

Some Amenability Properties on Segal Algebras

By

Peter Olamide Olanipekun



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As the candidate's supervisor I have approved this dissertation for submission.

Dr. O. T. Mewomo

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Dedication

This work is dedicated to God Almighty and to my beloved family.

Abstract

It has been realized that the definition of amenability given by B. E. Johnson in his Classical Memoir of American Mathematical Society in 1972 is too restrictive and does not allow for the development of a rich general theory. For this reason, by relaxing some of the constraints in the definition of amenability via restricting the class of bimodules in question or by relaxing the structure of the derivations, various notions of amenability have been introduced after the pioneering work of Johnson on amenability in Banach algebras. This dissertation is focused on six of these notions of amenability in Banach algebras, namely: contractibility, amenability, weak amenability, generalized amenability, character amenability and character contractibility. The first five of these notions are studied on arbitrary Banach algebras and the last two are studied on some classes of Segal algebras. In particular, results on hereditary properties and several characterizations of these notions are reviewed and discussed. Indeed, we discussed the equivalent of these notions with the existence of a bounded approximate diagonal, virtual diagonal, splitting of exact sequences of Banach bimodules and the existence of a certain Hahn-Banach extension property. Also, some relations that exist between these notions of amenability are also established. We show that approximate contractibility and approximate amenability are equivalent. Some conditions under which the amenability of the underlying group of a Segal algebra implies the character amenability of the Segal algebras are also given. Finally, some new results are obtained which serves as our contribution to knowledge.

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Declaration

This dissertation, in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used in the text, proper reference has been made.

Peter Olamide Olanipekun

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Chapter 1

Introduction

1.1 Background

The theory of Banach algebras is an active area of mathematical research which blends mathematical analysis and abstract algebra. Banach algebras are Banach spaces on which a binary operation of multiplication can be defined. These algebras were named after Stefan Banach (1892 - 1945) since the underlying topological structure of such algebras is a Banach space even though much of the foundation of Banach algebras was laid since 1940 by I. M. Gelfand (1913 - 2009). The name "Banach algebra" was first used by Warren Ambrose (1914 - 1995) in 1945 even though Mitio Nagumo seem to have first considered such algebras in 1936 in a paper where he considered the analytic aspect and ignored the algebraic part. Since then, the theory of Banach algebras has attracted much attention from several researchers for obvious reasons: Banach algebras are known to provide many examples and answers to questions of interest in abstract Harmonic analysis, approximation theory, topology and other related areas. For example, the Banach algebra of continuous linear operators generalizes the concept of a matrix which acts on a Euclidean space. There are many areas of research in Banach algebras. One of such is the notion of amenability, which is main focus and at the center of this dissertation.

Amenability can be traced to 1904 when Lebesgue considered the monotone convergence theorem for his integral. Lebesgue posed the following question: Does monotone convergence theorem follow from existing integral axioms? Also, in [23], Hausdorff considered a similar question: Does there exist a finitely additive set function which is invariant under certain group actions? The answer to these two

questions led to the Banach-Tarski paradox in 1924. There are many versions of this paradox but the strongest form is: For any two bounded sets A and B in three-dimensional space with non-empty interior, there is a partition of A into finitely many sets which can be reassembled to obtain B . This paradox seem to raise many questions and so in an attempt to explain the paradox, the class of amenable groups was introduced by von Neumann in [48]. A locally compact group is said to be amenable if it has a left invariant mean. Since 1940, attention was then shifted from finitely additive measures to means. Let μ be an Haar measure and m be a mean on a set X , then the connection between μ and m is given by $\mu(X) = m(\chi_X)$ and μ induces a left translation invariant linear functional on the space of μ integrable functions. Most of spaces of bounded functions on locally compact group G lack this translation invariant linear functional in $L^\infty(G)$. Thus, the groups possessing such positive invariant mean were first termed amenable by Day in [13]. Consequently, a locally compact group G for which there exists a linear functional $\mu : L^\infty(G) \rightarrow \mathbb{C}$ satisfying $\mu(1) = \|\mu\| = 1$ and $\mu(\delta_{x*}f) = \mu(f)$, ($x \in G, f \in L^\infty(G)$) is called amenable. Since then, the study of amenability shifted to locally compact groups (see [39], [40] and [44] for further details).

We remark that there is a distinction between group amenability and Banach algebras amenability. Amenability was first studied for Banach algebras by Johnson in his Classical Memoir of American Mathematical Society in 1972, see [24]. Johnson show that the Banach algebra $L^1(G)$ is amenable if and only if the locally compact group G is amenable as a group. This is equivalent to the statement that the first Hochschild cohomology group with coefficient in the dual group algebra vanishes. This then motivated his definition for the amenability of Banach algebras. According to him, Banach algebras satisfying this cohomological triviality conditions are amenable. It has been realized that the definition of amenability given by Johnson [24] is too restrictive and so does not allow for the development of a rich general theory and also too weak enough to include a variety of interesting examples. For this reason, by relaxing some of the constraints in the definition of amenability via

- (i) restricting the class of bimodules in question,
- (ii) relaxing the structure of the derivations,
- (iii) the combination of (i) and (ii) above,

various notions of amenability have been recently introduced after the pioneering work of Johnson in 1972. In this dissertation, we consider six of these notions of

amenability, namely: contractibility, amenability, weak amenability, generalized amenability, character amenability and character contractibility. The first five are studied for general Banach algebras while the last two are studied for a class of Banach algebra, called Segal algebras.

1.2 Research Motivation

Many notions of amenability on Banach algebras have been introduced in the literature by different authors and studied over different classes of Banach algebras. Johnson in [24], Dales in [9] and Runde in [44] proved many of the hereditary properties and characterizations of amenable Banach algebras and since then, other authors have shown that similar hereditary properties and characterizations hold for weaker notions of amenability. Thus it will be of interest to critically review and survey most of these results and see how they can be extended to other classes of Banach algebras where the philosophy have not been fully explored.

In [19], Ghahramani and Loy introduced the generalized notion of amenability with the aim of getting a Banach algebra without bounded approximate identity which still have this generalized notion. Samea [45] studied essential amenability and approximate essential amenability for Segal algebras. Also, in [38], Nasr-Isfahani and Nemati introduced and studied the essential character amenability of Banach algebras. Motivated by these works, we relate the amenability of a symmetric abstract Segal algebra with the amenability of the underlying Banach algebra and also study some notions of amenability on the Segal algebra $S^1(G)$ in relation to some properties of the locally compact group G .

1.3 Objectives

The main objectives of this research work are to:

1. review some known and relevant results on some notions of amenability in Banach algebras in literature,
2. investigate these notions of amenability for general Banach algebras,
3. characterize some of these notions of amenability for Segal algebras,
4. investigate some of these notions for a class of Banach algebra, called the Segal algebras in relation to the structure of the underlying groups.

Chapter 2

Preliminaries on Banach Algebras

In this chapter, we introduce and define some terms and concepts that are relevant to this work. Some basic and general results are also presented. Our standard references are [6], [9] and [12].

2.1 Basic Definitions

The following definitions are well known concepts in the theory of Banach spaces.

Definition 2.1.1. *Let \mathcal{A} be a vector space over the scalar field $\mathbb{F} = \mathbb{C}$ or \mathbb{R} . \mathcal{A} is called an algebra if there is a map*

$$\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}, \quad (a, b) \mapsto ab$$

called the product or multiplication such that for $a, b, c \in \mathcal{A}$ and $\alpha \in \mathbb{F}$:

$$(ab)c = a(bc), \quad a(b + c) = ab + ac, \quad (a + b)c = ac + bc, \quad (\alpha a)b = \alpha(ab) = a(\alpha b).$$

\mathcal{A} has the structure of both a vector space and a ring and is called a complex (real) algebra if $\mathbb{F} = \mathbb{C}$ ($\mathbb{F} = \mathbb{R}$). Unless otherwise stated, all algebras in this dissertation are over the complex field.

Definition 2.1.2. *Let \mathcal{A} be an algebra.*

- (i) An element $e \in \mathcal{A}$ is called a left (right) identity in \mathcal{A} if $ea = a$ ($ae = a$) for all $a \in \mathcal{A}$.*
- (ii) An element $a^{-1} \in \mathcal{A}$ is said to be the left (right) inverse of $a \in \mathcal{A}$ if $a^{-1}a = e$ ($aa^{-1} = e$).*

(iii) \mathcal{A} has an identity if $a = ea = ae$, that is, the left and right identities coincide. An element in \mathcal{A} is said to be invertible if it has an inverse.

Definition 2.1.3. An algebra norm is a map $\|\cdot\| : \mathcal{A} \rightarrow \mathbb{C}$ satisfying

(i) $\|a\| \geq 0$

(ii) $\|a\| = 0 \iff a = 0$

(iii) $\|\alpha a\| = |\alpha| \|a\|$

(iv) $\|a + b\| \leq \|a\| + \|b\|$

(v) $\|ab\| \leq \|a\| \|b\|$, for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

A normed algebra is the pair $(\mathcal{A}, \|\cdot\|)$.

Remark 2.1.4. A Banach algebra is a complete normed algebra. We shall simply write \mathcal{A} for $(\mathcal{A}, \|\cdot\|)$ whenever there would be no confusion. If $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{A}$, then \mathcal{A} is called a commutative Banach algebra. Observe that if e is the identity for \mathcal{A} , then by Property (v), $\|e\| = \|e \cdot e\| \leq \|e\| \|e\|$. That is, $\|e\| \geq 1$. \mathcal{A} is said to be a unital Banach algebra if it has the identity e such that $\|e\| = 1$. Although, some Banach algebras possess identity and as such the definition of a spectrum in terms of inverses of elements of the Banach algebra comes more naturally leading to elegant concepts in spectral analysis. Yet, certain Banach algebras \mathcal{A} which lack an identity can be isometrically embedded into another Banach algebra of the direct sum $\mathcal{A} \oplus \mathbb{C}$ in a process often referred to as the unitization of \mathcal{A} .

Remark 2.1.5. 1. It is easy to see that multiplication is jointly continuous in a Banach algebra \mathcal{A} . Let $(x_n), (y_n) \subset \mathcal{A}$ with $x, y \in \mathcal{A}$, such that $x_n \rightarrow x$, $y_n \rightarrow y$, as $n \rightarrow \infty$ then

$$\begin{aligned} \|x_n y_n - xy\| &= \|x_n y_n - x_n y + x_n y - xy\| = \|x_n(y_n - y) + (x_n - x)y\| \\ &\leq \|x_n(y_n - y)\| + \|(x_n - x)y\| \leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \\ &\rightarrow 0. \end{aligned}$$

2. Let $(\mathcal{A}, \|\cdot\|_1)$ be a Banach algebra. Then there exists a norm $\|\cdot\|_2$ which is equivalent to $\|\cdot\|_1$ for which $(\mathcal{A}, \|\cdot\|_2)$ is also a Banach algebra. Moreover, if $(\mathcal{A}, \|\cdot\|_1)$ is unital with the identity e , then $(\mathcal{A}, \|\cdot\|_2)$ is also unital and $\|e\|_2 = 1$.

2.2 Constructing New Banach Algebras from Old Ones

2.2.1 Direct sum of algebras

Let \mathcal{A} and \mathcal{B} be Banach algebras, the sum $\mathcal{A} + \mathcal{B}$ of \mathcal{A} and \mathcal{B} , is the set $\{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$. Suppose that $\mathcal{A} \cap \mathcal{B} = \{0\}$, then $\mathcal{A} + \mathcal{B}$ is denoted by $\mathcal{A} \oplus \mathcal{B}$ and called the internal direct sum of \mathcal{A} and \mathcal{B} . The algebra $\mathcal{A} \oplus \mathcal{B}$ is defined as the Cartesian product $\mathcal{A} \times \mathcal{B}$ with the algebra multiplication $(a, b) \cdot (a', b') = (aa', bb')$ where $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$ and norm $\|(a, b)\|_{\oplus} := \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}}$. It is easy to see that $\mathcal{A} \oplus \mathcal{B}$ is a Banach algebra since

$$\begin{aligned}
 \|(a, b)(a', b')\|_{\oplus} &= \|(aa', bb')\|_{\oplus} \\
 &= \|aa'\|_{\mathcal{A}} + \|bb'\|_{\mathcal{B}} \\
 &\leq \|a\|_{\mathcal{A}}\|a'\|_{\mathcal{A}} + \|b\|_{\mathcal{B}}\|b'\|_{\mathcal{B}} \\
 &\leq \|a\|_{\mathcal{A}}\|a'\|_{\mathcal{A}} + \|a\|_{\mathcal{A}}\|b'\|_{\mathcal{B}} + \|b\|_{\mathcal{B}}\|a'\|_{\mathcal{A}} + \|b\|_{\mathcal{B}}\|b'\|_{\mathcal{B}} \\
 &= (\|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}})(\|a'\|_{\mathcal{A}} + \|b'\|_{\mathcal{B}}) \\
 &= \|(a, b)\|_{\oplus} \|(a', b')\|_{\oplus}.
 \end{aligned}$$

The algebra $\mathcal{A} \oplus \mathcal{B}$ is unital if and only if \mathcal{A} and \mathcal{B} are unital. Suppose that e and e' are the identities of \mathcal{A} and \mathcal{B} respectively, then (e, e') is the identity for $\mathcal{A} \oplus \mathcal{B}$.

2.2.2 Unitization

Consider the direct sum $\mathcal{A}^{\#} = \mathcal{A} \oplus \mathbb{C}$ where \mathcal{A} is a non-unital Banach algebra. For $(a, \alpha), (b, \beta) \in \mathcal{A}^{\#}$, define the product by $(a, \alpha) \cdot (b, \beta) = (ab + \beta a + \alpha b, \alpha\beta)$ and the norm $\|\cdot\|_{\#}$ by $\|(a, \alpha)\|_{\#} = \|a\|_{\mathcal{A}} + |\alpha|$, then $\mathcal{A}^{\#}$ is a Banach algebra with identity $(0, 1)$. The sub-multiplicative property is satisfied as follows:

$$\begin{aligned}
 \|(a, \alpha)(b, \beta)\|_{\#} &= \|(ab + \beta a + \alpha b, \alpha\beta)\|_{\#} = \|(ab + \beta a + \alpha b)\| + |\alpha\beta| \\
 &\leq \|a\|\|b\| + \|a\|\|\beta\| + |\alpha|\|b\| + |\alpha|\|\beta\| \\
 &= \|a\|\|b\| + |\alpha|\|b\| + \|a\|\|\beta\| + |\alpha|\|\beta\| \\
 &= (\|a\| + |\alpha|)\|b\| + (\|a\| + |\alpha|)\|\beta\| \\
 &= (\|a\| + |\alpha|)(\|b\| + \|\beta\|) = \|(a, \alpha)\|_{\#} \|(b, \beta)\|_{\#}.
 \end{aligned}$$

$\mathcal{A}^{\#}$ is called the unitization of \mathcal{A} and is commutative if and only if \mathcal{A} is commutative. If \mathcal{A} has an identity then $\mathcal{A}^{\#} \simeq \mathcal{A}$.

2.2.3 Quotient and opposite algebras

Let \mathcal{A} be a Banach algebra. We recall that a subset I of \mathcal{A} is a two sided ideal of \mathcal{A} if I is a subspace of \mathcal{A} and

$$aI, Ia \subset I,$$

for all $a \in \mathcal{A}$. For an ideal I in \mathcal{A} ,

$$\mathcal{A}/I = \{a + I : a \in \mathcal{A}\}$$

is the quotient algebra. \mathcal{A}/I has a natural algebraic structure well defined by

$$(a + I)(b + I) = ab + I, \quad (a, b \in \mathcal{A}).$$

If \mathcal{A} is a Banach algebra and I is a closed ideal in \mathcal{A} , then \mathcal{A}/I is a Banach algebra with the quotient norm. The map $\varphi : \mathcal{A} \rightarrow \mathcal{A}/I$ defined by $\varphi(a) = a + I$ is a surjective homomorphism with $\ker(\varphi) = I$.

Theorem 2.2.1. *Let I be a proper closed two sided ideal of a Banach algebra \mathcal{A} . Let $[x]$ in \mathcal{A}/I denote the equivalence class of x . Define product by $[x][y] = [xy]$ and the quotient norm by $\|[x]\| = \inf_{r \in I} \|x + r\|$. Then*

(i) \mathcal{A}/I is a Banach algebra.

(ii) \mathcal{A}/I is unital if \mathcal{A} is unital.

(iii) The identity $[e]$ in \mathcal{A}/I is such that $\|[e]\| = 1$.

Proof. It is well known that \mathcal{A}/I is a Banach space.

(i) Let $s, t \in I$ and $x, y \in \mathcal{A}$, then $xs, ty, ts \in I$ and so there exists $r \in I$ such that $r = xs + ty + ts$. Now, it follows that

$$\begin{aligned} \|[x][y]\| = \|[xy]\| &= \inf_{r \in I} \|xy + r\| \\ &= \inf_{s, t \in I} \|xy + xs + ty + ts\| \\ &= \inf_{s, t \in I} \|x(y + s) + t(y + s)\| \\ &= \inf_{s, t \in I} \|(x + t)(y + s)\| \\ &\leq \inf_{s, t \in I} (\|x + t\| \|y + s\|) \\ &= \inf_{t \in I} \|x + t\| \cdot \inf_{s \in I} \|y + s\| \\ &= \|[x]\| \|[y]\|. \end{aligned}$$

(ii) Let e be the identity for \mathcal{A} then for all $a \in \mathcal{A}$, it follows that

$$[a] = [ae] = [a][e]$$

and

$$[a] = [ea] = [e][a].$$

Thus $[e]$ is the identity for \mathcal{A}/I .

(iii) Let $\|e\| = 1$, then

$$\|[e]\| = \inf_{r \in I} \|e + r\| \leq \|e\| = 1.$$

But $\|[e]\| \not\leq 1$. So $\|[e]\| = 1$. □

Let \mathcal{A} be a Banach algebra. The Banach algebra \mathcal{A}^{op} formed by reversing the product in \mathcal{A} is called the opposite algebra of \mathcal{A} . Both \mathcal{A} and \mathcal{A}^{op} have the same algebraic structure and in the case that \mathcal{A} is commutative, then $\mathcal{A} = \mathcal{A}^{\text{op}}$. Also, if \mathcal{A} is unital then \mathcal{A}^{op} is unital.

2.3 Examples in Different Classes

2.3.1 Algebras over locally compact groups

A topological group G is a group which also has the Hausdorff topological space structure for which the maps

$$(x, y) \mapsto xy : G \times G \rightarrow G$$

and

$$x \mapsto x^{-1} : G \rightarrow G \quad (x, y \in G)$$

relating the two structures are continuous. A locally compact group G is a topological group for which the topology on G is locally compact.

We recall that every locally compact group G has a left Haar measure. A left Haar measure on G is a positive Borel measure μ on G such that for every measurable subset U of G

H1 $\mu(xU) = \mu(U)$ for each $x \in G$.

H2 $\mu(U) = \inf\{\mu(V) : V \subset U, V \text{ is open}\}$ and $\mu(V) > 0$ for every non empty open set V .

H3 $\mu(U) = \sup\{\mu(W) : W \subset U, W \text{ is compact}\}$.

Property H1 implies that μ is left translation invariant while Properties H2 and H3 imply that μ is a regular Borel measure.

Example 2.3.1 (Group algebra). *Let G be a locally compact group and μ be a left Haar measure on G . The space $L^1(G)$ of all measurable functions on G satisfying*

$$\int_G |f| d\mu < \infty$$

equipped with the norm

$$\|f\|_1 = \left(\int_G |f| d\mu \right)$$

is a Banach space and with the product defined by the convolution

$$(f * g)(x) = \int_G f(y)g(y^{-1}x) d\mu(y),$$

where $f, g \in L^1(G)$, $x \in G$, it becomes a Banach algebra.

Indeed, let $x, y \in G$, then $y^{-1} \in G$. Set $y^{-1}x = z \in G$, so that

$$\begin{aligned} \|f * g\|_1 &= \int_G |(f * g)(x)| d\mu \\ &= \int_G \left| \int_G (f(y)g(y^{-1}x)) \right| d\mu(y) d\mu \\ &\leq \int_G \int_G |(f(y)g(y^{-1}x))| d\mu(y) d\mu \\ &\leq \int_G |(f(y)| d\mu(y) \int_G |g(y^{-1}x)| d\mu \\ &\leq \int_G |(f(y)| d\mu(y) \int_G |g(z)| d\mu(z) \\ &= \|f\|_1 \|g\|_1. \end{aligned}$$

Thus $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

It was shown in [47] that $L^1(G)$ is commutative as a Banach algebra if and only if G is an abelian group.

Example 2.3.2 (Measure algebra). *Let G be a locally compact group. The space $M(G)$, of all bounded regular complex Borel measures on G equipped with the total variation norm $\|\mu\| = |\mu|(G)$ for all $\mu \in M(G)$ is a Banach space. By defining the convolution product*

$$(\mu * \nu)(f) = \int_G \int_G f(xy) d\mu(x) d\nu(y) \quad (f \in C_0(G), \mu, \nu \in M(G), x, y \in G),$$

($M(G), \|\cdot\|$) becomes a Banach algebra. $M(G)$ can be identified with the dual space of $C_0(G)$ which is the set of all continuous functions from G to \mathbb{C} vanishing at infinity with duality given by $\mu(f) = \int_G f(x) d\mu(x)$ ($f \in C_0(G), \mu \in M(G)$).

Recall that a function f vanishes at infinity if for any $\varepsilon > 0$, there is a compact subset W of a locally compact Hausdorff space X such that $|f(x)| < \varepsilon$ for each $x \in X \setminus W$.

Example 2.3.3 (Fourier algebra). *Let G be a locally compact group and let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The space $A_p(G)$ of all functions $f \in C_0(G)$ defined by*

$$x \mapsto \sum_{n=1}^{\infty} (g_n * h_n(x^{-1})) \quad G \rightarrow \mathbb{C}$$

such that $\sum_{n=1}^{\infty} \|g_n\|_p \|h_n\|_q < \infty$, where $(g_n)_{n \in \mathbb{N}} \in L^p(G)$, $(h_n)_{n \in \mathbb{N}} \in L^q(G)$ and equipped with the norm

$$\|f\|_A = \inf \left\{ \sum_{n=1}^{\infty} \|g_n\|_p \|h_n\|_q : f = \sum_{n=1}^{\infty} (g_n * h_n) \right\}$$

is a Banach space. The Banach space $(A_p(G), \|\cdot\|_A)$ with pointwise product is a Banach algebra commonly referred to as the Figa-Talamanca Herz algebra. The Fourier algebra is a Figa-Talamanca Herz algebra when $p = 2$ and it is denoted by $A(G)$.

2.3.2 Semigroup algebras

A semigroup is a non empty set with an associative binary operation given by

$$(s_1, s_2) \mapsto s_1 s_2 \quad S \times S \rightarrow S.$$

Let S be a semigroup. The space $\ell^1(S)$ of all functions from S to \mathbb{C} defined by

$$\ell^1(S) = \{f : S \rightarrow \mathbb{C} : \sum_{x \in S} |f(x)| < \infty\},$$

with the norm $\|\cdot\|_1$ given by

$$\|f\|_1 = \sum_{x \in S} |f(x)| \quad (f \in \ell^1)$$

is a Banach space. $(\ell^1(S), \|\cdot\|_1)$ becomes a Banach algebra under the convolution product defined by

$$(f * g)(x) = \sum \{f(y)g(z) : y, z \in S, yz = x\} \quad (x \in S).$$

The sub-multiplicative property can easily be verified as follows: Let $f, g \in \ell^1(S)$ then

$$\begin{aligned} \|f * g\| &= \sum_{x \in S} |(f * g)(x)| \\ &= \sum_{x \in S} \left| \sum_{yz=x} f(y)g(z) \right| \\ &\leq \sum_{x \in S} \sum_{yz=x} |f(y)||g(z)| \\ &\leq \sum_{(y,z) \in S \times S} |f(y)||g(z)| \\ &= \sum_{y \in S} \sum_{z \in S} |f(y)||g(z)| \\ &= \sum_{y \in S} |f(y)| \sum_{z \in S} |g(z)| \\ &= \|f\| \|g\|. \end{aligned}$$

Every $f \in \ell^1(S)$ can be represented as

$$f = \sum_{y \in S} f(y)\delta_y,$$

where δ_y is the characteristic function of S by

$$\delta_y(x) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y. \end{cases}$$

$\ell^1(S)$ is a commutative Banach algebra if and only if S is commutative. Also, $\ell^1(S^\#) = \ell^1(S)^\#$. Suppose that S has an identity e_S , then δ_{e_S} is the identity of $\ell^1(S)$ and $\|\delta_{e_S}\| = 1$. It is still possible for $\ell^1(S)$ to possess an identity in the case where S is non-unital.

2.3.3 Operator algebras

Let X be a Banach space, the space $\mathfrak{B}(X)$ denotes the set of all bounded linear operators on X . Then $\mathfrak{B}(X)$ is a Banach space with the operator norm

$$\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}.$$

It becomes a Banach algebra with the product specified by the following composition

$$(ST)(x) = (S \circ T)(x) = S(Tx), \quad (x \in X, S, T \in \mathfrak{B}(X)).$$

Indeed,

$$\|TS(x)\| \leq \|T\|\|S(x)\| \leq \|T\|\|S\|\|x\|$$

which implies $\|TS\| \leq \|T\|\|S\|$. Thus, the Banach space $\mathfrak{B}(X)$ is a non-commutative, unital Banach algebra with identity operator as the unit.

Closed ideals of $\mathfrak{B}(X)$ such as ideals $A(X)$ of approximable operators, ideal $K(X)$ of compact operator and ideal $N(X)$ of nuclear operators are also Banach algebras.

2.3.4 Function algebras

There are several examples of Banach algebra whose element are functions on some specified set.

1. Let W be a compact Hausdorff space. The space $\mathfrak{C}(W)$ of all continuous functions on W , equipped with the norm

$$\|f\|_\infty := \sup_{x \in W} |f(x)|$$

is a Banach space. $(\mathfrak{C}(W), \|\cdot\|_\infty)$ becomes a commutative Banach algebra with pointwise product.

2. (Disc algebra). Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be an open unit disk and let $\mathcal{D} = \mathfrak{C}(\overline{\mathbb{D}})$. Then the space

$$\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{D} : f \text{ is analytic on } \mathbb{D}\}$$

is a closed subalgebra of \mathcal{D} consisting of all continuous functions on the closure $\overline{\mathbb{D}}$. $\mathcal{A}(\mathbb{D})$ is a commutative unital Banach algebra under the supremum norm and pointwise product.

3. Let \mathbb{D} be an open unit disk. The space $H^\infty(\mathbb{D})$ of all bounded and analytic functions from \mathbb{D} to \mathbb{C} is a commutative unital Banach algebra under pointwise product and supremum norm.
4. Let (X, τ) be a topological space. The space $\mathfrak{C}(X)$ of all continuous complex valued functions on X equipped with the norm

$$\|f\| = \sup_{x \in X} |f(x)|, \quad f \in \mathfrak{C}(X), x \in X$$

is a Banach space and a Banach algebra with the pointwise product. Indeed,

$$\begin{aligned} \|fg\| &= \sup_{x \in X} |fg(x)| = \sup_{x \in X} |f(x)g(x)| = \sup_{x \in X} |f(x)||g(x)| \\ &\leq \sup_{x \in X} |f(x)| \sup_{x \in X} |g(x)| = \|f\| \|g\|. \end{aligned}$$

$(\mathfrak{C}(X), \|\cdot\|)$ is a unital commutative Banach algebra, where the constant function is the identity.

2.4 Some Basic Concepts and General Results

2.4.1 Approximate identity

Definition 2.4.1. Let \mathcal{A} be a unital Banach algebra with identity e . An element $a \in \mathcal{A}$ is said to be invertible if there exists $a^{-1} \in \mathcal{A}$ such that $aa^{-1} = a^{-1}a = e$. The element a^{-1} is referred to as the inverse of a and is unique for every invertible element in \mathcal{A} .

Remark 2.4.2. The set of all invertible elements of \mathcal{A} is denoted by $\text{Inv}\mathcal{A}$. It can easily be shown that $\text{Inv}\mathcal{A}$ forms a group with multiplication defined by the group operation. Indeed, let $a, b \in \text{Inv}\mathcal{A}$. Since $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$ then it follows that $ab \in \text{Inv}\mathcal{A}$ and $a^{-1} \in \text{Inv}\mathcal{A}$.

Theorem 2.4.3. *Let \mathcal{A} be a unital Banach algebra with identity e . If $a \in \mathcal{A}$ and $\|a\| < 1$ then $(e - a) \in \text{Inv}(\mathcal{A})$. Moreover,*

$$(e - a)^{-1} = \sum_{n=0}^{\infty} a^n. \quad (2.1)$$

Proof. Since \mathcal{A} is a Banach algebra, then by the sub-multiplicative property of \mathcal{A} , it follows that $\|a^n\| \leq \|a\|^n$. Since $\|a\| \leq 1$, then $\sum_{n=0}^{\infty} \|a\|^n$ converges and consequently, $\sum_{n=0}^{\infty} \|a^n\|$ converges. Now,

$$\begin{aligned} (e - a) \sum_{n=0}^{\infty} a^n &= (e - a) \lim_{N \rightarrow \infty} \sum_{n=0}^N a^n = \lim_{N \rightarrow \infty} (e - a) \sum_{n=0}^{\infty} a^n \\ &= \lim_{N \rightarrow \infty} \left(\sum_{n=0}^N e a^n - \sum_{n=0}^N a^{n+1} \right) \\ &= \lim_{N \rightarrow \infty} (e - a + a - a^2 + a^2 - a^3 + \cdots + a^{N-1} - a^N + a^N - a^{N+1}) \\ &= \lim_{N \rightarrow \infty} (e - a^{N+1}) = e \end{aligned}$$

Similarly,

$$\left(\sum_{n=0}^{\infty} a^n \right) (e - a) = e.$$

Whence $\sum_{n=0}^{\infty} a^n = (e - a)^{-1}$. □

Corollary 2.4.4. *Let $a \in \text{Inv}\mathcal{A}$ and $b \in \mathcal{A}$ such that $\|a^{-1}\| = \frac{1}{\alpha}$ with $\|b\| = \beta < \alpha$. Then $a - b \in \text{Inv}\mathcal{A}$. Moreover,*

$$\|(a - b)^{-1} - a^{-1} + a^{-1} b a^{-1}\| \leq \frac{\beta^2}{\alpha^2(\beta - \alpha)}.$$

Proof. Let $\|a^{-1}b\| \leq \|a^{-1}\| \|b\| \leq \frac{\beta}{\alpha} < 1$. By Theorem 2.4.3, $(e - a^{-1}b) \in \text{Inv}\mathcal{A}$ and since $a - b = a(e - a^{-1}b)$, then $a - b \in \text{Inv}\mathcal{A}$. Also, $(a - b)^{-1} = (e - a^{-1}b)^{-1} a^{-1}$.

