N-5} - (N-3)\lambda_3 X_0^{2N-4} \\ - (N-3)\lambda_4 X_0^{2N-3} - (N-3)X_0 + 2(N-3) = 0,\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}(2N-7)\lambda_1 X_{AH}^{2N-7} - (N-3)\lambda_2 X_{AH}^{2N-6} - (N-3)\lambda_3 X_{AH}^{2N-5} \\ - (N-3)\lambda_4 X_{AH}^{2N-4} - (N-3) = 0,\end{aligned} \tag{4.33}$$

respectively. We can solve these equations numerically to get the values of X_0 and X_{AH} in different dimensions. For our calculations we took $\lambda_1 = 5.0$, $\lambda_2 = 0.01$, $\lambda_3 = 2.3$, $\lambda_4 = 0.05$. From Table 4.2 we can easily see that in 4 dimensions, this class of mass function leads to a naked singularity, as the trapped surfaces do not form early enough to shield the singularity from outside observers. However when we make the transition to higher dimensions we see that the value of the tangent to the outgoing null geodesic from the central singularity is greater than the slope of the apparent horizon curve at the central singularity. In this case the outgoing null direction is within the trapped region and hence the singularity is causally cut off from the external observer. By the continuity of the mass function considered above, this can be easily converted to a open set in the mass function space, where this scenario continues to be true and we shall explicitly prove this in the following subsection.

Table 4.2: Values of X_0 and Min$[X_{AH}]$ for different dimensions for $\lambda_1 = 5.0$, $\lambda_2 = 0.01$, $\lambda_3 = 2.3$, $\lambda_4 = 0.05$.

N	X_0	Min$[X_{AH}]$
4	0.204	1.472
5	1.806	0.526
6	1.902	0.672
7	1.948	0.751

4.5.2 Proof of existence of open set of mass functions with the above properties

Having found out a specific example of a mass function for which the naked singularities in 4D are eliminated when we go to higher dimensions, we are now required to prove that such a mass function is generic in the sense that there exists an open set of such mass functions in the function space. Since this problem of deducing the nature of the central singularity is reduced to finding and comparing real roots of polynomials (4.32) and (4.33), all we need to show here is the real roots of these polynomials are continuous functions of the coefficients.

To do this, first of all we observe that the roots that are given in the Table 4.2 are all of multiplicity one. This can be easily seen by differentiating the LHS of (4.32) and (4.33) and substituting the roots to find nonzero values. Now, for any complex polynomial $p(z)$ of degree $n \geq 1$ with m distinct roots $\{\alpha_1, \cdots, \alpha_m\}$, $(1 \leq m \leq n)$, let us define the quantity $R_0(p)$ as follows:

$$R_0(p) = \begin{cases} \frac{1}{2}, & \text{if } m = 1. \\ \frac{1}{2}\min|\alpha_i - \alpha_j|, i \leq j \leq m, & \text{if } m > 1. \end{cases} \tag{4.34}$$

We now state the well known result of complex analysis (Alen 2015):

Theorem 4.5.1. *Let $p(z)$ be a polynomial of degree $n \geq 1$, with real coefficients $\{\mu_k\}$. Suppose α be a real root of $p(z)$ of multiplicity one. Then for any ϵ with $0 \leq \epsilon \leq R_0(p)$, there exists a $\delta(\epsilon) > 0$ such that any polynomial $q(z)$ with real coefficients ν_k and $|\mu_k - \nu_k| \leq \delta$, has a real root β with $|\alpha - \beta| \leq \epsilon$.*

The above theorem shows that if a polynomial $p(z)$ with real coefficients has a real root α of multiplicity one, then any polynomial $q(z)$ obtained by small (real) perturbations to the coefficients of $p(z)$ will also have a real root in a neighbourhood of α. That is, not only the root depends continuously on coefficients, but it also remains real, under sufficiently small perturbations of coefficients.

This results directly translates to our problem of open set of mass functions in the mass function space. Once we have a specific example as shown in Table 4.2, any perturbations around that will have the same outcome as far as the nature of the singularities are concerned. Hence this class of mass functions is not fine tuned, but quite generic and the outcome is stable under perturbations.

