UNIVERSITY OF KWAZULU-NATAL

# A COVARIANT APPROACH TO LRS-II SPACETIME MATCHING

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Submitted in fulfilment of the academic requirements for the degree of Master in Science to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal.

As the candidate's supervisors, we have approved this thesis for submission.

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# In Memoriam

*Ronald Leslie Paul*

*(1930 - 2016)*

# Acknowledgments

Firstly I would like to thank the Lord God Almighty, the mathematical Creator of the universe, for my success in the completion of this thesis.

My sincerest thanks to my supervisors, Dr. Rituparno Goswami and Prof. Sunil Maharaj, for their constant support and mentoring. Without their expert guidance, this thesis would not exist.

My special thanks to the National Research Foundation for financial assistance in the form of the NRF Innovation Masters Scholarship.

I would like to thank Dr. Anne Marie Nzioki for helping explain the 1+3 and 1+1+2 covariant formalisms. Thank you also to my fellow colleagues for your support.

My deepest thanks to my parents, Colin and Sarah, for their love and support throughout my life. I truly appreciate all your hard work and sacrifices. Thank you for giving me the opportunity to follow my dreams. I must also thank my brother Nolan for his love and support.

Erwin Roderic Paul

January 2017

# Abstract

In this thesis we examine the spacetime matching conditions covariantly for Locally Rotationally Symmetric class II (LRS-II) spacetimes, of which spherical symmetry is a special case. We use the semi-tetrad 1+1+2 covariant formalism and look at two general spacetime regions in LRS-II and match them across a timelike hypersurface using the Israel-Darmois matching conditions. This gives a new and unique result which is transparently presented in terms of the matching of various geometrical quantities (e.g. the expansion, shear, acceleration). Thereafter we apply the new result to the case involving a general spherically symmetric spacetime, representing for instance the interior of a star, and the Schwarzschild spacetime, which could represent the exterior. It is shown that the matching conditions make the Misner-Sharp and Schwarzschild masses exactly the same at the boundary, and the pressure is zero on the boundary.

# Contents

# Chapter 1

# Introduction

General relativity is Einstein's theory which describes gravitation as the curvature of spacetime due to the presence of matter and energy in the spacetime. The *Einstein field equations* are the set of nonlinear partial differential equations that model gravitational interactions, and these equations relate the spacetime geometry to the energy and matter in the spacetime. In tensor form and using natural units ($c = 8\pi G = 1$), the Einstein field equations are given by

$$G_{ab} + \Lambda g_{ab} = T_{ab}, \tag{1.1}$$

where

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab}, \tag{1.2}$$

is the Einstein tensor, $\Lambda$ is the cosmological constant, $g_{ab}$ is the metric tensor and $T_{ab}$ is the energy momentum tensor. In (1.2), $R_{ab}$ is the Ricci tensor and $R$ is the Ricci scalar. Solutions to (1.1) are the components of the metric tensor $g_{ab}$, which is the essential object in general relativity as it characterizes a particular spacetime. Consider two intersecting spacetime regions with different matter and energy distributions, which of course have different metric tensors. These two spacetime regions need to be matched at their common boundary to ensure that there are no sudden kinks or singularities in the spacetime. This is the essence of spacetime matching.

The matching of different spacetimes allows us to gain a theoretical insight, which the individual models describing each spacetime might not allow (Fayos, Senovilla and Torres, 1996). Spacetime matching can be used to study some fascinating physical problems. Among these are the gravitational collapse or expansion of stars and the collision of gravtitational waves (Fayos, Senovilla and Torres, 1996). A simple class of spacetimes which enables this study is the Locally Rotationally Symmetric class II (LRS-II) spacetimes. Since LRS spacetimes have a preferred spatial direction locally, the semi-tetrad 1+1+2 covariant formalism will be needed (Betschart and Clarkson, 2004). A basic study is to match two general spacetime regions in LRS-II across a timelike hypersurface, using the Israel-Darmois (Israel 1966, Darmois 1927) matching conditions. This general study can then be expanded to explore more exotic physical problems.

## 1.1 Need for spacetime matching

Many astrophysical systems have more than one spacetime. An example is a spherically symmetric star immersed in vacuum. Inside the star, the spacetime is obtained by solving the Einstein field equations with the stellar matter. The exterior of the star is vacuum and so by Birkhoff's theorem, the exterior is the Schwarzschild spacetime. Now we have to smoothly match these two spacetimes at the boundary of the star, so that there are no sudden jumps in the field equations. If such jumps exist, then that will create surface stress energy tensors that can destabilize the whole system.

## 1.2 Israel-Darmois matching conditions

The Israel-Darmois matching conditions state that to smoothly match two spacetimes across a hypersurface, the following conditions must be satisfied:

1. The projected metric on the hypersurface should be equal on both the sides.

2. The extrinsic curvatures of both sides should be equal on the hypersurface.

These conditions are very similar to the conditions in electromagnetism, except that there the potentials and normal derivatives were matched.

## 1.3 Basic questions in spacetime matching

Fayos, Senovilla and Torres (1996) suppose that we are given two adjacent orientable spacetimes $\mathcal{V}$ and $\overline{\mathcal{V}}$. Then two basic practical questions we must ask are

1. Are these spacetimes matchable?

2. What are the possible matching hypersurfaces?

In practical problems it is important to select the correct direction for the normal vector to the hypersurface, which was also mentioned by Goldwirth and Katz (1995) and Fayos, Senovilla and Torres (1996). These choices will be discussed in detail in the next chapter, together with the necessary conditions for general matching, which will thereafter be applied to the spherically symmetric case. These elementary conditions enable us to determine the feasibility of matchings just by inspection of the conformal diagrams (Fayos, Senovilla and Torres, 1996).

## 1.4 Previous works

The subject of spacetime matching has been previously studied in literature. Fayos, Senovilla and Torres (1996) outlined some of the previous treatments, which we shall now mention. Oppenheimer and Snyder (1939) considered the matching of a closed collapsing dust Friedmann-Lemaître-Robertson-Walker spacetime with a Schwarzschild exterior spacetime. Einstein and Straus (1945) looked at the complementary matching problem to Oppenheimer and Snyder (1939), in a paper which considered the impact of the expansion of space on the gravitational field surrounding the star. The early efforts to describe primordial black holes in Hacyan (1979) and Reed and Henriksen

(1980) considered the generalization from the Schwarzschild spacetime to the Vaidya spacetime. Hacyan (1979) modified the Einstein and Straus (1945) model for a radiation filled universe and thus replaced the Schwarzschild metric with the Vaidya metric. Lake (1980) studied the general treatment of this problem. Using the same type of matching, Lake and Hellaby (1981) showed that the radiative version of the problem in Oppenheimer and Snyder (1939) resulted in naked singular spacetimes, a counterexample of the cosmic censorship conjecture. The general matching conditions for the matching of the Vaidya and general Friedmann-Lemaître-Robertson-Walker spacetimes were given in Fayos *et al* (1991). Fayos, Senovilla and Torres (1996) considered the general matching of two spherically symmetric spacetimes across a timelike hypersurface, and applied their results to the general matching of the Vaidya spacetime and the general flat Friedmann-Lemaître-Robertson-Walker spacetime with a linear equation of state $p = \gamma\rho$.

Another interesting paper which looked at spacetime matching is Santos (1985), which studied the matching conditions for a "shear-free isotropic fluid undergoing radial heat flow with outgoing unpolarized radiation". Santos (1985) found the relation

$$p_\Sigma = (qB)_\Sigma, \tag{1.3}$$

where $p$ is the isotropic pressure, $q$ is the radial heat flux and $\Sigma$ represents the boundary. This result is different to the result of Glass (1981), which was $p_\Sigma = 0$. According to Santos (1985), (1.3) tells us that a spherically symmetric shear-free distribution of a collapsing fluid, which is dissipating heat radially, has a nonzero isotropic pressure at the boundary. The isotropic pressure can only be zero at the boundary if $q_\Sigma = 0$, i.e. the fluid is not dissipating, and in such a situation there is no radiation and the exterior spacetime is the Schwarzscild spacetime (Santos, 1985).

## 1.5 This thesis

We explore spacetime matching for two general regions in Locally Rotationally Symmetric class II (LRS-II) spacetimes using the semi-tetrad 1+1+2 covariant formalism and the Israel-Darmois matching conditions. The outcome of this will be a new and unique result, namely the general matching conditions for LRS-II spacetimes. Thereafter we will apply the new result to the matching of a general spherically symmetric spacetime to the exterior Schwarzschild spacetime. This example represents a spherically symmetric star immersed in vacuum.

The thesis is organised as follows: In the first chapter we outline the main results concerning general matching conditions in general relativity. In the subsequent two chapters we briefly explain the 1+3 and 1+1+2 covariant formalisms. In chapter 4 we briefly describe LRS-II spacetimes and set out their field equations. The next chapter is where we will employ the Israel-Darmois matching conditions to match two general spacetime regions in LRS-II across a timelike hypersurface, which will give us the new and unique matching conditions for LRS-II spacetimes. In the penultimate chapter we will apply our results from the previous chapter to the matching of a spherically symmetric spacetime and the Schwarzschild spacetime. The final chapter is a discussion of the results and their use in possible future research.

# Chapter 2

# General matching conditions in general relativity

## 2.1 Introduction

We aim in this chapter to present the general results concerning the general matching of two spherically symmetric spacetimes across a timelike hypersurface. The reader is referred to Fayos, Senovilla and Torres (1996), on which this chapter has been based, for a comprehensive exposition.

We initially make a few standard assumptions about the matching problem. Let $\mathcal{V}^+$ and $\mathcal{V}^-$ be two $C^3$ orientable spacetimes with $C^2$ metrics $g^+$ and $g^-$, having boundaries $\mathcal{S}^+$ and $\mathcal{S}^-$ respectively. The reader is encouraged to consult Hawking and Ellis (1973) for the standard definitions.