Thus,

$$\begin{aligned}
\|(a-b)^{-1} - a^{-1} + a^{-1}ba^{-1}\| &= \|(e - a^{-1}b)^{-1}a^{-1} - a^{-1} + a^{-1}ba^{-1}\| \\
&= \left\| \left((e - a^{-1}b)^{-1} - e + a^{-1}b \right) a^{-1} \right\| \\
&\leq \left(\|a^{-1}b - e\| + \|(e - a^{-1}b)^{-1}\| \right) \|a^{-1}\| \\
&\leq \left(\|a^{-1}\| \|b\| + \|e\| + \|(e - a^{-1}b)\| \right) \|a^{-1}\| \\
&\leq \left(\frac{\beta}{\alpha} + 1 + \frac{1}{1 - \frac{\beta}{\alpha}} \right) \frac{1}{\alpha} = \frac{\beta^2}{\alpha^2(\beta - \alpha)}.
\end{aligned}$$

□

Corollary 2.4.5. *Inv \mathcal{A} is an open subset of \mathcal{A} .*

Many natural examples of Banach algebras are non-unital, but rather possess what is called an approximate identity which is a net that behaves like a multiplicative identity in the limit. Banach algebras which are known to have bounded approximate identity possess the factorization property. This is a theorem proved by Cohen in 1959 and was later extended to module form in 1970 by Hewitt and Ross (see [14] for a more detailed historical account). A unital algebra \mathcal{A} factors trivially since for all $a \in \mathcal{A}$, $a = ae = ea$. The existence of bounded approximate identity in a Banach algebra \mathcal{A} also accounts for the identity of the second dual \mathcal{A}'' with respect to the Arens product.

Definition 2.4.6. *Let Δ be a partially ordered set. The set Δ is called a directed set if for $\alpha_1, \alpha_2 \in \Delta$ there exists $\alpha \in \Delta$ such that $\alpha_k \leq \alpha$ for $k = 1, 2$. Let X be a set. Then the map*

$$x : \alpha \mapsto x(\alpha), \quad x : \Delta \rightarrow X$$

is called a net in X .

Lemma 2.4.7 (Zorn's Lemma). *Let X be a partially ordered non empty set such that every totally ordered subset of X has an upper bound, then X has a maximal element.*

Definition 2.4.8. *Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra. Then*

(i) *a left approximate identity for \mathcal{A} is a net $(e_\alpha) \subset \mathcal{A}$ such that $e_\alpha a$ converges in the norm to $a \in \mathcal{A}$, that is*

$$\lim_{\alpha} e_\alpha a = a \quad \text{or} \quad \|e_\alpha a - a\| \rightarrow 0 \quad (a \in \mathcal{A});$$

(ii) a right approximate identity for \mathcal{A} is a net $(e_\alpha) \subset \mathcal{A}$ such that ae_α converges in the norm to $a \in \mathcal{A}$, that is

$$\lim_{\alpha} ae_\alpha = a \quad \text{or} \quad \|ae_\alpha - a\| \rightarrow 0 \quad (a \in \mathcal{A});$$

(iii) an approximate identity for \mathcal{A} is a net $(e_\alpha) \subset \mathcal{A}$ which is both left and right approximate identity for \mathcal{A} ;

(iv) $K > 0$ is called a bound for a left or right approximate identity $(e_\alpha) \subset \mathcal{A}$ if $\sup_{\alpha} \|e_\alpha\| \leq K$;

(v) A left or right approximate identity $(e_\alpha) \subset \mathcal{A}$ is bounded if there exists some $K > 0$ satisfying the condition in (iv) above;

(vi) \mathcal{A} is said to have a bounded approximate identity if it has a left and a right bounded approximate identity;

(vii) \mathcal{A} is approximately unital if it has a bounded approximate identity;

(viii) \mathcal{A} has a left approximate unit if for all $a \in \mathcal{A}$ and $\epsilon > 0$ there exists $u \in \mathcal{A}$ (depending on a and ϵ) such that $\|ua - a\| < \epsilon$;

(ix) \mathcal{A} has a right approximate unit if for all $a \in \mathcal{A}$ and $\epsilon > 0$ there exists $u \in \mathcal{A}$ (depending on a and ϵ) such that $\|au - a\| < \epsilon$;

(x) \mathcal{A} has a left or right approximate unit bounded by $K > 0$, if the element $u \in \mathcal{A}$ is such that $\|u\| \leq K$.

Remark 2.4.9. If the net is a sequence, the approximate identity is said to be sequential.

Proposition 2.4.10. Let \mathcal{A} be a Banach algebra.

(i) Let $(e_\alpha)_{\alpha \in \Delta}$ be a left approximate identity for \mathcal{A} and suppose that $(f_\beta)_{\beta \in \Lambda}$ is a bounded net in \mathcal{A} , then $(f \circ e)_{(\alpha, \beta) \in \Delta \times \Lambda}$ is a left approximate identity for \mathcal{A} and is bounded if (e_α) is bounded.

(ii) Suppose that \mathcal{A} has a left approximate identity of bound m and right approximate identity of bound n . Then \mathcal{A} has an approximate identity of bound $m + n + mn$.

(iii) Let $(e)_{\alpha \in \Delta}$ and $(f)_{\beta \in \Lambda}$ be bounded left and right approximate identity for \mathcal{A} respectively. Then $(f \circ e)_{(\alpha, \beta) \in \Delta \times \Lambda}$ is a bounded two-sided approximate identity for \mathcal{A} .

2.4.2 Spectrum of a Banach algebra

Definition 2.4.11. Let \mathcal{A} be a unital Banach algebra with identity e . The spectrum of an element $a \in \mathcal{A}$ is defined as

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : \lambda e - a \notin \text{Inv } \mathcal{A}\}.$$

We denote the spectrum of $a \in \mathcal{A}$ by $\sigma(a)$, and we shall write λ for $\lambda e \in \mathbb{C}$ when no confusion can arise.

Example 2.4.12. (i) Consider the Banach algebra \mathbb{C} . The spectrum of $\lambda \in \mathbb{C}$ is given by $\sigma(\lambda) = \{\lambda\}$.

(ii) Let X be a finite dimensional Banach space and $T \in \mathfrak{B}(X)$. It was shown in Section 2.3.3 that $\mathfrak{B}(X)$ is a Banach algebra. Now the spectrum of $\mathfrak{B}(X)$ is precisely the set of all eigenvalues of T .

Proof. Recall that a linear map is bijective if and only if it is invertible. Let I be the identity map, then it follows that

$$\begin{aligned} \text{Inv}\mathfrak{B}(X) &= \{T \in \mathfrak{B}(X) : \ker T = \{0\}\} \\ \sigma(T) &= \{\lambda I \in \mathbb{C} : \lambda I - T \notin \text{Inv}\mathfrak{B}(X)\} \\ \sigma(T) &= \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \neq \{0\}\} \\ \sigma(T) &= \{\lambda \in \mathbb{C} : (\lambda I - T)(x) = 0\} \quad \text{for some } x \neq 0 \\ \sigma(T) &= \{\lambda \in \mathbb{C} : \lambda x - Tx = 0\} \\ \sigma(T) &= \{\lambda \in \mathbb{C} : \lambda x = Tx\}. \end{aligned}$$

□

Definition 2.4.13. Let \mathcal{A} be a unital Banach algebra.

(i) The resolvent set $\rho_{\mathcal{A}}(a)$ of $a \in \mathcal{A}$ is defined by

$$\rho_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : \lambda - a \in \text{Inv}\mathcal{A}\}.$$

That is $\rho_{\mathcal{A}}(a) = \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$.

(ii) The resolvent function of $a \in \mathcal{A}$ is the map $\varrho : \rho_{\mathcal{A}}(a) \rightarrow \mathcal{A}$ defined by $\varrho(\lambda) = (\lambda - a)^{-1}$.

We recall that a division algebra is an algebra \mathcal{A} such that every non-zero element $a \in \mathcal{A}$ is invertible. That is, $\text{Inv}\mathcal{A} = \mathcal{A} \setminus \{0\}$.

Theorem 2.4.14. *Let \mathcal{A} be a unital Banach algebra. If $a \in \mathcal{A}$, then the spectrum $\sigma_{\mathcal{A}}(a)$ is non-empty and compact.*

Proof. Suppose for contradiction that $\sigma_{\mathcal{A}}(a) = \emptyset$, then $\rho(a) = \mathbb{C}$ and so for $\lambda \in \mathbb{C}$, the resolvent function $\varrho : \mathbb{C} \rightarrow \mathcal{A}$ is given by $\varrho(\lambda) = (\lambda - a)^{-1}$.

For $\lambda_0 \in \rho(a)$, let $f \in \mathcal{A}'$, $\lambda \in \rho(a) \setminus \{\lambda_0\}$ and set $g = f \circ \varrho$. Then

$$\begin{aligned} \varrho(\lambda) - \varrho(\lambda_0) &= \frac{1}{\lambda - a} - \frac{1}{\lambda_0 - a} = \frac{\lambda_0 - a - \lambda + a}{(\lambda - a)(\lambda_0 - a)} = \frac{\lambda_0 - \lambda}{(\lambda - a)(\lambda_0 - a)} \\ &= (\lambda_0 - \lambda)(\lambda - a)^{-1}(\lambda_0 - a)^{-1} = (\lambda_0 - \lambda)\varrho(\lambda)\varrho(\lambda_0). \end{aligned}$$

Now,

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_0} \frac{f(\varrho(\lambda)) - f(\varrho(\lambda_0))}{\lambda - \lambda_0} &= \lim_{\lambda \rightarrow \lambda_0} f \left(\frac{\varrho(\lambda) - \varrho(\lambda_0)}{\lambda - \lambda_0} \right) \\ &= - \lim_{\lambda \rightarrow \lambda_0} f \left((\lambda - a)^{-1}(\lambda_0 - a)^{-1} \right) \\ &= -f \left((\lambda_0 - a)^{-2} \right) = -f(\varrho(\lambda))^2. \end{aligned}$$

Thus g is analytic. But

$$\lim_{|\lambda| \rightarrow \infty} \varrho(\lambda) = \lim_{|\lambda| \rightarrow \infty} (\lambda - a)^{-1} = \lim_{|\lambda| \rightarrow \infty} \frac{1}{\lambda - a} = 0.$$

Thus $\varrho(\lambda)$ converges and hence bounded. It then follows by Liouville's theorem that $gf \circ \varrho = 0$. In particular, $f(\varrho(0)) = 0$ for each $f \in \mathcal{A}'$. Thus by Hahn-Banach Theorem $\varrho(0) = 0$ implying that $\varrho(\lambda)$ is not invertible. This contradicts the fact that $\varrho(\lambda)$ is invertible and so we conclude that $\sigma_{\mathcal{A}}(a)$ is non-empty.

Next we show that $\sigma(a)$ is compact. It suffices to show that $\sigma(a)$ is closed and bounded. That $\sigma(a)$ is bounded is clear since $\sigma(a) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$. It remains to show that $\sigma(a)$ is closed or equivalently that the resolvent set $\rho(a)$ is open. This follows from the fact that the map $\varphi : \lambda \mapsto \lambda e - a, \mathbb{C} \rightarrow \mathcal{A}$ is continuous and $\text{Inv} \mathcal{A}$ is open. Whence the set $\rho(a) = \varphi^{-1}(\text{Inv} \mathcal{A})$ is open in \mathbb{C} . \square

Theorem 2.4.15 (Gelfand - Mazur). *Let \mathcal{A} be a division unital Banach algebra with an identity e , then there exists a unique isometric isomorphism of \mathcal{A} onto \mathbb{C} . That is, $\mathcal{A} = \mathbb{C}e$.*

Proof. Let $a \in \mathcal{A}$. Then by Theorem 2.4.14, $\sigma_{\mathcal{A}}(a) \neq \emptyset$. This implies that there exists $\lambda \in \sigma_{\mathcal{A}}(a)$ such that $\lambda e - a \notin \text{Inv} \mathcal{A}$. Since \mathcal{A} is a division algebra, then $\lambda e - a \notin \mathcal{A}$ and so $\lambda e - a = 0$. Thus $\lambda e = a$.

We now define a map

$$\varphi : \mathcal{A} \rightarrow \mathbb{C}e, \quad a \mapsto \lambda e.$$

That φ is bijective is clear. Also, for $a, b \in \mathcal{A}$, $\varphi(ab) = \varphi(a)\varphi(b)$ and so φ is an isomorphism. Finally, we show that $\|\varphi(a)\| = \|a\|$. This follows clearly from

$$\|a\| = \|\lambda e\| = \|\varphi(a)\|.$$

□

Theorem 2.4.16 (Spectral Mapping Property for Polynomials). *Let \mathcal{A} be a complex algebra with an identity and let $a \in \mathcal{A}$. If p is a complex polynomial. Then*

$$\sigma(p(a)) = \{p(\lambda) : \lambda \in \sigma(a)\}.$$

Definition 2.4.17 (Spectral radius). *Let \mathcal{A} be a unital Banach algebra and let $a \in \mathcal{A}$. The spectral radius $r_{\mathcal{A}}(a)$ of a is defined by the set*

$$r_{\mathcal{A}}(a) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}.$$

Theorem 2.4.18. *Let \mathcal{A} be a unital Banach algebra and let $a \in \mathcal{A}$. Then*

- (i) $\rho(a)$ is an open subset of \mathbb{C} .
- (ii) $r_{\mathcal{A}}(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}$.
- (iii) For each $n \in \mathbb{N}$ and $r_0 > r_{\mathcal{A}}(a)$,

$$a^n = \frac{1}{2\pi i} \int_{|\zeta|=r_0} \zeta^n (\zeta e - a)^{-1} d\zeta.$$

Remark 2.4.19. *Theorem 2.4.18(ii) is known as the Spectral radius formula.*

Proposition 2.4.20. *Let \mathcal{A} be a commutative unital Banach algebra and let $a, b \in \mathcal{A}$. Then*

- (i). $r_{\mathcal{A}}(\lambda a) = |\lambda|r_{\mathcal{A}}(a)$.
- (ii). $r_{\mathcal{A}}(ab) \leq r_{\mathcal{A}}(a)r_{\mathcal{A}}(b)$.

Proof. (i). It follows easily that

$$\begin{aligned} r_{\mathcal{A}}(\lambda a) &= \inf_{n \in \mathbb{N}} \|(\lambda a)^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|\lambda^n a^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} |\lambda^n|^{\frac{1}{n}} \|a\|^{\frac{1}{n}} \\ &= |\lambda| \inf_{n \in \mathbb{N}} \|a\|^{\frac{1}{n}} = |\lambda|r_{\mathcal{A}}(a). \end{aligned}$$

(ii). It is known that $(ab)^n = a^n b^n$ and $\|a^n\| \leq \|a\|^n$ for all $n \in \mathbb{N}$.

Now,

$$\begin{aligned} r_{\mathcal{A}}(ab) &= \inf_{n \in \mathbb{N}} \|(ab)^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|a^n b^n\|^{1/n} \\ &\leq \inf_{n \in \mathbb{N}} \|a^n\|^{1/n} \inf_{n \in \mathbb{N}} \|b^n\|^{1/n} = r_{\mathcal{A}}(a)r_{\mathcal{A}}(b). \end{aligned}$$

□

2.4.3 Ideals, quotient and homomorphism

Definition 2.4.21. Let \mathcal{A} be a Banach algebra.

1. A left ideal of \mathcal{A} is a subset I of \mathcal{A} such that
 - (a) \mathcal{A} is a vector subspace of \mathcal{A} ,
 - (b) $ab \in I$ for all $a \in \mathcal{A}$, $b \in I$.
2. A right ideal of \mathcal{A} is a subset I of \mathcal{A} such that
 - (a) \mathcal{A} is a vector subspace of \mathcal{A} ,
 - (b) $ba \in I$ for all $a \in \mathcal{A}$, $b \in I$.
3. A two-sided ideal I of \mathcal{A} is a subset of \mathcal{A} which is both a left and right ideal of \mathcal{A} . That is,
 - (a) I is a vector subspace of \mathcal{A} ,
 - (b) both $\mathcal{A}I \subseteq I$ and $IA \subseteq I$.

Example 2.4.22. (i) Let \mathcal{A} be a Banach algebra and $\mathcal{A}^\#$ be the unitization of \mathcal{A} . Then \mathcal{A} is a closed ideal of $\mathcal{A}^\#$.

(ii) Let Λ be a closed subset of a compact space Ω . The sets $I(\Lambda)$ and $K(\Lambda)$ defined by

$$\begin{aligned} I(\Lambda) &= \{f \in \mathfrak{C}(\Omega) : f|_{\Lambda} = 0\} \\ K(\Lambda) &= \{f \in \mathfrak{C}(\Omega) : f = 0 \text{ on a neighbourhood of } \Lambda\} \end{aligned}$$

are ideals in $\mathfrak{C}(\Omega)$. In fact, $I(\Lambda)$ is closed and $K(\Lambda)$ is dense in $I(\Lambda)$.

Definition 2.4.23. Let \mathcal{A} be a Banach algebra.

1. An ideal I in \mathcal{A} is called a proper ideal if I is not equal to \mathcal{A} . That is $I \subsetneq \mathcal{A}$.
2. An ideal I is called a maximal ideal of \mathcal{A} if I is a proper ideal such that there is no proper ideal that contains I . That is, there is no proper ideal K of \mathcal{A} such that $I \subsetneq K \subsetneq \mathcal{A}$.

Lemma 2.4.24. *Let \mathcal{A} be a unital Banach algebra and let I be an ideal of \mathcal{A} . Then I is a proper ideal if and only if $I \cap \text{Inv}\mathcal{A} = \emptyset$.*

Theorem 2.4.25. *Let \mathcal{A} be a unital Banach algebra.*

- (i). *The closure \bar{I} of I is also a proper ideal of \mathcal{A} if I is a proper ideal of \mathcal{A} .*
- (ii). *Any maximal ideal of \mathcal{A} is closed.*

Definition 2.4.26. *Let \mathcal{A} be a Banach algebra and I be a closed ideal in \mathcal{A} . The set $\mathcal{A}/I = \{a + I : a \in \mathcal{A}\}$ is called the quotient Banach algebra. The natural algebraic structure of \mathcal{A} is defined by the product*

$$(a + I)(b + I) = (ab + I), \quad (a, b \in \mathcal{A}).$$

Proposition 2.4.27. *Let \mathcal{A} be a Banach algebra and I be a closed ideal in \mathcal{A} . Then \mathcal{A}/I is a Banach algebra with the quotient norm.*

Definition 2.4.28. *Let \mathcal{A} and \mathcal{B} be Banach algebras.*

- (i) *A linear map $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is an algebra homomorphism if*

$$\varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}).$$

- (ii) *The kernel of φ is the set $\ker \varphi = \{a \in \mathcal{A} : \varphi(a) = 0\}$.*
- (iii) *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism. If φ is injective then φ is called a monomorphism.*
- (iv) *An epimorphism of \mathcal{A} onto \mathcal{B} is a surjective homomorphism.*
- (v) *An algebra isomorphism of \mathcal{A} onto \mathcal{B} is a bijective homomorphism of \mathcal{A} into \mathcal{B} .*

Remark 2.4.29. (i) *Suppose that \mathcal{A} and \mathcal{B} are both unital Banach algebras with identity $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ respectively. Then the homomorphism φ maps the identity $e_{\mathcal{A}}$ to $e_{\mathcal{B}}$, that is, $\varphi(e_{\mathcal{A}}) = e_{\mathcal{B}}$. This easily follows from the fact that for any $a \in \mathcal{A}$,*

$$\varphi(a) = \varphi(ae_{\mathcal{A}}) = \varphi(e_{\mathcal{A}}a) = \varphi(a)\varphi(e_{\mathcal{A}}) \Rightarrow \varphi(e_{\mathcal{A}}) = \varphi(a)(\varphi(a))^{-1} = e_{\mathcal{B}}.$$

- (ii) For $a \in \mathcal{A}$ and for all $b \in \text{Inv}\mathcal{A}$, $\varphi(a) \neq 0$.
- (iii) $\ker \varphi$ is a proper ideal of \mathcal{A} provided it is non-zero.
- (iv) The quotient map $\varphi : a \mapsto a + I$, $\mathcal{A} \rightarrow \mathcal{A}/I$ is a surjective homomorphism with $\ker \varphi = I$.

Example 2.4.30. (i) Let \mathcal{A} be a Banach algebra. Then the map $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\varphi(b) = b \quad (b \in \mathcal{A})$$

is an homomorphism of \mathcal{A} into \mathcal{A} .

- (ii) Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ and $\phi : \mathcal{B} \rightarrow \mathbb{C}$ be algebra homomorphisms. Then $\phi \circ \varphi : \mathcal{A} \rightarrow \mathbb{C}$ is also an algebra homomorphism.

2.4.4 Gelfand theory

Definition 2.4.31. Let \mathcal{A} be an algebra over the scalar field \mathbb{C} . Then a character on \mathcal{A} is a homomorphism $\varphi : \mathcal{A} \rightarrow \mathbb{C}$; that is, a non-zero linear functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ which satisfies $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. We denote by $\sigma(\mathcal{A})$ the set of all characters on \mathcal{A} .

Example 2.4.32. Let X be a compact topological space. Suppose that $\mathcal{A} = \mathfrak{C}(X)$. For some $x \in X$, define the map

$$\varphi_x : \mathcal{A} \rightarrow \mathbb{C}, \quad \varphi_x(f) = f(x) \quad (f \in \mathcal{A}).$$

Then φ_x is a character on \mathcal{A} .

Lemma 2.4.33. Let \mathcal{A} be a commutative unital Banach algebra and let I be a maximal ideal of \mathcal{A} .

- (i) If $\varphi \in \sigma(\mathcal{A})$, then $\ker \varphi = I$.
- (ii) The map $\phi : \lambda + I \mapsto \lambda$, $\mathcal{A}/I \rightarrow \mathbb{C}$ is an isometric isomorphism.

Proof. (i) Since $\varphi \in \sigma(\mathcal{A})$, then φ is a non-zero homomorphism. We know that $\ker \varphi$ is a proper ideal of \mathcal{A} (since for any $a \in \mathcal{A}$, $b \in \ker \varphi$ it follows that $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(a) \cdot 0 = 0$ so that $ab \in \ker \varphi$). Suppose that J is another ideal of \mathcal{A} such that $\ker \varphi \subsetneq J$. Let $a \in J \setminus \ker \varphi$ such that $\varphi(a) \neq 0$, then $b = \varphi(a)^{-1}a \in J$ and $\varphi(b) = 1$. Since $\varphi(e) = 1$, then it follows that $e - b \in \ker \varphi$ and so $e = b + e - b \in J$. By Lemma 2.4.24, $J = \mathcal{A}$. Thus, it must be that $\ker \varphi$ is a maximal ideal and so $\ker \varphi = I$.

(ii) Since I is a maximal ideal of \mathcal{A} then I is closed by Theorem 2.4.25. We know that \mathcal{A}/I is a commutative unital Banach algebra. To prove the result, it suffices to show that every non-zero element of \mathcal{A}/I is invertible and this easily follows from Theorem 2.4.15. For $a \in \mathcal{A}$, define the set $J = \{ab + x : x \in I, b \in \mathcal{A}\}$. Since I is a maximal ideal of \mathcal{A} and \mathcal{A}/I is a commutative Banach algebra, then J is an ideal and $I \subsetneq J$. It follows that $J = \mathcal{A}$. Thus, it holds that $e \in J$ and $ab + x = e$, for some $b \in \mathcal{A}, x \in I$. Now observe that for some $x \in I$,

$$(a + I)(b + I) = ab + I = ab + x + I = e + I.$$

Hence $(a+I)$ is invertible. Since every non-zero elements of \mathcal{A}/I is invertible, then by applying Theorem 2.4.15, the result follows. \square

Theorem 2.4.34. *Let \mathcal{A} be a unital Banach algebra. Then every maximal ideal of \mathcal{A} is closed.*

Proof. Let I be a maximal ideal of \mathcal{A} and \bar{I} be the closure of I . Since $I \neq \mathcal{A}$ then I does not contain invertible elements and so $I \subseteq \mathcal{A} \setminus (\text{Inv}\mathcal{A})$. By Corollary 2.4.5, $\text{Inv}\mathcal{A}$ is open and so the complement $(\text{Inv}\mathcal{A})^c = \mathcal{A} \setminus (\text{Inv}\mathcal{A})$ is closed. Since I is contained in \bar{I} , then the inclusion $I \subseteq \bar{I} \subseteq \mathcal{A} \setminus \text{Inv}\mathcal{A}$ holds. Since I is maximal, then either $I = \bar{I}$ or $\bar{I} = \mathcal{A} \setminus \text{Inv}\mathcal{A}$. But the latter is not the case and so I is closed. \square

Theorem 2.4.35. *Let \mathcal{A} be a Banach algebra, then the homomorphism $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is continuous.*

Corollary 2.4.36. *Let \mathcal{A} be a Banach algebra and let $\varphi \in \sigma(\mathcal{A})$. Then φ is continuous.*

It should be noted that not all Banach algebra have characters on them but many examples of Banach algebra satisfying commutative property are known to have characters defined on them. The following theorem therefore guarantees that the character space of commutative Banach algebra is non empty.

Theorem 2.4.37. *Let \mathcal{A} be a commutative Banach algebra then there exists some character for \mathcal{A} .*

Proof. Suppose that every element $a \in \mathcal{A}$ is invertible, then \mathcal{A} and \mathbb{C} are isomorphic and the isomorphism $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is a character. Suppose that there are some

elements of \mathcal{A} which are not invertible. Let I be a maximal ideal of \mathcal{A} and let $a \in \mathcal{A}$ be such that $a \in \mathcal{A} \setminus \text{Inv}\mathcal{A}$. Then by Zorn's Lemma $a\mathcal{A}$ is a proper ideal of \mathcal{A} and $a\mathcal{A} \subset I$. But by Lemma 2.4.33(i), it holds that $I = \ker \varphi$ for some $\varphi \in \sigma(\mathcal{A})$. Thus $\sigma(\mathcal{A}) \neq \emptyset$ and so \mathcal{A} has at least one character. \square

Example 2.4.38. For $n > 1$, let $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ and let (e_{ij}) be the $n \times n$ identity matrix, that is, every (i, j) entry is 1 for $i = j$ and 0 otherwise. It is clear that \mathcal{A} is non-commutative. Moreover, $\mathcal{M}_n(\mathbb{C})$ possesses no character.

Theorem 2.4.39. Let \mathcal{A} be a commutative unital Banach. Then the mapping $\varphi \mapsto \ker \varphi$ is a bijection from $\sigma(\mathcal{A})$ onto the set of maximal ideals of \mathcal{A} .

Corollary 2.4.40. Let \mathcal{A} be a commutative unital Banach algebra. Then for $a \in \mathcal{A}$,

- (i) $a \in \text{Inv}\mathcal{A}$ if and only if $\varphi(a) \neq 0$ for all $\varphi \in \sigma(\mathcal{A})$,
- (ii) $\sigma_{\mathcal{A}}(a) = \{\varphi(a) : \varphi \in \sigma(\mathcal{A})\}$,
- (iii) $r_{\mathcal{A}}(a) = \sup_{\varphi \in \sigma(\mathcal{A})} |\varphi(a)|$.

Definition 2.4.41. Let \mathcal{A} be a commutative unital Banach algebra. Let $a \in \mathcal{A}$ and $\varphi \in \sigma(\mathcal{A})$. Define the map

$$\hat{a} : \varphi \mapsto \varphi(a), \quad \sigma(\mathcal{A}) \rightarrow \mathbb{C}.$$

The map $\mathcal{G} : a \mapsto \hat{a}, \quad \mathcal{A} \rightarrow \mathfrak{C}(\sigma(\mathcal{A}))$ is an homomorphism and

$$\|\hat{a}\|_{\infty} \leq \|a\| \quad (a \in \mathcal{A}).$$

The map \mathcal{G} is called the Gelfand transform of \mathcal{A} and \hat{a} is called the Gelfand transform of a .

Theorem 2.4.42 (Gelfand representation theorem). Let \mathcal{A} be a commutative unital Banach algebra. Then

- (i) $\sigma_{\mathcal{A}}(a) = \hat{a}(\sigma(\mathcal{A}))$.
- (ii) $r_{\mathcal{A}}(a) = |\hat{a}|_{\mathcal{A}}$.
- (iii) $a \in \text{Inv}\mathcal{A}$ if and only if $\hat{a} \in \text{Inv}\mathfrak{C}(\sigma(\mathcal{A}))$.

2.4.5 Tensor product

There are different ways of defining the product on Banach algebras some of which was considered in Section 2.2. One of such ways is the tensor product which we now consider in this section. For more details on this section see [26].

Definition 2.4.43. *Let $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_3 be vector spaces. An algebraic tensor product of \mathcal{A}_1 and \mathcal{A}_2 is a pair (θ, \mathcal{A}_3) , where θ is a bilinear map from $\mathcal{A}_1 \times \mathcal{A}_2$ into \mathcal{A}_3 , called the tensor map such that if \mathcal{A}_4 is any vector space and if $f : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathcal{A}_4$ is any bilinear map, then there exists a unique $g : \mathcal{A}_3 \rightarrow \mathcal{A}_4$ satisfying $f = g \circ \theta$.*

For $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, we write $a_1 \otimes a_2$ for $\theta(a_1, a_2)$. Also the tensor product of \mathcal{A}_1 and \mathcal{A}_2 is written as $\mathcal{A}_1 \otimes \mathcal{A}_2$. A tensor is an element $t \in \mathcal{A}_1 \otimes \mathcal{A}_2$ which is given by

$$t = \sum_{i=1}^n a_i \otimes b_i \quad n \in \mathbb{N}, a_i \in \mathcal{A}_1, b_i \in \mathcal{A}_2.$$

Although there are different kinds of norm that can be defined on the tensor product of Banach algebras, but we shall consider what is called the projective tensor product norm.

Definition 2.4.44. *Let \mathcal{A}_1 and \mathcal{A}_2 be Banach algebras and let $t \in \mathcal{A}_1 \otimes \mathcal{A}_2$, the projective tensor norm is defined as*

$$\|t\|_\pi = \inf \left\{ \sum_{i=1}^n \|a_i\| \|b_i\| < \infty : t = \sum_{i=1}^n a_i \otimes b_i, a_i \in \mathcal{A}_1, b_i \in \mathcal{A}_2 \right\}.$$

Proposition 2.4.45. *Let \mathcal{A}_1 and \mathcal{A}_2 be Banach algebras. Then the vector space*

$$\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2 = \{a \otimes b : a \in \mathcal{A}_1, b \in \mathcal{A}_2\}$$

with the projective tensor norm and multiplication defined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ ($a, a' \in \mathcal{A}_1, b, b' \in \mathcal{A}_2$) is a Banach algebra.

Remark 2.4.46. (i) $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ is called the completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ under the projective tensor norm.

(ii) The product $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ is uniquely determined.

(iii) $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ is commutative if and only if both \mathcal{A} and \mathcal{B} are commutative.

(iv) $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ is unital if and only if both \mathcal{A}_1 and \mathcal{A}_2 are unital. Suppose that e_1 and e_2 are identities for \mathcal{A}_1 and \mathcal{A}_2 respectively, then $e_1 \otimes e_2$ is the identity for $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$.

Theorem 2.4.47. Let \mathcal{A}_1 and \mathcal{A}_2 be Banach algebras. Let $\phi : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathcal{B}$ be a bilinear map, then there exists a unique linear map $\psi : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{B}$ such that $\psi(a \otimes b) = \phi(a, b)$ ($a \in \mathcal{A}_1, b \in \mathcal{A}_2$).

Definition 2.4.48. Let \mathcal{A} be a Banach algebra. The linear map $\pi : a \otimes b \mapsto ab$, $\mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ is called the projective map. The dual projective map $\pi' : \mathcal{A}' \rightarrow (\mathcal{A} \hat{\otimes} \mathcal{A})'$ is defined by $\langle a \otimes b, \pi'(f) \rangle = \langle ab, f \rangle$ ($f \in \mathcal{A}', a, b \in \mathcal{A}$). Moreover, we can identify $((\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2)', \|\cdot\|_\pi)$ with $\mathfrak{B}(\mathcal{A}_1, \mathcal{A}_2')$ for the Banach algebras \mathcal{A}_1 and \mathcal{A}_2 .

2.4.6 Modules

In this section, we give a brief introduction to the concept of bimodule (see [3] for details). The following definitions are taken from [32].