Table 4.3: Range for X_0 and Min$[X_{AH}]$ for different dimensions: $\{\lambda_1 = 5.2, 0.009 < \lambda_2 < 0.012, \lambda_3 = 2.3, 0 \leq \lambda_4 < 0.4\}$

N	Range for X_0	Range for Min$[X_{AH}]$
4	$1.3934 < X_0 < 1.3941$	$1.5010 < \text{Min}[X_{AH}] < 1.5017$
5	$1.8406 < X_0 < 1.8412$	$0.5184 < \text{Min}[X_{AH}] < 0.5185$
6	$1.9387 < X_0 < 1.9393$	$0.6658 < \text{Min}[X_{AH}] < 0.6659$
7	$1.9865 < X_0 < 1.9872$	$0.7431 < \text{Min}[X_{AH}] < 0.7432$

4.5.3 Numerical verification

We would now like to verify explicitly, with the aid of numerical calculations, the results in the previous subsection. We numerically solve equations (4.32) and (4.33) to get the values of X_0 and X_{AH} in different dimensions to show that there exists a set of parameter intervals for which the mass function leads to a naked singularity in four dimensions and a black hole in higher dimensions. For example, some of the intervals are $\{4.8 < \lambda_1 < 5.25, 0.009 < \lambda_2 < 0.012, 2.25 < \lambda_3 < 2.38, \lambda_4 = 0.05\}$ with the range of values shown in Table 4.4 and $\{\lambda_1 = 5.2, 0.009 < \lambda_2 < 0.012, \lambda_3 = 2.3, 0 \leq \lambda_4 < 0.4\}$ as shown in Table 4.3, we can easily see that in four dimensions, these classes of mass function lead to a naked singularity, as the trapped surfaces do not form early enough. However when we make the transition to higher dimensions, the final outcome is a black hole.

Table 4.4: Range for X_0 and $\mathrm{Min}[X_{AH}]$ for different dimensions: $\{4.8 < \lambda_1 < 5.25, 0.009 < \lambda_2 < 0.012, 2.25 < \lambda_3 < 2.38, \lambda_4 = 0.05\}$

N	Range for X_0	Range for $\mathrm{Min}[X_{AH}]$
4	$0.194 < X_0 < 0.213$	$1.377 < \mathrm{Min}[X_{AH}] < 1.483$
5	$1.789 < X_0 < 1.818$	$0.517 < \mathrm{Min}[X_{AH}] < 0.534$
6	$1.884 < X_0 < 1.915$	$0.665 < \mathrm{Min}[X_{AH}] < 0.678$
7	$1.930 < X_0 < 1.962$	$0.745 < \mathrm{Min}[X_{AH}] < 0.756$

As a result of our detailed analytical and numerical investigations of the previous subsections, we can state the following proposition:

Proposition 4.5.2. *There exist classes of mass function in generalized Vaidya space-times, that produce a locally naked central singularity in four dimensions, but this naked singularity gets eliminated in higher dimensions due to temporal advancement of trapped surface formation.*

Chapter 5

Covariant description of generalized Vaidya spacetime

5.1 Introduction

The spacetime manifold can be described using different approaches such as: expressing the metric $g_{ab}(x^i)$ of the manifold in terms of coordinates x^i, with its connection given through the Christoffel symbols; using the tetrad formalisms, such as the Newman-Penrose null tetrad method (Newman & Penrose 1962), with the connection given through the Ricci rotation coefficients; and using the covariant approach developed by Ehlers (1961), Ellis (1971), and Ellis & Elst (1999), where variables are defined with respect to a partial frame formalism such as the $1+3$ decomposition and later extended to the $1+1+2$ decomposition of general relativity (Clarkson & Barret 2003).

While the first and second approaches are more useful for studying particular spacetimes by choosing coordinates with respect to the symmetries, the covariant approach has been proven to be a strong tool to describe spacetimes since it clearly and easily gives the physics or/and geometry of the spacetime by tensor quantities and relations, which are independent of the coordinate system. In this chapter, we calculate the quantities that describe the geometry of the generalized Vaidya spacetime using $1+1+2$ approach.