## 2.2 Matching conditions for general spacetimes

The reader should consult Israel (1966), Clarke and Dray (1987) and Mars and Senovilla (1993) as well for additional details. We assume that there is a $C^3$ diffeomorphism from $\mathcal{S}^-$ to $\mathcal{S}^+$. This means that there is a three times continuously differentiable invertible function which maps from $\mathcal{S}^-$ to $\mathcal{S}^+$. Let the disjoint union of $\mathcal{V}^+$ and $\mathcal{V}^-$, which have points that are related through the diffeomorphism identified, be the complete spacetime, which we shall denote as $\mathcal{V}_4$. The images of $\mathcal{S}^-$ and $\mathcal{S}^+$ in $\mathcal{V}_4$ shall be noted by $\mathcal{S}$. The issue now is if $\mathcal{V}^+$ and $\mathcal{V}^-$ can be joined in such a manner that $\mathcal{V}_4$ has a Lorentzian geometry with the Einstein field equations well defined. From Clarke and Dray (1987), this is possible if and only if $\mathcal{S}^+$ and $\mathcal{S}^-$ are isometrical with respect to their first fundamental forms $h^+$ and $h^-$ which have been derived from $\mathcal{V}^+$ and $\mathcal{V}^-$ respectively, as in this case there is a natural continuous extension $g$ of the metric to the entire $\mathcal{V}_4$.

There are two embeddings given, $x^\mu_\pm = x^\mu_\pm(\xi^a)$ of $\mathcal{S}$, where $\xi^a$ are intrinsic coordinates for $\mathcal{S}$ and $x^\mu_\pm$ are local coordinates for $\mathcal{V}^\pm$. The requirement that the first fundamental forms must match is

$$h^+_{ab} = h^-_{ab}, \tag{2.1}$$

where from Israel (1966), Clarke and Dray (1987) and Mars and Senovilla (1993)

$$h^\pm_{ab} \equiv g^\pm_{\mu\nu}(x_\pm(\xi))\frac{\partial x^\mu_\pm(\xi)}{\partial \xi^a}\frac{\partial x^\mu_\pm(\xi)}{\partial \xi^b}. \tag{2.2}$$

Note that $h_{ab}$ is the 3-space metric. Clarke and Dray (1987) and Mars and Senovilla (1993) note that it should be mentioned how the tangent spaces are to be identified. Hence consider

$$\vec{t}^{\,+}_a \equiv \frac{\partial x^\mu_+(\xi)}{\partial \xi^a}\frac{\partial}{\partial x^\mu_+}, \tag{2.3}$$

and

$$\vec{t}^{\,-}_a \equiv \frac{\partial x^\mu_-(\xi)}{\partial \xi^a}\frac{\partial}{\partial x^\mu_-}, \tag{2.4}$$

which are two different tangent vector fields to $\mathcal{S}$. The equation (2.1) tells us that the scalar products of $\{\vec{t}^{\,\pm}_a\}$ in $\mathcal{V}_4$ coincide. However we require the entire four dimensional tangent spaces at $\mathcal{S}$. Thus consider the spacelike unit vectors $\vec{n}^\pm$ which are orthogonal to $\mathcal{S}^{\pm}$ on $\mathcal{S}^{\pm}$. They are defined by

$$n^{+}_{\mu}t^{+\mu}_{a} = 0,\ n^{+}_{\mu}n^{+\mu} = 1, \tag{2.5}$$

and similarly for $\vec{n}^{-}$. These normal vectors are selected such that if $\vec{n}^{-}$ points from $\mathcal{V}^{-}$ inwards, then $\vec{n}^{+}$ points from $\mathcal{V}^{+}$ outwards and vice versa. Suppose that

$$\frac{d}{ds}\Big|^{+}_{p} = A\vec{n}^{+}\big|_{p} + B^{a}\vec{t}^{+}_{a}\big|_{p}, \tag{2.6}$$

is the tangent vector to the curve at p from the viewpoint of $\mathcal{V}^{+}$. Then $\vec{n}^{-}$ has to be chosen such that

$$\frac{d}{ds}\Big|^{-}_{p} = A\vec{n}^{-}\big|_{p} + B^{a}\vec{t}^{-}_{a}\big|_{p}, \tag{2.7}$$

is the tangent vector to the curve at p from the viewpoint of $\mathcal{V}^{-}$. This selection gives us two bases $\left\{\vec{n}^{+}, \vec{t}^{+}_{a}\right\}$ and $\left\{\vec{n}^{-}, \vec{t}^{-}_{a}\right\}$ for the tangent space of $\mathcal{V}_4$ at $\mathcal{S}$. After this selection has been completed, the $\pm$ can be omitted and we can thus write the basis for the tangent spaces at $\mathcal{S}$ as $\left\{\vec{n}, \vec{t}_{a}\right\}$. Now in the spacetime $\mathcal{V}_4$, we have a unique $C^1$ atlas $\mathcal{C}$ and a continuous extension $g$ of the metric to $\mathcal{V}_4$. The Einstein field equations are well defined in the distributional sense provided that the extension $g$ of the metric is continuous in $\mathcal{V}_4$ and (2.1) is satisfied (see Clarke and Dray 1987, Mars and Senovilla 1993). Equation (2.1) is an important condition for the calculation of the Riemann tensor distribution and its contractions (Mars and Senovilla, 1993). The singular part of a tensor distribution, being one of the two distinct components of these distributions, is proportional to the Dirac one-form distribution $\delta_{\mu}$ which is linked with $\mathcal{S}$ (Clarke and Dray 1987, Mars and Senovilla 1993). Therefore this singular part describes an infinite discontinuity at $\mathcal{S}$. These infinite discontinuities need to be avoided in the matter and curvature tensors, as only finite discontinuities are physically relevant for a timelike matching hypersurface. The removal of the singular part of the Riemann tensor distribution, for a general timelike hypersurface, is the same as the removal of the singular part of the Einstein tensor distribution (Clarke and Dray 1987, Mars and

Senovilla 1993). This occurs if and only if the second fundamental forms of $\mathcal{S}$ match, i.e.

$$K_{ab}^{-} = K_{ab}^{+}, \tag{2.8}$$

where

$$K_{ab}^{\pm} \equiv -n_{\mu}^{\pm} \left( \frac{\partial^2 x_{\pm}^{\mu}(\xi)}{\partial \xi^a \partial \xi^b} + \Gamma_{\rho\nu}^{\pm\mu} \frac{\partial x_{\pm}^{\rho}(\xi)}{\partial \xi^a} \frac{\partial x_{\pm}^{\nu}(\xi)}{\partial \xi^b} \right). \tag{2.9}$$

Note that $\Gamma^{\mu}{}_{\rho\nu}$ is the Christoffel symbol of the second kind, and it represents the metric connection coefficients which are given by

$$\Gamma^{\mu}{}_{\rho\nu} = \tfrac{1}{2} g^{\mu\lambda} \left( g_{\nu\lambda,\rho} + g_{\lambda\rho,\nu} - g_{\rho\nu,\lambda} \right), \tag{2.10}$$

where a comma denotes partial differentiation. Thus to match two spacetimes across their common boundary, the matching conditions (2.1) and (2.8) must be satisfied.

Consider the matching of two full spacetimes $\mathcal{V}$ and $\overline{\mathcal{V}}$. Let $S$ be a general timelike hypersurface which divides $\mathcal{V}$ into two complementary parts which shall be denoted as 1 and 2. Similarly let $\overline{S}$ divide $\overline{\mathcal{V}}$ into $\overline{1}$ and $\overline{2}$. From Goldwirth and Katz (1995), we see that the matching of the spacetimes $\mathcal{V}$ and $\overline{\mathcal{V}}$ can be done in four different ways: 1 with $\overline{1}$, 1 with $\overline{2}$, 2 with $\overline{1}$ and 2 with $\overline{2}$. The complete spacetime $\mathcal{V}_4$ is formed by the disjoint union of $\mathcal{V}$ and $\overline{\mathcal{V}}$. $\mathcal{S}$ shall be the image of both $S$ and $\overline{S}$ in $\mathcal{V}_4$. Note that due to the earlier considerations regarding the normal vector of the matching hypersurface, if $\mathcal{S}$ matches a part of $\mathcal{V}$ with a part of $\overline{\mathcal{V}}$, for example 1 and $\overline{2}$, then $\mathcal{S}$ also matches 2 and $\overline{1}$. For this reason 1-$\overline{2}$ and 2-$\overline{1}$ are called complementary matchings. Hence there are just two distinct matchings 1-$\overline{2}$ and 1-$\overline{1}$. We will now provide a necessary condition that will enable us to determine which matchings are valid. If we sssume that one of the possible matchings has been carried out, then there exists a local coordinate system of $\mathcal{V}_4$ where the metric is $C^1$ for every point $p$ on $\mathcal{S}$ (Mars and Senovilla 1993, Lichnerowicz 1955, Bonner and Vickers 1981). We call these admissible coordinates

(Lichnerowicz, 1955).