Definition 2.4.49. Let \mathcal{A} be an algebra. A right \mathcal{A} -module X is an additive group X together with a map $\cdot : X \times \mathcal{A} \rightarrow X$ which satisfies the following conditions:

- (i) $(x + y) \cdot a = x \cdot a + y \cdot a$,
- (ii) $x \cdot (a + b) = x \cdot a + x \cdot b$,
- (iii) $x \cdot (ab) = (x \cdot a) \cdot b$, ($x \in X, a, b \in \mathcal{A}$).

Definition 2.4.50. Let \mathcal{A} be an algebra, a left \mathcal{A} -module X is an additive group X together with a map $\cdot : \mathcal{A} \times X \rightarrow X$ which satisfies the following conditions:

- (i) $a \cdot (x + y) = a \cdot x + a \cdot y$,
- (ii) $(a + b) \cdot x = a \cdot x + b \cdot x$,
- (iii) $a \cdot (b \cdot x) = (ab) \cdot x$, ($x \in X, a, b \in \mathcal{A}$).

Definition 2.4.51. Let \mathcal{A} be an algebra and let X be an additive commutative group, X is called an \mathcal{A} -bimodule, if X is both a left \mathcal{A} -module and a right \mathcal{A} -module, and also satisfies $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ ($a, b \in \mathcal{A}, x \in X$).

Definition 2.4.52. Let \mathcal{A} be a Banach algebra. A Banach space which is also a right \mathcal{A} -module is called a right Banach \mathcal{A} -module if there is a constant $k > 0$ such that $\|x \cdot a\| \leq k\|a\|\|x\|$ ($x \in X, a \in \mathcal{A}$).

Definition 2.4.53. Let \mathcal{A} be a Banach algebra. A Banach space X which is also a left \mathcal{A} -module is called a left Banach \mathcal{A} -module if there exists a constant $k > 0$ such that $\|a \cdot x\| \leq k\|a\|\|x\|$ ($x \in X, a \in \mathcal{A}$).

Definition 2.4.54. Let \mathcal{A} be a Banach algebra and let X be an additive commutative group. If X is both left and right Banach \mathcal{A} -module, then X is called a Banach \mathcal{A} -bimodule.

Remark 2.4.55. By renorming X , we may take $k = 1$.

Definition 2.4.56. Let \mathcal{A} be a unital Banach algebra with identity e .

- (i) The left \mathcal{A} -module X is called unital if $e \cdot x = x$ ($x \in X$).
- (ii) The right \mathcal{A} -module X is called unital if $x \cdot e = x$ ($x \in X$).
- (iii) A unital Banach \mathcal{A} bimodule is a left and right Banach \mathcal{A} -bimodule if $e \cdot x = x = x \cdot e$ ($x \in X$).

Definition 2.4.57. (i) Let \mathcal{A} be a Banach algebra. The Banach \mathcal{A} -bimodule X is called commutative or symmetric if $a \cdot x = x \cdot a$, ($a \in \mathcal{A}, x \in X$).

- (ii) For a commutative Banach algebra \mathcal{A} , a Banach \mathcal{A} -bimodule is a commutative \mathcal{A} -bimodule.

Example 2.4.58. (i) Let \mathcal{A} be a Banach algebra, then we can consider \mathcal{A} as a Banach \mathcal{A} -bimodule. The left and right module actions are specified by

$$a \cdot x = ax, \quad x \cdot a = xa \quad (a, x \in \mathcal{A}).$$

Thus the product in \mathcal{A} defines the module operation.

- (ii) Let \mathcal{A} and \mathcal{B} be Banach algebras. Suppose that X and Y are Banach \mathcal{A} -bimodule and Banach \mathcal{B} -bimodule respectively. Then the projective tensor product $X \hat{\otimes} Y$ of X and Y is an \mathcal{A} -bimodule with module operation specified by

$$a \cdot (x \otimes y) = a \cdot x \otimes y, \quad (x \otimes y) \cdot a = x \otimes y \cdot a \quad (a \in \mathcal{A}, x \in X, y \in Y).$$

- (iii) Let \mathcal{A} be a Banach algebra.

- (i) Suppose that X is a right Banach \mathcal{A} -bimodule, then X' is a left Banach \mathcal{A} -bimodule. The module action is specified by

$$\langle x, a \cdot f \rangle = \langle x \cdot a, f \rangle \quad (a \in \mathcal{A}, x \in X, f \in X').$$

(ii) Suppose that X is a left Banach \mathcal{A} -bimodule, then X' is a right Banach \mathcal{A} -bimodule. The module action is specified by

$$\langle x, f \cdot a \rangle = \langle a \cdot x, f \rangle \quad (a \in \mathcal{A}, x \in X, f \in X').$$

(iii) The dual space X' of the Banach \mathcal{A} -bimodule X is also a Banach \mathcal{A} -bimodule with module action specified in (i) and (ii) above. X' is called the dual Banach \mathcal{A} -bimodule.

(iv) Let \mathcal{A} and \mathcal{B} be Banach algebras, and let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a continuous homomorphism. Then \mathcal{B} is a Banach \mathcal{A} -bimodule with operation specified by

$$a \cdot b = \varphi(a)b, \quad b \cdot a = b\varphi(a) \quad (a \in \mathcal{A}, b \in \mathcal{B}).$$

2.4.7 Second dual of a Banach algebra and Arens products

Let \mathcal{A} be a Banach algebra, the second dual space of \mathcal{A} denoted by \mathcal{A}'' is a Banach space. Recall that for $\Phi \in \mathcal{A}''$ and $\phi \in \mathcal{A}'$, the canonical image of $a \in \mathcal{A}$ in \mathcal{A}'' is given by $\langle \phi, \Phi \rangle = \langle a, \phi \rangle$ ($a \in \mathcal{A}$). The dual Banach space \mathcal{A}'' can be made into a Banach algebra by defining certain products on it. These products were defined by Arens in [3] and are denoted by \square and \diamond . They are called the first and second Arens product on the second dual of Banach algebras respectively. We now define Arens products on the second dual \mathcal{A}'' of \mathcal{A} and establish that \mathcal{A}'' is a Banach algebra with respect to both first and second Arens products (see also [2], [5], [12], [15] and [16] for more details).

Definition 2.4.59. (Arens products). *Let \mathcal{A} be a Banach algebra. Then*

1. for $\phi \in \mathcal{A}'$,

$$\langle ba, \phi \rangle = \langle b, a \cdot \phi \rangle, \quad \langle b, \phi \cdot a \rangle = \langle ab, \phi \rangle \quad (a, b \in \mathcal{A}),$$

2. for $\Phi \in \mathcal{A}''$, and $\phi \in \mathcal{A}'$ define $\phi \cdot \Phi$ and $\Phi \cdot \phi$ in \mathcal{A}' by

$$\langle \Phi, a \cdot \phi \rangle = \langle a, \phi \cdot \Phi \rangle, \quad \langle a, \Phi \cdot \phi \rangle = \langle \Phi, \phi \cdot a \rangle \quad (a \in \mathcal{A}).$$

3. for $\Phi, \Psi \in \mathcal{A}''$,

$$\langle \Phi \square \Psi, \phi \rangle = \langle \Phi, \Psi \cdot \phi \rangle, \quad \langle \Phi \diamond \Psi, \phi \rangle = \langle \Psi, \phi \cdot \Phi \rangle \quad (\phi \in \mathcal{A}').$$

Equivalently, Arens products can also be defined in terms of some convergent nets in \mathcal{A} as follows:

Let (a_α) and (b_β) be convergent nets in \mathcal{A} such that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$. Then

$$\Phi \square \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta, \quad \Phi \diamond \Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta$$

where all the limits are taken in the $\sigma(\mathcal{A}'', \mathcal{A}')$ -topology on \mathcal{A}'' .

In the case that the first and second Arens products are equal, then \mathcal{A} is said to be Arens regular.

Theorem 2.4.60. *Let \mathcal{A} be a Banach algebra, then both (\mathcal{A}'', \square) and $(\mathcal{A}'', \diamond)$ are Banach algebras containing \mathcal{A} as a closed subalgebra.*

Remark 2.4.61. 1. *For a commutative Banach algebra \mathcal{A} , it is easy to see that $\phi \cdot \Phi = \Phi \cdot \phi$ ($\phi \in \mathcal{A}', \Phi \in \mathcal{A}''$) and $\Phi \square \Psi = \Psi \diamond \Phi$ ($\Phi, \Psi \in \mathcal{A}''$) and so $(\mathcal{A}'', \diamond) = (\mathcal{A}'', \square)^{op}$.*

2. *The map $R_\Phi : \Psi \mapsto \Psi \square \Phi$ is continuous on $(\mathcal{A}'', \sigma(\mathcal{A}'', \mathcal{A}))$ for each $\Phi \in \mathcal{A}''$. Also, the map $L_a : \Psi \mapsto a \square \Psi$ is continuous on $(\mathcal{A}'', \sigma(\mathcal{A}'', \mathcal{A}))$ for each $a \in \mathcal{A}$. Thus the first Arens product is continuous.*

3. *Suppose that \mathcal{A} has no identity element, then it is possible to identify $(\mathcal{A}^{\#\#}, \square)$ with $(\mathcal{A}'', \square)^\#$.*

4. *The first and second Arens product \square and \diamond respectively are both associative on \mathcal{A}'' .*

2.4.8 Topological centers

. Recall that the center of Banach algebra \mathcal{A} is the collection of all element of \mathcal{A} that commutes with every other elements in \mathcal{A} . For the second dual \mathcal{A}'' of \mathcal{A} , we talk about the left and right topological center which we now define as follows (for more details on this section, see [11] and [17]).

Definition 2.4.62. *The left and right topological center of \mathcal{A}'' denoted by $\mathfrak{Z}_l(\mathcal{A}'')$ and $\mathfrak{Z}_r(\mathcal{A}'')$ respectively are*

$$\mathfrak{Z}_l(\mathcal{A}'') := \{\Phi \in \mathcal{A}'' : \Phi \square \Psi = \Phi \diamond \Psi \quad (\Psi \in \mathcal{A}'')\}$$

$$\mathfrak{Z}_r(\mathcal{A}'') := \{\Phi \in \mathcal{A}'' : \Psi \square \Phi = \Psi \diamond \Phi \quad (\Psi \in \mathcal{A}'')\}.$$

Remark 2.4.63. 1. It can be shown that $\mathfrak{Z}(\mathcal{A}'', \square) \subset \mathfrak{Z}_r(\mathcal{A}'')$. Let $\Phi \in \mathfrak{Z}(\mathcal{A}'', \square)$ then $\phi \cdot \Phi = \Phi \cdot \phi$ ($\phi \in \mathcal{A}'$). Thus, for $\Psi \in \mathcal{A}''$ and $\phi \in \mathcal{A}'$, it follows that

$$\langle \Psi \square \Phi, \phi \rangle = \langle \Phi \square \Psi, \phi \rangle = \langle \Phi, \Psi \cdot \phi \rangle = \langle \Phi, \phi \cdot \Psi \rangle = \langle \Psi \diamond \Phi, \phi \rangle,$$

which gives $\Psi \square \Phi = \Psi \diamond \Phi$. Thus $\Phi \in \mathfrak{Z}_r(\mathcal{A}'')$. Similarly, by interchanging the position of Φ and Ψ , we can show that $\mathfrak{Z}(\mathcal{A}'', \square) \subset \mathfrak{Z}_l(\mathcal{A}'')$.

2. The left and right topological centers are both norm-closed subalgebra of \mathcal{A}'' endowed with the Arens products.
3. The following inclusions easily hold

$$\mathcal{A} \subset \mathfrak{Z}_l(\mathcal{A}'') \subset \mathcal{A}'', \quad \mathcal{A} \subset \mathfrak{Z}_r(\mathcal{A}'') \subset \mathcal{A}''.$$

4. The Banach algebra \mathcal{A} is Arens regular if $\mathfrak{Z}_l(\mathcal{A}'') = \mathcal{A}'' = \mathfrak{Z}_r(\mathcal{A}'')$, whence it follows that the two Arens product coincide.
5. If the Banach algebra \mathcal{A} is commutative then the centers of both (\mathcal{A}'', \square) and $(\mathcal{A}'', \diamond)$ coincide. Indeed,

$$\mathfrak{Z}_l(\mathcal{A}'') = \mathfrak{Z}_r(\mathcal{A}'') = \mathfrak{Z}(\mathcal{A}'', \square).$$

Theorem 2.4.64. Let \mathcal{A} be a Banach algebra and let $\Phi \in \mathcal{A}''$. Then the following statements are equivalent.

1. $\Phi \in \mathfrak{Z}_l(\mathcal{A}'')$.
2. $L_\Phi : \Psi \mapsto \Phi \square \Psi$ is continuous on $(\mathcal{A}'', \sigma(\mathcal{A}'', \mathcal{A}'))$.
3. $\Phi \cdot a_\alpha \rightarrow \Phi \square \Psi$ whenever (a_α) is a net in \mathcal{A} such that $\lim_\alpha a_\alpha = \Psi$.

Remark 2.4.65. Statement 2 in Theorem 2.4.64 holds if and only if \mathcal{A} is Arens regular.

We conclude this chapter by introducing some important definitions and results on Banach spaces which are relevant to this work.

Definition 2.4.66. Let X be a Banach space. The weak topology on X is the smallest topology that makes every map $f \in X'$ continuous. Similarly, the weak* topology is the smallest topology on X that makes every map

$$\Lambda_x : f \mapsto \langle x, f \rangle, \quad \Lambda_x : X' \rightarrow \mathbb{C}$$

to be continuous.

The following important properties of weak* topology are useful in this work.

Lemma 2.4.67 (Goldstine). *Let X be a Banach space and $i : (X, \|\cdot\|) \rightarrow (X'', \|\cdot\|)$ be a linear isometry. Then for each Λ in X'' , there is a bounded net (x_α) in X such that $\|x_\alpha\| \leq \|\Lambda\|$ and $i(x_\alpha) \rightarrow \Lambda$ in $\sigma(X'', X')$ with the limit taken in $\sigma(X'', X')$ on X'' . That is, $\hat{x}_\alpha \rightarrow \Lambda$ in the weak* topology on X'' .*

Theorem 2.4.68 (Mazur). *Let $(X, \|\cdot\|)$ be a Banach space, then for every convex set $U \subset X$, the closures of U in $(X, \|\cdot\|)$ and $(X, \sigma(X, X''))$ are the same.*

Theorem 2.4.69. *Let X be a Banach space, then the set $\mathcal{B}_{X'} = \{f \in X' : \|f\| \leq 1\}$ is compact in the weak* topology.*

Theorem 2.4.70 (Open Mapping Theorem). *Let X and Y be Banach spaces and let $T : X \rightarrow Y$ be a continuous linear homomorphism. Then,*

- (i) *T is open if T is a surjection.*
- (ii) *T is a linear homeomorphism if T is an injection.*

Chapter 3

Notions of Amenability in Banach Algebras

In this chapter, four notions of amenability on an arbitrary Banach algebra are studied. Some nice hereditary properties and characterizations of these amenability notions are discussed and reviewed.

3.1 Definitions with Examples

Definition 3.1.1. *Let \mathcal{A} be a Banach algebra and let X be a Banach \mathcal{A} -bimodule. A linear map $D : \mathcal{A} \rightarrow X$ is called a derivation if for $a, b \in \mathcal{A}$,*

$$D(ab) = D(a) \cdot b + a \cdot D(b).$$

Example 3.1.2. *Let \mathcal{A} be a Banach algebra and X be a Banach \mathcal{A} -bimodule. For $x \in X$, define the linear map*

$$\delta_x : a \mapsto a \cdot x - x \cdot a, \quad \mathcal{A} \rightarrow X.$$

Then for $a, b \in \mathcal{A}$ and $x \in X$,

$$\begin{aligned} \delta_x(ab) &= (ab) \cdot x - x \cdot (ab) = (ab) \cdot x - a \cdot (x \cdot b) + a \cdot (x \cdot b) - x \cdot (ab) \\ &= a \cdot (b \cdot x - x \cdot b) + (a \cdot x - x \cdot a) \cdot b = \delta_x(a) \cdot b + a \cdot \delta_x(b). \end{aligned}$$

Thus the linear map $\delta_x : \mathcal{A} \rightarrow X$ is a derivation. Derivations of this form are called inner derivations.

Definition 3.1.3. Let \mathcal{A} be a Banach algebra and let X be a Banach \mathcal{A} -bimodule. Then

- (i) \mathcal{A} is contractible if every continuous derivation $D : \mathcal{A} \rightarrow X$ is inner for every Banach \mathcal{A} -bimodule X .
- (ii) \mathcal{A} is amenable if every continuous derivation $D : \mathcal{A} \rightarrow X'$ is inner for every Banach \mathcal{A} -bimodule X .
- (iii) \mathcal{A} is weakly amenable if every continuous derivation $D : \mathcal{A} \rightarrow \mathcal{A}'$ is inner.

The above notions of amenability can also be defined in terms of the triviality of the first Hochschild cohomology group. Let \mathcal{A} be a Banach algebra and let X be a Banach \mathcal{A} -bimodule. Denote by $Z^1(\mathcal{A}, X)$ the space of all continuous derivations from \mathcal{A} into X and $N^1(\mathcal{A}, X)$ the space of all inner derivations from \mathcal{A} into X . The first Hochschild cohomology group of \mathcal{A} with coefficients in X is the quotient space $H^1(\mathcal{A}, X)$ given by

$$H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X).$$

Clearly, $Z^1(\mathcal{A}, X') = N^1(\mathcal{A}, X') \iff H^1(\mathcal{A}, X') = \{0\}$.

Definition 3.1.4. (i) \mathcal{A} is contractible if $H^1(\mathcal{A}, X) = \{0\}$ for every Banach \mathcal{A} -bimodule X .

(ii) \mathcal{A} is amenable if $H^1(\mathcal{A}, X') = \{0\}$ for every Banach \mathcal{A} -bimodule X .

(iii) \mathcal{A} is weakly amenable if $H^1(\mathcal{A}, \mathcal{A}') = \{0\}$.

Example 3.1.5. (i) The group algebra $L^1(G)$ is amenable if and only if the locally compact group G is amenable.

(ii) $L^1(G)$ is weakly amenable for any group G .

(iii) A C^* -algebra is amenable if and only if it is nuclear.

(iv) Every C^* -algebra is weakly amenable.

(v) The measure algebra $M(G)$ on a locally compact group G is amenable if and only if G is discrete.

(vi) For a locally compact group G , the Fourier algebra $A(G)$ is amenable if and only if G admits an abelian subgroup of finite index.

(vii) The Banach algebra $B(l^p)$ of bounded linear operators on the Banach space l^p is not amenable for $1 \leq p \leq \infty$.

For details on the above examples, see [30].

3.2 Contractible Banach Algebras

3.2.1 Characterizations

We shall need the following in the proof of the next theorem.

Proposition 3.2.1. *Let \mathcal{A} be a Banach algebra, then the following are equivalent:*

- (i) $H^1(\mathcal{A}, X) = \{0\}$ for each Banach \mathcal{A} -bimodule X .
- (ii) $H^1(\mathcal{A}, X) = \{0\}$ for each neo-unital Banach \mathcal{A} -bimodule X .

Theorem 3.2.2. *Let \mathcal{A} be a Banach algebra. Then the following statements are equivalent:*

- (i) \mathcal{A} is contractible.
- (ii) \mathcal{A} is unital and has a projective diagonal.

Proof. Assume that \mathcal{A} is contractible. We first show that \mathcal{A} is unital. Let $X = \mathcal{A} \times \mathcal{A}$ be a Banach \mathcal{A} -bimodule with left and right operations defined by $a \cdot (b, c) = (ab, 0)$ and $(b, c) \cdot a = (0, ca)$ respectively for $a, b, c \in \mathcal{A}$. The linear map $D : \mathcal{A} \rightarrow X$ defined by $D(b) = (b, b)$ is a derivation. Since for any $a, b \in \mathcal{A}$,

$$D(ab) = (ab, ab) = (0, ab) + (ab, 0) = (a, a)b + a(b, b) = D(a)b + aD(b).$$

By assumption, \mathcal{A} is contractible and so D must be inner, that is, there exists $(x, -x) \in X$ such that for all $b \in \mathcal{A}$,

$$(b, b) = D(b) = b \cdot (x, -x) - (x, -x) \cdot b = (bx, 0) - (0, -xb) = (bx, xb).$$

Thus $(b, b) = (bx, xb)$ which implies that $bx = b = xb$ and so x is both left and right identity for \mathcal{A} . Hence \mathcal{A} is unital.

We now show that \mathcal{A} possess a projective diagonal. Let $X = \ker \pi$, since \mathcal{A} itself is a Banach \mathcal{A} -bimodule and $\ker \pi$ is a closed subalgebra of \mathcal{A} , then

$\ker \pi$ is also a Banach \mathcal{A} -bimodule. Define a linear map $D : \mathcal{A} \rightarrow \ker \pi$ by $D(a) = a \otimes e - e \otimes a$. It is easy to see that

$$\begin{aligned} a \otimes e - a \cdot x &= e \otimes a - x \cdot a \\ a \cdot (e \otimes e - e \cdot x) &= (e \otimes e - x \cdot e) \cdot a. \end{aligned}$$

Now set $u = e \otimes e - x$, since $x \in \ker \pi$ and π is linear, then $a \cdot \pi(u) = a \cdot \pi(e \otimes e - x) = a \cdot (\pi(e \otimes e) - \pi(x)) = a \cdot (e \cdot e - 0) = a \cdot e = a$. That is, $a \cdot \pi(u) = a$. Hence u is a projective diagonal for \mathcal{A} .

Conversely, we assume a unit and a projective diagonal for \mathcal{A} and then show that every derivation $D : \mathcal{A} \rightarrow X$ is inner for every Banach \mathcal{A} -bimodule X . Let $u = \sum_{i=1}^n a_i \otimes b_i$ be a projective diagonal for \mathcal{A} . Let e be an identity for \mathcal{A} , since $a \cdot \pi(u) = a = \pi(u) \cdot a$, then $\pi(u) = e$. By Proposition 3.2.1, we shall show that the derivation $D : \mathcal{A} \rightarrow X$ is inner for every neo-unital Banach \mathcal{A} -bimodule X . We recall from [9] that for a projective diagonal $u = \sum_{i=1}^n a_i \otimes b_i$, there is a linear map $\mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow X$ given by $a \otimes b \mapsto a \cdot T(b)$ (where $T \in \mathfrak{C}(\mathcal{A}, X)$) which satisfies

$$\sum_{i=1}^n a a_i \cdot T(b_i) = \sum_{i=1}^n a_i T(b_i a) \quad (a \in \mathcal{A}).$$

Now, set $T = D$ and let $x = \sum_{i=1}^n a_i \cdot D(b_i)$, then

$$\begin{aligned} a \cdot x - x \cdot a &= \sum_{i=1}^n a a_i \cdot D(b_i) - \sum_{i=1}^n a_i \cdot D(b_i) \cdot a \\ &= \sum_{i=1}^n a a_i \cdot D(b_i) - a_i \cdot D(b_i) \cdot a \\ &= \sum_{i=1}^n a_i D(b_i \cdot a) - a_i \cdot D(b_i) \cdot a \\ &= \sum_{i=1}^n a_i (D(b_i) \cdot a + b_i \cdot D(a)) - a_i \cdot D(b_i) \cdot a \\ &= \sum_{i=1}^n a_i \cdot D(b_i) \cdot a - a_i \cdot D(b_i) \cdot a + a_i \cdot b_i \cdot D(a) \\ &= \sum_{i=1}^n a_i b_i \cdot D(a) = \pi(u) \cdot D(a) = e \cdot D(a) = D(a). \end{aligned}$$

The last equality holds since X is neo-unital. Thus $D(a) = a \cdot x - x \cdot a$ and \mathcal{A} is contractible. \square

Corollary 3.2.3. *Let \mathcal{A} be Banach algebra and I be a closed ideal of \mathcal{A} . Then the following are equivalent.*

- (i) I is contractible.
- (ii) I has an identity.
- (iii) I is complemented in \mathcal{A} .

3.2.2 Hereditary properties

We give the following nice hereditary properties whose proofs are much similar to those of amenable Banach algebra to be considered in Section 3.3. However, we give a proof for Proposition 3.2.4(iv) via the existence of a projective diagonal which is different from the proof given for its analogue on amenable Banach algebra.

Proposition 3.2.4. *Let \mathcal{A} be a Banach algebra,*

- (i) *If \mathcal{A} is contractible, \mathcal{B} is another Banach algebra and $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is continuous homomorphism with dense range, then \mathcal{B} is contractible.*
- (ii) *If I is a closed ideal of \mathcal{A} such that both I and \mathcal{A}/I are contractible, then \mathcal{A} is contractible.*
- (iii) *\mathcal{A} is contractible if and only if $\mathcal{A}^\#$ is contractible.*
- (iv) *If \mathcal{A} is contractible and \mathcal{B} is also a contractible Banach algebra, then $\mathcal{A} \hat{\otimes} \mathcal{B}$ is contractible.*

Proof. The proofs of (i), (ii) and (iii) are similar to the proofs of their corresponding amenability analogues in Section 3.3. We prove (iv). By Theorem 3.2.2, \mathcal{A} and \mathcal{B} both possess projective diagonal since they are contractible. It suffices by Theorem 3.2.2 to show that $\mathcal{A} \hat{\otimes} \mathcal{B}$ has a projective diagonal and is unital. Let $u = \sum_{i=1}^n r_i \otimes s_i$ and $v = \sum_{k=1}^m x_k \otimes y_k$ be projective diagonals for \mathcal{A} and \mathcal{B} respectively. Also, let $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ be the projective maps on \mathcal{A} and \mathcal{B} respectively, then it is clear that for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $a \cdot \pi_{\mathcal{A}}(u) = a$, $b \cdot \pi_{\mathcal{B}}(v) = b$ and $a \cdot u = u \cdot a$, $b \cdot v = v \cdot b$. It is well known that $(\mathcal{A} \hat{\otimes} \mathcal{A}) \hat{\otimes} (\mathcal{B} \hat{\otimes} \mathcal{B}) \cong (\mathcal{A} \hat{\otimes} \mathcal{B}) \hat{\otimes} (\mathcal{A} \hat{\otimes} \mathcal{B})$. Now $u \otimes v \in (\mathcal{A} \hat{\otimes} \mathcal{A}) \hat{\otimes} (\mathcal{B} \hat{\otimes} \mathcal{B})$ and for $a \in \mathcal{A}$, $b \in \mathcal{B}$,

$$(a \otimes b) \cdot (u \otimes v) = au \otimes bv = ua \otimes vb = (u \otimes v) \cdot (a \otimes b)$$

and

$$\begin{aligned}
(a \otimes b) \cdot \pi_{\mathcal{A} \hat{\otimes} \mathcal{B}}(u \otimes v) &= (a \otimes b) \pi_{\mathcal{A} \hat{\otimes} \mathcal{B}} \left(\sum_{i=1}^n r_i \otimes s_i \otimes \sum_{k=1}^m x_k \otimes y_k \right) \\
&= (a \otimes b) \pi_{\mathcal{A} \hat{\otimes} \mathcal{B}} \left(\sum_{i=1}^n \sum_{k=1}^m (r_i \otimes x_k) \otimes (s_i \otimes y_k) \right) \\
&= (a \otimes b) \left(\sum_{i=1}^n \sum_{k=1}^m (r_i \otimes x_k) \cdot (s_i \otimes y_k) \right) \\
&= (a \otimes b) \left(\sum_{i=1}^n \sum_{k=1}^m r_i s_i \otimes x_k y_k \right) \\
&= \left(\sum_{i=1}^n a r_i s_i \otimes \sum_{k=1}^m b x_k y_k \right) \\
&= a \pi_{\mathcal{A}} \left(\sum_{i=1}^n r_i \otimes s_i \right) \otimes b \pi_{\mathcal{B}} \left(\sum_{k=1}^m x_k \otimes y_k \right) \\
&= a \pi_{\mathcal{A}}(u) \otimes b \pi_{\mathcal{B}}(v) = a \otimes b
\end{aligned}$$

Thus $u \otimes v$ is a projective diagonal for $\mathcal{A} \hat{\otimes} \mathcal{B}$.

It is clear that \mathcal{A} and \mathcal{B} are unital since they are both contractible. Let $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ be identities for \mathcal{A} and \mathcal{B} respectively, then $e_{\mathcal{A}} \otimes e_{\mathcal{B}}$ is the identity for $\mathcal{A} \hat{\otimes} \mathcal{B}$. Clearly, for $a \otimes b \in \mathcal{A} \hat{\otimes} \mathcal{B}$,

$$(a \otimes b) \cdot (e_{\mathcal{A}} \otimes e_{\mathcal{B}}) = a \cdot e_{\mathcal{A}} \otimes b \cdot e_{\mathcal{B}} = a \otimes b.$$

□

3.3 Amenable Banach Algebras

3.3.1 Characterizations

There are several characterizations of amenability in the literature. We gave some of these characterizations in this subsection.

We recall that a virtual diagonal for a Banach algebra \mathcal{A} is an element $M \in (\mathcal{A} \hat{\otimes} \mathcal{A})''$ such that

$$a \cdot M = M \cdot a, \quad a \cdot \pi''(M) = a, \quad (a \in \mathcal{A}).$$

Also, a net $(m_\alpha) \in \mathcal{A} \hat{\otimes} \mathcal{A}$ is called an approximate diadonal for if

$$a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0, \quad a \cdot \pi(m_\alpha) \rightarrow a \quad (a \in A),$$

where $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow A$ is the diagonal operator. The approximate diadonal $(m_\alpha) \in \mathcal{A} \hat{\otimes} \mathcal{A}$ is called a bounded approximate diadonal if it is bounded.

Theorem 3.3.1. *Let \mathcal{A} be a Banach algebra. The following are equivalent:*

- (i) \mathcal{A} is amenable.
- (ii) \mathcal{A} has an approximate diagonal.
- (iii) \mathcal{A} has a virtual diagonal.

Proof. (ii) \implies (iii). Let (m_α) be an approximate diagonal for \mathcal{A} and (\hat{m}_α) be a bounded net in $(\mathcal{A} \hat{\otimes} \mathcal{A})''$. Then by Alaoglu theorem (\hat{m}_α) has a weak* limit point. Let $M \in (\mathcal{A} \hat{\otimes} \mathcal{A})''$ be a weak* limit point for (\hat{m}_α) , that is, $M = \text{weak}^* \text{-}\lim_\alpha \hat{m}_\alpha$, then for all $a \in \mathcal{A}$,

$$M \cdot a - a \cdot M = \text{weak}^* \text{-}\lim_\alpha \hat{m}_\alpha \cdot a - a \cdot \hat{m}_\alpha = \text{weak-}\lim_\alpha m_\alpha \cdot a - a \cdot m_\alpha = 0$$

and

$$\pi''_{\mathcal{A}}(M) \cdot a = \text{weak}^* \text{-}\lim_\alpha \pi''(\hat{m}_\alpha) \cdot a = \text{weak-}\lim_\alpha \pi(m_\alpha) \cdot a = a.$$

Thus $M \in (\mathcal{A} \hat{\otimes} \mathcal{A})''$ is a virtual diagonal for \mathcal{A} .