5.2 Covariant formalisms

In $1+3$ formalism, the spacetime manifold is decomposed into 'time' and 'space' by means of a fundamental observer. This approach is more useful for investigating small deviations from homogeneity and isotropy in cosmological models. In this method, spacetime is entirely described in terms of scalars, 3-vectors and projected symmetric trace-free (PSTF) 3-tensors and their naturally associated equations obtained by using the Ricci and Bianchi identities (Ehlers 1961; Ellis 1971; Gerold & Clarkson 2004; Maartens 1997; MacCallum 1973; Trumper 1965). A spacetime $(\mathcal{M}, g)$ is split into space and time relative to a congruence of observers with a 4-velocity defined by

$$u^a = \frac{dx^a}{d\tau}, \qquad u_a u^a = -1, \tag{5.1}$$

where τ is proper time measured along the observers' worldlines. These observers are referred to as 'fundamental observers' if they represent the average motion of matter.

The $1+3$ formalism has been a strong and useful tool for understanding different aspects of relativistic cosmology and fluid flows such as the gauge invariant or covariant perturbation formalism (Bruni *et al.* 1992; Dunsby *et al.* 1992; Ellis *et al.* 1990). In the treatment of Dunsby *et al.* (1992), the kinetic and dynamical variables are employed to describe nature with both physical and geometric significance that remain valid in all coordinate systems. This is different from the metric approach which is based on the choice of a reference coordinate system. Recently, the $1+3$ approach was used to develop the linear pertubation theory for fourth order theories of gravity (FOG) (Ananda *et al.* 2008, 2009; Carloni *et al.* 2008). This approach has also been used to study the physics of the cosmic microwave background (CMB) (Challinor & Lasenby 1998; Dunsby 1997; Maartens *et al.* 1999).

The $1+1+2$ covariant formulation of spacetimes on the other hand is a natural extension of the $1+3$ formalism, that suits spherically symmetric spacetimes, including the Schwarzschild solution, Lemaître-Tolman-Bond (LTB) models, Vaidya spacetimes and other classes of Bianchi models. This approach was studied and developed by Clarkson and Barret (Clarkson & Barret 2003), and involves a 'semi-tetrad' where, in addition to the timelike vector field, u^a of the $1+3$ approach, an arbitrary unit vector n^a orthogonal to u^a is introduced such that:

$$n^a n_a = 1, \qquad u^a n_a = 0. \tag{5.2}$$

The projection tensor

$$N_a{}^b \equiv h_a{}^b - n_a n^b = g_a{}^b + u_a u^b - n_a n^b \tag{5.3}$$

then projects vectors orthogonal to n^a and u^a, ($n^a N_{ab} = 0 = u^a N_{ab}$) onto 2-spaces ($N_a{}^a = 2$), called the *sheet*. This sheet carries a natural 2-volume element (Levi-Civita 2-tensor) defined by

$$\varepsilon_{ab} \equiv \varepsilon_{abc} n^c = u^d \eta_{dabc} n^c \qquad \varepsilon_{(ab)} = 0 = \varepsilon_{ab} n^b, \tag{5.4}$$

where ε_{abc} is the volume element of $3-$spaces. The following relations can be worked out using equations (5.3) and (5.4):

$$\varepsilon_{abc} = n_a \varepsilon_{bc} + n_b \varepsilon_{ca} + n_c \varepsilon_{ab}, \tag{5.5}$$

$$\varepsilon_{ab} \varepsilon^{cd} = N_a{}^c N_b{}^d - N_a{}^d N_b{}^c, \tag{5.6}$$

$$\varepsilon_a{}^c \varepsilon_{bc} = N_{ab}, \tag{5.7}$$

$$\varepsilon^{ab} \varepsilon_{ab} = 2. \tag{5.8}$$

Any $3-$vector ψ can be split into a scalar, Ψ, which is part of the vector parallel to n^a, and a 2-vector, Ψ^a, lying in the sheet orthogonal to n^a:

$$\psi^a = \Psi n^a + \Psi^a, \tag{5.9}$$

where $\Psi \equiv \psi_a n^a$ and $\Psi^a \equiv N^{ab}\psi_b \equiv \psi^{\bar{a}}$. Here a bar over the index denotes projection with N_{ab}. Any projected symmetric trace free (PSTF) tensor ψ_{ab}, can therefore be split into scalar, a 2−vector, and 2−tensor parts given as

$$\psi_{ab} = \psi_{\langle ab\rangle} = \Psi\left(n_a n_b - \frac{1}{2}N_{ab}\right) + 2\Psi_{(a}n_{b)} + \Psi_{ab}, \tag{5.10}$$

where

$$\begin{aligned}
\Psi &= n^a n^b \psi_{ab} = -N^{ab}\psi_{ab}, && (5.11a)\\
\Psi_a &= N_a{}^b n^c \psi_{bc} = \psi_{\bar{a}}, && (5.11b)\\
\Psi_{ab} &= \Psi_{\{ab\}} \equiv \left(N_{(a}{}^c N_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right)\psi_{cd}. && (5.11c)
\end{aligned}$$

Curly brackets represent the part of a tensor which is PSTF with respect to n^a. For any object $\psi_{\cdots}{}^{\cdots}$, two new derivatives are defined:

$$\begin{aligned}
\hat{\psi}_{a\ldots b}{}^{c\ldots d} &\equiv n^e D_e \psi_{a\ldots b}{}^{c\ldots d}, && (5.12)\\
\delta_e \psi_{a\ldots b}{}^{c\ldots d} &\equiv N_e{}^j N_a{}^f \cdots N_b{}^g N_h{}^c \cdots N_i{}^d D_j \psi_{f\ldots g}{}^{h\ldots i}, && (5.13)
\end{aligned}$$

where the hat-derivative is the derivative along the vector field n^a in the space orthogonal to u^a and the δ-derivative is a projected derivative on the sheet, with projection on every free index.

The $1 + 1 + 2$ formalism is suitable for studying the pertubations of the so-called local rotationally symmetric (LRS) spacetimes (Betschart & Clarkson 2004; Clarkson & Barret 2003; Clarkson *et al.* 2004). Recently, this covariant approach was used to study exact solutions and perturbations of rotationally symmetric spacetimes in f(R) gravity in FOG (Nzioki, 2013; and references therein). The $1 + 1 + 2$ formalism has also been used to review and study the cosmic censorship conjecture of spherically symmetric spacetimes (Aymen *et al.* 2014), dynamics of black holes and black hole entropy (Giovanni *et al.* 2015).

5.2.1 LRS spacetimes

In this section, we discuss the local rotationally symmetric (LRS) spacetimes basing on (Clarkson & Barret 2003; Ellis 1967). These are spacetimes in which each point has a unique preferred spatial direction that constitutes a local axis of symmetry, that is, all observations are identical under rotations about it and are the same in all directions perpendicular to it. LRS spacetimes may be characterized covariantly by the following scalar quantities (Gerold & Clarkson 2004)

$$\{\mathcal{A}, \theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, Q, \Pi, \Lambda\}, \tag{5.14}$$

where $\mathcal{A}$ is the observer's acceleration, θ is expansion of the spacetime, ϕ represents the sheet expansion, Σ is the shear, Ω is the vorticity, $\mathcal{E}$ represents the Weyl curvature, ξ is the twisting of the sheet, $\mathcal{H}$ is the magnetic Weyl curvature, μ is the energy density, p represents the isotropic pressure of matter, Q is the heat flux, Π is the anisotropic pressure and Λ is the cosmological constant.