## 2.3 Matching of spherically symmetric spacetimes

A spherically symmetric spacetime (Hawking and Ellis 1973, Kramer *et al* 1980) is one which admits the group of rotations SO(3) as an isometry group, where the group orbits are spacelike two-surfaces. Thus the orbits are two-surfaces of constant positive curvature, called two-spheres, and there are two surfaces orthogonal to the orbits (Kramer *et al*, 1980). Two angular coordinates $\{x^2, x^3\} \equiv \{\theta, \phi\}$ can be selected to describe the orbits while two other coordinates $\{x^0, x^1\}$ describe the orthogonal surfaces. Note that $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$, and every two-sphere is characterised by constant values of the $\left\{x^A\right\} \equiv \{x^0, x^1\}$ coordinates where $(A, B, ... = 0, 1)$. A positive function $R(x^A)$ can be defined such that the total area of a two-sphere is $4\pi R^2$. The line element of a general spherically symmetric spacetime is therefore

$$ds^2 = g_{BC}\left(x^A\right) dx^B dx^C + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.11}$$

where the two-metric $g_{BC}$ has Lorentzian signature. There are two preferred congruences of null geodesics for these spherically symmetric spacetimes, defined as those invariant by the isometry group and the two principal null directions of the type-D Weyl tensor (see Kramer *et al* 1980). A congruence is a set of curves such that through each point in a region there passes only one curve. Geodesics are the extremal paths, along which particles travel, on a manifold. Particles with mass travel along timelike geodesics whereas massless particles such as photons travel along null geodesics. The product of the expansions $\kappa_1$ and $\kappa_2$ of the two affinely parametrized congruences is given by

$$\kappa_1\kappa_2 = -\frac{\chi}{2R^2}, \tag{2.12}$$

where

$$\chi \equiv g^{\mu\nu}\partial_\mu R \partial_\nu R. \tag{2.13}$$

Thus the two expansions have the same sign when $\chi < 0$, which is the region of the closed trapped surfaces (Hawking and Ellis, 1973), and they differ in sign when $\chi > 0$. When one of the expansions is zero, $\chi = 0$, and the resulting hypersurface is known as the apparent horizon (Hawking and Ellis, 1973). We can easily observe that

$$m \equiv \frac{R}{2}\left(1 - \chi\right), \tag{2.14}$$

is the normal mass function introduced by Hernández and Misner (Hernández and Misner 1966, Cahill and McVittie 1970, Zannias 1990).

We now look at the general matching of two spherically symmetric spacetimes $\mathcal{V}$ and $\overline{\mathcal{V}}$ which are time oriented. We need to ascertain which of the four possible matchings are allowed. Taking into account the required continuity of the signs of the expansions for the two preferred null geodesic congruences across $\mathcal{S}$, we have that the signs of $\chi\big|_p$ and $\overline{\chi}\big|_p$ have to be the same for every point $p$ in $\mathcal{S}$. When $\chi\big|_p > 0$ the two expansions have opposite signs in $\mathcal{V}$ and the same for $\overline{\mathcal{V}}$.

Now if $\text{sign}(\chi) = \text{sign}(\overline{\chi}) = +1$ in a region of $\mathcal{S}$, then only one of the possible distinct matchings and its complementary is permitted in this region, while the other distinct matching cannot occur.

It is slightly different when $\text{sign}(\chi) = \text{sign}(\overline{\chi}) = -1$ in a region of $\mathcal{S}$. In this case both expansions have the same sign in $\mathcal{V}$ and $\overline{\mathcal{V}}$, and by the correct choice of the time orientations all can be regarded as positive. We thus have that all four matchings are permitted for this case.

For the case in which $\chi = \overline{\chi} = 0$ in a region of $\mathcal{S}$, one of the expansions is zero while the other could be zero or nonzero. The first possibility gives us the same type of scenario as $\text{sign}(\chi) = \text{sign}(\overline{\chi}) = -1$, while the second possibility results in the same type of scenario as $\text{sign}(\chi) = \text{sign}(\overline{\chi}) = +1$.

Summarizing the above, the following theorem has been proven:

**Theorem:** *If there is at least one point $p$ in $\mathcal{S}$ where $\chi\big|_p > 0$ (or where $\chi\big|_p = 0$ with one of the expansions nonzero), then only one matching and its complementary is possible in principle.*

Note that we must still verify whether the full set of matching conditions are satisfied.

## 2.4 Discussion

In this chapter we have looked at the matching of two general spacetimes and discovered the allowable ways in which the matching can be performed. It was found that the matching can be done in four different ways, but due to complementary matchings only two are distinct. Thereafter we considered the general matching of two spherically symmetric spacetimes, which resulted in the above theorem.

# Chapter 3

# 1+3 Covariant formalism

## 3.1 Introduction

Spacetime is described by Einstein's theory of general relativity. Solving problems in general relativity involves solving the Einstein field equations directly using a metric description of the spacetime. This approach relies on choosing a coordinate system relevant to the symmetry of the problem at the outset. An alternative coordinate independent approach is the 1+3 covariant formalism whereby the spacetime geometry and physics is described by scalars, 3-vectors and projected symmetric trace-free 3-tensors. In this formalism the spacetime is split, through a timelike vector, into time and space, where the 3-space is orthogonal to the timelike vector. The 1+3 formalism has been successful in applications using the Friedmann-Lemaître-Robertson-Walker cosmological model which is homogeneous and isotropic (Betschart and Clarkson, 2004).

## 3.2 Formalism

The reader is encouraged to look at Ellis and van Elst (1998) for a more in depth study of the 1+3 formalism, on which this summary has been based. Consider a timelike congruence parametrised by the proper time, $\tau$, with $u^a$ as the timelike tangent vector to the congruence. Through the use of this timelike vector

$$u^a = \frac{dx^a}{d\tau},\ (u^a u_a = -1), \tag{3.1}$$

we split the spacetime into time and space. This results in the following projection tensors

$$U^a{}_b = -u^a u_b, \tag{3.2}$$

$$h^a{}_b = g^a{}_b + u^a u_b, \tag{3.3}$$

where $U^a{}_b$ projects parallel to $u^a$, and $h^a{}_b$ projects orthogonal to $u^a$ onto the 3-space. With this choice of timelike vector, we can naturally define two derivatives: the *covariant time derivative* (denoted by a dot) for any tensor $T^{a..b}{}_{c..d}$, given by

$$\dot{T}^{a..b}{}_{c..d} = u^e \nabla_e T^{a..b}{}_{c..d}, \tag{3.4}$$

and through use of the tensor $h_{ab}$, the fully orthogonally *projected covariant derivative* $D$ for any tensor $T^{a..b}{}_{c..d}$, given by

$$D_e T^{a..b}{}_{c..d} = h^a{}_f h^p{}_c ... h^b{}_g h^q{}_d h^r{}_e \nabla_r T^{f..g}{}_{p..q}\ , \tag{3.5}$$

with total projection on all free indices. As a result of the spacetime splitting, the spatial 3-volume element is given by

$$\epsilon_{abc} = u^d \eta_{abcd} \implies \epsilon_{abc} = \epsilon_{[abc]},\ \epsilon_{abc} u^c = 0, \tag{3.6}$$

where $\eta_{abcd}$ is the 4-dimensional volume element. The 3-volume element satisfies the following identities:

$$\epsilon^{abc}\epsilon_{def} = 3!h^{[a}{}_{d}h^{b}{}_{e}h^{c]}{}_{f}, \tag{3.7}$$

$$\epsilon^{abc}\epsilon_{cde} = 2!h^{[a}{}_{d}h^{b]}{}_{e}, \tag{3.8}$$

$$\epsilon^{abc}\epsilon_{bcd} = 2!h^{a}{}_{d}, \tag{3.9}$$

$$\epsilon^{abc}\epsilon_{abc} = 3!\,. \tag{3.10}$$

The covariant derivative of $u^a$ can now be decomposed into its irreducible parts as

$$\nabla_a u_b = -u_a A_b + \frac{1}{3}h_{ab}\Theta + \sigma_{ab} + \omega_{ab}, \tag{3.11}$$

where $A_b = \dot{u}_b$ is the acceleration, $\Theta = D_a u^a$ is the expansion, $\sigma_{ab} = D_{\langle a}u_{b\rangle}$ is the shear tensor and $\omega_{ab}$ is the vorticity tensor. The *Weyl curvature tensor* $C_{abcd}$, which is the trace-free part of the *Riemann curvature tensor* $R_{abcd}$, is defined by the equation

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} + \tfrac{1}{3}Rg_{a[c}g_{d]b}. \tag{3.12}$$

The Weyl tensor can similarly be decomposed into its irreducible *electric* and *magnetic* parts as

$$E_{ab} = C_{abcd}u^c u^d, \tag{3.13}$$

and

$$H_{ab} = \tfrac{1}{2}\epsilon_{ade}C^{de}{}_{bc}u^c. \tag{3.14}$$

The energy momentum tensor can likewise be decomposed as

$$T_{ab} = \mu u_a u_b + q_a u_b + q_b u_a + p h_{ab} + \pi_{ab}\ , \tag{3.15}$$

where $\mu = T_{ab}u^a u^b$ is the energy density, $q_a = q_{\langle a\rangle} = -h^c{}_a T_{cd}u^d$ is the 3-vector that defines the heat flux, $p = (1/3)h^{ab}T_{ab}$ is the isotropic pressure and $\pi_{ab} = \pi_{\langle ab\rangle}$ is the anisotropic pressure.

### 3.2.1 Propagation and constraint equations

The set of variables that fully describes a spacetime under the 1+3 formalism is

$$\{\Theta, \dot{u}^a, \omega_{ab}, \sigma_{ab}, E_{ab}, H_{ab}, \mu, p, q^a, \pi_{ab}\}\,. \tag{3.16}$$

The Einstein field equations and its integrability conditions give rise to certain propagation and constraint equations which relate the above variables.