Conversely, let M be a virtual diagonal for \mathcal{A} . Let $(m_\alpha) \in \mathcal{A} \hat{\otimes} \mathcal{A}$ be a bounded net. By Goldstine's theorem, the image of (m_α) under the canonical map converges to M in the weak* topology on $(\mathcal{A} \hat{\otimes} \mathcal{A})''$. That is, $M = \text{weak}^* \text{-}\lim_\alpha \hat{m}_\alpha$. Then it holds that for all $a \in \mathcal{A}$,

$$\text{weak-}\lim_\alpha (m_\alpha \cdot a - a \cdot m_\alpha) = \text{weak}^* \text{-}\lim_\alpha (\hat{m}_\alpha \cdot a - a \cdot \hat{m}_\alpha) = M \cdot a - a \cdot M = 0$$

and

$$\text{weak-}\lim_\alpha \pi(m_\alpha) \cdot a = \text{weak}^* \text{-}\lim_\alpha \pi''(\hat{m}_\alpha) \cdot a = \pi''(M) \cdot a = a.$$

We now construct an approximate diagonal for \mathcal{A} . For each $K = \{a_1, a_2, \dots, a_n\} \subset \mathcal{A}$ and $\epsilon > 0$, the bounded net

$$((m_\alpha \cdot a_1 - a_1 \cdot m_\alpha, \pi(m_\alpha)a_1 - a_1), \dots, (m_\alpha \cdot a_n - a_n \cdot m_\alpha, \pi(m_\alpha) \cdot a_n - a_n))$$

in the space $((\mathcal{A} \hat{\otimes} \mathcal{A}) \times \mathcal{A})^n$ converges to 0 in the weak topology. Let \mathcal{H} be the convex hull of $\{m_\alpha : \alpha \in \Delta\}$, then

$$0 \in \left(\overline{\mathcal{H}}^{\text{weak}} \cdot a_i - a_i \cdot \overline{\mathcal{H}}^{\text{weak}} \right) \cap \left(\pi(\overline{\mathcal{H}}^{\text{weak}}) \cdot a_i - a_i \right) \quad (i \in \mathbb{N}).$$

By Mazur's theorem, $\overline{\mathcal{H}}^{\text{weak}} = \overline{\mathcal{H}}$ and there exists $v_{K,\epsilon} \in \mathcal{H}$ such that $\|v_{K,\epsilon} \cdot a - a \cdot v_{K,\epsilon}\| < \epsilon$ and $\|\pi_{\mathcal{A}}(v_{K,\epsilon}) \cdot a - a\| < \epsilon$ for $a \in K$. Finally, $K \times \mathbb{R}^+$ is a directed set for which the partial order is given by $(K_1, \epsilon_1) \preceq (K_2, \epsilon_2)$ if and only if $K_1 \subset K_2$ and $\epsilon_1 \geq \epsilon_2$. Thus the net $(v_{K,\epsilon})$ is an approximate diagonal for \mathcal{A} .

(iii) \implies (i). Suppose that M is a virtual diagonal for \mathcal{A} . Then it is clear that \mathcal{A} has an approximate diagonal. Let (m_α) be an approximate diagonal for \mathcal{A} and X be a neo-unital Banach \mathcal{A} -bimodule and let $D \in Z^1(\mathcal{A}, X')$. By Proposition 3.3.9, it suffices to show that $H^1(\mathcal{A}, X') = \{0\}$. For $x \in X$, let μ_x be the continuous linear functional on $\mathcal{A} \hat{\otimes} \mathcal{A}$ given by

$$\mu_x(a \otimes b) = (a \cdot Db)(x) \quad (a, b \in \mathcal{A})$$

which we shall write as $\mu_x(a \otimes b) = \langle x, a \cdot Db \rangle$. Now, set $f(x) = M(\mu_x)$, then $f \in X'$. Let $a \in \mathcal{A}, x \in X$ then for $b, c \in \mathcal{A}$, it follows that

$$\begin{aligned} \mu_{x \cdot a - a \cdot x}(b \otimes c) &= \langle x \cdot a - a \cdot x, b \cdot Dc \rangle = \langle x \cdot a, b \cdot Dc \rangle - \langle a \cdot x, b \cdot Dc \rangle \\ &= \langle x, a \cdot b \cdot Dc \rangle - \langle x, b \cdot Dc \cdot a \rangle = \langle x, a \cdot b \cdot Dc - b \cdot Dc \cdot a \rangle \\ &= \langle x, a \cdot b \cdot Dc \rangle - \langle x, b \cdot Dc \cdot a \rangle \end{aligned}$$

and

$$\begin{aligned} (\mu_x \cdot a - a \cdot \mu_x)(b \otimes c) &= \langle b \otimes c, \mu_x \cdot a - a \cdot \mu_x \rangle = \langle b \otimes c, \mu_x \cdot a \rangle - \langle b \otimes c, a \cdot \mu_x \rangle \\ &= \langle a \cdot (b \otimes c), \mu_x \rangle - \langle (b \otimes c) \cdot a, \mu_x \rangle \\ &= \langle ab \otimes c, \mu_x \rangle - \langle b \otimes ca, \mu_x \rangle \\ &= \langle x, ab \cdot Dc \rangle - \langle x, b \cdot Dca \rangle \\ &= \langle x, a \cdot b \cdot Dc \rangle - \langle x, b \cdot Dc \cdot a + b \cdot c \cdot da \rangle \\ &= \langle x, a \cdot b \cdot Dc \rangle - \langle x, b \cdot Dc \cdot a \rangle - \langle x, b \cdot c \cdot Da \rangle \end{aligned}$$

and so

$$\begin{aligned} \mu_{x \cdot a - a \cdot x}(b \otimes c) &= (\mu_x \cdot a - a \cdot \mu_x)(b \otimes c) + (bcDa)(x) \\ &= (\mu_x \cdot a - a \cdot \mu_x)(b \otimes c) + (\pi(b \otimes c)Da)(x). \end{aligned}$$

Thus, for $m \in \mathcal{A} \hat{\otimes} \mathcal{A}$,

$$\mu_{x \cdot a - a \cdot x}(m) = (\mu_x \cdot a - a \cdot \mu_x)(m) + (\pi(m)Da)(x).$$

Thus, for (m_α) converging weak* to M such that $M(\mu_x) = \lim_\alpha \mu_x(m_\alpha)$,

$$\begin{aligned} (a \cdot f - f \cdot a)(x) &= \langle x, a \cdot f - f \cdot a \rangle = \langle x, a \cdot f \rangle - \langle x, f \cdot a \rangle \\ &= \langle x \cdot a, f \rangle - \langle a \cdot x, f \rangle = \langle x \cdot a - a \cdot x, f \rangle \\ &= f(x \cdot a - a \cdot x) = M(\mu_{x \cdot a - a \cdot x}) = \lim_\alpha (\mu_{x \cdot a - a \cdot x})(m_\alpha) \\ &= \lim_\alpha (\mu_x \cdot a - a \cdot \mu_x)(m_\alpha) + (\lim_\alpha \pi(m_\alpha)Da)(x) \\ &= M(\mu_x \cdot a - a \cdot \mu_x) + (\lim_\alpha \pi(m_\alpha)Da)(x) \\ &= (a \cdot M - M \cdot a)(\mu_x) + (\pi''(M)Da)(x). \end{aligned}$$

Observe that

$$\|\mu_x(a \otimes b)\| \leq \|\mu_x\| \|a \otimes b\| = \|\mu_x\| \|a\| \|b\| \quad (3.1)$$

$$\|\mu_x(a \otimes b)\| \leq \|(aDb)(x)\| = \|a\| \|D\| \|b\| \|x\|. \quad (3.2)$$

It follows from (3.1) and (3.2) that $\|\mu_x\| \leq \|D\| \|x\|$. Now,

$$\begin{aligned} &\|(a \cdot f - f \cdot a) - (Da)(x)\| \\ &= \|(a \cdot M - M \cdot a)(\mu_x) + (\pi''(M)Da)(x) - (Da)(x)\| \\ &\leq \|(a \cdot M - M \cdot a)(\mu_x)\| + \|(\pi''(M)Da)(x) - (Da)(x)\| \\ &= \|(a \cdot M - M \cdot a)\| \|\mu_x\| + \|(\pi''(M)Da)(x) - (Da)(x)\| \\ &= \|(a \cdot M - M \cdot a)\| \|D\| \|x\| + \|(\pi''(M)Da - Da)(x)\| \\ &= \|(a \cdot M - M \cdot a)\| \|D\| \|x\| + \|[(\pi''(M) - e) \cdot (Da)](x)\| \\ &\leq \|(a \cdot M - M \cdot a)\| \|D\| \|x\| + \|(\pi''(M) - e)\| \|Da\| \|x\| \\ &\rightarrow 0. \end{aligned}$$

Whence $(Da)(x) = (a \cdot f - f \cdot a)(x)$ and $D = \delta_f$ or $H^1(\mathcal{A}, X') = \{0\}$. Hence \mathcal{A} is amenable.

Conversely, let \mathcal{A} be amenable then by Proposition 3.3.10, \mathcal{A} has a bounded approximate identity. Suppose that (e_α) is a bounded approximate identity for \mathcal{A} , and let $(e_\alpha \otimes e_\alpha)''$ be a bounded net in $(\mathcal{A} \hat{\otimes} \mathcal{A})''$. Let N be a weak* limit point of $(e_\alpha \otimes e_\alpha)''$, then without loss of generality, we may suppose that $N = \text{weak}^* \lim_\alpha (e_\alpha \otimes e_\alpha)''$. The map $\delta_N : \mathcal{A} \rightarrow (\mathcal{A} \hat{\otimes} \mathcal{A})''$ given by $\delta_N(a) = N \cdot a - a \cdot N$ is a

derivation into $(\mathcal{A} \hat{\otimes} \mathcal{A})''$. To see this, let $a, b \in \mathcal{A}$, then

$$\begin{aligned}\delta_N(ab) &= N \cdot (ab) - (ab) \cdot N = N \cdot (ab) - a \cdot (N \cdot b) + a \cdot (N \cdot b) - (ab) \cdot N \\ &= (N \cdot a - a \cdot N) \cdot b + a \cdot (N \cdot b - b \cdot N) = \delta_N(a) \cdot b + a \cdot \delta_N(b).\end{aligned}$$

Then,

$$\begin{aligned}\pi''(\delta_N(a)) &= \text{weak}^* \text{-} \lim_{\alpha} \pi''((e_{\alpha} \otimes e_{\alpha})'' \cdot a - a \cdot (e_{\alpha} \otimes e_{\alpha})'') \\ &= \text{weak-} \lim_{\alpha} \pi((e_{\alpha} \otimes e_{\alpha}) \cdot a - a \cdot (e_{\alpha} \otimes e_{\alpha})) \\ &= \text{weak-} \lim_{\alpha} \pi(e_{\alpha} \otimes e_{\alpha} a - a e_{\alpha} \otimes e_{\alpha}) \\ &= \text{weak-} \lim_{\alpha} \pi(e_{\alpha} \otimes e_{\alpha} a) - \pi(a e_{\alpha} \otimes e_{\alpha}) \\ &= \text{weak-} \lim_{\alpha} (e_{\alpha}^2 a - a e_{\alpha}^2) = 0.\end{aligned}$$

The last equality is due to the fact that $e_{\alpha}^2 a \rightarrow a$ and $a e_{\alpha}^2 \rightarrow a$, that is,

$$\|e_{\alpha}^2 a - a\| = \|e_{\alpha}^2 a - e_{\alpha} a + e_{\alpha} a - a\| = \|(e_{\alpha} + 1)(e_{\alpha} a - a)\| \leq \|e_{\alpha} + 1\| \|e_{\alpha} a - a\| \rightarrow 0$$

and

$$\|a e_{\alpha}^2 - a\| = \|a e_{\alpha}^2 + a e_{\alpha} - a e_{\alpha} - a\| = \|(e_{\alpha} + 1)(a e_{\alpha} - a)\| \leq \|e_{\alpha} + 1\| \|a e_{\alpha} - a\| \rightarrow 0.$$

Since \mathcal{A} has a bounded approximate identity then by Cohen's factorization theorem, \mathcal{A} factors, that is, $\mathcal{A} = \mathcal{A} \cdot \mathcal{A}$. This implies that $\pi(\mathcal{A} \hat{\otimes} \mathcal{A}) = \mathcal{A} \cdot \mathcal{A} = \mathcal{A}$ and so π is surjective. Hence $\ker \pi'' \cong (\ker \pi)''$. Furthermore, since π is a bounded bimodule homomorphism then $\ker \pi$ is a closed submodule of $\mathcal{A} \hat{\otimes} \mathcal{A}$ and so $\ker \pi''$ is a closed submodule of $(\mathcal{A} \hat{\otimes} \mathcal{A})''$. In fact, $\ker \pi''$ is a Banach \mathcal{A} -bimodule. Thus \mathcal{A} is amenable implies that there exists $M \in \ker \pi''$ such that $\delta_N(a) = \delta_M(a)$. Set $K = N - M$, then

$$\begin{aligned}a \cdot K - K \cdot a &= a \cdot (N - M) - (N - M) \cdot a = a \cdot N - N \cdot a - a \cdot M + M \cdot a \\ &= (M \cdot a - a \cdot M) - (N \cdot a - a \cdot N) = \delta_M(a) - \delta_N(a) = 0\end{aligned}$$

and for $a \in \mathcal{A}$,

$$\begin{aligned}\pi''(K) \cdot a &= \pi''(N - M) \cdot a = (\pi''(N) - \pi''(M)) \cdot a = \pi''(N) \cdot a \\ &= \text{weak}^* \text{-} \lim_{\alpha} \pi''(e_{\alpha} \otimes e_{\alpha})'' \cdot a = \text{weak-} \lim_{\alpha} \pi(e_{\alpha} \otimes e_{\alpha}) \cdot a \\ &= \text{weak-} \lim_{\alpha} e_{\alpha} e_{\alpha} \cdot a = a.\end{aligned}$$

Thus K is a virtual diagonal for \mathcal{A} . □

We next give the characterization of amenability in terms of the splitting of short exact sequences.

Let \mathcal{A} be a Banach algebra and let X_1, X_2, X_3, \dots be a left, right or two sided \mathcal{A} -module. Suppose that $f_1 : X_1 \rightarrow X_2, f_2 : X_2 \rightarrow X_3, f_3 : X_3 \rightarrow X_4, \dots$ are Banach \mathcal{A} -module homomorphisms. The sequence of Banach \mathcal{A} -module given by

$$X_1 \rightarrow X_2 \rightarrow X_3 \cdots \rightarrow X_n \rightarrow X_{n+1} \rightarrow \cdots$$

is called exact at X_n if $\text{Im}f_{n-1} = \ker f_n$. In particular, the following short sequence of Banach \mathcal{A} -module

$$\Sigma : 0 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow 0$$

is called short exact sequence at X_3 if $\text{Im}f_2 = \ker f_3$. The short sequence Σ

- (i) is exact if f_2 is one-to-one, $\text{Im}f_3 = f_4$ and $\text{Im}f_2 = \ker f_3$
- (ii) is admissible if it is exact and $f_3 : X_3 \rightarrow X_4$ has a bounded right inverse
- (iii) splits if it is admissible and the bounded right inverse in (ii) is a Banach \mathcal{A} -module homomorphism.

The next result is a characterization of amenability in terms of the splitting of admissible short exact sequences of Banach \mathcal{A} -modules (see [8]).

Theorem 3.3.2. *Let \mathcal{A} be a Banach algebra. Consider the following short exact sequences of \mathcal{A} -bimodules*

$$\Pi : 0 \rightarrow K \xrightarrow{i} \mathcal{A} \hat{\otimes} \mathcal{A} \xrightarrow{\pi} \mathcal{A} \rightarrow 0$$

and its dual

$$\Pi' : 0 \rightarrow \mathcal{A}' \xrightarrow{\pi'} (\mathcal{A} \hat{\otimes} \mathcal{A})' \xrightarrow{i'} K \rightarrow 0$$

where π is given by $\pi(a \otimes b) = ab$ and i is the natural injection of its kernel K . Then the following are equivalent:

1. \mathcal{A} is amenable
2. \mathcal{A} has a bounded approximate identity and for each essential \mathcal{A} -bimodule X , any admissible short exact sequence Π' of \mathcal{A} -bimodules splits.

Proof. Let \mathcal{A} be amenable, then by Propositions 3.3.1 and 3.3.10, \mathcal{A} has a virtual diagonal and bounded approximate identity respectively. Let M be a virtual diagonal for \mathcal{A} . For $a \in \mathcal{A}$, $f \in (\mathcal{A} \hat{\otimes} \mathcal{A})'$, we define $\langle a, \theta f \rangle = \langle f \cdot a, M \rangle$. Let I be the identity map on \mathcal{A}' . We claim that $\theta\pi' = I$. Let $\lambda \in \mathcal{A}'$, then

$$\begin{aligned} \langle a, \theta\pi'\lambda \rangle &= \langle (\pi'\lambda) \cdot a, M \rangle = \langle \pi'(\lambda \cdot a), M \rangle \\ &= \langle \lambda \cdot a, \pi''M \rangle = \langle \lambda, a \cdot \pi''M \rangle = \langle \lambda, a \rangle. \end{aligned}$$

Next is to show that θ is an \mathcal{A} -bimodule homomorphism, let $b \in \mathcal{A}$ then

$$\begin{aligned} \langle a, \theta(b \cdot f) \rangle &= \langle (b \cdot f) \cdot a, M \rangle = \langle f \cdot a, M \cdot b \rangle \\ &= \langle f \cdot a, b \cdot M \rangle = \langle f \cdot (ab), M \rangle \\ &= \langle ab, \theta f \rangle = \langle a, b \cdot \theta f \rangle \end{aligned}$$

and therefore $\theta(b \cdot f) = b \cdot \theta(f)$. Similarly $\theta(f \cdot b) = \theta(f) \cdot b$, since

$$\begin{aligned} \langle a, \theta(f \cdot b) \rangle &= \langle (f \cdot b) \cdot a, M \rangle = \langle f \cdot (ba), M \rangle \\ &= \langle ba, \theta f \rangle = \langle a, \theta f \cdot b \rangle. \end{aligned}$$

Conversely, suppose that \mathcal{A} has a bounded approximate identity (e_α) and θ is an \mathcal{A} -bimodule homomorphism with $\theta\pi' = I$. By passing to a subnet, let $u \in (\mathcal{A} \hat{\otimes} \mathcal{A})''$ be the weak* limit point of $(e_\alpha \otimes e_\alpha)$. Set $M = \theta'\pi''u$. Then M is a virtual diagonal for \mathcal{A} . To see this, let $a \in \mathcal{A}$, $f \in (\mathcal{A} \hat{\otimes} \mathcal{A})'$, then

$$\begin{aligned} \langle f, a \cdot M \rangle &= \langle f, a\theta'\pi''u \rangle \\ &= \langle \pi'\theta(f \cdot a), u \rangle = \lim_\alpha \langle \pi'\theta(f \cdot a), e_\alpha \otimes e_\alpha \rangle \\ &= \lim_\alpha \langle \theta f, ae_\alpha^2 \rangle = \langle \theta f, a \rangle \\ &= \lim_\alpha \langle \theta f, e_\alpha^2 a \rangle = \lim_\alpha \langle \pi'\theta(a \cdot f), e_\alpha \otimes e_\alpha \rangle \\ &= \langle \pi'\theta(a \cdot f), u \rangle = \langle a \cdot f, \theta'\pi''u \rangle = \langle f, M \cdot a \rangle. \end{aligned}$$

Also, we have

$$\pi''(M)a = \pi''\theta'\pi''ua = \pi''ua = a.$$

Thus \mathcal{A} has a virtual diagonal and by Proposition 3.3.1, \mathcal{A} is amenable. \square

Another interesting way of characterizing amenable Banach algebras is to show that they have certain Hahn-Banach extension property. It was observed by Lau in [28] that a Banach algebra \mathcal{A} is amenable if and only if any of the following holds:

- (i) Whenever X is a Banach \mathcal{A} -bimodule and Y is an \mathcal{A} -submodule of X , then for each $f \in Y'$ such that $a \cdot f = f \cdot a$ $a \in \mathcal{A}$, there exists $\bar{f} \in X'$ which extends f and $a \cdot \bar{f} = \bar{f} \cdot a$ for all $a \in \mathcal{A}$.
- (ii) Whenever X is a Banach \mathcal{A} -bimodule, there exists a bounded projection P from X' onto $\{f \in X' : a \cdot f = f \cdot a \forall a \in \mathcal{A}\}$ such that $T \cdot P = P \cdot T$ for any weak*-continuous bounded linear operator T from X' into X' commuting with the action of \mathcal{A} on X' .

Let X be a Banach \mathcal{A} -bimodule and denote by

$$Z(\mathcal{A}, X') = \bigcap_{a \in \mathcal{A}} \{f \in X' : a \cdot f = f \cdot a\},$$

the closed linear subspace of X' , which is invariant under each bounded linear operator from X' into X' commuting with the action of \mathcal{A} . The next result is another nice characterization of the amenability of \mathcal{A} .

Theorem 3.3.3. *Let \mathcal{A} be a Banach algebra. The following are equivalent:*

1. \mathcal{A} is amenable.
2. For any Banach \mathcal{A} -bimodule X and any Banach \mathcal{A} -submodule Y of X , each linear functional in $Z(\mathcal{A}, Y')$ has an extension to a linear functional in $Z(\mathcal{A}, X')$.
3. For any Banach \mathcal{A} – bimodule, there exists a bounded projection from X' onto $Z(\mathcal{A}, X')$ which commutes with any weak*-continuous bounded linear operator from X' into X' commuting with the action of \mathcal{A} on X' .

Proof. See [28]. □

3.3.2 Hereditary properties

Proposition 3.3.4. *Let \mathcal{A} and \mathcal{B} be Banach algebras and $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a continuous homomorphism with dense range in \mathcal{B} . If \mathcal{A} is amenable, then \mathcal{B} is amenable.*

Proof. It suffices to show that every continuous derivation $D : \mathcal{B} \rightarrow X'$ for every Banach \mathcal{B} -bimodule X is inner. We make X into a Banach \mathcal{A} -bimodule by the multiplication

$$ax = \varphi(a)x, \quad xa = x\varphi(a), \quad a \in \mathcal{A}, x \in X.$$

Consider the map $(D \circ \varphi) : \mathcal{A} \rightarrow X'$. We claim that this is a continuous derivation since the composition of continuous maps is continuous and for any $a, b \in \mathcal{A}$,

$$\begin{aligned} (D \circ \varphi)(ab) &= D(\varphi(ab)) \\ &= D(\varphi(a)\varphi(b)) \\ &= D(\varphi(a))\varphi(b) + \varphi(a)D(\varphi(b)) \\ &= (D \circ \varphi)(a)b + a(D \circ \varphi)(b). \end{aligned}$$

Since \mathcal{A} is amenable then the continuous derivation $(D \circ \varphi) : \mathcal{A} \rightarrow X'$ is inner. That is, there exist $f \in X'$ satisfying

$$D(\varphi(a)) = (D \circ \varphi)(a) = af - fa = \varphi(a)f - f\varphi(a), \quad a \in \mathcal{A}.$$

Also since φ has a dense range in \mathcal{B} . That is, $\overline{\varphi(\mathcal{A})} = \mathcal{B}$, then for $(s_\alpha)_{\alpha \in \Delta} \subset \varphi(\mathcal{A})$ such that $s_\alpha \rightarrow s \in \mathcal{B}$, it follows that

$$\begin{aligned} D_f(s) = D_f(\lim_\alpha s_\alpha) &= \lim_\alpha D_f(s_\alpha) \\ &= \lim_\alpha (s_\alpha f - f s_\alpha) = \lim_\alpha (s_\alpha f) - \lim_\alpha (f s_\alpha) \\ &= s f - f s. \end{aligned}$$

Thus, $D : \mathcal{B} \rightarrow X'$ is inner and \mathcal{B} is amenable. □

Let \mathcal{A} be an amenable Banach algebra and I a closed ideal of \mathcal{A} . Let X be a Banach \mathcal{A} -bimodule and a Banach \mathcal{A}/I -bimodule. Define $D \in H^1(\mathcal{A}, X')$ by $D(a) = (d \circ \varphi)(a)$, where $d : \mathcal{A}/I \rightarrow X'$ and φ is the natural canonical map from \mathcal{A} onto the quotient space \mathcal{A}/I , this map is a surjective homomorphism. Then by following the proof of Proposition 3.3.4, it becomes clear that $d \in H^1(\mathcal{A}/I, X')$. Indeed, the following Corollary holds.

Corollary 3.3.5. *Let I be a closed two sided ideal of an amenable Banach algebra \mathcal{A} . Then the quotient Banach algebra \mathcal{A}/I is amenable.*

Proposition 3.3.6. *Let \mathcal{A} be a Banach algebra and I an amenable closed ideal of \mathcal{A} . If \mathcal{A}/I is amenable, then \mathcal{A} is amenable.*

Proof. Let X be a Banach \mathcal{A} -bimodule then X is also a Banach I -bimodule. For $D \in Z^1(\mathcal{A}, X')$, the restriction map $D|_I$ is a derivation into X' , that is, $D|_I \in Z^1(I, X')$. Let $\phi_0 \in X'$, define a continuous derivation $\overline{D} : \mathcal{A} \rightarrow X'$ by

$\overline{D} = D - \delta_{\phi_0}$. Since I is amenable, there exist $\phi_0 \in X'$ such that $D|_I = \delta_{\phi_0}$. Thus $\overline{D} = 0$ and in particular $\overline{D}|_I = 0$. Let

$$Y = cl\{a \cdot x + y \cdot b \mid a, b \in I, x, y \in X\}$$

and

$$Y^0 = \{\varphi \in X' \mid a \cdot \varphi = \varphi \cdot a = 0 \forall a \in I\}.$$

Then Y is a closed submodule of X and X/Y is a Banach A/I -bimodule. Y^0 is a dual Banach A/I -bimodule, that is, $Y^0 \cong (X/Y)'$. Let $a \in \mathcal{A}$, $b \in I$ then

$$0 = \overline{D}(ab) = \overline{D}(a)b + a\overline{D}(b) = \overline{D}(a)b + 0 = \overline{D}(a)b.$$

Thus for every $x \in X$,

$$\langle xb, \overline{D}(a) \rangle = \langle xb, (D - \delta_{\phi_0})(a) \rangle = \langle x, b \cdot (D - \delta_{\phi_0})(a) \rangle = 0$$

$$\langle bx, \overline{D}(a) \rangle = \langle bx, (D - \delta_{\phi_0})(a) \rangle = \langle x, (D - \delta_{\phi_0})(a) \cdot b \rangle = 0.$$

That is, $\langle xb, \overline{D}(a) \rangle = \langle bx, \overline{D}(a) \rangle = 0$ and so there exist $\varphi \in X'$ such that

$$0 = \overline{D}(a)(xb) = \varphi(xb)$$

and

$$0 = \overline{D}(a)(bx) = \varphi(bx).$$

Thus $\overline{D} : \mathcal{A} \rightarrow Y^0$, that is, $\overline{D}(\mathcal{A}) \subset Y^0$.

Now, define $D_1 : \mathcal{A}/I \rightarrow Y^0$ by $D_1 : a + I \mapsto \overline{D}(a)$, that is, $D_1(a + I) = (D - \delta_{\phi_0})(a)$. Since \mathcal{A}/I is amenable then there exist $\varphi \in Y^0$ such that $D_1 = \delta_{\varphi}$. Thus $\delta_{\varphi} = D - \delta_{\phi_0}$ or $D = \delta_{\varphi + \phi_0}$. Hence \mathcal{A} is amenable. \square

Lemma 3.3.7. *Let \mathcal{A} be an amenable Banach algebra. A closed ideal I of \mathcal{A} is amenable if I has a finite codimension.*

Proposition 3.3.8. *A Banach algebra \mathcal{A} is amenable if and only if $\mathcal{A}^{\#}$ is amenable.*

Proof. Let $\mathcal{A}^{\#}$ be amenable and X be a Banach $\mathcal{A}^{\#}$ -bimodule. Then X is also a Banach \mathcal{A} -bimodule. The right and left module action of $\mathcal{A}^{\#}$ on X is specified by

$$x(a, \alpha) = x \cdot a \quad (a, \alpha)x = a \cdot x, \quad x \in X, (a, \alpha) \in \mathcal{A}^{\#}.$$

Let $\overline{D} \in Z^1(\mathcal{A}^\#, X')$. Define the map $\overline{D} : (a, \alpha) \mapsto D(a)$, then for $(a, \alpha), (b, \beta) \in \mathcal{A}^\#$,

$$\begin{aligned} D(ab) &= \overline{D}(ab, \alpha\beta) = \overline{D}((a, \alpha)(b, \beta)) = \overline{D}(a, \alpha)(b, \beta) + (a, \alpha)\overline{D}(b, \beta) \\ &= \overline{D}(a, \alpha) \cdot b + a \cdot \overline{D}(b, \beta) = D(a) \cdot b + a \cdot D(b). \end{aligned}$$

Thus the restriction $D : \mathcal{A} \rightarrow X'$ of $\overline{D} \in Z^1(\mathcal{A}^\#, X')$ is a derivation. Next is to show that $D \in Z^1(\mathcal{A}, X')$ is inner. Since $\mathcal{A}^\#$ is amenable then there exists $\lambda \in X'$ such that for $(a, \alpha) \in \mathcal{A}^\#$,

$$\begin{aligned} D(a, \alpha) &= (a, \alpha) \cdot \lambda - \lambda(a, \alpha) = (a\lambda, \alpha\lambda) - (\lambda a, \lambda\alpha) = (a\lambda - \lambda a, \alpha\lambda - \lambda\alpha) \\ &= (a\lambda - \lambda a, 0). \end{aligned}$$

Thus $D(a) = a\lambda - \lambda a$, that is, D is inner and \mathcal{A} is amenable.

Alternatively, by identifying \mathcal{A} with the ideal $\{(a, 0) \mid a \in \mathcal{A}\}$ in $\mathcal{A}^\#$ via the isometric isomorphism $a \mapsto (a, 0)$, it becomes clear that \mathcal{A} is a (maximal) closed ideal of $\mathcal{A}^\#$ with codimension of one. By applying Lemma 3.3.7, the result follows.

Conversely, let \mathcal{A} be amenable, it suffices to show that $H^1(\mathcal{A}^\#, X') = \{0\}$ for a neo-unital Banach $\mathcal{A}^\#$ -bimodule which is also a neo-unital Banach \mathcal{A} -bimodule. Let $\overline{D} : \mathcal{A}^\# \rightarrow X'$ be a derivation then $\overline{D}(e) = \overline{D}(e \cdot e) = e \cdot \overline{D}(e) + \overline{D}(e) \cdot e = \overline{D}(e) + \overline{D}(e) = 2\overline{D}(e)$. This implies that $\overline{D}(e) = 0$. Thus we can consider \overline{D} as a derivation from \mathcal{A} into X' and since \mathcal{A} is amenable, then \overline{D} is inner on \mathcal{A} . This coincides with D being inner on $\mathcal{A}^\#$ since $\overline{D}(e)$ is trivial. Thus $\mathcal{A}^\#$ is amenable. □

Proposition 3.3.9. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity, the following are equivalent:*

- (i). $H^1(\mathcal{A}, X') = \{0\}$ for each Banach \mathcal{A} -bimodule X .
- (ii). $H^1(\mathcal{A}, X') = \{0\}$ for each neo-unital Banach \mathcal{A} -bimodule X .

Proposition 3.3.10. *Let \mathcal{A} be a Banach algebra. If \mathcal{A} is amenable then it has a bounded approximate identity.*

Theorem 3.3.11. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity which is contained as closed ideal in a Banach algebra \mathcal{B} . Let X be a neo-unital Banach \mathcal{A} -bimodule and let $D \in Z^1(\mathcal{A}, X')$. Then X is a Banach \mathcal{B} -bimodule in a canonical fashion and there exist a unique extension $\overline{D} \in Z^1(\mathcal{B}, X')$ such that*

- (i) $\overline{D}|_{\mathcal{A}} = D$.
- (ii) \overline{D} is continuous with respect to the strict topology on \mathcal{B} .