5.2.2 LRS class II spacetimes

In LRS class II spacetimes, the vorticity, Ω and hence the twisting ξ are considered to be zero. This also causes the magnetic Weyl curvature $\mathcal{H}$ to vanish. Thus, the LRS class II spacetimes are described by the scalars

$$\{\mathcal{A}, \theta, \phi, \Sigma, \mathcal{E}, \mu, p, Q, \Pi, \Lambda\}, \tag{5.15}$$

defined by the following systems of equations

Propagation:

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \left(\frac{1}{3}\theta + \Sigma\right)\left(\frac{2}{3}\theta - \Sigma\right) - \frac{2}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi, \tag{5.16}$$

$$\hat{\Sigma} - \frac{2}{3}\hat{\theta} = -\frac{3}{2}\phi\Sigma - Q, \tag{5.17}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)Q; \tag{5.18}$$

Evolution:

$$\dot{\phi} \;=\; -\left(\Sigma - \frac{2}{3}\theta\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + Q, \tag{5.19}$$

$$\dot{\Sigma} - \frac{2}{3}\dot{\theta} \;=\; -\mathcal{A}\phi + 2\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)^2 + \frac{1}{3}\left(\mu + 3p - 2\Lambda\right) - \mathcal{E} + \frac{1}{2}\Pi, \tag{5.20}$$

$$\dot{\mathcal{E}} - \frac{1}{3}\dot{\mu} + \frac{1}{2}\dot{\Pi} = \left(\frac{3}{2}\Sigma - \theta\right)\mathcal{E} + \frac{1}{4}\left(\Sigma - \frac{2}{3}\theta\right)\Pi + \frac{1}{2}\phi Q - \frac{1}{2}(\mu + p)\left(\Sigma - \frac{2}{3}\theta\right); \tag{5.21}$$

Propagation/evolution:

$$\hat{\mathcal{A}} - \dot{\theta} \;=\; -\left(\mathcal{A} + \phi\right)\mathcal{A} + \frac{1}{3}\theta^2 + \frac{3}{2}\Sigma^2 + \frac{1}{2}\left(\mu + 3p - 2\Lambda\right), \tag{5.22}$$

$$\dot{\mu} + \hat{Q} \;=\; -\theta\left(\mu + p\right) - \left(\phi + 2\mathcal{A}\right)Q - \frac{3}{2}\Sigma\Pi, \tag{5.23}$$

$$\dot{Q} + \hat{p} + \hat{\Pi} \;=\; -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\frac{4}{3}\theta + \Sigma\right)Q - \left(\mu + p\right)\mathcal{A}. \tag{5.24}$$

The intrinsic Ricci-curvature is given by (Gerold & Clarkson 2004)

$$\begin{aligned} {}^3R_{ab} = & \left[\tfrac{2}{3}(\mu + \Lambda) + \mathcal{E} + \tfrac{1}{2}\Pi + \Sigma^2 - \tfrac{1}{3}\theta\Sigma - \tfrac{2}{9}\theta^2\right] n_a n_b \\ & + \left[\tfrac{2}{3}(\mu + \Lambda) - \tfrac{1}{2}\mathcal{E} - \tfrac{1}{4}\Pi + \tfrac{1}{4}\Sigma^2 + \tfrac{1}{6}\theta\Sigma - \tfrac{2}{9}\theta^2\right] N_{ab}. \end{aligned} \tag{5.25}$$

This then gives the intrinsic Ricci-scalar of the 3-surface as

$${}^3R = -2\left[\mu + \Lambda - \frac{1}{3}\theta^2 + \frac{3}{4}\Sigma^2\right]. \tag{5.26}$$

The vanishing of the sheet distortion, ξ, implies that the sheet is a genuine 2-surface. Using the the Gauss equation for n^a and the 3-Ricci identities, the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to u^a is found to be

$${}^3R_{ab} = -\left[\hat{\phi} + \frac{1}{2}\phi^2\right] n_a n_b - \left[\frac{1}{2}\hat{\phi} + \frac{1}{2}\phi^2 - K\right] N_{ab}, \tag{5.27}$$

where K is the Gaussian curvature of the sheet defined as ${}^2R_{ab} = K N_{ab}$. The 3-Ricci scalar is therefore given as

$${}^3R = -2\left[\hat{\phi} + \frac{3}{4}\phi^2 - K\right]. \tag{5.28}$$

Using equations (5.16), (5.27) and (5.28), the Gaussian curvature, K can be written in the form (Gerold & Clarkson 2004)

$$K = \frac{1}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2 - \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)^2. \tag{5.29}$$