Using the Ricci identity for $u^a$

$$2\nabla_{[a}\nabla_{b]}u^c = R_{ab}{}^c{}_d u^d, \tag{3.17}$$

we can obtain three propagation and three constraint equations. The propagation equations are

$$\dot{\Theta} - D_a\dot{u}^a = -\tfrac{1}{3}\Theta^2 + (\dot{u}_a\dot{u}^a) - 2\sigma^2 + 2\omega^2 - \tfrac{1}{2}(\mu + 3p) + \Lambda, \tag{3.18}$$

$$\dot{\omega}^{\langle a\rangle} - \tfrac{1}{2}\epsilon^{abc}D_b\dot{u}_c = -\tfrac{2}{3}\Theta\omega^a + \sigma^a{}_b\omega^b, \tag{3.19}$$

and

$$\dot{\sigma}^{\langle ab} - D^{\langle a}\dot{u}^{b\rangle} = -\tfrac{2}{3}\Theta\sigma^{ab} + \dot{u}^{\langle a}\dot{u}^{b\rangle} - \sigma^{\langle a}{}_c\,\sigma^{b\rangle c} - \omega^{\langle a}\omega^{b\rangle} - \left(E^{ab} - \tfrac{1}{2}\pi^{ab}\right). \tag{3.20}$$

Equation (3.18) is the famous Raychaudhuri equation (Raychaudhuri, 1955) which describes gravitational attraction. Its importance lies in its use in proving the Hawking-Penrose singularity theorems in general relativity. The equations (3.19) and (3.20) are the vorticity and shear propagation equations respectively. The constraint equations are

$$0 = (C_1)^a = D_b\sigma^{ab} - \tfrac{2}{3}D^a\Theta + \epsilon^{abc}\left[D_b\omega_c + 2\dot{u}_b\omega_c\right] + q^a, \tag{3.21}$$

$$0 = (C_2) = D_a\omega^a - (\dot{u}_a\omega^a)\,, \tag{3.22}$$

and

$$0 = (C_3)^{ab} = H^{ab} + 2\dot{u}^{\langle a}\omega^{b\rangle} + D^{\langle a}\omega^{b\rangle} - (\mathrm{curl}\,\sigma)^{ab}\,. \tag{3.23}$$

Equations (3.21), (3.22) and (3.23) are the $(0\alpha)$ equation, the vorticity divergence identity and the $H_{ab}$ equation respectively.

Using the twice-contracted Bianchi identity, we can obtain two propagation equations and a constraint equation. The propagation equations are

$$\dot{\mu} + D_a q^a = -\Theta\left(\mu + p\right) - 2\left(\dot{u}_a q^a\right) - \left(\sigma_{ab}\pi^{ab}\right), \tag{3.24}$$

and

$$\dot{q}^{\langle a\rangle} D^a p + D_b\pi^{ab} = -\tfrac{4}{3}\Theta q^a - \sigma^a{}_b q^b - \left(\mu + p\right)\dot{u}^a - \dot{u}_b\pi^{ab} - \epsilon^{abc}\omega_b q_c, \tag{3.25}$$

where (3.24) and (3.25) are the energy conservation and momentum conservation equations respectively. The constraint equation is

$$0 = D_a p + \left(\mu + p\right)\dot{u}_a. \tag{3.26}$$

Finally using the once-contracted Bianchi identity, we obtain two propagation and two constraint equations. The propagation equations are

$$\begin{aligned}\dot{E}^{\langle ab\rangle} + \tfrac{1}{2}\dot{\pi}^{\langle ab\rangle} - (\mathrm{curl}\,H)^{ab} + \tfrac{1}{2}D^{\langle a}q^{b\rangle} = &-\tfrac{1}{2}\left(\mu + p\right)\sigma^{ab} - \Theta\left(E^{ab} + \tfrac{1}{6}\pi^{ab}\right)\\ &+3\sigma^{\langle a}{}_c\left(E^{b\rangle c} - \tfrac{1}{6}\pi^{b\rangle c}\right) - \dot{u}^{\langle a}q^{b\rangle}\\ &+\epsilon^{cd\langle a}\left[2\dot{u}_c H^{b\rangle}{}_d + \omega_c\left(E^{b\rangle}{}_d + \tfrac{1}{2}\pi^{b\rangle}{}_d\right)\right],\end{aligned} \tag{3.27}$$

and

$$\dot{H}^{\langle ab\rangle} + (\mathrm{curl}\, E)^{ab} - \tfrac{1}{2}\,(\mathrm{curl}\, \pi)^{ab} = -\Theta H^{ab} + 3\sigma^{\langle a}{}_{c}H^{b\rangle c} + \tfrac{3}{2}\omega^{\langle a}q^{b\rangle}$$

$$-\epsilon^{cd\langle a}\left[2\dot{u}_c E^{b\rangle}{}_{d} - \tfrac{1}{2}\sigma^{b\rangle}{}_{c}\, q_d - \omega_c H^{b\rangle}{}_{d}\right], \qquad (3.28)$$

where (3.27) and (3.28) are the $\dot{E}$ and $\dot{H}$ equations respectively. The constraint equations are

$$0 = (C_4)^a = D_b\left(E^{ab} + \tfrac{1}{2}\pi^{ab}\right) - \tfrac{1}{3}D^a\mu + \tfrac{1}{3}\Theta q^a - \tfrac{1}{2}\sigma^a{}_b q^b - 3\omega_b H^{ab}$$

$$-\epsilon^{abc}\left[\sigma_{bd}H_c{}^d - \tfrac{3}{2}\omega_b q_c\right], \qquad (3.29)$$

and

$$0 = (C_5)^a = D_b H^{ab} + (\mu + p)\,\omega^a + 3\omega_b\left(E^{ab} - \tfrac{1}{6}\pi^{ab}\right)$$

$$+\epsilon^{abc}\left[\tfrac{1}{2}D_b q_c + \sigma_{bd}\left(E_c{}^d + \tfrac{1}{2}\pi_c{}^d\right)\right], \qquad (3.30)$$

where (3.29) is the (div$E$)-equation and (3.30) is the (div$H$)-equation.

## 3.3 Discussion

We have presented in this chapter a summary of the 1+3 covariant formalism which splits the spacetime into time and space through the use of a timelike vector $u^a$. Our choice of timelike vector enabled two derivatives to be defined: the covariant time derivative and the fully orthogonally projected covariant derivative. Using this formalism, certain quantites, namely the covariant derivative of $u^a$, the Weyl tensor and the energy momentum tensor, were decomposed into their irreducible parts. Finally we presented the propagation and constraint equations which relate the set of variables under the 1+3 covariant formalism.

# Chapter 4

# 1+1+2 Covariant formalism

## 4.1 Introduction

The 1+3 covariant formalism is not well suited for spacetimes that are not homogeneous and isotropic, such as Schwarzschild black holes, as it doesn't take into account spacetimes with a preferred spatial direction (Clarkson and Barrett, 2003). For this we need the 1+1+2 covariant formalism which was recently developed by Clarkson and Barrett (2003) and applied to spherically symmetric spacetimes.

## 4.2 Formalism

The reader is encouraged to look at Clarkson and Barrett (2003), Betschart and Clarkson (2004) and Clarkson (2007), on which this chapter has been based, for a more thorough treatment of the 1+1+2 formalism. Here we split the spacetime again but now through the use of a preferred spatial vector $e^a$ which is orthogonal to $u^a$. The vector $e^a$ satisfies

$$e^a e_a = 1, \tag{4.1}$$

and

$$e^a u_a = 0. \tag{4.2}$$

We now have a new projection tensor

$$N^a{}_b \equiv h^a{}_b - e^a e_b = g^a{}_b + u^a u_b - e^a e_b, \tag{4.3}$$

which projects vectors orthogonal to $u^a$ and $e^a$ onto 2-spaces called sheets. The sheets have the following volume element

$$\eta_{ab} \equiv \eta_{abc} e^c = u^d \eta_{dabc} e^c \implies \eta_{ab} e^b = 0 = \eta_{(ab)}. \tag{4.4}$$

We can now irreducibly split any 3-vector $\psi^a$ into a scalar, $\Psi$, which is the component of the vector parallel to $e^a$, and a 2-vector, $\Psi^a$, which lies in the sheet orthogonal to $e^a$:

$$\begin{aligned} \psi^a &= \Psi e^a + \Psi^a, \quad \text{where} \quad \Psi \equiv \psi_a e^a \ , \\ & \qquad \text{and} \quad \Psi^a \equiv N^{ab} \psi_b \equiv \psi^{\bar{a}}. \end{aligned} \tag{4.5}$$

Note that a bar over an index denotes projection with $N_{ab}$. Similarly for any 3-tensor, $\psi_{ab}$,

$$\psi_{ab} = \psi_{\langle ab \rangle} = \Psi \left( e_a e_b - \tfrac{1}{2} N_{ab} \right) + 2\Psi_{(a} e_{b)} + \Psi_{ab} \ , \tag{4.6}$$

where

$$\begin{aligned} \Psi &\equiv e^a e^b \psi_{ab} = -N^{ab} \psi_{ab} \ , \\ \Psi_a &\equiv N_a{}^b e^c \psi_{bc} = \Psi_{\bar{a}} \ , \\ \Psi_{ab} &\equiv \left( N_{(a}{}^c N_{b)}{}^d - \tfrac{1}{2} N_{ab} N^{cd} \right) \psi_{cd} \equiv \Psi_{\{ab\}} \ . \end{aligned} \tag{4.7}$$

We now define two new derivatives: the hat-derivative which is the derivative along the vector field $e^a$ in the surfaces orthogonal to $u^a$, and the $\delta$-derivative which is the derivative projected onto the sheet, with projection on all free indices. For any tensor

$\psi_{a...b}{}^{c...d}$, these are given by

$$\hat{\psi}_{a..b}{}^{c..d} \equiv e^f D_f \psi_{a..b}{}^{c..d} , \quad (4.8)$$

$$\delta_f \psi_{a..b}{}^{c..d} \equiv N_a{}^f ... N_b{}^g N_h{}^c .. N_i{}^d N_f{}^j D_j \psi_{f..g}{}^{i..j} . \quad (4.9)$$