Proof. See [44]. □

Proposition 3.3.12. *Let \mathcal{A} be an amenable Banach algebra and let I be a closed ideal of \mathcal{A} . I is amenable if and only if it has a bounded approximate identity.*

Proof. Assume that I is amenable, then by Proposition 3.3.10, I has a bounded approximate identity. Conversely, suppose that I has a bounded approximate identity, then it suffices to show by Proposition 3.3.9 that every derivation from I into a neo-unital Banach I -bimodule is inner. Let X be a neo-unital Banach I -bimodule and $D \in Z^1(I, X')$, then by Theorem 3.3.11, X is Banach \mathcal{A} -bimodule in a canonical fashion and there is an extension $\overline{D} \in Z^1(\mathcal{A}, X')$ such that $\overline{D}|_I = D$. Since \mathcal{A} is amenable, then there exists $\phi \in X'$ such that $D = \overline{D}|_I(a) = a\phi - \phi a$, $a \in I$. Thus I is amenable. □

Proposition 3.3.13. *Let \mathcal{A} and \mathcal{B} be amenable Banach algebras. Then $\mathcal{A} \hat{\otimes} \mathcal{B}$ is amenable.*

Proof. By Theorem 3.3.1, it suffices to show that the Banach algebra $\mathcal{A} \hat{\otimes} \mathcal{B}$ has a bounded approximate diagonal. Since \mathcal{A} and \mathcal{B} are amenable, then they both possess bounded approximate diagonal. Let $u_\alpha := \sum_{i=1}^n r_i^{(\alpha)} \otimes s_i^{(\alpha)}$ and $v_\beta := \sum_{k=1}^m x_k^{(\beta)} \otimes y_k^{(\beta)}$ such that $(u_\alpha) \subset \mathcal{A} \hat{\otimes} \mathcal{A}$ and $(v_\beta) \subset \mathcal{B} \hat{\otimes} \mathcal{B}$ are bounded approximate diagonal for \mathcal{A} and \mathcal{B} respectively with

$$\lim_{\alpha} (u_\alpha \cdot a - a \cdot u_\alpha) = 0, \quad \lim_{\alpha} \pi_{\mathcal{A}}(u_\alpha)a = a$$

and

$$\lim_{\beta} (v_\beta \cdot b - b \cdot v_\beta) = 0, \quad \lim_{\beta} \pi_{\mathcal{B}}(v_\beta)b = b$$

respectively. It is known that $(\mathcal{A} \hat{\otimes} \mathcal{A}) \hat{\otimes} (\mathcal{B} \hat{\otimes} \mathcal{B}) \cong (\mathcal{A} \hat{\otimes} \mathcal{B}) \hat{\otimes} (\mathcal{A} \hat{\otimes} \mathcal{B})$ and so, $(u_\alpha \otimes v_\beta) \subset (\mathcal{A} \hat{\otimes} \mathcal{B}) \hat{\otimes} (\mathcal{A} \hat{\otimes} \mathcal{B})$ is a bounded approximate diagonal for $\mathcal{A} \hat{\otimes} \mathcal{B}$. To see this, let $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then

$$\begin{aligned}
& \lim_{\alpha} \lim_{\beta} \pi_{\mathcal{A} \hat{\otimes} \mathcal{B}}(u_{\alpha} \otimes v_{\beta}) \cdot (a \otimes b) \\
&= \lim_{\alpha} \lim_{\beta} \pi_{\mathcal{A} \hat{\otimes} \mathcal{B}} \left(\sum_{i=1}^n r_i^{(\alpha)} \otimes s_i^{(\alpha)} \otimes \sum_{k=1}^m x_k^{(\beta)} \otimes y_k^{(\beta)} \right) \cdot (a \otimes b) \\
&= \lim_{\alpha} \lim_{\beta} \sum_{k=1}^m \sum_{i=1}^n \pi_{\mathcal{A} \hat{\otimes} \mathcal{B}} \left(r_i^{(\alpha)} \otimes x_k^{(\beta)} \right) \otimes \left(s_i^{(\alpha)} \otimes y_k^{(\beta)} \right) \cdot (a \otimes b) \\
&= \lim_{\alpha} \lim_{\beta} \sum_{k=1}^m \sum_{i=1}^n \left(r_i^{(\alpha)} s_i^{(\alpha)} \otimes x_k^{(\beta)} y_k^{(\beta)} \right) \cdot (a \otimes b) \\
&= \sum_{k=1}^m \sum_{i=1}^n \lim_{\alpha} \lim_{\beta} r_i^{(\alpha)} s_i^{(\alpha)} a \otimes x_k^{(\beta)} y_k^{(\beta)} b \\
&= \sum_{k=1}^m \sum_{i=1}^n \lim_{\alpha} \lim_{\beta} \left(\pi_{\mathcal{A}} \left(r_i^{(\alpha)} \otimes s_i^{(\alpha)} \right) \cdot a \otimes \pi_{\mathcal{B}} \left(x_k^{(\beta)} \otimes y_k^{(\beta)} \right) \cdot b \right) \\
&= \lim_{\alpha} \pi_{\mathcal{A}} \left(\sum_{i=1}^n r_i^{(\alpha)} \otimes s_i^{(\alpha)} \right) \cdot a \otimes \lim_{\beta} \pi_{\mathcal{B}} \left(\sum_{k=1}^m x_k^{(\beta)} \otimes y_k^{(\beta)} \right) \cdot b \\
&= \lim_{\alpha} \pi_{\mathcal{A}}(u_{\alpha}) \cdot a \otimes \lim_{\beta} \pi_{\mathcal{B}}(v_{\beta}) \cdot b = a \otimes b.
\end{aligned}$$

Also,

$$\begin{aligned}
& \lim_{\alpha} \lim_{\beta} ((u_{\alpha} \otimes v_{\beta}) \cdot (a \otimes b) - (a \otimes b) \cdot (u_{\alpha} \otimes v_{\beta})) = \lim_{\alpha} \lim_{\beta} (u_{\alpha} a \otimes v_{\beta} b - a u_{\alpha} \otimes b v_{\beta}) \\
&= \lim_{\alpha} \lim_{\beta} u_{\alpha} a \otimes v_{\beta} b - \lim_{\alpha} \lim_{\beta} a u_{\alpha} \otimes b v_{\beta} = \lim_{\alpha} \lim_{\beta} a u_{\alpha} \otimes b v_{\beta} - \lim_{\alpha} \lim_{\beta} a u_{\alpha} \otimes b v_{\beta} = 0.
\end{aligned}$$

Finally, the boundedness of $(u_{\alpha} \otimes v_{\beta})$ follows from the boundedness of (u_{α}) and (v_{β}) . \square

Corollary 3.3.14. *Let \mathcal{A} and \mathcal{B} be unital Banach algebras. If \mathcal{A} and \mathcal{B} are amenable, then $\mathcal{A}^{\#} \hat{\otimes} \mathcal{B}^{\#}$ is amenable.*

Proof. Since \mathcal{A} and \mathcal{B} are amenable, then by Proposition 3.3.13 $\mathcal{A} \hat{\otimes} \mathcal{B}$ is amenable. This implies by Proposition 3.3.8 that $(\mathcal{A} \hat{\otimes} \mathcal{B})^{\#} = \mathcal{A}^{\#} \hat{\otimes} \mathcal{B}^{\#}$ is amenable. \square

We conclude this section by showing that the first Hochschild cohomology group of \mathcal{A} is trivial for Banach \mathcal{A} -bimodules with trivial left module action. First, we note the following for the dual Banach \mathcal{A} -bimodule X . Let \mathcal{A} be a Banach algebra.

(i) If X is a left Banach \mathcal{A} -module, then X' becomes a right Banach \mathcal{A} -module through

$$\langle x, \phi \cdot a \rangle = \langle a \cdot x, \phi \rangle \quad a \in \mathcal{A}, x \in X, \phi \in X'.$$

(ii) If X is a right Banach \mathcal{A} -module, then X' becomes a left Banach \mathcal{A} -module through

$$\langle x, a \cdot \phi \rangle = \langle x \cdot a, \phi \rangle \quad a \in \mathcal{A}, x \in X', \phi \in X'.$$

(iii) If X is a Banach \mathcal{A} -bimodule, then X' equipped with the left and right modules actions from (ii) and (i) respectively is a Banach \mathcal{A} -bimodule.

Proposition 3.3.15. *Let \mathcal{A} be a Banach algebra with a bounded right approximate identity and X be a Banach \mathcal{A} -bimodule such that $\mathcal{A} \cdot X = \{0\}$. Then $H^1(\mathcal{A}, X') = \{0\}$.*

Proof. It is clear that X' is a right Banach \mathcal{A} -module. Since the left action \mathcal{A} on X is trivial, then the right action $X' \cdot \mathcal{A}$ is also trivial. That is, $\phi \cdot a = 0$ for all $\phi \in X'$ and $a \in \mathcal{A}$. Thus for a continuous derivation $D : \mathcal{A} \rightarrow X'$, it follows that

$$D(ab) = D(a)b + aD(b) = aD(b) + 0 = aD(b).$$

Let $(e_\alpha)_{\alpha \in \Delta}$ be a bounded approximate identity for \mathcal{A} and let $\phi \in X'$ be a weak*-accumulation point of $(De_\alpha)_{\alpha \in \Delta}$. Without loss of generality, we may suppose that $\phi = \text{weak}^* \text{-} \lim_\alpha D(e_\alpha)$. Then,

$$\begin{aligned} D(a) &= \text{weak}^* \text{-} \lim_\alpha D(ae_\alpha) = \text{weak}^* \text{-} \lim_\alpha (D(a)e_\alpha + aD(e_\alpha)) \\ &= \text{weak}^* \text{-} a \lim_\alpha D(e_\alpha) + \text{weak}^* \text{-} \lim_\alpha D(a)e_\alpha \\ &= a \text{weak}^* \text{-} \lim_\alpha D(e_\alpha) + 0 \\ &= a \cdot \phi. \end{aligned}$$

That is,

$$D(a) = a \cdot \phi = a \cdot \phi - 0 = a \cdot \phi - \phi \cdot a$$

Thus $D = \text{ad}_\phi$. □

3.4 Weak Amenability of Banach Algebras

3.4.1 Definition and some general results

The notion of weak amenability was introduced by Bade et al. in [4] for commutative Banach algebras with the aim of studying classes of Banach algebra which satisfy $H^1(\mathcal{A}, \mathcal{A}') = \{0\}$ but not amenable. Such Banach algebras are referred to as weakly amenable. For example, $L^1(G)$ is amenable if and only if G is amenable as a locally compact group but $L^1(G)$ is weakly amenable irrespective of whether G is amenable or not. Thus the class of weakly amenable Banach algebras is larger than the class of amenable Banach algebras. Clearly, all amenable Banach algebras are weakly amenable but the converse is not true. A counter example is the Banach algebra l_p ($1 \leq p < \infty$) with pointwise multiplication which lacks a bounded approximate identity. A commutative Banach algebra \mathcal{A} is weakly amenable if the first Hochschild cohomology $H^1(\mathcal{A}, X)$ with coefficients in symmetric modules vanishes. That is every bounded inner derivation $D : \mathcal{A} \rightarrow X$ is zero for every commutative Banach \mathcal{A} -bimodule X . Since this definition makes no sense for non-commutative Banach algebra, Johnson in [25] gave a definition for general Banach algebra. We recall from [25] that a Banach algebra \mathcal{A} is weakly amenable if every continuous derivation $D : \mathcal{A} \rightarrow \mathcal{A}'$ is inner.

Let X be a Banach \mathcal{A} -bimodule. Then we can make X into a Banach $\mathcal{A}^\#$ -bimodule by defining $e \cdot x = x \cdot e = x$ for all $x \in X$. Now, consider the following short exact sequence of linear maps

$$\Pi : 0 \rightarrow X \rightarrow \mathfrak{B}(\mathcal{A}^\#, X) \rightarrow \mathfrak{B}(\mathcal{A}^\#, X)/\text{Im } D_1 \rightarrow 0,$$

where $\mathfrak{B}(\mathcal{A}^\#, X)$ is the set of all bounded linear operators from $\mathcal{A}^\#$ into X , $D_1 : X \rightarrow \mathfrak{B}(\mathcal{A}^\#, X)$ is defined by $D_1(x) : a \mapsto x \cdot a$, $a \in \mathcal{A}^\#$ and $D_2 : \mathfrak{B}(\mathcal{A}^\#, X) \rightarrow \mathfrak{B}(\mathcal{A}^\#, X)/\text{Im } D_1$.

For all $a \in \mathcal{A}$ and $\bar{D} \in \mathfrak{B}(\mathcal{A}^\#, X)$, if we define the left action by

$$a \cdot \bar{D} : b \rightarrow a \cdot \bar{D}(b), \quad \mathcal{A}^\# \rightarrow X$$

and the right action by

$$\bar{D} \cdot a : b \rightarrow \bar{D}(ab), \quad \mathcal{A}^\# \rightarrow X,$$

then

- (i) $\mathfrak{B}(\mathcal{A}^\#, X)$ is a Banach \mathcal{A} -bimodule,
- (ii) D_1 and D_2 are Banach \mathcal{A} -bimodule homomorphisms.

This follows from the fact that $D_1(x) \in \mathfrak{B}(\mathcal{A}^\#, X)$ and hence satisfies $D_1(xy) = D_1(x)D_1(y)$ for all $x, y \in X$. Also, from the definition of D_2 , it follows that

$$\begin{aligned} D_2(D_1(xy)) &= D_1(xy) + \text{Im } D_1 &= D_1(x)D_1(y) + \text{Im } D_1 \\ & &= (D_1(x) + \text{Im } D_1)(D_1(y) + \text{Im } D_1) \\ & &= D_2(D_1(x))D_2(D_1(y)). \end{aligned}$$

Let $D \in \mathfrak{B}(\mathcal{A}, X)$ and e be the identity element in $\mathcal{A}^\#$, then the bounded linear map $\bar{D} : \mathcal{A}^\# \rightarrow X$ is an extension of $D : \mathcal{A} \rightarrow X$ whenever $\bar{D}(e) = 0$. With this in mind, we recapture the definition of continuous derivations from \mathcal{A} into X in the following proposition.

Proposition 3.4.1. *Let \mathcal{A} be a Banach algebra, $D : \mathcal{A} \rightarrow X$ be a bounded linear map and let Π be the short exact sequence defined above. Suppose D_1 and D_2 are Banach \mathcal{A} -bimodule homomorphisms defined above. Then the following statements are equivalent:*

- (i) D is a derivation;
- (ii) $D_1 \circ D = \delta_{\bar{D}}$;
- (iii) $D_2 \circ \delta_{\bar{D}} = 0$.

Proof. We shall first show that (ii) \iff (i) and then (ii) \iff (iii).

Assume that (ii) holds. Since $D_1(x) : a \mapsto x \cdot a$ for $a \in \mathcal{A}^\#$ and $x \in X$, then

$$(D_1 \circ D) \cdot a(b) = D_1((D \cdot a)(b)) = (D \cdot a)(b).$$

Also,

$$\delta_{\bar{D}} = \bar{D} \cdot a(b) - a \cdot \bar{D}(b) = \bar{D}(ab) - a \cdot \bar{D}(b).$$

Thus statement (ii) implies that $(D \cdot a)(b) = \bar{D}(ab) - a \cdot \bar{D}(b)$. That is $\bar{D}(ab) = (D \cdot a)(b) + a \cdot \bar{D}(b)$.

Case 1. Suppose that $b = e \in \mathcal{A}^\#$, then by using the right module operation

$$\begin{aligned} \bar{D}(a) = \bar{D}(ab) &= \bar{D}(a \cdot e) = (D \cdot a)(e) + a \cdot \bar{D}(e) = (D \cdot a)(e) + a \cdot 0 \\ &= (D \cdot a)(e) + 0 = \bar{D}(ae) = \bar{D}(a). \end{aligned}$$

Whence $\bar{D} = D$ and so (i) holds.

Case 2. If $b \neq e \in \mathcal{A}^\#$, then D is necessarily the restriction of \bar{D} on \mathcal{A} . Thus (i) holds.

Conversely, assume that $D : \mathcal{A} \rightarrow X$ is a derivation. Since $\overline{D}(a) = D(a)$ for $a \in \mathcal{A}$, and $\overline{D}(e) = 0$ for $e \in \mathcal{A}^\#$, then \overline{D} is also a derivation from $\mathcal{A}^\#$ to X . Thus for all $a \in \mathcal{A}, b \in \mathcal{A}^\#$, it then follows that $\overline{D}(ab) = \overline{D}(a)b + a\overline{D}(b)$. By using the left and right action of \mathcal{A} on the Banach \mathcal{A} -bimodule $\mathfrak{B}(\mathcal{A}, X)$, then for $a, b \in \mathcal{A}^\#$,

$$\begin{aligned} \overline{D} \cdot a(b) - a \cdot \overline{D}(b) &= \overline{D}(ab) - a \cdot \overline{D}(b) = \overline{D}(a) \cdot b + a \cdot \overline{D}(b) - a \cdot \overline{D}(b) \\ &= \overline{D}(a) \cdot b = D_1(\overline{D}(a))(b) = (D_1 \circ \overline{D})(a)(b). \end{aligned}$$

Whence $\overline{D} \cdot a - a \cdot \overline{D} = (D_1 \circ \overline{D})(a)$ which gives $\delta_{\overline{D}} = D_1 \circ D$ and so (ii) holds.

(ii) \implies (iii). Let $D_1 \circ D = \delta_{\overline{D}}$. Then for $D_2(D_1 \circ D) = D_2(\delta_{\overline{D}}) = D_2(\overline{D} \cdot b - b \cdot \overline{D})$, ($b \in \mathcal{A}^\#$). Since the sequence Π is exact, then

(E1) D_2 is onto and D_1 is one-to-one,

(E2) $\text{Im}D_1 = \ker D_2$.

By the definition of \overline{D} , we can assume that $\overline{D} \cdot b - b \cdot \overline{D}$ is in the image of D_1 and then show that it is also in the kernel of D_2 . Let $\overline{D} \cdot b - b \cdot \overline{D} \in \text{Im}D_1$, then by (E2),

$$\overline{D} \cdot b - b \cdot \overline{D} = D_2(\overline{D} \cdot b - b \cdot \overline{D}) = 0.$$

Thus $D_2 \circ \delta_{\overline{D}} = 0$.

Conversely, assume that (iii) holds. Then by exactness of Π , it follows that for each $a \in \mathcal{A}$, there is a unique $x_a \in X$ such that $x_a \cdot b = \overline{D}(ab) - a \cdot \overline{D}(b)$ for all $b \in \mathcal{A}^\#$. Applying this to $b = e$, we obtain $x_a = D(a)$, so that (ii) holds. \square

3.4.2 Hereditary properties

The following important propositions are from [9].

Proposition 3.4.2. *Let \mathcal{A} be a weakly amenable Banach algebra. Then*

(i) \mathcal{A} is essential,

(ii) there are no non-zero continuous point derivations on \mathcal{A} ,

(iii) suppose that \mathcal{A} is commutative, then $Z^1(\mathcal{A}, X) = \{0\}$ for every symmetric Banach \mathcal{A} -bimodule X .

Proposition 3.4.3. *Let \mathcal{A} be a weakly amenable, commutative Banach algebra, let I be a closed ideal in \mathcal{A} , and let X be a Banach I -module. Then $D|_{I^4} = 0$ for each $D \in Z^1(I, X)$.*

The following characterization of weakly amenable commutative Banach algebra from [4] is important for the remaining part of this section:

Let \mathcal{A} be a commutative Banach algebra. Let $D \in Z^1(\mathcal{A}, \mathcal{A}')$. Then \mathcal{A} is weakly amenable if and only if $D = 0$.

Proposition 3.4.4. *Let \mathcal{A} be a Banach algebra.*

- (i) *Suppose that \mathcal{A} is commutative and \mathcal{B} is another commutative Banach algebra such that $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a continuous homomorphism with dense range. Then \mathcal{B} is weakly amenable if \mathcal{A} is weakly amenable.*
- (ii) *Let I be a closed ideal of \mathcal{A} . If I and \mathcal{A}/I are both weakly amenable, then \mathcal{A} is weakly amenable.*
- (iii) *The closed ideal I of a commutative weakly amenable Banach algebra \mathcal{A} is weakly amenable if and only if $\overline{I^2} = I$.*
- (iv) *\mathcal{A} is weakly amenable if and only if $\mathcal{A}^\#$ is weakly amenable.*
- (v) *Let \mathcal{A} and \mathcal{B} be weakly amenable commutative Banach algebras. Then $\mathcal{A} \hat{\otimes} \mathcal{B}$ is weakly amenable.*

Proof. (i) We make \mathcal{B} into a Banach \mathcal{A} -bimodule with module operation specified by

$$a \cdot b = \varphi(a)b, \quad b \cdot a = b \cdot \varphi(a), \quad (a \in \mathcal{A}, b \in \mathcal{B}).$$

We shall show that every continuous derivation $D : \mathcal{B} \rightarrow \mathcal{B}'$ is inner. The linear map $D \circ \varphi : \mathcal{A} \rightarrow \mathcal{B}'$ is a derivation, since for $a, a' \in \mathcal{A}$,

$$\begin{aligned} (D \circ \varphi)(aa') &= D(\varphi(aa')) = D(\varphi(a)\varphi(a')) = D(\varphi(a))\varphi(a') + \varphi(a)D(\varphi(a')) \\ &= (D \circ \varphi)(a)a' + a(D \circ \varphi)(a'). \end{aligned}$$

Since \mathcal{A} is commutative and weakly amenable, then by Proposition 3.4.2(iii), $D \circ \varphi = 0$. Now, \mathcal{B} is commutative also implies that $b \cdot f = f \cdot b$ for $b \in \mathcal{B}$, $f \in \mathcal{B}'$. Since $\overline{\varphi(\mathcal{A})} = \mathcal{B}$, then for $(b_\alpha)_{\alpha \in \Delta} \subset \varphi(\mathcal{A})$ such that $b_\alpha \rightarrow b \in \mathcal{B}$, it follows that

$$\begin{aligned} 0 = b \cdot f - f \cdot b &= \lim_{\alpha} (b_\alpha f) - \lim_{\alpha} (f b_\alpha) = \lim_{\alpha} (b_\alpha f - f b_\alpha) \\ &= \lim_{\alpha} \delta_f(b_\alpha) = \delta_f(\lim_{\alpha} b_\alpha) = \delta_f(b). \end{aligned}$$

So necessarily $D = b \cdot f - f \cdot b = 0$. Whence $H^1(\mathcal{B}, \mathcal{B}') = \{0\}$. Thus, \mathcal{B} is weakly amenable.

- (ii) We run the standard argument of the proof of Proposition 3.3.6 with modifications where necessary. Let $D \in Z^1(\mathcal{A}, \mathcal{A}')$, then the restriction map $D|_I$ is a derivation into \mathcal{A}' , that is, $D|_I \in Z^1(I, \mathcal{A}')$. Let $\lambda_1 \in \mathcal{A}'$, define a continuous derivation $\tilde{D} : \mathcal{A} \rightarrow \mathcal{A}'$ by $\tilde{D} = D - \delta_{\lambda_1}$. Indeed, \tilde{D} is a derivation since the sum or difference of any two derivation is a derivation. Since I is weakly amenable, there exists $\lambda_1 \in \mathcal{A}'$ such that $D|_I = \delta_{\lambda_1}$. Thus $\tilde{D} = 0$ when restricted to I . Next, we define

$$F := \text{cl}\{a \cdot a' + b \cdot b' \mid a, b \in I, a', b' \in \mathcal{A}\}$$

and

$$F^0 := \{\lambda \in \mathcal{A}' \mid a \cdot \lambda = \lambda \cdot a = 0 \forall a \in I\}.$$

Clearly, F is closed ideal of the Banach algebra \mathcal{A} and also a closed submodule of the Banach \mathcal{A} -bimodule \mathcal{A} (that is, \mathcal{A} is considered a Banach bimodule over itself) and so \mathcal{A}/F is a Banach \mathcal{A}/I bimodule. F^0 is a dual Banach \mathcal{A}/I bimodule and so $F^0 \cong (\mathcal{A}/I)'$. Let $a \in \mathcal{A}$, $b \in I$ then

$$0 = \tilde{D}|_I(ab) = \tilde{D}(ab) = \tilde{D}(a)b + a\tilde{D}(b) = \tilde{D}(a)b + 0 = \tilde{D}(a)b.$$

Thus for every a' in the Banach \mathcal{A} -bimodule \mathcal{A} , a in the Banach algebra \mathcal{A} and $b \in I$, we have

$$\begin{aligned} \langle a' \cdot b, \tilde{D}(a) \rangle &= \langle a' \cdot b, (D - \delta_{\lambda_1})(a) \rangle = \langle a', b \cdot (D - \delta_{\lambda_1})(a) \rangle = 0, \\ \langle b \cdot a', \tilde{D}(a) \rangle &= \langle b \cdot a', (D - \delta_{\lambda_1})(a) \rangle = \langle a', (D - \delta_{\lambda_1})(a) \cdot b \rangle = 0. \end{aligned}$$

That is $\langle a' \cdot b, \tilde{D}(a) \rangle = \langle b \cdot a', \tilde{D}(a) \rangle = 0$ and so there exists $\lambda \in \mathcal{A}'$ such that $0 = \tilde{D}(a)(a' \cdot b) = \lambda(a' \cdot b)$ and $0 = \tilde{D}(a)(b \cdot a') = \lambda(b \cdot a')$, that is, $\tilde{D}(a)(a' \cdot b) = \tilde{D}(a)(b \cdot a') = 0$. Thus $\tilde{D} : \mathcal{A} \rightarrow F^0$, that is $\tilde{D}(\mathcal{A}) \subset F^0$. Now, define $D_1 : \mathcal{A}/I \rightarrow F^0$ by $D_1 : a+I \mapsto \tilde{D}(a)$, that is, $D_1(a+I) = (D - \delta_{\lambda_1})(a)$. Since \mathcal{A}/I is weakly amenable, then there exists $\lambda \in F^0$ such that $D_1 = \delta_{\lambda}$. Thus $\delta_{\lambda} = D - \delta_{\lambda_1}$ which implies $D = \delta_{\lambda} + \delta_{\lambda_1} = \delta_{\lambda + \lambda_1}$ where $\lambda + \lambda_1 \in \mathcal{A}'$. Thus \mathcal{A} is weakly amenable

- (iii) Assume that I is weakly amenable, then by Proposition 3.4.2(i), I is essential, that is, $\overline{I^2} = I$.

Conversely, assume that $\overline{I^2} = I$, then $\overline{I^4} = \overline{I^2} = I$. Let $D \in Z^1(I, I')$, then $D|_{I^4} = 0$ by Proposition 3.4.3. Thus $D|_I = 0$, that is D is the zero derivation when restricted to I . Since \mathcal{A} is commutative, then I is also commutative. Now, I is commutative also implies that $a \cdot \lambda = \lambda \cdot a$ for $a \in I$ and $\lambda \in I'$. So, $D(a) = a \cdot \lambda - \lambda \cdot a = 0$ ($a \in I$). Thus D is inner and I is weakly amenable.

- (iv) The proof is much similar to that of Proposition 3.3.8. Let $\mathcal{A}^\#$ be weakly amenable and \mathcal{A} be a Banach $\mathcal{A}^\#$ -bimodule. Clearly, \mathcal{A} is also a Banach \mathcal{A} -bimodule. The right and left module action of $\mathcal{A}^\#$ on \mathcal{A} is specified by

$$x \cdot (a, \alpha) = xa \quad (a, \alpha) \cdot x = ax, \quad x \in \mathcal{A}, (a, \alpha) \in \mathcal{A}^\#.$$

Let $\tilde{D} \in Z^1(\mathcal{A}^\#, \mathcal{A}')$. Define the map $\tilde{D} : (a, \alpha) \mapsto D(a)$, then for $(a, \alpha), (b, \beta) \in \mathcal{A}^\#$, it follows that

$$\begin{aligned} D(ab) = \tilde{D}(ab, \alpha\beta) &= \tilde{D}((a, \alpha)(b, \beta)) = \tilde{D}(a, \alpha)(b, \beta) + (a, \alpha)\tilde{D}(b, \beta) \\ &= D(a)(b, \beta) + (a, \alpha)D(b) = D(a)b + aD(b). \end{aligned}$$

Thus the restriction $D : \mathcal{A} \rightarrow \mathcal{A}'$ of $\tilde{D} \in Z^1(\mathcal{A}^\#, \mathcal{A}')$ is a derivation. Next is to show that $D \in Z^1(\mathcal{A}, \mathcal{A}')$ is inner. Since $\mathcal{A}^\#$ is weakly amenable then there exists $\lambda \in \mathcal{A}'$ such that $(a, \alpha) \in \mathcal{A}^\#$,

$$\begin{aligned} D(a, \alpha) &= (a, \alpha) \cdot \lambda - \lambda(a, \alpha) = (a\lambda, \alpha\lambda) - (\lambda a, \lambda\alpha) = (a\lambda - \lambda a, \alpha\lambda - \lambda\alpha) \\ &= (a\lambda - \lambda a, 0). \end{aligned}$$

Thus $D(a) = a\lambda - \lambda a$, that is, D is inner and \mathcal{A} is weakly amenable.

Conversely, let \mathcal{A} be weakly amenable and let $D \in Z^1(\mathcal{A}^\#, \mathcal{A}')$. Since \mathcal{A} is unital, then for $\lambda \in \mathcal{A}'$, $\lambda \cdot e = e \cdot \lambda = \lambda$. Now, for $D \in Z^1(\mathcal{A}^\#, \mathcal{A}')$,

$$D(e) = D(e \cdot e) = D(e)e + eD(e) = D(e) + D(e) = 2D(e).$$

Thus $D(e) = 0$ and so D can be considered as a derivation from \mathcal{A} into \mathcal{A}' and since \mathcal{A} is weakly amenable, then D is inner on \mathcal{A} . This coincides with D being inner on $\mathcal{A}^\#$ since $D(e)$ is trivial and so $\mathcal{A}^\#$ is weakly amenable.

- (v) We adopt the proof in [10]. Let X be a Banach $(\mathcal{A}^\# \hat{\otimes} \mathcal{B}^\#)$ -bimodule. Let $D \in Z^1(\mathcal{A}^\# \hat{\otimes} \mathcal{B}^\#, X)$, then we can make X into a Banach \mathcal{A} -bimodule. The module operation is specified by

$$a \cdot x = (a \otimes e_{\mathcal{B}})x, \quad x \cdot a = x(a \otimes e_{\mathcal{B}}).$$

Define $D(a \otimes e_{\mathcal{B}}) = D(a)$, then for $(a \otimes e_{\mathcal{B}}), (a' \otimes e_{\mathcal{B}}) \in \mathcal{A}^\# \hat{\otimes} \mathcal{B}^\#$, it holds that

$$\begin{aligned} D(aa') &= D((a \otimes e_{\mathcal{B}})(a' \otimes e_{\mathcal{B}})) = D(a \otimes e_{\mathcal{B}})(a' \otimes e_{\mathcal{B}}) + (a \otimes e_{\mathcal{B}})D(a' \otimes e_{\mathcal{B}}) \\ &= D(a)(a' \otimes e_{\mathcal{B}}) + (a \otimes e_{\mathcal{B}})D(a') = D(a)a' + aD(a'). \end{aligned}$$

Thus $D \in Z^1(\mathcal{A}^\#, X)$ and so the restriction $D|_{\mathcal{A}^\# \otimes e_{\mathcal{B}}}$ is in $Z^1(\mathcal{A}^\#, X)$. By (iv) above, $\mathcal{A}^\#$ is weakly amenable and commutative, and so $D|_{\mathcal{A}^\# \otimes e_{\mathcal{B}}} = 0$. Similarly, $D|_{e_{\mathcal{A}} \otimes \mathcal{B}^\#} = 0$. Thus $D = 0$ and $\mathcal{A}^\# \hat{\otimes} \mathcal{B}^\#$ is weakly amenable. Since \mathcal{A} and \mathcal{B} are weakly amenable, then by Proposition 3.4.2(i), $\overline{\mathcal{A}^2} = \mathcal{A}$ and $\overline{\mathcal{B}^2} = \mathcal{B}$. Set $I = \mathcal{A} \hat{\otimes} \mathcal{B}$. Then I is closed ideal in $\mathcal{A}^\# \hat{\otimes} \mathcal{B}^\#$, and $\overline{I^2} = I$ and so I is weakly amenable by (iii) above. \square

Corollary 3.4.5. *Let \mathcal{A} be a commutative Banach algebra. If \mathcal{A} is weakly amenable then $\mathcal{A} \hat{\otimes} \mathcal{A}$ is weakly amenable.*

Proof. The result follows easily by setting $\mathcal{B} = \mathcal{A}$ in Proposition 3.4.4(v). \square

Corollary 3.4.6. *Let \mathcal{A} and \mathcal{B} be commutative Banach algebras.*

- (i) *If $\mathcal{A}^\#$ and $\mathcal{B}^\#$ are weakly amenable, then $\mathcal{A}^\# \hat{\otimes} \mathcal{B}^\#$ is weakly amenable.*
- (ii) *If $\mathcal{A}^\#$ and $\mathcal{B}^\#$ are weakly amenable, then $\mathcal{A} \hat{\otimes} \mathcal{B}$ is weakly amenable.*
- (iii) *If \mathcal{A} and \mathcal{B} are weakly amenable, then $\mathcal{A}^\# \hat{\otimes} \mathcal{B}^\#$ is weakly amenable.*

Proof. (i) Follows directly from the proof of Proposition 3.4.4(v).