The evolution and propagation equations for K are given by (Gerold & Clarkson 2004)

$$\dot{K} = -\left(\frac{2}{3}\theta - \Sigma\right)K, \tag{5.30}$$

$$\hat{K} = -\phi K. \tag{5.31}$$

Every scalar ϕ in LRS class II spacetimes has to satisfy the commutation relation

$$\dot{\hat{\psi}} - \hat{\dot{\psi}} = -\mathcal{A}\dot{\psi} + \left(\frac{1}{3}\theta + \Sigma\right)\hat{\psi}. \tag{5.32}$$

5.3 Generalized Vaidya spacetime

The generalized Vaidya spacetime, also known as the generalised Vaidya solution, describes the geometry outside a radiating spherically symmetric star. It was given by Wang and Yu (1999), and includes all the known solutions of Einstein field equations with combination of Type I and Type II matter fields.

For the incoming radiation, the metric of the generalized Vaidya spacetime is defined by equation (3.4). The corresponding energy momentum tensor (EMT) can be written as (Husain 1996; Wang & Wu 1999)

$$T_{ab} = \mu k_a k_b + (\rho + \varrho)(k_a l_b + k_b l_a) + \varrho g_{ab}. \tag{5.33}$$

If we define the null vectors k_a and l_a as

$$k_a = \frac{1}{\sqrt{2}}(u_a + n_a) \qquad \text{and} \qquad l_a = \frac{1}{\sqrt{2}}(u_a - n_a), \tag{5.34}$$

then, using the generalized Vaidya metric in equation (3.4) and the energy momentum tensor (5.33), an observer in a static frame ($\dot{r} = 0$) will experience the radiation fluid with the energy density μ, the isotropic pressure $p = \frac{1}{3}(\mu - \rho + 2\varrho)$, the radial heat flux $Q = \mu$ and the anisotropic sheet pressure $\Pi = \frac{2}{3}(\mu - \rho - \varrho)$.

If we consider a static frame ($\dot{r} = 0$), for nonstationary spherically symmetric generalized Vaidya spacetime, the above equations give the following constraints:

$$\Sigma = \frac{2}{3}\theta, \tag{5.35a}$$

$$\mu = -\theta\phi, \tag{5.35b}$$

$$\mathcal{E} = \mu - \mathcal{A}\phi - \frac{1}{3}(\rho - \varrho), \tag{5.35c}$$

$$K = \frac{1}{4}\phi^2 - \mathcal{E} + \frac{1}{3}(2\rho + \varrho). \tag{5.35d}$$

We observe that equation (5.35) consists of four equations in six unknowns, $\mathcal{A}, \phi, \Sigma, \theta, \mathcal{E}$ and μ. If we choose the splitting vectors defined by (Gerold & Clarkson 2004)

$$u^a = \left(1 - \frac{2m(v,r)}{r}\right)^{-\frac{1}{2}} \left(\frac{\partial}{\partial u}\right)^a \tag{5.36}$$

and

$$n^a = -\left(1 - \frac{2m(v,r)}{r}\right)^{-\frac{1}{2}} \left(\frac{\partial}{\partial u}\right)^a + \left(1 - \frac{2m(v,r)}{r}\right)^{\frac{1}{2}} \left(\frac{\partial}{\partial r}\right)^a, \tag{5.37}$$

then we can solve for $\mathcal{A}$ using the formula (Clarkson & Barret 2003)

$$\mathcal{A} = \mathcal{A}^a n_a, \tag{5.38}$$

where $\mathcal{A} = u^b \nabla_a u^a$, and the metric (3.4). We find that

$$\mathcal{A} = \left(\frac{m(v,r)}{r^2} - \frac{m'(v,r)}{r}\right)\left(1 - \frac{2m(v,r)}{r}\right)^{-\frac{1}{2}} - \frac{\dot{m}(v,r)}{r}\left(1 - \frac{2m(v,r)}{r}\right)^{-\frac{3}{2}}. \tag{5.39}$$