Now using (4.5) and (4.6), the 1+3 kinematical and Weyl quantities can be split into the irreducible set $\{\Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}$ as follows:

$$\dot{u}^a = \mathcal{A} e^a + \mathcal{A}^a, \quad (4.10)$$

$$\omega^a = \Omega e^a + \Omega^a, \quad (4.11)$$

$$\sigma_{ab} = \Sigma \left( e_a e_b - \tfrac{1}{2} N_{ab} \right) + 2\Sigma_{(a} e_{b)} + \Sigma_{ab}, \quad (4.12)$$

$$E_{ab} = \mathcal{E} \left( e_a e_b - \tfrac{1}{2} N_{ab} \right) + 2\mathcal{E}_{(a} e_{b)} + \mathcal{E}_{ab}, \quad (4.13)$$

$$H_{ab} = \mathcal{H} \left( e_a e_b - \tfrac{1}{2} N_{ab} \right) + 2\mathcal{H}_{(a} e_{b)} + \mathcal{H}_{ab}. \quad (4.14)$$

The fluid variables, $q^a$ and $\pi_{ab}$, may similarly be split

$$q^a = Q e^a + Q^a, \quad (4.15)$$

$$\pi_{ab} = \Pi \left( e_a e_b - \tfrac{1}{2} N_{ab} \right) + 2\Pi_{(a} e_{b)} + \Pi_{ab}. \quad (4.16)$$

The covariant derivatives of $u^a$ and $e^a$ can now be decomposed into

$$\begin{aligned} \nabla_a u_b &= -u_a \left( \mathcal{A} e_b + \mathcal{A}_b \right) + e_a e_b \left( \tfrac{1}{3}\Theta + \Sigma \right) + \Omega \epsilon_{ab} \\ &+ e_a \left( \Sigma_b + \epsilon_{bc} \Omega^c \right) + e_b \left( \Sigma_a - \epsilon_{ac} \Omega^c \right) \\ &+ N_{ab} \left( \tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma \right) + \Sigma_{ab}, \quad (4.17) \\ \nabla_a e_b &= -\mathcal{A} u_a u_b - u_a \alpha_b + e_a u_b \left( \tfrac{1}{3}\Theta + \Sigma \right) + \xi \epsilon_{ab} \\ &+ u_b \left( \Sigma_a - \epsilon_{ac} \Omega^c \right) + e_a a_b + \tfrac{1}{2} \phi N_{ab} + \zeta_{ab}. \quad (4.18) \end{aligned}$$

## 4.3 Discussion

We have presented an overview of the 1+1+2 covariant formalism developed by Clarkson and Barrett (2003). Here the spacetime was split further through the use of a preferred spatial vector $e^a$ which is orthogonal to $u^a$. Using this formalism, we showed that any 3-vector can be irreducibly split into a scalar component parallel to $e^a$ and a 2-vector which lies in the sheet orthogonal to $e^a$. We defined two further derivatives: the hat-derivative which is the derivative along the vector field $e^a$ in the surfaces orthogonal to $u^a$, and the $\delta$-derivative which is the derivative projected onto the sheet, with projection on all free indices. We thereafter presented the decomposition into their irreducible parts of certain relevant quatities, such as the 1+3 kinematical and Weyl quantities and the covariant derivatives of $u^a$ and $e^a$.

# Chapter 5

# LRS-II spacetimes

## 5.1 Introduction

Locally Rotationally Symmetric (LRS) spacetimes were classified in Ellis (1967), Stewart and Ellis (1968) and van Elst and Ellis (1996). LRS spacetimes are defined by the property that, in an open neighbourhood of each point, there is a continuous subgroup of the Lorentz group which leaves invariant the Riemann tensor and its covariant derivatives to the third order (Kăspar, Vrba and Svítek 2014). Thus LRS spacetimes are spacetimes which have a unique preferred spatial direction at each point (Betschart and Clarkson, 2004). This direction results in a local axis of symmetry, such that all observations are identical under rotations about it (Betschart and Clarkson, 2004). Only scalar quantities are required to describe an LRS spacetime under the 1+1+2 formalism, as all 2-tensors and 2-vectors vanish, due to these spacetimes being isotropic about the axis of symmetry (Betschart and Clarkson, 2004). An LRS-II spacetime is a spacetime free of rotation (Acquaviva *et al*, 2015) as it has no vorticity terms (Betschart and Clarkson, 2004). The LRS-II quantities in (5.1) satisfy certain covariant propagation and/or evolution equations which are obtained from the Bianchi and Ricci identities for the vectors $u^a$ and $e^a$ (Betschart and Clarkson, 2004).

## 5.2 Set of variables

The set of variables that fully describe an LRS spacetime is $\{\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \Pi, Q\}$ (Betschart and Clarkson 2004, Acquaviva *et al* 2015). Since an LRS-II spacetime is free of rotation, this results in the variables $\xi$, $\Omega$ and $\mathcal{H}$ vanishing (Betschart and Clarkson 2004, Acquaviva *et al* 2015). Thus for LRS-II spacetimes, we have the following smaller set of variables which describe the spacetime,

$$\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}, \mu, p, \Pi, Q\}\,. \tag{5.1}$$

## 5.3 Field equations

From Betschart and Clarkson (2004) and Acquaviva *et al* (2015), the field equations are,

*Propagation*:

$$\begin{aligned}
\hat{\phi} &= -\tfrac{1}{2}\phi^2 + \left(\tfrac{1}{3}\Theta + \Sigma\right)\left(\tfrac{2}{3}\Theta - \Sigma\right) \\
&\quad -\tfrac{2}{3}\left(\mu + \Lambda\right) - \mathcal{E} - \tfrac{1}{2}\Pi, & (5.2)\\
\hat{\Sigma} - \tfrac{2}{3}\hat{\Theta} &= -\tfrac{3}{2}\phi\Sigma - Q\ , & (5.3)\\
\hat{\mathcal{E}} - \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\hat{\Pi} &= -\tfrac{3}{2}\phi\left(\mathcal{E} + \tfrac{1}{2}\Pi\right) + \left(\tfrac{1}{2}\Sigma - \tfrac{1}{3}\Theta\right)Q. & (5.4)
\end{aligned}$$

*Evolution*:

$$\begin{aligned}
\dot{\phi} &= -\left(\Sigma - \tfrac{2}{3}\Theta\right)\left(\mathcal{A} - \tfrac{1}{2}\phi\right) + Q\ , & (5.5)\\
\dot{\Sigma} - \tfrac{2}{3}\dot{\Theta} &= -\mathcal{A}\phi + 2\left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)^2 \\
&\quad +\tfrac{1}{3}\left(\mu + 3p - 2\Lambda\right) - \mathcal{E} + \tfrac{1}{2}\Pi\,, & (5.6)\\
\dot{\mathcal{E}} - \tfrac{1}{3}\dot{\mu} + \tfrac{1}{2}\dot{\Pi} &= +\left(\tfrac{3}{2}\Sigma - \Theta\right)\mathcal{E} + \tfrac{1}{4}\left(\Sigma - \tfrac{2}{3}\Theta\right)\Pi \\
&\quad +\tfrac{1}{2}\phi Q - \tfrac{1}{2}\left(\mu + p\right)\left(\Sigma - \tfrac{2}{3}\Theta\right)\ . & (5.7)
\end{aligned}$$

*Propagation/evolution*:

$$
\begin{aligned}
\hat{\mathcal{A}} - \dot{\Theta} &= -\left(\mathcal{A} + \phi\right)\mathcal{A} + \tfrac{1}{3}\Theta^2 + \tfrac{3}{2}\Sigma^2 \\
&\quad + \tfrac{1}{2}\left(\mu + 3p - 2\Lambda\right) \ , \qquad (5.8)\\
\dot{\mu} + \hat{Q} &= -\Theta\left(\mu + p\right) - \left(\phi + 2\mathcal{A}\right)Q - \tfrac{3}{2}\Sigma\Pi, \qquad (5.9)\\
\dot{Q} + \hat{p} + \hat{\Pi} &= -\left(\tfrac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\tfrac{4}{3}\Theta + \Sigma\right)Q \\
&\quad - \left(\mu + p\right)\mathcal{A} \ . \qquad (5.10)
\end{aligned}
$$

From Betschart and Clarkson (2004) and Acquaviva *et al* (2015) the Gaussian curvature $K$ of the 2-sheet, defined by ${}^2R_{ab} = KN_{ab}$, can be written in terms of the covariant scalars as

$$K = \tfrac{1}{3}\left(\mu + \Lambda\right) - \mathcal{E} - \tfrac{1}{2}\Pi + \tfrac{1}{4}\phi^2 - \left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)^2 \ . \qquad (5.11)$$

Betschart and Clarkson (2004) and Acquaviva *et al* (2015) state that the evolution and propagation equations for the Gaussian curvature $K$ are

$$
\begin{aligned}
\dot{K} &= -\left(\tfrac{2}{3}\Theta - \Sigma\right)K \ , \qquad (5.12)\\
\hat{K} &= -\phi K \ . \qquad (5.13)
\end{aligned}
$$

From Acquaviva *et al* (2015), the Misner-Sharp mass (which is the mass in a spherically symmetric region) for LRS-II under the 1+1+2 covariant formalism takes the form

$$\mathcal{M}(r,t) = \frac{1}{2K^{3/2}}\left(\frac{1}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi\right). \qquad (5.14)$$

## 5.4 Discussion

In this chapter we have defined what is an LRS spacetime and we have briefly described an important class of these spacetimes, namely LRS-II spacetimes. We showed that the set of variables that fully describe an LRS-II spacetime is $\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}, \mu, p, \Pi, Q\}$. Finally the field equations for LRS-II spacetimes were presented.

# Chapter 6

# Spacetime matching for LRS-II

We consider two general regions in LRS-II and match them, using the Israel-Darmois matching conditions, across a timelike hypersurface $S$, which will be referred to as the boundary.