- (ii) Since $\mathcal{A}^\#$ and $\mathcal{B}^\#$ are weakly amenable, then \mathcal{A} and \mathcal{B} are weakly amenable by Proposition 3.4.4(iv), and so the result holds by 3.4.4(v).
- (iii) Since \mathcal{A} and \mathcal{B} are weakly amenable, then $\mathcal{A}^\#$ and $\mathcal{B}^\#$ are weakly amenable by 3.4.4(iv), and so the result holds by Corollary 3.4.6(i). \square

Corollary 3.4.7. *Let \mathcal{A} and \mathcal{B} be commutative Banach algebras. Suppose that I and K are closed two sided ideals of \mathcal{A} and \mathcal{B} respectively. If \mathcal{A} and \mathcal{B} are both weakly amenable. Then*

- (i) *The Banach algebra \mathcal{A}/I is commutative and weakly amenable.*
- (ii) *The Banach algebra \mathcal{B}/K is commutative and weakly amenable.*
- (iii) *The Banach algebra $\mathcal{A}/I \hat{\otimes} \mathcal{B}/K$ is weakly amenable.*

Proof. (i) There is a natural canonical map φ from \mathcal{A} onto \mathcal{A}/I given by

$$\varphi : a \mapsto a + I : \mathcal{A} \rightarrow \mathcal{A}/I.$$

This map is surjective, has a dense range in \mathcal{A}/I , continuous and an homomorphism. Indeed, for $a \in \mathcal{A}$,

$$\varphi(ab) = ab + I = (a + I)(b + I) = \varphi(a)\varphi(b).$$

Now, since \mathcal{A} is weakly amenable, then \mathcal{A}/I is also weakly amenable by Proposition 3.4.4(i).

That \mathcal{A}/I is commutative follows easily from the assumption that \mathcal{A} is commutative. To see this, let $a_1, a_2 \in \mathcal{A}$ then $a_1 \cdot a_2 = a_2 \cdot a_1$ implies that $(a_1 + I)(a_2 + I) = a_1a_2 + I = a_2a_1 + I = (a_2 + I)(a_1 + I)$.

- (ii) Proof is similar to the argument of the proof of Corollary 3.4.7(i). Set $\mathcal{B} = \mathcal{A}$ and $K = I$ in (i) above and then the result follows.
- (iii) By (i) and (ii) above, \mathcal{A}/I and \mathcal{B}/K are both weakly amenable and commutative. The remaining part of the proof then follows from Proposition 3.4.4(iv).

□

3.5 Character Amenability of Banach Algebras

3.5.1 Definitions

As earlier discussed, several modifications of the original notion of amenability in Banach algebras are introduced after the pioneering work of Johnson [24]. One of such modification was initiated by Lau for a class of F -algebras in [29] and was later generalized by Kaniuth, Lau and Pym in [27]. Monfared in [37] extended these notions and initiated the notion of character amenability. This is stronger than left amenability and also relaxes the constraint in the definition of amenability given by Johnson in [24]. This is because the class of Banach \mathcal{A} -bimodules that are required in the definition of character amenability are those in which either the left or right module actions are defined by the characters on \mathcal{A} . Thus character amenability is weaker than amenability introduced in [24].

For a Banach algebra \mathcal{A} , we recall that a character on \mathcal{A} is an algebra homomorphism $\varphi : \mathcal{A} \rightarrow \mathbb{C}$. The space of all characters on \mathcal{A} , called the character space of \mathcal{A} is denoted by $\sigma(\mathcal{A})$.

Denote by $M_{\varphi_l}^{\mathcal{A}}$ the collection of all Banach \mathcal{A} -bimodule X whose right module action is defined by $x \cdot a = \varphi(a)x$ ($a \in \mathcal{A}, x \in X, \varphi \in \sigma(A)$) and let $M_{\varphi_r}^{\mathcal{A}}$ be the collection of all Banach \mathcal{A} -bimodule X for which the left module action is given by $a \cdot x = \varphi(a)x$ ($a \in \mathcal{A}, x \in X, \varphi \in \sigma(A)$).

Let \mathcal{A} be a Banach algebra and let the right module action of \mathcal{A} on X be as given above. Then for $f \in X', a \in \mathcal{A}, \varphi \in \sigma(A)$,

$$\langle x, a \cdot f \rangle = \langle x \cdot a, f \rangle = \langle \varphi(a)x, f \rangle = \langle x, \varphi(a)f \rangle.$$

The last equality follows from the linearity of f . Thus $a \cdot f = \varphi(a)f$ which gives the left module action of \mathcal{A} on X' . Similarly, from the left module action of \mathcal{A} on X , we can obtain the right module action of \mathcal{A} on X' to be $f \cdot a = \varphi(a)x$. The following definition are taken from [36] and [37].

Definition 3.5.1. *Let \mathcal{A} be a Banach algebra and $\varphi \in \sigma(A)$.*

- (i) \mathcal{A} is left φ -amenable if every continuous derivation $D : \mathcal{A} \rightarrow X'$ is inner for every $X \in M_{\varphi_r}^{\mathcal{A}}$.
- (ii) \mathcal{A} is right φ -amenable if every continuous derivation $D : \mathcal{A} \rightarrow X'$ is inner for every $X \in M_{\varphi_l}^{\mathcal{A}}$.
- (iii) \mathcal{A} is left character amenable if it is left φ -amenable for every $\varphi \in \sigma(A)$.
- (iv) \mathcal{A} is right character amenable if it is right φ -amenable for every $\varphi \in \sigma(A)$.
- (v) \mathcal{A} is character amenable if it is both left and right character amenable.

3.5.2 Some general results

We now give explicit proofs of some important general results on character amenability in [34], [33], [35] and [37]. We begin with the following much needed characterization whose proof will be appropriate for the next section.

Proposition 3.5.2. *Let I be a closed two sided ideal of a character amenable Banach algebra \mathcal{A} . Then I is character amenable if and only if I has a bounded approximate identity.*

An immediate implication of Proposition 3.5.2 is that if \mathcal{A} is character amenable then it has a bounded approximate identity and hence factors. Thus, the much desired factorization property is extended from the class of amenable Banach algebra to the class of character amenable ones. The following lemma is an analogue of Proposition 3.3.9 for character amenable Banach algebras.

Lemma 3.5.3. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity and φ be a character on \mathcal{A} , then the following statements are equivalent:*

- (i) $H^1(\mathcal{A}, X') = \{0\}$ for each Banach \mathcal{A} -bimodule $X \in M_{\varphi_r}^{\mathcal{A}}$.
- (ii) $H^1(\mathcal{A}, X') = \{0\}$ for each left neo-unital Banach \mathcal{A} -bimodule $X \in M_{\varphi_r}^{\mathcal{A}}$.

Proposition 3.5.4. *Let \mathcal{A} be a character amenable Banach algebra and I a closed two sided ideal of codimension one in \mathcal{A} .*

1. *Then*

- (i) \mathcal{A} has a bounded approximate identity and hence factors;
- (ii) the unitization algebra $\mathcal{A}^\#$ is character amenable;
- (iii) I has a bounded approximate identity;
- (iv) I is character amenable.

2. *Suppose \mathcal{B} is another Banach algebra and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a continuous homomorphism with dense range, then*

- (i) then \mathcal{B} is character amenable;
- (ii) if I is a closed ideal of \mathcal{A} , then \mathcal{A}/I is character amenable;
- (iii) if both I and \mathcal{A}/I are character amenable then \mathcal{A} is also character amenable ;
- (iv) $\mathcal{A} \times \mathcal{B}$ is character amenable if and only if both \mathcal{A} and \mathcal{B} are character amenable.

Proof. (i) Without loss of generality, let $\varphi \in \sigma(\mathcal{A})$ and X be a Banach \mathcal{A} -bimodule with the left and right actions given by $a \cdot x = \varphi(a)x$ and $x \cdot a = 0$ respectively for all $x \in X$. Since \mathcal{A} is character amenable, then in particular, \mathcal{A} is φ -amenable and by (Theorem 1.1 of [27]), $H^1(\mathcal{A}, X') = \{0\}$ for every X . Moreover, by (Propositions 1.5 and 1.6 of [24]), the statement that $H^1(\mathcal{A}, X') = \{0\}$ for every X for which the right module action is trivial is equivalent to the existence of a bounded left approximate identity for \mathcal{A} . Similar argument also holds for the existence of a right bounded approximate identity for \mathcal{A} . That \mathcal{A} factors is clear.

(ii) Let \mathcal{A} be character amenable, then by Lemma 3.5.3, it suffices to show that $H^1(\mathcal{A}^\#, X') = \{0\}$ for $X \in M_{\varphi_r}^{\mathcal{A}^\#}$, where X is neo-unital. Let $\tilde{D} \in Z^1(\mathcal{A}^\#, X')$

then $\tilde{D}(e) = \tilde{D}(e \cdot e) = \tilde{D}(e) \cdot e + e \cdot \tilde{D}(e) = \tilde{D}(e) + \tilde{D}(e) = 2\tilde{D}(e)$. This implies that $\tilde{D}(e) = 0$. For $X \in M_{\varphi_r}^{\mathcal{A}^\#}$, it is clear that $X \in M_{\varphi_r}^{\mathcal{A}}$ and so we can consider \tilde{D} as a derivation from \mathcal{A} into X' and since \mathcal{A} is character amenable, then \tilde{D} is inner. This coincides with \tilde{D} being inner on $\mathcal{A}^\#$ since $\tilde{D}(e)$ is trivial. Thus, $H^1(\mathcal{A}^\#, X') = \{0\}$ for each neo-unital $X \in M_{\varphi_r}^{\mathcal{A}^\#}$ and by Lemma 3.5.3, $H^1(\mathcal{A}^\#, X') = \{0\}$ for each $X \in M_{\varphi_r}^{\mathcal{A}^\#}$. Thus $\mathcal{A}^\#$ is left character amenable. Similar argument holds for the right character amenability of $\mathcal{A}^\#$ and so the character amenability of \mathcal{A} immediately follows.

Conversely, assume that $\mathcal{A}^\#$ is right character amenable. For $\varphi \in \sigma(\mathcal{A})$, let $X \in M_{\varphi_l}^{\mathcal{A}}$. We shall show that $H^1(\mathcal{A}, X') = \{0\}$ for every $X \in M_{\varphi_l}^{\mathcal{A}}$. Define $\tilde{\varphi} : \mathcal{A}^\# \rightarrow \mathbb{C}$ by

$$\tilde{\varphi}(a, z) = \varphi(a) + z \quad (z \in \mathbb{C}, \tilde{\varphi} \in \sigma(\mathcal{A}^\#)).$$

Indeed $\tilde{\varphi} \in \sigma(\mathcal{A}^\#)$ and is linear. Since for any $(a, z), (a', z') \in \mathcal{A}^\#$,

$$\begin{aligned} \tilde{\varphi}((a, z)(a', z')) &= \tilde{\varphi}(aa' + az' + za', zz') = \varphi(aa' + az' + za') + zz' \\ &= \varphi(aa') + \varphi(az') + \varphi(za') + zz' \\ &= \varphi(a)\varphi(a') + \varphi(a)z' + z\varphi(a') + zz' \\ &= (\varphi(a) + z)(\varphi(a') + z') = \tilde{\varphi}(a, z)\tilde{\varphi}(a', z'). \end{aligned}$$

Also,

$$\begin{aligned} \tilde{\varphi}((a, z) + (a', z')) &= \tilde{\varphi}(a + a', z + z') = \varphi(a + a') + z + z' \\ &= \varphi(a) + z + \varphi(a') + z' = \tilde{\varphi}(a, z) + \tilde{\varphi}(a', z') \end{aligned}$$

and for any $\alpha \in \mathbb{C}$,

$$\tilde{\varphi}(\alpha(a, z)) = \tilde{\varphi}(\alpha a, \alpha z) = \varphi(\alpha a) + \alpha z = \alpha(\varphi(a) + z) = \alpha\tilde{\varphi}(a, z).$$

Moreover, X is also a Banach $\mathcal{A}^\#$ -bimodule, in fact $X \in M_{\tilde{\varphi}_l}^{\mathcal{A}^\#}$. The left and right module action of $\mathcal{A}^\#$ on X are given by

$$(a, z) \cdot x = a \cdot x + zx = \tilde{\varphi}(a, z)x, \quad x \cdot (a, z) = x \cdot a + zx = \tilde{\varphi}(a, z)x \quad ((a, z) \in \mathcal{A}^\#, x \in X).$$

Let $\tilde{D} \in Z^1(\mathcal{A}^\#, X')$. Define the map $\tilde{D} : (a, z) \mapsto D(a)$. Then we show that D is an inner derivation from \mathcal{A} into X' . For $a, b \in \mathcal{A}$, $z, z' \in \mathbb{C}$,

$$\begin{aligned} D(ab) = \tilde{D}(ab, zz') &= \tilde{D}((a, z)(b, z')) = \tilde{D}(a, z) \cdot (b, z') + (a, z) \cdot \tilde{D}(b, z') \\ &= \tilde{D}(a, z) \cdot b + a \cdot \tilde{D}(b, z') = D(a) \cdot b + a \cdot D(b). \end{aligned}$$

Hence the restriction $D : \mathcal{A} \rightarrow X'$ of \tilde{D} is a derivation. Now, since $\mathcal{A}^\#$ is character amenable by assumption, then there exists $\lambda \in X' \in M_{\hat{\varphi}}^{\mathcal{A}}$ such that $\tilde{D}(a, z) = (a, z) \cdot \lambda - \lambda \cdot (a, z) = a \cdot \lambda - \lambda \cdot a \quad (a, z) \in \mathcal{A}^\#$. Whence $D(a) = \tilde{D}(a, z) = a \cdot \lambda - \lambda \cdot a$ and D is inner.

Thus \mathcal{A} is right character amenable. Similar argument holds for the left character amenability of \mathcal{A} . Hence \mathcal{A} is character amenable.

(iii) From (i) above, \mathcal{A} has a bounded approximate identity, say (e_α) . Also, since I is a closed ideal of codimension one in \mathcal{A} , then we can write $I = \ker \varphi$ for some $\varphi \in \sigma(\mathcal{A})$. By adjoining a unit e to \mathcal{A} , we can write the unitization $\mathcal{A}^\# = \mathcal{A} \oplus \mathbb{C}e$. Now, for $\varphi \in \sigma(\mathcal{A})$, we define the extension $\hat{\varphi} \in \sigma(\mathcal{A}^\#)$ by

$$\hat{\varphi}(\hat{a}) = \hat{\varphi}(a + ze) = \varphi(a) + z \quad (\hat{a} \in \mathcal{A}^\#, a \in \mathcal{A}, z \in \mathbb{C}).$$

Thus, $J = \{\hat{a} \in \mathcal{A}^\# : z = -\varphi(a)\} = \ker \hat{\varphi}$ has codimension one as a subspace of $\mathcal{A}^\#$ so that $\mathcal{A}^\# = \mathcal{A} \oplus \mathbb{C}e = J \oplus \mathbb{C}e \cong J^\#$. Since $\mathcal{A}^\#$ is character amenable by (ii) above, then it follows that $J^\#$ is character amenable. Thus J is also character amenable and so has a bounded approximate identity (s_α) by (i). For each β , set

$$s_\beta = a_\beta + c_\beta e$$

for some $a_\beta \in \mathcal{A}$ and $c_\beta \in \mathbb{C}$ such that $c_\beta = -\varphi(a_\beta)$, that is, $s_\beta \in \ker \hat{\varphi}$. We now construct a bounded approximate identity for I . Observe that $f_\alpha = \varphi(e_\alpha)^{-1}e_\alpha$ is a bounded approximate identity for \mathcal{A} and $\varphi(e_\alpha) \rightarrow 1$. Indeed,

$$\varphi(a) = \varphi(e_\alpha \cdot a) = \varphi(a \cdot e_\alpha) = \varphi(e_\alpha)\varphi(a).$$

Thus $\varphi(e_\alpha) \rightarrow 1$. Also, $f_\alpha \cdot a = \varphi(e_\alpha)^{-1}e_\alpha \cdot a = \varphi(e_\alpha)^{-1} \cdot a \rightarrow a$ and $a \cdot f_\alpha = a \cdot \varphi(e_\alpha)^{-1}e_\alpha = a \cdot \varphi(e_\alpha)^{-1} \rightarrow a$. Moreover, $\varphi(f_\alpha) \rightarrow 1$ since (f_α) is also a bounded approximate identity for \mathcal{A} . Set

$$i_{\beta, \alpha} = a_\beta + c_\beta f_\alpha,$$

then $(i_{\beta, \alpha}) \subset \ker \varphi = I$ since $\varphi(i_{\beta, \alpha}) = \varphi(a_\beta) + \varphi(c_\beta f_\alpha) = \varphi(a_\beta) + c_\beta \varphi(f_\alpha) = \varphi(a_\beta) + c_\beta \cdot 1 = \varphi(a_\beta) - \varphi(a_\beta) = 0$. So $(i_{\beta, \alpha})$ is a bounded net in I . Also, since (s_β) is a bounded approximate identity for J , then for all $b \in I \subset J$, $s_\beta b \rightarrow b$ and $b s_\beta \rightarrow b$. Finally,

$$i_{\beta, \alpha} b = a_\beta b + c_\beta f_\alpha b = s_\beta b - c_\beta e \cdot b + c_\beta f_\alpha b = s_\beta b + c_\beta (f_\alpha b - b) \rightarrow b + c_\beta (b - b) = b$$

and

$$b \cdot i_{\beta, \alpha} = b a_\beta + b c_\beta f_\alpha = b s_\beta - b c_\beta e + b c_\beta f_\alpha = b s_\beta + c_\beta (b f_\alpha - b) \rightarrow b + c_\beta (b - b) = b.$$

Thus $(i_{\beta,\alpha})$ is a bounded approximate identity for I .

(iv) Since I has a bounded approximate identity by (i), then it is character amenable by Proposition 3.5.2. \square

3.5.3 Characterizations

In this subsection, we characterize character amenability in terms of φ -approximate diagonal and φ -virtual diagonal. The following definitions would be useful in the sequel. For more details, see [27], [36] and [37]. Definition 3.5.6 is due to Hu et al. in [36].

Definition 3.5.5. *Let φ be a character on a Banach algebra \mathcal{A} and $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ be the diagonal operator. An element M in the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ satisfying the following conditions*

$$\varphi(\pi(M)) = 1 \quad \text{and} \quad a \cdot M = \varphi(a)M = M \cdot a$$

is called a φ -diagonal for \mathcal{A} .

Definition 3.5.6. *Let \mathcal{A} be a Banach algebra and let φ be in the character space of \mathcal{A} . Let $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ be the diagonal operator and let $\pi'' : (\mathcal{A} \hat{\otimes} \mathcal{A})'' \rightarrow \mathcal{A}''$ be the second dual of π . Then*

(i) *a net $(m_\alpha) \subset \mathcal{A} \hat{\otimes} \mathcal{A}$ is called a φ -approximate diagonal for \mathcal{A} if*

$$(i) \quad \|m_\alpha \cdot a - \varphi(a)m_\alpha\| \rightarrow 0, \quad \|a \cdot m_\alpha - \varphi(a)m_\alpha\| \rightarrow 0, \quad (a \in \mathcal{A})$$

$$(ii) \quad \langle m_\alpha, \varphi \otimes \varphi \rangle = \varphi(\pi(m_\alpha)) \rightarrow 1.$$

(ii) *an element $M \in (\mathcal{A} \hat{\otimes} \mathcal{A})''$ satisfying*

$$(i) \quad M \cdot a = \varphi(a)M, \quad a \cdot M = \varphi(a)M, \quad (a \in \mathcal{A})$$

$$(ii) \quad \langle \varphi \otimes \varphi, M \rangle = \pi''(M)\varphi = 1$$

is called a φ -virtual diagonal for \mathcal{A} .

The next result characterizes φ -amenability in terms of φ -approximate diagonal and φ -virtual diagonal and so it is an analogue of characterization of amenability in terms of bounded approximate diagonal and virtual diagonal.

Theorem 3.5.7. *Let \mathcal{A} be a Banach algebra and $\varphi \in \sigma(\mathcal{A})$. The following statements are equivalent:*

- (i) \mathcal{A} is φ -amenable.
- (ii) \mathcal{A} has a bounded φ -approximate diagonal.
- (iii) \mathcal{A} has a φ -virtual diagonal.
- (iv) There exists $f \in \mathcal{A}''$ such that $f(\varphi) = 1$ and $a \cdot f = \varphi(a)f$ ($a \in \mathcal{A}$).
- (v) There exists a bounded net (a_α) in \mathcal{A} such that $\varphi(a_\alpha) = 1$ for all α and

$$\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0 \quad (a \in \mathcal{A}).$$

Remark 3.5.8. We remark that φ -amenability can also be characterized in terms of the existence of a certain Hahn-Banach extension property as shown in [27]. The proof is a variant of the proof given in [28].

Theorem 3.5.9. Let φ be in the character space of the Banach algebra \mathcal{A} , then the following statements are equivalent:

- (i) \mathcal{A} is φ -amenable.
- (ii) For any Banach \mathcal{A} -bimodule $X \in \mathcal{M}_{\varphi l}^{\mathcal{A}}$ and any Banach submodule Y of X , each linear function in $Z(\mathcal{A}, Y')$ extends to some element of $Z(\mathcal{A}, X)$.
- (iii) For any Banach \mathcal{A} -bimodule $X \in \mathcal{M}_{\varphi l}^{\mathcal{A}}$, there exists a continuous projection from X' onto $Z(\mathcal{A}, X')$ which commutes with every other weak*-continuous bounded linear operator from X' into X' commuting with the action of \mathcal{A} on X' .

Chapter 4

Generalized Notions of Amenability

In this chapter, the approximate versions of the notions of amenability studied in the previous chapter are considered. These approximate versions have been found to contain larger classes of Banach algebras than their corresponding original notions. For instance, amenable Banach algebras are approximately amenable but the converse is not generally true. A counter example is found in ([19], Example 8.2). The generalized notions of amenability were initiated in [19] with the hope of finding a Banach algebra without bounded approximate identity but satisfy any of these generalized notions. The reason for this is that bounded approximate identity is a necessary condition for amenability and so Banach algebras without bounded approximate identity can not be amenable. In this chapter, we studied some of these generalized notions of amenability, discussed some important general results, hereditary properties and characterization on them.

4.1 Definitions with Examples

Definition 4.1.1. *Let \mathcal{A} be a Banach algebra and let X be a Banach \mathcal{A} -bimodule.*

- (i) *A derivation $D : \mathcal{A} \rightarrow X$ is said to be approximately inner if there exists a net (x_α) in X such that*

$$D(a) = \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a), \quad (a \in \mathcal{A}).$$

- (ii) *\mathcal{A} is said to be approximately amenable if every continuous derivation $D : \mathcal{A} \rightarrow X'$ is approximately inner for every Banach \mathcal{A} -bimodule X .*

- (iii) \mathcal{A} is said to be approximately contractible if every continuous derivation $D : \mathcal{A} \rightarrow X$ is approximately inner for every Banach \mathcal{A} -bimodule X .
- (iv) \mathcal{A} is said to be approximately weakly amenable if every continuous derivation $D : \mathcal{A} \rightarrow \mathcal{A}'$ is approximately inner.

Example 4.1.2. (i) The group algebra $L^1(G)$ is approximately amenable if and only if the locally compact group G is amenable.

- (ii) The measure algebra $M(G)$ on a locally compact group G is approximately amenable if and only if G is amenable.
- (iii) If the group G has an open abelian subgroup, then the Fourier algebra $A(G)$ is approximately amenable if G is amenable.
- (iv) The Fourier algebra $A(G)$ is not approximately amenable for $G = F_2$, where F_2 is the free group on two generators.
- (v) The second dual algebra $L^1(G)''$ is approximately amenable if and only if G is a finite group.

For details on the above example see [30].

4.1.1 Some general results

Proposition 4.1.3. Let \mathcal{A} and \mathcal{B} be Banach algebras. Suppose that $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a continuous epimorphism. If \mathcal{A} is approximately amenable, then so is \mathcal{B} .

Proof. We shall show that every continuous derivation $D : \mathcal{B} \rightarrow Y'$ is approximately inner for every Banach \mathcal{B} -bimodule Y with the module action given by

$$a \cdot y = \varphi(a)y, \quad y \cdot a = y\varphi(a), \quad a \in \mathcal{A}, \quad y \in Y.$$

The map $D \circ \varphi : \mathcal{A} \rightarrow Y'$ is a continuous derivation since for $a, b \in \mathcal{A}$,

$$\begin{aligned} (D \circ \varphi)(ab) &= D(\varphi(ab)) = D(\varphi(a)\varphi(b)) \\ &= D(\varphi(a))\varphi(b) + \varphi(a)D(\varphi(b)) \\ &= D(\varphi(a))b + aD(\varphi(b)) \\ &= (D \circ \varphi)(a)b + a(D \circ \varphi)(b). \end{aligned}$$

Since D and φ are both continuous, then their composition $D \circ \varphi$ is a continuous derivation. Next is to show that this derivation is approximately inner. In fact, this follows

easily from the hypothesis that \mathcal{A} is approximately amenable. Thus, there exist a net $(s_\nu) \subset X'$ satisfying

$$D(\varphi(a)) = (D \circ \varphi)(a) = \lim_{\nu} (\varphi(a) \cdot s_\nu - s_\nu \cdot \varphi(a)), \quad a \in \mathcal{A}.$$

Thus D is approximately inner and so \mathcal{B} is approximately amenable. \square

Remark 4.1.4. *If I is a closed ideal of the Banach algebra \mathcal{A} , we know that there is a natural epimorphism between \mathcal{A} and \mathcal{A}/I . Thus, by Proposition 4.1.3, we expect that the approximate amenability of \mathcal{A} can be transferred to \mathcal{A}/I . The following Corollary immediately follows.*

Corollary 4.1.5. *Let I be a closed two-sided ideal of a Banach algebra \mathcal{A} . Then*

- (i) \mathcal{A}/I is approximately amenable if \mathcal{A} is approximately amenable
- (ii) \mathcal{A} is approximately amenable if I is amenable and \mathcal{A}/I is approximately amenable.

We omit the proof of the following Proposition which is a slight variation of standard argument which hold for amenable Banach algebras considered in Section 3.3.

Proposition 4.1.6. *Let \mathcal{A} be an approximately amenable Banach algebra with a bounded approximate identity and \mathcal{B} an amenable Banach algebra. Then $\mathcal{A} \hat{\otimes} \mathcal{B}$ is approximately amenable.*

Lemma 4.1.7. *Let \mathcal{A} be a unital Banach algebra with identity e , X an \mathcal{A} -bimodule, $D : \mathcal{A} \rightarrow X'$ a derivation. Then*

1. *there is a derivation $D_1 : \mathcal{A} \rightarrow e \cdot X' \cdot e$ and $\eta \in X'$ such that*
 - (i) $\|\eta\| \leq 2C_X \|D\|$,
 - (ii) $D = D_1 + \delta_\eta$,
2. *there is a net $(s_\nu) \subset e \cdot X' \cdot e$ and $\eta \in X'$ such that*
 - (i) $\|\eta\| \leq 2C_X \|D\|$
 - (ii) $D = \delta_\eta + \lim \delta_{s_\nu}$.

Proof. See Lemmas 2.3 and 2.4 of [19]. \square

Proposition 4.1.8. *\mathcal{A} is approximately amenable if and only if $\mathcal{A}^\#$ is approximately amenable.*

Proof. Let $\mathcal{A}^\#$ be approximately amenable and $D : \mathcal{A}^\# \rightarrow X'$ be a derivation where X is an \mathcal{A} -bimodule. By Lemma 4.1.7, $D = D_1 + \delta_\eta$. Now, for the identity $e \in \mathcal{A}^\#$, it follows that $D_1(e) = 0$. Indeed, since $D_1 : \mathcal{A}^\# \rightarrow e \cdot X' \cdot e$, then there exists $\phi \in X'$ such that $D_1(a) = e \cdot \phi \cdot e$ for all $a \in \mathcal{A}^\#$. In particular, $D_1(e) = e \cdot \phi \cdot e$ and

$$D_1(e) = D_1(e \cdot e) = D_1(e) \cdot e + e \cdot D_1(e) = (e \cdot \phi \cdot e) \cdot e + e \cdot (e \cdot \phi \cdot e) = e \cdot \phi \cdot e + e \cdot \phi \cdot e = 2D_1(e)$$

which gives $2D_1(e) - D_1(e) = 0$. That is, $D_1(e) = 0$. Thus, the restriction of D_1 on \mathcal{A} is approximately inner and so \mathcal{A} is approximately amenable.

Conversely, assume that \mathcal{A} is approximately amenable. Let $D : \mathcal{A} \rightarrow X'$ be a derivation which we extend to $\mathcal{A}^\#$ by setting $D(e) = 0$. This extension is approximately inner on $\mathcal{A}^\#$ and so $\mathcal{A}^\#$ is approximately amenable. \square

Let \mathcal{A} and \mathcal{B} be approximately amenable Banach algebras. We ask if the direct sum $\mathcal{A} \oplus \mathcal{B}$ will also be approximately amenable and if so, under what condition? A few results have been obtained about this in the literature. In fact, it was shown in [19] that if \mathcal{A} and \mathcal{B} each has a bounded approximate identity then $\mathcal{A} \oplus \mathcal{B}$ would also be approximately amenable. The result was then extended to a more general case of $\mathcal{A}^\# \oplus \mathcal{B}^\#$ from where the approximate amenability of the closed ideal $\mathcal{A} \oplus \mathcal{B}$ can be obtained from Corollary 4.1.14 which we shall prove in the next section. An improvement on this result was given in [20]. Indeed the authors in [20] proved the sufficiency of the existence of bounded approximate identity for only one of \mathcal{A} or \mathcal{B} . By observing a close relationship between the existence of a two sided approximate identity in approximately amenable Banach algebras and the approximate amenability of the direct sum of approximately amenable Banach algebras, they obtained the following result in [20].