The energy density is calculated from equation (5.33) as $\mu = T_{ab}l^a l^b$. This then gives

$$\mu = \left(\frac{2\dot{m}(v,r)}{r^2} + \frac{6m''(v,r)}{r}\right)\left(1 - \frac{2m(v,r)}{r}\right)^{-1}. \tag{5.40}$$

Now, having calculated $\mathcal{A}$, μ and knowing also that $K = r^{-2}$ (for spherical 2-geometry), all other quantities in equation (5.35) can directly be obtained as

$$\phi = 2\left[\left(1 - \frac{2m(v,r)}{r}\right)^{-3} \mathcal{C}^2(v,r) + \mathcal{B}(v,r)\right]^{1/2} - 2\mathcal{A}, \tag{5.41}$$

$$\mathcal{E} = \left(\frac{2\dot{m}(v,r)}{r^2} + \frac{6m''(v,r)}{r}\right)\left(1 - \frac{2m(v,r)}{r}\right)^{-1} - \frac{1}{3}\left(\frac{2m'(v,r)}{r^2} + \frac{m''(v,r)}{r}\right) - \mathcal{A}\phi, \tag{5.42}$$

$$\theta = -\left(\frac{2\dot{m}(v,r)}{r^2} + \frac{6m''(v,r)}{r}\right)\left(1 - \frac{2m(v,r)}{r}\right)^{-1}\phi^{-1}, \tag{5.43}$$

where

$$\mathcal{C} = \left(\frac{m(v,r)}{r^2} - \frac{m'(v,r)}{r}\right)\left(1 - \frac{2m(v,r)}{r}\right) - \frac{\dot{m}(v,r)}{r},$$

$$\mathcal{B} = \left(\frac{2\dot{m}(v,r)}{r^2} + \frac{6m''(v,r)}{r}\right)\left(1 - \frac{2m(v,r)}{r}\right)^{-1} - \frac{2m'(v,r)}{r^2} + r^{-2}.$$

It can easily be shown that for $m(v,r) = m(v)$, these quantities simplify to those of the Vaidya spacetime with outgoing radiation (Gerold & Clarkson 2004)

$$\mu = -\frac{\dot{m}(v)}{r^2}\left(1 - \frac{2m(v)}{r}\right)^{-1}, \tag{5.44}$$

$$\mathcal{A} = \frac{m(v)}{r^2}\left(1 - \frac{2m(v)}{r}\right)^{-1/2} - \frac{\dot{m}(v)}{r}\left(1 - \frac{2m(v)}{r}\right)^{-3/2}, \tag{5.45}$$

$$\phi = \frac{2}{r}\sqrt{1 - \frac{2m(v)}{r}}, \tag{5.46}$$

$$\mathcal{E} = -\frac{2m(v)}{r^3}, \tag{5.47}$$

$$\theta = \frac{\dot{m}(v)}{r}\left(1 - \frac{2m(v)}{r}\right)^{-3/2}. \tag{5.48}$$

The quantities in the above equations, [(5.39) - (5.43)] describe the physical and geometrical properties of the generalized Vaidya spacetime with respect to gravitational collapse, with the following meaning: μ is the energy density of the radiation fluid experienced by an observer in a static frame, $\mathcal{A}$ is the observer's acceleration, ϕ represents the sheet expansion, $\mathcal{E}$ is the electric Weyl curvature (tidal forces) experienced by a star during gravitational collapse, and θ is the expansion of the spacetime.

Chapter 6

Conclusions

In this thesis we developed a general mathematical formalism to study the gravitational collapse of the generalized Vaidya spacetime in the context of the cosmic censorship conjecture.