Using (4.3), the metric tensor in Region 1 is given by

$$g_{ab} = -u_a u_b + e_a e_b + N_{ab}. \tag{6.1}$$

We shall denote Region 2 with a tilde. Thus its metric tensor is given by

$$\widetilde{g}_{ab} = -\widetilde{u}_a \widetilde{u}_b + \widetilde{e}_a \widetilde{e}_b + \widetilde{N}_{ab}. \tag{6.2}$$

Let $n_a$ be the unit normal in Region 1 to the matching timelike hypersurface $S$. It is given as

$$n_a = \alpha u_a + \beta e_a. \tag{6.3}$$

Similarly in Region 2, let $\widetilde{n}_a$ be the unit normal to $S$. Thus we have

$$\widetilde{n}_a = \widetilde{\alpha} \widetilde{u}_a + \widetilde{\beta} \widetilde{e}_a. \tag{6.4}$$

Since $S$ is timelike, like the boundary of a star, $n_a$ and $\widetilde{n}_a$ are spacelike. Making use of (6.1) and (6.3), we have that in Region 1, the first fundamental form is given by

$$\begin{aligned} h_{ab} &= g_{ab} - n_a n_b && (6.5)\\ &= -u_a u_b + e_a e_b + N_{ab} - \left[\alpha^2 u_a u_b + \beta^2 e_a e_b + \alpha\beta\left(u_a e_b + e_a u_b\right)\right] \\ &= -\left(1+\alpha^2\right) u_a u_b + \left(1-\beta^2\right) e_a e_b - \alpha\beta u_a e_b - \alpha\beta e_a u_b + N_{ab}. && (6.6) \end{aligned}$$

Likewise, using (6.2) and (6.4), the first fundamental form in Region 2 is

$$\begin{aligned} \widetilde{h}_{ab} &= \widetilde{g}_{ab} - \widetilde{n}_a \widetilde{n}_b && (6.7)\\ &= -\left(1+\widetilde{\alpha}^2\right) \widetilde{u}_a \widetilde{u}_b + \left(1-\widetilde{\beta}^2\right) \widetilde{e}_a \widetilde{e}_b - \widetilde{\alpha}\widetilde{\beta} \widetilde{u}_a \widetilde{e}_b - \widetilde{\alpha}\widetilde{\beta} \widetilde{e}_a \widetilde{u}_b + \widetilde{N}_{ab}. && (6.8) \end{aligned}$$

The second fundamental form in Region 1 is

$$K_{ab} = h^c{}_a h^d{}_b \nabla_d n_c. \tag{6.9}$$

Similarly in Region 2, the second fundamental form is

$$\widetilde{K}_{ab} = \widetilde{h}^c{}_a \widetilde{h}^d{}_b \nabla_d \widetilde{n}_c. \tag{6.10}$$

A summary of the 2 regions is shown on the next page in Table 5.1.

The Israel-Darmois matching conditions require the matching of the first and second fundamental forms on the boundary (Madhav, Goswami and Joshi, 2005), i.e.

$$\begin{aligned} h_{ab} &= \widetilde{h}_{ab}, && (6.11)\\ K_{ab} &= \widetilde{K}_{ab}. && (6.12) \end{aligned}$$

| **Region 1** | **Region 2** |
|---|---|
| $g_{ab} = -u_a u_b + e_a e_b + N_{ab}$ | $\widetilde{g}_{ab} = -\widetilde{u}_a \widetilde{u}_b + \widetilde{e}_a \widetilde{e}_b + \widetilde{N}_{ab}$ |
| Let $n_a$ be the unit normal to $S$. | Let $\widetilde{n}_a$ be the unit normal to $S$. |
| $n_a = \alpha u_a + \beta e_a$ | $\widetilde{n}_a = \widetilde{\alpha}\widetilde{u}_a + \widetilde{\beta}\widetilde{e}_a$ |
| Since $S$ is timelike, $n_a$ is spacelike. | Since $S$ is timelike, $\widetilde{n}_a$ is spacelike. |
| First fundamental form on the boundary<br>$h_{ab} = g_{ab} - n_a n_b$<br>$= -\left(1+\alpha^2\right) u_a u_b + \left(1-\beta^2\right) e_a e_b$<br>$-\alpha\beta u_a e_b - \alpha\beta e_a u_b + N_{ab}$ | First fundamental form on the boundary<br>$\widetilde{h}_{ab} = \widetilde{g}_{ab} - \widetilde{n}_a \widetilde{n}_b$<br>$= -\left(1+\widetilde{\alpha}^2\right) \widetilde{u}_a \widetilde{u}_b + \left(1-\widetilde{\beta}^2\right) \widetilde{e}_a \widetilde{e}_b$<br>$-\widetilde{\alpha}\widetilde{\beta}\widetilde{u}_a \widetilde{e}_b - \widetilde{\alpha}\widetilde{\beta}\widetilde{e}_a \widetilde{u}_b + \widetilde{N}_{ab}$ |
| Second fundamental form on the boundary<br>$K_{ab} = h^c{}_a h^d{}_b \nabla_d n_c$ (see (6.17) and (6.23)) | Second fundamental form on the boundary<br>$\widetilde{K}_{ab} = \widetilde{h}^c{}_a \widetilde{h}^d{}_b \nabla_d \widetilde{n}_c$ (see (6.18) and (6.27)) |

Table 6.1: A summary of Region 1 and Region 2.

Taking the covariant derivative of (6.3) yields

$$\begin{aligned} \nabla_d n_c &= \nabla_d \left(\alpha u_c + \beta e_c\right) \\ &= \left(\nabla_d \alpha\right) u_c + \alpha \left(\nabla_d u_c\right) + \left(\nabla_d \beta\right) e_c + \beta \left(\nabla_d e_c\right). \end{aligned} \tag{6.13}$$

Applying (4.17) and (4.18) to (6.13) gives

$$\begin{aligned} \nabla_d n_c &= \left(\nabla_d \alpha\right) u_c + \alpha \left[-\mathcal{A} u_d e_c + e_d e_c \left(\tfrac{1}{3}\Theta + \Sigma\right) + N_{cd}\left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)\right] \\ &+ \left(\nabla_d \beta\right) e_c + \beta \left[-\mathcal{A} u_c u_d + \left(\tfrac{1}{3}\Theta + \Sigma\right) e_d u_c + \tfrac{1}{2}\phi N_{cd}\right]. \end{aligned} \tag{6.14}$$

Using (6.3) and (6.5) gives

$$\begin{aligned} h^c{}_a &= g^c{}_a - n^c n_a \\ &= \delta^c{}_a - \left(\alpha u^c + \beta e^c\right)\left(\alpha u_a + \beta e_a\right) \\ &= \delta^c{}_a - \alpha^2 u^c u_a - \beta^2 e^c e_a - \alpha\beta u^c e_a - \beta\alpha e^c u_a. \end{aligned} \tag{6.15}$$

Now acting (6.15) on (6.14) yields

$$h^c{}_a \nabla_d n_c = \nabla_d n_a + \alpha^2 u_a \nabla_d \alpha - \beta^2 e_a \nabla_d \beta + \alpha\beta e_a \nabla_d \alpha - \alpha\beta u_a \nabla_d \beta. \tag{6.16}$$

From (6.9), we see that we need to act $h^d{}_b$, given by (6.15), on (6.16). Doing so, we obtain

$$\begin{aligned}
K_{ab} &= \nabla_b n_a + \alpha^2 u_a \nabla_b \alpha - \beta^2 e_a \nabla_b \beta + \alpha\beta e_a \nabla_b \alpha - \alpha\beta u_a \nabla_b \beta \\
&- \alpha^2 u_b u^d \nabla_d n_a - \alpha^4 u_b u_a u^d \nabla_d \alpha + \alpha^2\beta^2 u_b e_a u^d \nabla_d \beta \\
&- \alpha^3 \beta u_b e_a u^d \nabla_d \alpha + \alpha^3 \beta u_b u_a u^d \nabla_d \beta - \beta^2 e_b e^d \nabla_d n_a \\
&- \alpha^2 \beta^2 e_b u_a e^d \nabla_d \alpha + \beta^4 e_b e_a e^d \nabla_d \beta - \alpha\beta^3 e_b e_a e^d \nabla_d \alpha \\
&+ \alpha\beta^3 e_b u_a e^d \nabla_d \beta - \alpha\beta e_b u^d \nabla_d n_a - \alpha^3 \beta e_b u_a u^d \nabla_d \alpha \\
&+ \alpha\beta^3 e_b e_a u^d \nabla_d \beta - \alpha^2\beta^2 e_b e_a u^d \nabla_d \alpha + \alpha^2\beta^2 e_b u_a u^d \nabla_d \beta \\
&- \alpha\beta u_b e^d \nabla_d n_a - \alpha^3 \beta u_b u_a e^d \nabla_d \alpha + \alpha\beta^3 u_b e_a e^d \nabla_d \beta \\
&- \alpha^2\beta^2 u_b e_a e^d \nabla_d \alpha + \alpha^2\beta^2 u_b u_a e^d \nabla_d \beta. \qquad (6.17)
\end{aligned}$$