Proposition 4.1.9. *Let \mathcal{A} and \mathcal{B} be approximately amenable Banach algebras. Suppose that one of \mathcal{A} or \mathcal{B} has a bounded approximate identity, then $\mathcal{A} \oplus \mathcal{B}$ is approximately amenable.*

The result proved by Ghahramani and Read in [21] shows that the condition of the existence of bounded approximate identity for either \mathcal{A} or \mathcal{B} cannot be removed. The authors in [21] constructed a Banach algebra \mathcal{A} that has no bounded approximate identity and then showed that the direct sum $\mathcal{A} \oplus \mathcal{A}^{op}$ (where \mathcal{A}^{op} is the opposite algebra) is not approximately amenable.

Let \mathcal{A} be an approximately amenable Banach algebra, a problem of interest is to determine whether or not a closed ideal of \mathcal{A} can inherit the approximate

amenability property of \mathcal{A} . A major result proved in the following section would be needed to establish this. Hence, we deal with this interesting problem in the later part of the next section.

4.1.2 Characterizations

The following propositions are important for this section.

Proposition 4.1.10. *The Banach algebra \mathcal{A} has a bounded approximate identity if and only if \mathcal{A}'' has a right identity.*

Proof. See Proposition 28.7 of [6]. □

Proposition 4.1.11. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity. \mathcal{A} is approximately amenable if and only if every derivation into the dual of any neo-unital bimodule is approximately inner.*

Proof. See Proposition 2.5 of [19]. □

Let $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ be the diagonal operator, we recall that an approximate diagonal for \mathcal{A} is a net (m_α) in $\mathcal{A} \hat{\otimes} \mathcal{A}$ such that for $a \in \mathcal{A}$, $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ and $a\pi(m_\alpha) \rightarrow a$. In the same way, $(\mathcal{A} \hat{\otimes} \mathcal{A})'' \cong \mathcal{A}'' \hat{\otimes} \mathcal{A}''$ and so $\pi'' : (\mathcal{A} \hat{\otimes} \mathcal{A})'' \rightarrow \mathcal{A}''$. Thus $(M_\nu) \subset (\mathcal{A} \hat{\otimes} \mathcal{A})''$ can be considered an approximate diagonal for \mathcal{A}'' if $a \cdot M_\nu - M_\nu \cdot a \rightarrow 0$ and $\pi''(M_\nu) \rightarrow e$. We now characterize approximately amenable Banach algebras in terms of the existence of this net (M_ν) behaving like an approximate diagonal for \mathcal{A}'' .

Theorem 4.1.12. *Let \mathcal{A} be a Banach algebra. The following statements are equivalent:*

- (i) \mathcal{A} is approximately amenable.
- (ii) There exists a net (M_ν) in $(\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)''$ such that $a \cdot M_\nu - M_\nu \cdot a \rightarrow 0$ and $\pi''(M_\nu) \rightarrow e$ for each $a \in \mathcal{A}^\#$.
- (iii) There exists a net (M'_ν) in $(\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)''$ such that $a \cdot M'_\nu - M'_\nu \cdot a \rightarrow 0$ and $\pi''(M'_\nu) \rightarrow e$ for each $a \in \mathcal{A}^\#$ and every ν .

Proof. (i) \implies (iii). Let \mathcal{A} be approximately amenable, then $\mathcal{A}^\#$ is approximately amenable by Proposition 4.1.8. First is to note that $(\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)$ is a Banach $\mathcal{A}^\#$ -bimodule with the module operation defined by

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in \mathcal{A}^\#).$$

Also, the second dual $(\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)''$ is a Banach \mathcal{A} -bimodule and in particular $\ker \pi'' \subset (\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)$ is a Banach $\mathcal{A}^\#$ bimodule. The linear map $D_u : \mathcal{A}^\# \rightarrow \ker \pi''$ defined by $D_u(a) = a \cdot u - u \cdot a$ is a derivation since for $a, b \in \mathcal{A}^\#$,

$$\begin{aligned} D_u(ab) &= (ab) \cdot u - u \cdot (ab) = a \cdot (b \cdot u) - a \cdot (u \cdot b) + (a \cdot u) \cdot b - (u \cdot a) \cdot b \\ &= a \cdot (b \cdot u - u \cdot b) + (a \cdot u - u \cdot a) \cdot b = a \cdot D_u(b) + D_u(a) \cdot b \quad (a, b \in \mathcal{A}^\#). \end{aligned}$$

Since $\mathcal{A}^\#$ is approximately amenable, then there exists $(e_\alpha) \subset \ker \pi''$ satisfying

$$D_u(a) = \lim_{\alpha} (a \cdot e_\alpha - e_\alpha \cdot a) \quad (a \in \mathcal{A}^\#).$$

Now, write $M'_\nu = u - e_\alpha$ so that for $a \in \mathcal{A}^\#$,

$$\begin{aligned} a \cdot M'_\nu - M'_\nu \cdot a &= a \cdot (u - e_\alpha) - (u - e_\alpha) \cdot a = a \cdot u - a \cdot e_\alpha - u \cdot a + e_\alpha \cdot a \\ &= (a \cdot u - u \cdot a) - (a \cdot e_\alpha - e_\alpha \cdot a) = D_u(a) - (a \cdot e_\alpha - e_\alpha \cdot a) \rightarrow 0. \end{aligned}$$

Also, since e_α is in $\ker \pi''$, then for every α ,

$$\pi''(M'_\nu) = \pi''(u - e_\alpha) = \pi''(u) - \pi''(e_\alpha) = \pi''(u) - 0 = \pi''(e \otimes e) = e \cdot e = e.$$

Thus $a \cdot M'_\nu - M'_\nu \cdot a \rightarrow 0$ and $\pi''(M'_\nu) \rightarrow e$ for every ν which gives (iii).

(ii) \implies (i). Assume that (ii) holds. Let $D : \mathcal{A}^\# \rightarrow X'$, then by Proposition 4.1.11, we may suppose that X is neo-unital. For $x \in X$, let μ_x be the continuous linear function on $\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ defined by

$$\mu_x(a \otimes b) = (a \cdot Db)(x) \quad (a, b \in \mathcal{A}^\#)$$

which we shall write in the form

$$\mu_x(a \otimes b) = \langle x, a \cdot Db \rangle.$$

Now, set $f_\nu(x) = M'_\nu(\mu_x)$ for all ν . To show approximate amenability, it suffices to prove that $D(a) = \lim_{\nu} (a \cdot f_\nu - f_\nu \cdot a)$. Given $a \in \mathcal{A}^\#$, $x \in X$, then for $b, c \in \mathcal{A}^\#$,

$$\begin{aligned} \mu_{x \cdot a - a \cdot x}(b \otimes c) &= \langle x \cdot a - a \cdot x, b \cdot Dc \rangle = \langle x \cdot a, b \cdot Dc \rangle - \langle a \cdot x, b \cdot Dc \rangle \\ &= \langle x, a \cdot b \cdot Dc \rangle - \langle x, b \cdot Dc \cdot a \rangle = \langle x, a \cdot b \cdot Dc - b \cdot Dc \cdot a \rangle \\ &= \langle x, a \cdot b \cdot Dc \rangle - \langle x, b \cdot Dc \cdot a \rangle \end{aligned}$$

and

$$\begin{aligned}
(\mu_x \cdot a - a \cdot \mu_x)(b \otimes c) &= \langle b \otimes c, \mu_x \cdot a - a \cdot \mu_x \rangle = \langle b \otimes c, \mu_x \cdot a \rangle - \langle b \otimes c, a \cdot \mu_x \rangle \\
&= \langle a \cdot (b \otimes c), \mu_x \rangle - \langle (b \otimes c) \cdot a, \mu_x \rangle \\
&= \langle ab \otimes c, \mu_x \rangle - \langle b \otimes ca, \mu_x \rangle = \langle x, ab \cdot Dc \rangle - \langle x, b \cdot Dca \rangle \\
&= \langle x, a \cdot b \cdot Dc \rangle - \langle x, b \cdot Dc \cdot a + b \cdot c \cdot Da \rangle \\
&= \langle x, a \cdot b \cdot Dc \rangle - \langle x, b \cdot Dc \cdot a \rangle - \langle x, b \cdot c \cdot Da \rangle.
\end{aligned}$$

And so,

$$\mu_{x \cdot a - a \cdot x}(b \otimes c) = (\mu_x \cdot a - a \cdot \mu_x)(b \otimes c) + (bcDa)(x).$$

Thus for $(m_\nu^\alpha) \subset \mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ converging weak* to M_ν and for $m \in \mathcal{A}^\# \otimes \mathcal{A}^\#$, we can write

$$\mu_{x \cdot a - a \cdot x}(m) = (\mu_x \cdot a - a \cdot \mu_x)(m) + (\pi(m)Da)(x).$$

We recall that for $a \in \mathcal{A}$, $f \in \mathcal{A}'$, the mapping $\hat{a} \in \mathcal{A}''$ is given by $\hat{a}(f) = f(a)$ and so $M_\nu(\mu_x) = \lim_\alpha (m_\nu^\alpha)$. Thus,

$$\begin{aligned}
(a \cdot f_\nu - f_\nu \cdot a)(x) &= \langle x, a \cdot f_\nu - f_\nu \cdot a \rangle = \langle x, a \cdot f_\nu \rangle - \langle x, f_\nu \cdot a \rangle \\
&= \langle x \cdot a, f_\nu \rangle - \langle a \cdot x, f_\nu \rangle = \langle x \cdot a - a \cdot x, f_\nu \rangle \\
&= f_\nu(x \cdot a - a \cdot x) = M_\nu(\mu_{x \cdot a - a \cdot x}) = \lim_\alpha (\mu_{x \cdot a - a \cdot x})(m_\nu^\alpha) \\
&= \lim_\alpha (\mu_x \cdot a - a \cdot \mu_x)(m_\nu^\alpha) + \lim_\alpha (\pi(m_\nu^\alpha)Da)(x) \\
&= M_\nu(\mu_x \cdot a - a \cdot \mu_x) + \lim_\alpha (\pi(m_\nu^\alpha)Da)(x) \\
&= (a \cdot M_\nu - M_\nu \cdot a)(\mu_x) + (\pi''(M_\nu)Da)(x).
\end{aligned}$$

Observe that

$$\|\mu_x(a \otimes b)\| = \|\mu_x\| \|a \otimes b\| = \|\mu_x\| \|a\| \|b\| \quad (4.1)$$

and

$$\|\mu_x(a \otimes b)\| = \|(aDb)(x)\| = \|a\| \|D\| \|b\| \|x\|. \quad (4.2)$$

From (4.1) and (4.2), we obtain that $\|\mu_x\| = \|D\| \|x\|$. Now,

$$\begin{aligned}
\|a \cdot f_\nu - f_\nu \cdot a - (Da)(x)\| &= \|(a \cdot M_\nu - M_\nu \cdot a)(\mu_x) + (\pi''(M_\nu)Da)(x) - (Da)(x)\| \\
&\leq \|(a \cdot M_\nu - M_\nu \cdot a)(\mu_x)\| + \|(\pi''(M_\nu)Da)(x) - (Da)(x)\| \\
&= \|(a \cdot M_\nu - M_\nu \cdot a)\| \|\mu_x\| + \|(\pi''(M_\nu)Da)(x) - (Da)(x)\| \\
&= \|(a \cdot M_\nu - M_\nu \cdot a)\| \|D\| \|x\| + \|(\pi''(M_\nu)Da - Da)(x)\| \\
&= \|(a \cdot M_\nu - M_\nu \cdot a)\| \|D\| \|x\| + \|[(\pi''(M_\nu) - e) \cdot (Da)](x)\| \\
&\leq \|(a \cdot M_\nu - M_\nu \cdot a)\| \|D\| \|x\| + \|(\pi''(M_\nu) - e)\| \|Da\| \|x\| \\
&\rightarrow 0.
\end{aligned}$$

Thus $D = \lim_{\nu}(a \cdot f_{\nu} - f_{\nu} \cdot a)$. Thus $\mathcal{A}^{\#}$ is approximately amenable and by Proposition 4.1.8, \mathcal{A} is approximately amenable.

It is clear that (iii) implies (ii). □

We now show that direct consequences of Theorem 4.1.12 can be obtained to avoid the use of an adjoined identity $e \in \mathcal{A}^{\#}$ or by introducing two nets from \mathcal{A}'' which behave like a one-sided approximate identity for \mathcal{A} . Indeed, the following Corollary holds.

Corollary 4.1.13. *The Banach algebra \mathcal{A} is approximately amenable if and only if there exists nets $(M_{\nu}'') \subset (\mathcal{A} \hat{\otimes} \mathcal{A})''$, $(F_{\nu}), (G_{\nu}) \subset \mathcal{A}''$ which satisfy the condition that for each $a \in \mathcal{A}$*

$$(i) \ a \cdot M_{\nu}'' - M_{\nu}'' \cdot a + F_{\nu} \otimes a - a \otimes G_{\nu} \rightarrow 0,$$

$$(ii) \ a \cdot F_{\nu} \rightarrow a, \quad G_{\nu} \cdot a \rightarrow a \quad \text{and}$$

$$(iii) \ \pi''(M_{\nu}'') \cdot a - F_{\nu} \cdot a - G_{\nu} \cdot a \rightarrow 0.$$

Proof. Let $(M_{\nu}'') \subset (\mathcal{A} \hat{\otimes} \mathcal{A})''$, $(F_{\nu}), (G_{\nu}) \subset \mathcal{A}''$ and $(c_{\nu}) \subset \mathbb{C}$ be nets such that

$$M_{\nu} = M_{\nu}'' - F_{\nu} \otimes e - e \otimes G_{\nu} + c_{\nu}e \otimes e \tag{4.3}$$

where (M_{ν}) is the same net in Theorem 4.1.12. Since $\pi''(M_{\nu}) \rightarrow e$ by Theorem 4.1.12, then applying π'' to Equation (4.3), it follows that

$$\begin{aligned} \pi''(M_{\nu}) &= \pi''(M_{\nu}'' - F_{\nu} \otimes e - e \otimes G_{\nu} + c_{\nu}e \otimes e) \\ &= \pi''(M_{\nu}'') - \pi''(F_{\nu} \otimes e) - \pi''(e \otimes G_{\nu}) + \pi''(c_{\nu}e \otimes e) \\ &= \pi''(M_{\nu}'') - F_{\nu} \cdot e - e \cdot G_{\nu} + c_{\nu}e \cdot e = \pi''(M_{\nu}'') - F_{\nu} - G_{\nu} + c_{\nu}. \end{aligned}$$

Thus $\pi''(M_{\nu}'') - F_{\nu} - G_{\nu} + c_{\nu}e \rightarrow e$ from where $c_{\nu} \rightarrow 1$ and

$$\pi''(M_{\nu}'') \cdot a - F_{\nu} \cdot a - G_{\nu} \cdot a \rightarrow 0.$$

Also, from Theorem 4.1.12, it follows that for $a \in \mathcal{A}$,

$$\begin{aligned} a \cdot M_{\nu} - M_{\nu} \cdot a &= a \cdot (M_{\nu}'' - F_{\nu} \otimes e - e \otimes G_{\nu} + c_{\nu}e \otimes e) \\ &\quad - (M_{\nu}'' - F_{\nu} \otimes e - e \otimes G_{\nu} + c_{\nu}e \otimes e) \cdot a \\ &= a \cdot M_{\nu}'' - a \cdot F_{\nu} \otimes e - a \otimes G_{\nu} + a \cdot c_{\nu}e \otimes e \\ &\quad - M_{\nu}'' \cdot a + F_{\nu} \otimes a + e \otimes G_{\nu} \cdot a - c_{\nu}e \otimes a \\ &= a \cdot M_{\nu}'' - M_{\nu}'' \cdot a + F_{\nu} \otimes a - a \otimes G_{\nu} \\ &\quad - a \cdot F_{\nu} \otimes e + e \otimes G_{\nu} \cdot a + a \cdot c_{\nu}e \otimes e - c_{\nu}e \otimes a \\ &\rightarrow 0 \end{aligned}$$

whence $c_\nu \rightarrow 1$,

$$a \cdot M_\nu'' - M_\nu'' \cdot a + F_\nu \otimes a - a \otimes G_\nu \rightarrow 0,$$

and

$$a \cdot F_\nu \rightarrow a, \quad G_\nu \cdot a \rightarrow a.$$

Conversely, assume that (i), (ii) and (iii) hold, then we prove that \mathcal{A} is approximately amenable. Define $M_\nu = M_\nu'' - F_\nu \otimes e - e \otimes G_\nu + c_\nu e \otimes e$ where $(M_\nu'') \subset (\mathcal{A} \hat{\otimes} \mathcal{A})''$, $(F_\nu), (G_\nu) \subset \mathcal{A}''$ and $(c_\nu) \subset \mathbb{C}$ with $c_\nu \rightarrow 1$. It suffices by Theorem 4.1.12 to show that there exists a net $(M_\nu) \subset (\mathcal{A} \hat{\otimes} \mathcal{A})''$ such that for each $a \in \mathcal{A}$, $a \cdot M_\nu - M_\nu \cdot a \rightarrow 0$ and $\pi''(M_\nu) \rightarrow e$. Now, by using (i) and (ii) and the fact that $c_\nu \rightarrow 1$, it follows that

$$\begin{aligned} a \cdot M_\nu - M_\nu \cdot a &= a \cdot M_\nu'' - a \cdot F_\nu \otimes e - a \cdot e \otimes G_\nu + a \cdot c_\nu e \otimes e \\ &\quad - (M_\nu'' \cdot a - F_\nu \otimes e \cdot a - e \otimes G_\nu \cdot a + c_\nu e \otimes e \cdot a) \\ &= a \cdot M_\nu'' - a \cdot F_\nu \otimes e - a \otimes G_\nu + a \cdot c_\nu \otimes e \\ &\quad - M_\nu'' \cdot a + F_\nu \otimes a + e \otimes G_\nu \cdot a - c_\nu e \otimes a \\ &= a \cdot M_\nu'' - M_\nu'' \cdot a + F_\nu \otimes a - a \otimes G_\nu \\ &\quad - a \cdot F_\nu \otimes e + e \otimes G_\nu \cdot a - c_\nu e \otimes a + a \cdot c_\nu \otimes e \\ &\rightarrow 0 - a \otimes e + e \otimes a - e \otimes a + a \otimes e = 0. \end{aligned}$$

Moreso, applying π'' on M_ν and $c_\nu \rightarrow 1$ gives

$$\begin{aligned} \pi''(M_\nu) &= \pi''(M_\nu'' - F_\nu \otimes e - e \otimes G_\nu + c_\nu e \otimes e) \\ &= \pi''(M_\nu'') - \pi''(F_\nu \otimes e) - \pi''(e \otimes G_\nu) + \pi''(c_\nu e \otimes e) \\ &= \pi''(M_\nu'') - F_\nu \cdot e - e \cdot G_\nu + c_\nu e \cdot e \\ &= \pi''(M_\nu'') - F_\nu - G_\nu + e. \end{aligned}$$

That is, $\pi''(M_\nu) - e = \pi''(M_\nu'') - F_\nu - G_\nu$. Multiplying by a from the right and using (iii), it follows that

$$\pi''(M_\nu) \cdot a - e \cdot a = \pi''(M_\nu'') \cdot a - F_\nu \cdot a - G_\nu \cdot a \rightarrow 0.$$

Thus, $\pi''(M_\nu) \cdot a \rightarrow e \cdot a = a$ or $\pi''(M_\nu) \rightarrow e$. □

Observe that we have shown $\pi''(M_\nu) \rightarrow e$ and not that $a = \pi''(M_\nu) \cdot a$ for all $a \in \mathcal{A}$ which would have implied by the continuity of the first Arens product that it holds for all $a \in \mathcal{A}''$, so that \mathcal{A}'' would then have a right identity and necessarily by Proposition 4.1.10, \mathcal{A} would have a bounded left approximate identity. Analogous

argument would have been made for a bounded right approximate identity for \mathcal{A} and then we would have concluded that every approximately amenable Banach algebra has a bounded approximate identity. But this may not be true. In fact many authors have given this as open problems and conjectures, for example see [19] and [21]. We now turn to our question of interest: Can a closed ideal of an approximately amenable Banach algebra be approximately amenable? The following Corollary gives an affirmative answer for closed ideals with bounded approximate identity.

Corollary 4.1.14. *Let \mathcal{A} be an approximately amenable Banach algebra and let I be a closed two-sided ideal with a bounded approximate identity. Then I is approximately amenable.*

Proof. By our hypothesis, there is a bounded approximate identity for I . So by Proposition 4.1.10, I'' has a right identity E such that $a \cdot E = E \cdot a = a$ for all $a \in I$. We only need to show that there exists nets (M'_ν) , (F'_ν) , (G'_ν) which satisfies the conditions (i) to (iii) in Corollary 4.1.13. Define $M'_\nu = E \cdot M_\nu \cdot E \in (I \hat{\otimes} I)''$, $F'_\nu = E \cdot F_\nu \in I''$, $G'_\nu = G_\nu \cdot E \in I''$ where the nets (M_ν) , (F_ν) and (G_ν) are as given in Corollary 4.1.13 for \mathcal{A} . That is for all $a \in \mathcal{A}$,

$$(i) \quad a \cdot M_\nu - M_\nu \cdot a + F_\nu \otimes a - a \otimes G_\nu \rightarrow 0,$$

$$(ii) \quad a \cdot F_\nu \rightarrow a, \quad G_\nu \cdot a \rightarrow a \quad \text{and}$$

$$(iii) \quad \pi''(M_\nu) \cdot a - F_\nu \cdot a - G_\nu \cdot a \rightarrow 0.$$

Indeed, for $a \in I$,

$$\begin{aligned} a \cdot M'_\nu - M'_\nu \cdot a + a \cdot F'_\nu \otimes a - a \otimes G'_\nu &= a \cdot E \cdot M_\nu \cdot E - E \cdot M_\nu \cdot E \cdot a \\ &\quad + E \cdot F_\nu \otimes a - a \otimes G_\nu \cdot E \\ &= E \cdot (a \cdot M_\nu - M_\nu \cdot a + F_\nu \otimes a - a \otimes G_\nu) \cdot E \\ &\rightarrow 0. \end{aligned}$$

Also,

$$a \cdot F'_\nu = a \cdot E \cdot F_\nu = a \cdot F_\nu \rightarrow a, \quad G'_\nu = G_\nu \cdot E \cdot a = G_\nu \cdot a \rightarrow a$$

and

$$\begin{aligned}
\pi''(M'_\nu) \cdot a - E \cdot F_\nu \cdot a - G_\nu \cdot E \cdot a &= \pi''(E \cdot M_\nu \cdot E) \cdot a - E \cdot F_\nu \cdot a - G_\nu \cdot E \cdot a \\
&= E \cdot \pi''(M_\nu) \cdot E \cdot a - E \cdot F_\nu \cdot a - G_\nu \cdot E \cdot a \\
&= E \cdot \pi''(M_\nu) \cdot a - E \cdot F_\nu \cdot a - G_\nu \cdot E \cdot a \\
&= E \cdot \pi''(M_\nu) \cdot a - E \cdot F_\nu \cdot a - E \cdot G_\nu \cdot a \\
&\quad + E \cdot G_\nu \cdot a - G_\nu \cdot E \cdot a \\
&= E \cdot (\pi''(M_\nu) \cdot a - F_\nu \cdot a - G_\nu \cdot a) \\
&\quad + (E - 1)G_\nu \cdot a \\
&\rightarrow 0.
\end{aligned}$$

□

4.2 Relationship between some Notions of Amenability

The following theorem due to Gourdeau [22] shows that every amenable Banach algebra is approximately contractible and conversely.

Theorem 4.2.1. *Let \mathcal{A} be a Banach algebra and let X be a Banach \mathcal{A} -bimodule. Then the following statements are equivalent:*

- (i) \mathcal{A} is amenable.
- (ii) $H^1(\mathcal{A}, X'') = \{0\}$ for every X .
- (iii) Every derivation from \mathcal{A} into X is the limit of a net of inner derivation such that the implementing nets are bounded. That is, there exists a bounded net $(x_\alpha) \subset X$ such that

$$D(a) = \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a), \quad (a \in \mathcal{A}).$$

Proof. See [9].

□

Theorem 4.2.2. *Let \mathcal{A} be a commutative Banach algebra. If \mathcal{A} is approximately amenable then \mathcal{A} is weakly amenable.*

Proof. Let D be a continuous derivation on \mathcal{A} and let X be a Banach \mathcal{A} -bimodule. Since \mathcal{A} is approximately amenable then there exists a net $(s_\alpha) \subset X'$ such that $D(a) = \lim_\alpha (a \cdot s_\alpha - s_\alpha \cdot a)$. Since \mathcal{A} is commutative, then by Proposition 3.4.2, it follows that $D = 0$. This happens for every Banach \mathcal{A} -bimodule and so by setting $X = \mathcal{A}$, we obtain that \mathcal{A} is approximately weakly amenable. But an approximately weakly Banach algebra is weakly amenable. Hence, \mathcal{A} is weakly amenable. \square

The following theorem and proof are due to [20].

Theorem 4.2.3. *Let \mathcal{A} be a Banach algebra, then the following are equivalent:*

- (i) \mathcal{A} is approximately contractible.
- (ii) \mathcal{A} is approximately amenable.
- (iii) \mathcal{A} approximately weakly amenable.
- (iv) \mathcal{A} is weak*-approximately amenable.

Proof. It suffices to show that (iv) \implies (i).

Suppose that (iv) holds, that is, $\mathcal{A}^\#$ is weak*-approximately amenable. By following the classical argument of Theorem 4.1.12, there is a net $(M_\nu) \subset (\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)''$ such that for each $a \in \mathcal{A}$, $a \cdot M_\nu - M_\nu \cdot a \rightarrow 0$ and $\pi''(M_\nu) \rightarrow e$ in the weak*-topology of $(\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)''$ and \mathcal{A}'' respectively.

Now take $\epsilon > 0$, and finite sets $F \subset \mathcal{A}^\#$, $\Phi \subset (\mathcal{A}^\#)'$, and $N \subset (\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)'$. Then there is ν such that

$$|\langle a \cdot f - f \cdot a, M_\nu \rangle| = |\langle f, a \cdot M_\nu - M_\nu \cdot a \rangle| < \epsilon$$

and

$$|\langle \phi, \pi''(M_\nu) - e \rangle| < \epsilon$$

for all $a \in F$, $\phi \in \Phi$ and $f \in N$.

By Goldstine's theorem, and the weak*-continuity of π'' , there is $m \in \mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ such that

$$|\langle f, a \cdot m - m \cdot a \rangle| = |\langle a \cdot f - f \cdot a, m \rangle| < \epsilon \quad \text{and} \quad |\langle \phi, \pi(m) - e \rangle| < \epsilon$$

for all $a \in F$, $\phi \in \Phi$ and $f \in N$.

Thus there is a net $(m_\lambda) \subset \mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ such that for every $a \in \mathcal{A}$, $a \cdot m_\lambda - m_\lambda \cdot a \rightarrow 0$ and $\pi(m_\lambda) \rightarrow e$, weakly in $\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ and $\mathcal{A}^\#$ respectively.

Finally, for each finite set $F \subset \mathcal{A}^\#$, say $F = \{a_1, \dots, a_n\}$,

$$(a_1 \cdot m_\lambda - m_\lambda \cdot a_1, \dots, a_n \cdot m_\lambda - m_\lambda \cdot a_n, \pi(m_\lambda)) \rightarrow (0, \dots, 0, e)$$

weakly in $(\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)$. Thus

$$(0, \dots, 0, e) \in \overline{c\mathcal{O}}^{\text{weak}} \{(a_1 \cdot m_\lambda - m_\lambda \cdot a_1, \dots, a_n \cdot m_\lambda - m_\lambda \cdot a_n, \pi(m_\lambda))\}.$$

The Hahn-Banach theorem now gives that for each $\epsilon > 0$, there is $u_{\epsilon, F} \in \text{co}\{m_\lambda\}$, such that

$$\|a \cdot u_{\epsilon, F} - u_{\epsilon, F} \cdot a\| < \epsilon \quad \text{and} \quad \|\pi(u_{\epsilon, F}) - e\| < \epsilon$$

for $a \in F$. Thus, (i) holds. □

Chapter 5

Some Notions of Amenability on Segal Algebras

The notions of amenability discussed in the previous two chapters are for general Banach algebras. Several authors have studied these notions of amenability for different classes of Banach algebras. For examples, group algebra, measure algebra, Fourier algebra, semigroup algebra, Banach algebra of bounded linear operators on Banach space and some of its closed ideals which are also Banach algebras and Segal algebras. There are relationship between the structures of these algebras and these notions of amenability. In this chapter, we consider a class of Banach algebra called the Segal algebras and study some of these notions of amenability on them and also give some relationship that exist between these notions of amenability and the structures of this class of Banach algebras.

5.1 Segal Algebras

Segal algebras which are subalgebras of the group algebra $L^1(G)$ were first introduced and extensively studied by Reiter in [41], [42] and [43]. Although notable examples of what is now called Segal algebras were first introduced by Wiener in 1932 but it was in 1947 that Segal observed their intrinsic abstract structure.

Definition 5.1.1. *Let \mathcal{A} be a Banach algebra and \mathcal{B} a subalgebra of \mathcal{A} such that*

(S1) *\mathcal{B} is a left dense ideal in \mathcal{A} ;*

(S2) *$(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach algebra;*

(S3) *there exists $K > 0$ such that $\|b\|_{\mathcal{A}} \leq K\|b\|_{\mathcal{B}}$ for each $b \in \mathcal{B}$;*

(S₄) there exists $C > 0$ such that $\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$ for each $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Then \mathcal{B} is called an abstract segal algebra on \mathcal{A} or with respect to \mathcal{A} . Let G be a locally compact group. Suppose that $\mathcal{A} = L^1(G)$ then the subspace $S(G)$ of $L^1(G)$ is called a Segal algebra if the following conditions are satisfied:

- (s₁) $S(G)$ is dense in $L^1(G)$;
- (s₂) $(S(G), \|\cdot\|_{S(G)})$ is a Banach space and there exists $K > 0$ such that $\|f\|_{L^1(G)} \leq K\|f\|_{S(G)}$ for all $f \in S(G)$;
- (s₃) $S(G)$ is left translation invariant ($f \in S(G) \implies L_x * f \in S(G) \forall x \in G$) and the map $x \mapsto L_x * f$ from G into $S(G)$ is continuous for all $f \in S(G)$ where $L_x * f(y) = f(x^{-1}y)$, $y \in G$;
- (s₄) the Segal algebra norm is left invariant, that is, $\|L_x * f\|_{S(G)} = \|f\|_{S(G)}$ for all $f \in S(G)$ and $x \in G$.

Remark 5.1.2. The following can be deduced from the above definition.