In Chapter 2, we began by defining the spacetime manifold and some quantities characterizing it. We gave the meaning of a differentiable manifold and defined the metric tensor and its signature. We also defined the connection (Christoffel symbols) and covariant derivatives. We gave the meaning of geodesics and introduced the Riemann, Ricci and Einstein tensors. We then defined the energy momentum tensor which is the source of energy density and momentum in the Einstein field equations. In this chapter, we also showed how the Einsteins field equations were derived using the already defined tensors. Since the end state of gravitational collapse depends on the initial mass function of the collapsing matter and dynamics of the Einstein field equations, it was therefore important to explicitly show how these equations were obtained.

In Chapter 3, we introduced the generalized Vaidya spacetime which is considered as the generalization of the Vaidya spacetime or the radiating Schwarzschild spacetime. We calculated all the associated tensors and defined the energy conditions. We defined the tangent to the nonspacelike geodesics. We studied the structure of the central singularity to show that it can be a node with outgoing radial null geodesics emerging from the singular point with a definite value of the tangent, depending on the nature of the generalized Vaidya mass function and the parameters in the problem.

We calculated the apparent horizon and gave the condition for which the central singularity will be locally naked. We clearly showed that given any realistic mass function, there always exists an open set in the parameter space for which the central singularity is naked and CCC is violated. A similar result is well known for pure Type I matter fields. Hence we can conclude that the occurrence of a naked singularity is a "stable" phenomenon even when the nature of matter field changes by combining a radiation-like field along with a collapsing perfect fluid.

It is also evident that for an open set in the parameter space, these naked central singularities are strong and they cannot be regularised anyway by extension of spacetime through them. This has far reaching consequences as their presence will no longer make the global spacetime future asymptotically simple, and the proofs of black hole dynamics and thermodynamics have to be reformulated.

In Chapter 4, we extended our analysis of the gravitational collapse of generalized Vaidya spacetime in four dimensions, to spacetimes of arbitrary dimensions, in the context of the cosmic censorship conjecture. We defined the generalized Vaidya spacetime in higher dimensions and calculated all the required tensors and variables. Using the same techniques as in Chapter 3, we found the sufficient conditions on the generalized Vaidya mass function, that generates a locally naked central singularity that can causally communicate with an external observer. We carefully investigated the effect of the number of dimensions on the dynamics of the trapped regions, by studying the slope of the apparent horizon curve at the central singularity.

By considering specific examples, we showed that there exist classes of mass functions for which a naked singularity in four dimensions gets covered as we make the transition to higher dimensional spacetimes. Interestingly, the reason for this is same as in the case of dust collapse. From our analysis here, we can easily see that for a wide class of matter fields, a transition to higher dimensions favours trapped surface formation and the epoch of trapping advances as we go to higher dimensions. This makes the vicinity of the central singularity trapped even before the singularity is formed, and hence it is necessarily covered.

Therefore, we can safely conclude that for a large class of matter fields, which include both Type I and Type II matter, transition to higher dimensions does indeed restrict the set of physically realistic initial data, that leads to the formation of a locally naked singularity.

In Chapter 5, we described and analyzed the generalized Vaidya spacetime covariantly, where we started by giving a review to the covariant methods ($1+3$ and $1+2+2$) of describing the spacetime manifold. In the $1+3$ approach, a timelike vector u^a which splits spacetime into 'time' and 'space' is introduced. On the other hand, the $1+1+2$ decomposes the '3-space' relative to a preferred spatial vector n^a. The system of field equations (evolution, propagation and their corresponding constraints) of spacetime is derived from the Bianchi and Ricci identities in these formalisms in a gauge invariant (coordinate independent) manner. From the structure of these equations some important information about the spacetime can be obtained because the covariant decomposition of the spacetime introduces quantities that have a clear physical or geometrical meaning. This then gives an easy and better way of understanding the physics behind the spacetime manifold than using the normal metric approach which depends on the chosen coordinate system. We calculated the scalar quantities that define the spacetime in its entirety.

Finally, the generalized Vaidya spacetime is a more realistic spacetime than pure dust-like matter or perfect fluid, during the later stages of gravitational collapse of a massive star. A collapsing star should always radiate and hence there should be a combination of light-like matter along with a perfect fluid. Therefore a violation of censorship in these models should have novel astrophysical signatures which are yet to be properly deciphered.

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