In a similar way we can obtain

$$\begin{aligned}
\widetilde{K}_{ab} &= \nabla_b \widetilde{n}_a + \widetilde{\alpha}^2 \widetilde{u}_a \nabla_b \widetilde{\alpha} - \widetilde{\beta}^2 \widetilde{e}_a \nabla_b \widetilde{\beta} + \widetilde{\alpha}\widetilde{\beta} \widetilde{e}_a \nabla_b \widetilde{\alpha} - \widetilde{\alpha}\widetilde{\beta} \widetilde{u}_a \nabla_b \widetilde{\beta} \\
&- \widetilde{\alpha}^2 \widetilde{u}_b \widetilde{u}^d \nabla_d \widetilde{n}_a - \widetilde{\alpha}^4 \widetilde{u}_b \widetilde{u}_a \widetilde{u}^d \nabla_d \widetilde{\alpha} + \widetilde{\alpha}^2 \widetilde{\beta}^2 \widetilde{u}_b \widetilde{e}_a \widetilde{u}^d \nabla_d \widetilde{\beta} \\
&- \widetilde{\alpha}^3 \widetilde{\beta} \widetilde{u}_b \widetilde{e}_a \widetilde{u}^d \nabla_d \widetilde{\alpha} + \widetilde{\alpha}^3 \widetilde{\beta} \widetilde{u}_b \widetilde{u}_a \widetilde{u}^d \nabla_d \widetilde{\beta} - \widetilde{\beta}^2 \widetilde{e}_b \widetilde{e}^d \nabla_d \widetilde{n}_a \\
&- \widetilde{\alpha}^2 \widetilde{\beta}^2 \widetilde{e}_b \widetilde{u}_a \widetilde{e}^d \nabla_d \widetilde{\alpha} + \widetilde{\beta}^4 \widetilde{e}_b \widetilde{e}_a \widetilde{e}^d \nabla_d \widetilde{\beta} - \widetilde{\alpha} \widetilde{\beta}^3 \widetilde{e}_b \widetilde{e}_a \widetilde{e}^d \nabla_d \widetilde{\alpha} \\
&+ \widetilde{\alpha} \widetilde{\beta}^3 \widetilde{e}_b \widetilde{u}_a \widetilde{e}^d \nabla_d \widetilde{\beta} - \widetilde{\alpha}\widetilde{\beta} \widetilde{e}_b \widetilde{u}^d \nabla_d \widetilde{n}_a - \widetilde{\alpha}^3 \widetilde{\beta} \widetilde{e}_b \widetilde{u}_a \widetilde{u}^d \nabla_d \widetilde{\alpha} \\
&+ \widetilde{\alpha} \widetilde{\beta}^3 \widetilde{e}_b \widetilde{e}_a \widetilde{u}^d \nabla_d \widetilde{\beta} - \widetilde{\alpha}^2 \widetilde{\beta}^2 \widetilde{e}_b \widetilde{e}_a \widetilde{u}^d \nabla_d \widetilde{\alpha} + \widetilde{\alpha}^2 \widetilde{\beta}^2 \widetilde{e}_b \widetilde{u}_a \widetilde{u}^d \nabla_d \widetilde{\beta} \\
&- \widetilde{\alpha}\widetilde{\beta} \widetilde{u}_b \widetilde{e}^d \nabla_d \widetilde{n}_a - \widetilde{\alpha}^3 \widetilde{\beta} \widetilde{u}_b \widetilde{u}_a \widetilde{e}^d \nabla_d \widetilde{\alpha} + \widetilde{\alpha} \widetilde{\beta}^3 \widetilde{u}_b \widetilde{e}_a \widetilde{e}^d \nabla_d \widetilde{\beta} \\
&- \widetilde{\alpha}^2 \widetilde{\beta}^2 \widetilde{u}_b \widetilde{e}_a \widetilde{e}^d \nabla_d \widetilde{\alpha} + \widetilde{\alpha}^2 \widetilde{\beta}^2 \widetilde{u}_b \widetilde{u}_a \widetilde{e}^d \nabla_d \widetilde{\beta}. \qquad (6.18)
\end{aligned}$$

For our purposes we require $K_{ab}$ in terms of $u_a u_b$, $e_a e_b$, $u_a e_b$, $e_a u_b$ and $N_{ab}$. We likewise require $\widetilde{K}_{ab}$ in terms of $\widetilde{u}_a \widetilde{u}_b$, $\widetilde{e}_a \widetilde{e}_b$, $\widetilde{u}_a \widetilde{e}_b$, $\widetilde{e}_a \widetilde{u}_b$ and $\widetilde{N}_{ab}$. Note that we can write

$$\nabla_a \lambda = -\dot{\lambda} u_a + \hat{\lambda} e_a, \qquad (6.19)$$

where $\lambda$ is any scalar function such as $\alpha$ or $\beta$. Acting on (6.19) with $u^a$ gives

$$u^a \nabla_a \lambda = \dot{\lambda}, \tag{6.20}$$

while acting on (6.19) with $e^a$ gives

$$e^a \nabla_a \lambda = \hat{\lambda}. \tag{6.21}$$

Using (6.14), (6.19), (6.20) and (6.21), we can write $\nabla_b n_a$ in (6.17) as

$$\begin{aligned} \nabla_b n_a &= \left(-\dot{\alpha} - \beta\mathcal{A}\right) u_a u_b + \left[\alpha\left(\tfrac{1}{3}\Theta + \Sigma\right) + \hat{\beta}\right] e_a e_b + \left[\beta\left(\tfrac{1}{3}\Theta + \Sigma\right) + \hat{\alpha}\right] u_a e_b \\ &+ \left(-\dot{\beta} - \alpha\mathcal{A}\right) e_a u_b + \left[\alpha\left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right) + \tfrac{1}{2}\beta\phi\right] N_{ab}. \end{aligned} \tag{6.22}$$

We can now proceed and re-write (6.17). Using (3.1), (4.1), (4.2), (4.10) (noting that $\mathcal{A}^a = 0$), (6.19), (6.20), (6.21) and (6.22), (6.17) becomes

$$\begin{aligned} K_{ab} &= u_a u_b \Big[ -\left(1+\alpha^2\right)^2 \dot{\alpha} + \alpha\beta\left(1+\alpha^2\right)\dot{\beta} - \alpha\beta\left(1+\alpha^2\right)\hat{\alpha} \\ &+ \alpha^2\beta^2\hat{\beta} - \beta\mathcal{A}\left(1+\alpha^2\right) - \alpha\beta^2\left(\tfrac{1}{3}\Theta + \Sigma\right)\Big] + e_a e_b \Big[ -\alpha^2\beta^2\dot{\alpha} \\ &- \alpha\beta\left(1-\beta^2\right)\dot{\beta} + \alpha\beta\left(1-\beta^2\right)\hat{\alpha} + \left(1-\beta^2\right)^2\hat{\beta} \\ &+ \alpha\left(1-\beta^2\right)\left(\tfrac{1}{3}\Theta + \Sigma\right) - \alpha^2\beta\mathcal{A}\Big] + u_a e_b \Big[ -\alpha\beta\left(1+\alpha^2\right)\dot{\alpha} \\ &+ \alpha^2\beta^2\dot{\beta} + \left(1+\alpha^2\right)\left(1-\beta^2\right)\hat{\alpha} - \alpha\beta\left(1-\beta^2\right)\hat{\beta} \\ &+ \beta\left(1-\beta^2\right)\left(\tfrac{1}{3}\Theta + \Sigma\right) - \alpha\beta^2\mathcal{A}\Big] + e_a u_b \Big[ -\alpha\beta\left(1+\alpha^2\right)\dot{\alpha} \\ &- \alpha^2\beta^2\hat{\alpha} - \left(1+\alpha^2\right)\left(1-\beta^2\right)\dot{\beta} - \alpha\beta\left(1-\beta^2\right)\hat{\beta} - \alpha\mathcal{A}\left(1+\alpha^2\right) \\ &- \alpha^2\beta\left(\tfrac{1}{3}\Theta + \Sigma\right)\Big] + N_{ab}\Big[\alpha\left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right) + \tfrac{1}{2}\beta\phi\Big]. \end{aligned} \tag{6.23}$$

We can obtain a similar result for Region 2. First we need to replace the dot and hat derivative with the circle and bar derivative respectively. The circle derivative arises from using the operator $\widetilde{u}^a \nabla_a$ while the bar derivative arises from using the operator

$\widetilde{e}^a D_a$. Thus for Region 2 (6.19), (6.20) and (6.21) become

$$\nabla_a \widetilde{\lambda} = -\mathring{\widetilde{\lambda}} \widetilde{u}_a + \bar{\widetilde{\lambda}} \widetilde{e}_a, \tag{6.24}$$

$$\widetilde{u}^a \nabla_a \widetilde{\lambda} = \mathring{\widetilde{\lambda}}, \tag{6.25}$$

and

$$\widetilde{e}^a \nabla_a \widetilde{\lambda} = \bar{\widetilde{\lambda}} \tag{6.26}$$

respectively. Taking into account (6.24), (6.25), (6.26) and the process in going from (6.17) to (6.23), (6.18) can be rewritten as