- (i) Every Segal algebra is an abstract Segal algebra with respect to $L^1(G)$ but the converse is not true. Thus the class of abstract Segal algebras generalizes the class of classical Segal algebras. Consequently, most results which are proved for abstract Segal algebras can be applied to the classical class. We give an example of an abstract Segal algebra which is not a classical Segal algebra in Example 5.1.4(v).
- (ii) $S(G)$ is called a proper Segal algebra on G if it is a proper subalgebra of $L^1(G)$.
- (iii) In the case that G is discrete, then the only Segal algebra $S(G)$ is $L^1(G)$ itself.
- (iv) $S(G)$ has an essentially unique norm, that is, if $\|\cdot\|$ is any other norm on $S(G)$ satisfying (s₂) and (s₄) above, then there exists a constant $M > 0$ such that $M^{-1}\|f\|_{S(G)} \leq \|f\| \leq M\|f\|_{S(G)} \forall f \in S(G)$.
- (v) Suppose that $S_1(G)$ and $S_2(G)$ are Segal algebras such that $S_1(G) \subset S_2(G)$, then by using (s₂), there exists $k > 0$ such that $\|f\|_{S_2(G)} \leq k\|f\|_{S_1(G)}$. In general, let $S_i(G)$, ($i = 1, 2, \dots, n$) be Segal algebras on a locally compact group G such that

$$S_1(G) \subset S_2(G) \subset S_3(G) \subset \dots \subset S_n(G) \subset L^1(G)$$

then there exists, by induction, $k_i > 0$, ($i = 1, 2, \dots, n$) such that

$$\|f\|_{L^1(G)} \leq k_1 \|f\|_{S_n(G)} \leq k_2 \|f\|_{S_{n-1}(G)} \leq \dots \leq k_n \|f\|_{S_1(G)}.$$

(vi) A symmetric Segal algebra is a Segal algebra which satisfies the following additional properties:

- (s3') $S(G)$ is right translation invariant, that is, for all $x \in G$ and each $f \in S(G)$, $f \in S(G) \implies R_x * f \in S(G)$ and the map $x \mapsto R_x * f$ of G into $S(G)$ is continuous
- (s4') the Segal algebra norm is right translation invariant, that is, $\|R_x * f\|_{S(G)} = \|f\|_{S(G)}$ for all $f \in S(G)$ and $x \in G$.

By changing L_x to R_x and the order of convolution in the proof of a result which holds for a symmetric Segal algebra, we can obtain corresponding results for a (not necessarily symmetric) Segal algebra as the following proposition from [41] shows.

Proposition 5.1.3. (i) Segal algebras $S(G)$ are left ideals of $L^1(G)$ and

$$\|g * f\|_{S(G)} \leq \|g\|_{L^1(G)} \cdot \|f\|_{S(G)}, \quad (g \in L^1(G), f \in S(G)).$$

(ii) $(S(G), \|\cdot\|_{S(G)})$ is a Banach algebra.

(iii) A symmetric segal algebra is both left and right ideal of $L^1(G)$ and

$$\|f * g\|_{S(G)} \leq \|g\|_{L^1(G)} \cdot \|f\|_{S(G)}, \quad (g \in L^1(G), f \in S(G)).$$

Example 5.1.4. Let G be a locally compact group.

- (i) $L^1(G)$ is a trivial Segal algebra.
- (ii) For $1 \leq p < \infty$, the Banach space $(L^1(G) \cap L^p(G), \|\cdot\|_{L^1(G)} + \|\cdot\|_{L^p(G)})$ is a Segal algebra.
- (iii) Let $f \in L^1(\mathbb{R})$. Denote by \widehat{f} (for $1 \leq p < \infty$) the Fourier transform of f . Then, the space A_p consisting of all $f \in L^1(\mathbb{R})$ such that $\widehat{f} \in L^p(\mathbb{R})$ equipped with the norm $\|f\|_{L^1(\mathbb{R})} + \|\widehat{f}\|_{L^p(\mathbb{R})}$ is a Segal algebra.

(iv) Suppose that G is an abelian group and H is a discrete subgroup of G such that the quotient G/H is compact, that is, $G = KH$ for some compact set $K \subset G$. Then the space S_0 of all continuous functions on G defined by

$$S_0 = \{f \in L^1(G) : f(u) = \sum_{h \in H} \max_{x \in K} |f(uxy)|, u \in G \text{ and } f \text{ is bounded}\}$$

and equipped with the norm

$$\|f\|_{S_0} = \sup_{u \in G} \sum_{h \in H} \max_{x \in K} |f(uxy)|$$

is a Segal algebra.

(v) Let \mathcal{A} be the algebra of all completely continuous operators on $L^2(\mathbb{R})$ with the usual operator norm and let \mathcal{B} be the algebra of all operators of Hilbert-Schmidt type. That is, $T \in \mathcal{B}$ if there exists $K(x, y) \in L^2(\mathbb{R} \times \mathbb{R})$ such that $K(x, y) = \overline{K(y, x)}$ and

$$Tf(x) = \int_{\mathbb{R}} K(x, y)f(y) dy, \quad \text{almost everywhere.}$$

For $T \in \mathcal{B}$ let $\|T\|_{\mathcal{B}}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |K(x, y)|^2 dx dy$. Then \mathcal{B} is an abstract Segal algebra but not a (classical) segal algebra. For details, see [7]([46], p. 34).

It is important to note that Segal algebras are not amenable except for the trivial case when the $S(G) = L^1(G)$ where G is amenable (see [30]).

5.2 Some Results on Segal Algebras

In this section, we discuss the character amenability and character contractibility on some Segal algebras.

We have defined abstract Segal algebras in Section 5.1. We give detailed proofs of the following results from [1].

Theorem 5.2.1. *Let \mathcal{A} be a Banach algebra and let \mathcal{B} be an abstract Segal algebra with respect to \mathcal{A} . Then the following statements are equivalent:*

- (i) \mathcal{B} is character amenable.
- (ii) $\mathcal{A} = \mathcal{B}$ and \mathcal{A} is character amenable.

(iii) \mathcal{A} is Banach algebra isomorphic to \mathcal{B} and \mathcal{A} is character amenable.

Proof. (i) \implies (ii). Since \mathcal{B} is character amenable, then by Proposition 3.5.41(i), \mathcal{B} has a bounded approximate identity (e_α) . Since \mathcal{B} is an abstract Segal algebra then by definition 5.1.1, then for each $b \in \mathcal{B}$, there exists $K > 0$ such that

$$\|b\|_{\mathcal{A}} \leq K\|b\|_{\mathcal{B}} \quad (5.1)$$

and for each $a \in \mathcal{A}$, $b \in \mathcal{B}$, there exists $C > 0$ such that

$$\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}.$$

Thus, for each $a \in \mathcal{B}$,

$$\|a\|_{\mathcal{B}} = \lim_{\alpha} \|ae_{\alpha}\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}} \liminf_{\alpha} \|e_{\alpha}\|_{\mathcal{B}} \leq C \left(\sup_{\alpha} \|e_{\alpha}\|_{\mathcal{B}} \right) \|a\|_{\mathcal{A}}. \quad (5.2)$$

It then follows from (5.1) and (5.2) that the two norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$ are equivalent on \mathcal{B} . And since \mathcal{B} is dense in \mathcal{A} , then $\mathcal{A} = \mathcal{B}$. Since \mathcal{B} is character amenable, then \mathcal{A} is also character amenable.

(ii) \implies (iii). Since \mathcal{B} is an abstract Segal algebra with respect to \mathcal{A} implies that there exists $K > 0$ such that $\|a\|_{\mathcal{A}} \leq K\|a\|_{\mathcal{B}}$ for each $a \in \mathcal{A}$. Then by the open mapping theorem, the map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is an isomorphism.

(iii) \implies (i). The isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a continuous homomorphism with dense range and since \mathcal{A} is character amenable then by Proposition 3.5.42(i), \mathcal{B} is also character amenable. \square

Let \mathcal{A} be a Banach algebra and \mathcal{B} be an abstract Segal algebra on \mathcal{A} . Since \mathcal{B} is a left ideal of \mathcal{A} by definition, then the set $\sigma(\mathcal{B})$ of all non-zero characters on \mathcal{B} is a subset of $\sigma(\mathcal{A})$. In fact, the characters on \mathcal{B} are precisely the restriction of the characters on \mathcal{A} to \mathcal{B} . Indeed, the following holds.

Lemma 5.2.2. [1] *Let \mathcal{A} be a Banach algebra and let \mathcal{B} be an abstract Segal algebra with respect to \mathcal{A} . Then $\sigma(\mathcal{B}) = \{\varphi|_{\mathcal{B}} : \varphi \in \sigma(\mathcal{A})\}$.*

Proof. \mathcal{B} is dense in \mathcal{A} implies $\overline{\mathcal{B}} = \mathcal{A}$, that is $\varphi(\overline{\mathcal{B}}) = \varphi(\mathcal{A}) \neq 0$. Consequently, $\varphi|_{\mathcal{B}} \neq 0$, that is, $\varphi|_{\mathcal{B}} \in \sigma(\mathcal{B})$. It remains to show that for any $\phi \in \sigma(\mathcal{B})$, there is a unique extension $\varphi \in \sigma(\mathcal{A})$. Let $\phi \in \sigma(\mathcal{B})$, then for each $b \in \mathcal{B}$ there is a $b' \in \mathcal{B}$ such that $\phi(b') = 1$ and $\phi(b) = \phi(bb')$. Since \mathcal{B} is a Segal algebra, there exists $C > 0$ such that $\|bb'\|_{\mathcal{B}} \leq C\|b\|_{\mathcal{A}}\|b'\|_{\mathcal{B}}$ for each $b \in \mathcal{B}$ and consequently

$$|\phi(b)| = |\phi(b)\phi(b')| = |\phi(bb')| \leq \|\phi\| \|bb'\| \leq C\|b\|_{\mathcal{A}}\|b'\|_{\mathcal{B}}.$$

Thus ϕ is continuous on $(\mathcal{B}, \|\cdot\|_{\mathcal{A}})$. Since \mathcal{B} is a dense subspace of \mathcal{A} , then the extension of ϕ to $\varphi \in \sigma(\mathcal{A})$ is unique. \square

Remark 5.2.3. Lemma 5.2.2 shows that every character on \mathcal{B} is also a character on \mathcal{A} , that is $\sigma(\mathcal{B}) \subset \sigma(\mathcal{A})$. Thus, for each $\varphi \in \mathcal{B}$, if \mathcal{A} is φ -amenable, then necessarily \mathcal{B} is φ -amenable. The following Proposition 5.2.5 gives a characterization of the character amenability of \mathcal{B} in terms of the character amenability of \mathcal{A} . We shall need the following Lemma.

Lemma 5.2.4. [27] Let \mathcal{A} be a Banach algebra and $\varphi \in \sigma(\mathcal{A})$. Then \mathcal{A} is φ -amenable if and only if there exists a bounded net $(u_\alpha) \subset \mathcal{A}$ such that $\|au_\alpha - \varphi(a)u_\alpha\| \rightarrow 0$ for all $a \in \mathcal{A}$ and $\varphi(u_\alpha) = 1$ for all α .

Proposition 5.2.5. Let \mathcal{A} be a Banach algebra and let \mathcal{B} be an abstract Segal algebra with respect to \mathcal{A} . Then \mathcal{A} is φ -amenable if and only if \mathcal{B} is $\varphi|_{\mathcal{B}}$ -amenable for each $\varphi \in \sigma(\mathcal{A})$.

Proof. Let \mathcal{A} be φ -amenable. Then by Lemma 5.2.4 there exists a bounded net (a_α) in \mathcal{A} such that for all $a \in \mathcal{A}$ and α ,

$$\|aa_\alpha - \varphi(a)a_\alpha\| \rightarrow 0, \quad \varphi(a_\alpha) = 1. \quad (5.3)$$

Fix $b_0 \in \mathcal{B}$ such that $\varphi(b_0) = 1$ and set

$$b_\alpha := a_\alpha b_0^2 \in \mathcal{B}$$

for all α . Since \mathcal{B} is an abstract Segal algebra with respect to \mathcal{A} , then $(b - \varphi(b))a_\alpha \in \mathcal{A}$ and there exists $C > 0$ and $K > 0$ such that for each $b \in \mathcal{B}$,

$$\|b_0 b_0\|_{\mathcal{B}} \leq C \|b_0\|_{\mathcal{A}} \|b_0\|_{\mathcal{B}} \quad (5.4)$$

and

$$\|b_0\|_{\mathcal{A}} \leq K \|b_0\|_{\mathcal{B}}. \quad (5.5)$$

Consequently, using (5.4), (5.5) and Lemma 5.2.4, we have

$$\begin{aligned} \|bb_\alpha - \varphi(b)b_\alpha\|_{\mathcal{B}} &= \|ba_\alpha b_0^2 - \varphi(b)a_\alpha b_0^2\|_{\mathcal{B}} = \|(b - \varphi(b))a_\alpha b_0^2\|_{\mathcal{B}} \\ &\leq \|(b - \varphi(b))a_\alpha\|_{\mathcal{A}} \|b_0 b_0\|_{\mathcal{B}} \leq C \|(b - \varphi(b))a_\alpha\|_{\mathcal{A}} \|b_0\|_{\mathcal{A}} \|b_0\|_{\mathcal{B}} \\ &\leq CK \|(b - \varphi(b))a_\alpha\|_{\mathcal{A}} \|b_0\|_{\mathcal{B}} \|b_0\|_{\mathcal{B}} \rightarrow 0. \end{aligned}$$

and

$$\varphi(b_\alpha) = \varphi(a_\alpha b_0^2) = \varphi(a_\alpha)\varphi(b_0 b_0) = \varphi(a_\alpha)\varphi(b_0)\varphi(b_0) = \varphi(a_\alpha) \rightarrow 1$$

Since (a_α) is $\|\cdot\|_{\mathcal{A}}$ -bounded, it follows that (b_α) is $\|\cdot\|_{\mathcal{B}}$ -bounded. Thus, there exists a bounded net (b_α) such that for all $a \in \mathcal{A}$, $\|bb_\alpha - \varphi(b)b_\alpha\|_{\mathcal{B}} \rightarrow 0$ and $\varphi(b_\alpha) \rightarrow 1$ and by Lemma 5.2.4, it follows that \mathcal{B} is $\varphi_{\mathcal{B}}$ -amenable.

Conversely, similarly we assume that \mathcal{B} is $\varphi|_{\mathcal{B}}$ -amenable and then show that there exists a bounded net satisfying the conditions in Lemma 5.2.4. Let \mathcal{B} be $\varphi|_{\mathcal{B}}$ -amenable, then there is a bounded net (b_α) in \mathcal{B} . Fix $b_0 \in \mathcal{B}$ such that $\varphi(b_0) = 1$ and set

$$a_\alpha := b_0 b_\alpha$$

for all α . Since \mathcal{B} is a Segal algebra, there exists $K > 0$ such that for each $a \in \mathcal{A}$,

$$\begin{aligned} \|aa_\alpha - \varphi(a)a_\alpha\| &= \|ab_0 b_\alpha - \varphi(a)b_0 b_\alpha\| \\ &= \|ab_0 b_\alpha - \varphi(a)\varphi|_{\mathcal{B}}(b_0)b_\alpha + \varphi(a)\varphi|_{\mathcal{B}}(b_0)b_\alpha - \varphi(a)b_0 b_\alpha\|_{\mathcal{A}} \\ &\leq \|ab_0 b_\alpha - \varphi(a)\varphi|_{\mathcal{B}}(b_0)b_\alpha\|_{\mathcal{A}} + \|\varphi(a)\varphi|_{\mathcal{B}}(b_0)b_\alpha - \varphi(a)b_0 b_\alpha\|_{\mathcal{A}} \\ &= \|ab_0 b_\alpha - \varphi(a)\varphi|_{\mathcal{B}}(b_0)b_\alpha\|_{\mathcal{A}} + |\varphi(a)|\|\varphi|_{\mathcal{B}}(b_0)b_\alpha - b_0 b_\alpha\|_{\mathcal{A}} \\ &= \|ab_0 b_\alpha - \varphi|_{\mathcal{B}}(ab_0)b_\alpha\|_{\mathcal{A}} + |\varphi(a)|\|\varphi|_{\mathcal{B}}(b_0)b_\alpha - b_0 b_\alpha\|_{\mathcal{A}} \\ &\leq K(\|ab_0 b_\alpha - \varphi|_{\mathcal{B}}(ab_0)b_\alpha\|_{\mathcal{B}}) + |\varphi(a)|\|\varphi|_{\mathcal{B}}(b_0)b_\alpha - b_0 b_\alpha\|_{\mathcal{A}} \\ &\leq K^2(\|ab_0 b_\alpha - \varphi|_{\mathcal{B}}(ab_0)b_\alpha\|_{\mathcal{B}}) + |\varphi(a)|\|b_0 b_\alpha - \varphi|_{\mathcal{B}}(b_0)b_\alpha\|_{\mathcal{B}} \\ &\rightarrow 0. \end{aligned}$$

Also,

$$\varphi(a_\alpha) = \varphi(b_0 b_\alpha) = \varphi|_{\mathcal{B}}(b_0)\varphi|_{\mathcal{B}}(b_\alpha) = \varphi|_{\mathcal{B}}(b_\alpha) \rightarrow 1.$$

Since \mathcal{B} is a Segal algebra over \mathcal{A} , then $\|\cdot\|_{\mathcal{A}} \leq K\|\cdot\|_{\mathcal{B}}$ so that $(a_\alpha) \in \mathcal{A}$ is bounded net in the norm $\|\cdot\|_{\mathcal{A}}$. The result then follows from Lemma 5.2.4. \square

Corollary 5.2.6. *Let \mathcal{A} be a character amenable Banach algebra and \mathcal{B} a Segal algebra with respect to \mathcal{A} . Then*

- (i) \mathcal{B} is character amenable if and only if $\mathcal{A} = \mathcal{B}$.
- (ii) for every $\varphi \in \sigma(B)$, \mathcal{B} is φ -amenable.
- (iii) $\|\varphi\| \leq \|\phi\|$ for all $\varphi \in \sigma(B)$ and $\phi \in \sigma(A)$.

We now establish an analogue of Proposition 5.2.5 for the character contractible Banach algebras with a different method of proof. We begin with the following basic definition.

Definition 5.2.7. Let \mathcal{A} be a Banach algebra and φ be in the character space of \mathcal{A} . We say that \mathcal{A} is φ -contractible if \mathcal{A} has a φ -diagonal.

Proposition 5.2.8. Let \mathcal{A} be a Banach algebra and \mathcal{B} be a Segal algebra with respect to \mathcal{A} . For each $\varphi \in \sigma(A)$, \mathcal{A} is φ -contractible if and only if \mathcal{B} is $\varphi|_{\mathcal{B}}$ -contractible.

Proof. Let \mathcal{A} be a φ -contractible Banach algebra. Then \mathcal{A} has a φ -diagonal. Let $M \in \mathcal{A} \hat{\otimes} \mathcal{A}$ be a φ -diagonal for \mathcal{A} , then for all $a \in \mathcal{A}$,

$$\varphi(\pi(M)) = 1 \quad \text{and} \quad a\pi(M) = \varphi(a)\pi(M) \quad (5.6)$$

where $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ is the diagonal operator. \mathcal{B} is a Segal algebra implies that \mathcal{B} is a left dense ideal in \mathcal{A} . Thus there exists b_0 in $\mathcal{B} \subset \mathcal{A}$ satisfying $\varphi(b_0) = 1$. Define

$$b_1 := b_0\pi(M) \in \mathcal{B}. \quad (5.7)$$

Then, by using (5.6) and (5.7), we can write

$$bb_1 = bb_0\pi(M) = \varphi(b)b_0\pi(M) = \varphi(b)b_1.$$

Also,

$$\varphi(b_1) = \varphi(b_0\pi(M)) = \varphi(b_0)\varphi(\pi(M)) = \varphi(\pi(M)) = 1.$$

Clearly, $\varphi(\pi(b_1 \otimes b_1)) = \varphi(b_1b_1) = \varphi(b_1)\varphi(b_1) = 1$ and $b \cdot (b_1 \otimes b_1) = b \cdot b_1 \otimes b_1 = \varphi(b)b_1 \otimes b_1 = \varphi(b) \otimes b_1$. Thus, $(b_1 \otimes b_1) \in \mathcal{B} \hat{\otimes} \mathcal{B}$ is a $\varphi|_{\mathcal{B}}$ -diagonal for \mathcal{B} and so \mathcal{B} is $\varphi|_{\mathcal{B}}$ contractible.

Conversely, we assume that \mathcal{B} is $\varphi|_{\mathcal{B}}$ -contractible. Let $m \in \mathcal{B} \hat{\otimes} \mathcal{B}$ be a $\varphi|_{\mathcal{B}}$ -diagonal for \mathcal{B} , then $\pi(m) \cdot m = m$ and $\varphi(\pi(m)) = 1$. Since \mathcal{B} is a left ideal of \mathcal{A} then $a\pi(m) \in \mathcal{B}$ for all $a \in \mathcal{A}$. Now,

$$a \cdot m = a \cdot (\pi(m) \cdot m) = a\pi(m) \cdot m = \varphi(a)m$$

for all $a \in \mathcal{A}$. Thus m is a φ -diagonal for \mathcal{A} . □

Proposition 5.2.9. Let G be an amenable locally compact group and $S(G)$ be an (abstract) Segal algebra with respect to $L^1(G)$. Then

1. $S(G)$ is character amenable if and only if $S(G) = L^1(G)$.
2. $S(G)$ is φ -amenable for all $\varphi \in \sigma(S(G))$.

Proof. G is amenable implies that $L^1(G)$ is amenable ([24] Theorem 2.5, p. 32). Since every amenable Banach algebra is character amenable, then $L^1(G)$ is character amenable. The result then follows from Corollary 5.2.6 since every Segal algebra is an abstract Segal algebra. \square

Theorem 5.2.10. *Let G' the dual of a locally compact group G . Then the following are equivalent statements:*

- (i) G is compact.
- (ii) Every abstract Segal algebra on G is φ_ρ -contractible, $\rho \in G'$.
- (iii) There exists an abstract Segal algebra on G which is φ_ρ -contractible, $\rho \in G'$.

Let G be a locally compact group and let G' be the dual of G . For $1 \leq p < \infty$, the space $S_p(G)$ of all functions $f \in L^1(G)$ defined by

$$S_p(G) := \{f \in L^1(G) : \hat{f} \in L^p(G')\}$$

and equipped with the norm

$$\|f\|_p = \|f\|_1 + \|\hat{f}\|_p, \quad (f \in S_p(G))$$

where \hat{f} denotes the Fourier transform of f on G' is a Segal algebra.

Proposition 5.2.11. *Let G and H be locally compact abelian groups, and let $1 \leq p < \infty$. Assume that the Segal algebras $S_p(G)$ and $S_p(H)$ are both character amenable. Then $S_p(G \times H)$ is character amenable.*

Chapter 6

Collection of Results

In this chapter, we give a collection of results on some of the notions of amenability discussed in third and fourth chapters of this dissertation for general Banach algebras and for Segal algebras. These results serve as our contribution to knowledge.

6.1 Results on General Banach Algebras

The following lemma and theorems which are due to Curtis and Loy [8] are important in the sequel.

Lemma 6.1.1. *Let \mathcal{A} be a unital Banach algebra. Then the short exact sequence*

$$\prod : 0 \rightarrow \ker \pi \xrightarrow{i} \mathcal{A} \hat{\otimes} \mathcal{A} \xrightarrow{\pi} \mathcal{A} \rightarrow 0$$

is admissible. Suppose \mathcal{A} has a bounded left or right approximate identity, then \prod' is admissible.

Theorem 6.1.2. *Let \mathcal{A} be an amenable Banach algebra, and let*

$$\Sigma : 0 \rightarrow X' \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

be an admissible short exact sequence of left or right \mathcal{A} -modules with X' a dual \mathcal{A} -module. Then Σ splits.

Proposition 6.1.3. *Let \mathcal{A} be a Banach algebra with bounded approximate identity, X a left (or right) Banach \mathcal{A} -module, f a left (or right) \mathcal{A} -module homomorphism of \mathcal{A} onto X with kernel J . Then the exact sequence*

$$\Sigma : 0 \rightarrow X' \xrightarrow{f'} \mathcal{A}' \xrightarrow{i'} J' \rightarrow 0$$

splits as a sequence of left (or right) \mathcal{A} -modules if and only if the left (or right) ideal J has a bounded right (or left) approximate identity.

Theorem 6.1.4. *Let \mathcal{A} be a Banach algebra and $\pi : \mathcal{A} \hat{\otimes} \mathcal{A}^{op} \rightarrow \mathcal{A}$ be the diagonal operator. Then \mathcal{A} is amenable if and only if*

- (i) \mathcal{A} possesses a bounded approximate identity and
- (ii) the closed left ideal $\ker \pi$ possesses a bounded right approximate identity.

We now establish the main theorems of this section.

Theorem 6.1.5. *The Banach algebras \mathcal{A} and \mathcal{B} are amenable if their direct sum $\mathcal{A} \oplus \mathcal{B}$ is amenable.*

Proof. Fix $b \in \mathcal{B}$ and define a canonical projection mapping $\phi : \mathcal{A} \oplus \mathcal{B} \rightarrow \mathcal{A}$ by $\phi(a, b) = a$. Then ϕ is a continuous homomorphism with dense range. To see this, let (a_α) , $\alpha \in \Delta$ be a net in \mathcal{A} such that there exists an $a \in \mathcal{A}$ for which $\|a_\alpha - a\| \rightarrow 0$. Then

$$\|\phi(a_\alpha, b) - \phi(a, b)\| = \|a_\alpha - a\| \rightarrow 0.$$

Thus $\phi_b(a_\alpha, b) \rightarrow a$ and so ϕ_b is continuous. By the definition of ϕ_b , $\overline{\phi(\mathcal{A} \oplus \mathcal{B})} = \mathcal{A}$. Also, for $(a, b), (c, d) \in \mathcal{A} \oplus \mathcal{B}$,

$$\phi((a, b)(c, d)) = \phi(ac, bd) = ac = \phi(a, b)\phi(c, d)$$

so that ϕ is an homomorphism. It then follows that \mathcal{A} is amenable by Proposition 3.2.4. The proof of the amenability of \mathcal{B} is similar if one considers an homomorphism $\phi : (a, b) \mapsto b$. \square

Remark 6.1.6. *Let $\oplus_{i=1}^n \mathcal{A}_i$ be the direct sum of the Banach algebras \mathcal{A}_i ($i = 1, 2, \dots, n$). If $\oplus_{i=1}^n \mathcal{A}_i$ is amenable, then for each $i = 1, 2, \dots, n$, \mathcal{A}_i is amenable. This follows easily by induction.*

Theorem 6.1.7. *Let I be an amenable closed subalgebra of a Banach algebra \mathcal{A} . Then for any Banach \mathcal{A} -bimodule X , there exists a derivation $\overline{D} : \mathcal{A} \rightarrow X'$ such that $\overline{D}(I) = \{0\}$.*

Proof. Let $D : \mathcal{A} \rightarrow X'$ be a derivation. Since I is amenable, then there exists $\zeta \in X'$ such that the restriction $D|_I = \delta_\zeta^I$. Define $\overline{D} = D - \delta_\zeta^{\mathcal{A}}$, where $\delta_\zeta^{\mathcal{A}} : a \mapsto$

$a \cdot \zeta - \zeta \cdot a$. Then $\delta_\zeta^{\mathcal{A}}$ is a derivation and so for $a, b \in \mathcal{A}$, it holds that

$$\begin{aligned}
\overline{D}(ab) &= (D - \delta_\zeta^{\mathcal{A}})(ab) = D(ab) - \delta_\zeta^{\mathcal{A}}(ab) \\
&= (D(a)b + aD(b)) - (\delta_\zeta^{\mathcal{A}}(a)b + a\delta_\zeta^{\mathcal{A}}(b)) \\
&= (D(a) - \delta_\zeta^{\mathcal{A}}(a))b + a(D(b) - \delta_\zeta^{\mathcal{A}}(b)) \\
&= (D - \delta_\zeta^{\mathcal{A}})(a)b + a(D - \delta_\zeta^{\mathcal{A}})(b) \\
&= \overline{D}(a)b + a\overline{D}(b).
\end{aligned}$$

Thus \overline{D} is a derivation on \mathcal{A} . Finally, we show that $\overline{D}(I) = 0$. Since I is amenable, it follows that

$$\begin{aligned}
\overline{D}(I) &= (D - \delta_\zeta^{\mathcal{A}})(I) = D(I) - \delta_\zeta^{\mathcal{A}}(I) \\
&= \delta_\zeta^I(I) - \delta_\zeta^I(I) = \{0\}.
\end{aligned}$$

□

6.2 Results on Segal Algebras

The following results from [45] are useful in establishing our main result in this section. We shall denote by $\mathcal{A}^{\|\cdot\|_{\mathcal{A}}}$ a Banach algebra \mathcal{A} with respect to norm $\|\cdot\|_{\mathcal{A}}$ and write \mathcal{A}^2 for $\mathcal{A} \cdot \mathcal{A}$, where $\mathcal{A} \cdot \mathcal{A} = \{bb : b \in \mathcal{A}\}$.

Lemma 6.2.1. *Let \mathcal{A} be a Banach algebra such that $\mathcal{A} \cdot \mathcal{A} = \mathcal{A}$ and \mathcal{B} be an abstract Segal algebra with respect to \mathcal{A} . Then the closure $\overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{B}}}$ is an abstract Segal algebra with respect to \mathcal{A} .*

Theorem 6.2.2. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity and let \mathcal{B} be a symmetric abstract Segal algebra with respect to \mathcal{A} . Then $\overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{B}}}$ is a symmetric Segal algebra with respect to \mathcal{A} such that there exists an approximate identity for $\overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{B}}}$ which is also a bounded approximate identity for \mathcal{A} .*

Theorem 6.2.3. *Let \mathcal{A} be an amenable Banach algebra and \mathcal{B} a symmetric abstract Segal algebra on \mathcal{A} . Then $\overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{B}}}$ is a symmetric abstract Segal algebra on \mathcal{A} and is amenable.*

Proof. By Theorem 6.2.2, $\overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{B}}}$ has a bounded approximate identity and hence factors by Cohen factorization theorem. Also since \mathcal{A} is amenable, it has a bounded approximate identity, and so $\mathcal{A} = \mathcal{A} \cdot \mathcal{A}$. Then by Lemma 6.2.1, $\overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{B}}}$ is an abstract Segal algebra which is symmetric since \mathcal{B} is symmetric. Now, \mathcal{B}^2 is dense in \mathcal{A} since $\mathcal{A} = \mathcal{A} \cdot \mathcal{A} = \overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{A}}} \overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{A}}} = \overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{A}}}$. Thus $\overline{\mathcal{B}^2}^{\|\cdot\|_{\mathcal{B}}}$ is a dense ideal in \mathcal{A} . The result then follows from Proposition 3.3.12. \square

We recall that a Banach algebra \mathcal{A} is pseudo-amenable if has an approximate diagonal.

Theorem 6.2.4. *Every amenable SIN group is compact.*

Proof. It suffices to show that every amenable SIN group is compact. Let G be an amenable SIN group, then from standard results in [30], the Segal algebra $S^1(G)$ is pseudo-amenable if and only if G is compact. Also, $S^1(G)$ is pseudo-amenable if G is a SIN group. Hence the result. \square

6.3 Concluding Remarks

Generally, this dissertation presents a comprehensive study on the following major areas: the general theory of Banach algebras, some notions of amenability in Banach algebras, some characterizations and hereditary properties of these notions for arbitrary Banach algebras and for Segal algebras. Chapter 2 contains adequate background material on Banach algebra theory. Chapter 3 contains a discussion on some notions of amenability for an arbitrary Banach algebra. In particular, various characterizations and hereditary properties of amenable Banach algebras, contractible Banach algebras, weakly amenable Banach algebras, character amenable Banach algebras and character contractible Banach algebras are studied. Theorems 3.3.1 and 3.3.2 characterize the notion of amenability in terms of the existence of bounded approximate diagonal, virtual diagonal and splitting of a short exact sequence. In Chapter 4, important characterizations of generalized notions of amenability for general Banach algebras via the existence of certain nets satisfying some special conditions are considered. Some relationships between the following notions: approximate weak amenability, approximate contractibility, approximate amenability and weak amenability are also established. The conditions under which some weaker notions of amenability imply the stronger ones are also presented in this chapter. Chapter 5 contains an elegant discussion on Segal algebras. We studied some notions of amenability for a class of

Banach algebras called Segal algebras. In Chapter 6, some new results for general Banach algebras and Segal algebras are obtained and discussed. The results in this chapter serve as our contribution to knowledge.

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