$$\begin{aligned}
\widetilde{K}_{ab} &= \widetilde{u}_a \widetilde{u}_b \Big[ -\left(1+\widetilde{\alpha}^2\right)^2 \mathring{\widetilde{\alpha}} + \widetilde{\alpha}\widetilde{\beta}\left(1+\widetilde{\alpha}^2\right) \mathring{\widetilde{\beta}} - \widetilde{\alpha}\widetilde{\beta}\left(1+\widetilde{\alpha}^2\right) \bar{\widetilde{\alpha}} \\
&+ \widetilde{\alpha}^2 \widetilde{\beta}^2 \bar{\widetilde{\beta}} - \widetilde{\beta}\widetilde{\mathcal{A}}\left(1+\widetilde{\alpha}^2\right) - \widetilde{\alpha}\widetilde{\beta}^2 \left(\tfrac{1}{3}\widetilde{\Theta} + \widetilde{\Sigma}\right) \Big] + \widetilde{e}_a \widetilde{e}_b \Big[ -\widetilde{\alpha}^2 \widetilde{\beta}^2 \mathring{\widetilde{\alpha}} \\
&- \widetilde{\alpha}\widetilde{\beta}\left(1-\widetilde{\beta}^2\right) \mathring{\widetilde{\beta}} + \widetilde{\alpha}\widetilde{\beta}\left(1-\widetilde{\beta}^2\right) \bar{\widetilde{\alpha}} + \left(1-\widetilde{\beta}^2\right)^2 \bar{\widetilde{\beta}} \\
&+ \widetilde{\alpha}\left(1-\widetilde{\beta}^2\right)\left(\tfrac{1}{3}\widetilde{\Theta} + \widetilde{\Sigma}\right) - \widetilde{\alpha}^2 \widetilde{\beta}\widetilde{\mathcal{A}} \Big] + \widetilde{u}_a \widetilde{e}_b \Big[ -\widetilde{\alpha}\widetilde{\beta}\left(1+\widetilde{\alpha}^2\right) \mathring{\widetilde{\alpha}} \\
&+ \widetilde{\alpha}^2 \widetilde{\beta}^2 \mathring{\widetilde{\beta}} + \left(1+\widetilde{\alpha}^2\right)\left(1-\widetilde{\beta}^2\right) \bar{\widetilde{\alpha}} - \widetilde{\alpha}\widetilde{\beta}\left(1-\widetilde{\beta}^2\right) \bar{\widetilde{\beta}} \\
&+ \widetilde{\beta}\left(1-\widetilde{\beta}^2\right)\left(\tfrac{1}{3}\widetilde{\Theta} + \widetilde{\Sigma}\right) - \widetilde{\alpha}\widetilde{\beta}^2 \widetilde{\mathcal{A}} \Big] + \widetilde{e}_a \widetilde{u}_b \Big[ -\widetilde{\alpha}\widetilde{\beta}\left(1+\alpha^2\right) \mathring{\widetilde{\alpha}} \\
&- \widetilde{\alpha}^2 \widetilde{\beta}^2 \bar{\widetilde{\alpha}} - \left(1+\widetilde{\alpha}^2\right)\left(1-\widetilde{\beta}^2\right) \mathring{\widetilde{\beta}} - \widetilde{\alpha}\widetilde{\beta}\left(1-\widetilde{\beta}^2\right) \bar{\widetilde{\beta}} - \widetilde{\alpha}\widetilde{\mathcal{A}}\left(1+\widetilde{\alpha}^2\right) \\
&- \widetilde{\alpha}^2 \widetilde{\beta}\left(\tfrac{1}{3}\widetilde{\Theta} + \widetilde{\Sigma}\right) \Big] + \widetilde{N}_{ab} \Big[ \widetilde{\alpha}\left(\tfrac{1}{3}\widetilde{\Theta} - \tfrac{1}{2}\widetilde{\Sigma}\right) + \tfrac{1}{2}\widetilde{\beta}\widetilde{\phi} \Big].
\end{aligned} \tag{6.27}$$

Keeping with the symmetry of LRS spacetimes, let us assume that

$$N_{ab} = \widetilde{N}_{ab}, \tag{6.28}$$

so that the spherical 2-surfaces are the same on the boundary. Applying (6.28) on (6.12) (this gives $N^{ab} K_{ab} = \widetilde{N}^{ab} \widetilde{K}_{ab}$) and using (6.23) and (6.27), gives the first matching condition (6.29).

Using (6.11), (6.12) and (6.28) gives the valid result, $(h_{ab} - N_{ab}) K^{ab} = (\widetilde{h}_{ab} - \widetilde{N}_{ab}) \widetilde{K}^{ab}$, which can be computed using (6.6), (6.8), (6.23) and (6.27), to give the second matching condition (6.30).

The matching conditions for LRS-II spacetimes become

$$\begin{aligned} \beta\phi - \alpha\left(\Sigma - \tfrac{2}{3}\Theta\right) &= \widetilde{\beta}\widetilde{\phi} - \widetilde{\alpha}\left(\widetilde{\Sigma} - \tfrac{2}{3}\widetilde{\Theta}\right), & (6.29)\\ f_1\dot{\alpha} - \alpha\beta f_2\dot{\beta} + \alpha\beta f_2\hat{\alpha} + f_3\hat{\beta} &= \widetilde{f}_1\mathring{\widetilde{\alpha}} - \widetilde{\alpha}\widetilde{\beta}\widetilde{f}_2\mathring{\widetilde{\beta}} + \widetilde{\alpha}\widetilde{\beta}\widetilde{f}_2\bar{\widetilde{\alpha}} + \widetilde{f}_3\bar{\widetilde{\beta}}, & (6.30) \end{aligned}$$

where

$$f_1 = \left(1+\alpha^2\right)^3 - \alpha^2\beta^2\left(1-\beta^2\right) - 2\alpha^2\beta^2\left(1+\alpha^2\right), \tag{6.31}$$

$$f_2 = \left(1+\alpha^2\right)^2 + \left(1-\beta^2\right)^2 + \left(1+\alpha^2\right)\left(1-\beta^2\right) - \alpha^2\beta^2, \tag{6.32}$$

$$f_3 = \left(1-\beta^2\right)^3 - \alpha^2\beta^2\left(1+\alpha^2\right) - 2\alpha^2\beta^2\left(1-\beta^2\right), \tag{6.33}$$

$$\widetilde{f}_1 = \left(1+\widetilde{\alpha}^2\right)^3 - \widetilde{\alpha}^2\widetilde{\beta}^2\left(1-\widetilde{\beta}^2\right) - 2\widetilde{\alpha}^2\widetilde{\beta}^2\left(1+\widetilde{\alpha}^2\right), \tag{6.34}$$

$$\widetilde{f}_2 = \left(1+\widetilde{\alpha}^2\right)^2 + \left(1-\widetilde{\beta}^2\right)^2 + \left(1+\widetilde{\alpha}^2\right)\left(1-\widetilde{\beta}^2\right) - \widetilde{\alpha}^2\widetilde{\beta}^2, \tag{6.35}$$

$$\widetilde{f}_3 = \left(1-\widetilde{\beta}^2\right)^3 - \widetilde{\alpha}^2\widetilde{\beta}^2\left(1+\widetilde{\alpha}^2\right) - 2\widetilde{\alpha}^2\widetilde{\beta}^2\left(1-\widetilde{\beta}^2\right). \tag{6.36}$$

Thus after using the Israel-Darmois matching conditions to match two general spacetime regions in LRS-II across a timelike hypersurface using the semi-tetrad 1+1+2 covariant formalism, we have found the new and unique matching conditions for LRS-II spacetimes.

# Chapter 7

# An example of LRS-II spacetime matching

## 7.1 Introduction

The matching condition, (6.29), needs to be tested to see whether it makes sense. We apply the matching condition to an example involving the matching of a general spherically symmetric spacetime (the interior spacetime), and the Schwarzschild spacetime (the exterior spacetime) across a timelike hypersurface. This represents a spherically symmetric star immersed in vacuum.

## 7.2 Example

From Madhav, Goswami and Joshi (2005), the metric for a general spherically symmetric spacetime is given as

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)\,d\Omega^2, \tag{7.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the two-sphere line element. The Schwarzschild metric is given as

$$ds^2 = -\left(1 - \tfrac{2M}{\mathcal{R}}\right) dT^2 + \left(1 - \tfrac{2M}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 + \mathcal{R}^2 d\Omega^2, \tag{7.2}$$

At the boundary they become

$$ds_-^2 = -e^{2\nu(t,r)} dt^2 + R^2(t,r)\, d\Omega^2, \tag{7.3}$$

and

$$ds_+^2 = -\left(1 - \tfrac{2M}{\mathcal{R}}\right) dT^2 + \left(1 - \tfrac{2M}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 + \mathcal{R}^2 d\Omega^2, \tag{7.4}$$

respectively. From (7.1), we obtain using (3.1) and (4.1) that

$$u^a = \left(e^{-\nu}, 0, 0, 0\right), \tag{7.5}$$

and

$$e^a = \left(0, e^{-\psi}, 0, 0\right). \tag{7.6}$$

The normal to the boundary hypersurface for the interior spacetime is given by

$$n_r^- = \left(0, e^{\psi}, 0, 0\right). \tag{7.7}$$

Using (7.1) and (7.7), we find that for the interior spacetime

$$n_a = e_a. \tag{7.8}$$

Thus from (6.3)

$$\alpha = 0, \beta = 1, \tag{7.9}$$

for the interior. Thus from (6.29), for the interior spacetime we have $\phi$. Now $\phi$ is given by

$$\phi = N^{ab} D_a e_b. \tag{7.10}$$

From (3.5)

$$D_a e_b = h^c{}_a h^d{}_b \nabla_c e_d. \tag{7.11}$$

Since $h^a{}_b = \delta^a{}_b$ for the interior spacetime and

$$\nabla_a e_b = \delta_a e_b - \Gamma^i{}_{ab} e_i, \tag{7.12}$$

(7.11) becomes

$$D_a e_b = \delta_a e_b - \Gamma^i{}_{ab} e_i. \tag{7.13}$$

We have from (4.1) and (7.6) that

$$e_a = \left(0, e^{\psi}, 0, 0\right). \tag{7.14}$$

Thus $a = 1$ gives the only nonzero value in (7.14). Therefore $i = 1$ in (7.13), which now becomes

$$D_a e_b = \delta_a e_b - \Gamma^1{}_{ab} e_1. \tag{7.15}$$

Using (4.3), (7.1), (7.5) and (7.6), we obtain

$$N^{00} = 0, \tag{7.16}$$

$$N^{11} = 0, \tag{7.17}$$

$$N^{22} = \frac{1}{R^2}, \tag{7.18}$$

$$N^{33} = \frac{1}{R^2 \sin^2\theta}. \tag{7.19}$$

Using (7.14), (7.15), (7.16), (7.17), (7.18) and (7.19), (7.10) yields

$$\begin{aligned}
\phi &= N^{ab} D_a e_b \\
&= N^{ab}\left(\delta_a e_b - \Gamma^1{}_{ab} e_1\right) \\
&= N^{22} D_2 e_2 + N^{33} D_3 e_3 \\
&= N^{22}\left(\delta_2 e_2 - \Gamma^1{}_{22} e_1\right) + N^{33}\left(\delta_3 e_3 - \Gamma^1{}_{33} e_1\right) \\
&= N^{22}\left(-\Gamma^1{}_{22} e_1\right) + N^{33}\left(-\Gamma^1{}_{33} e_1\right) \\
&= \tfrac{1}{R^2}\left(e^{-2\psi} R \tfrac{\partial R}{\partial r}\right) e^{\psi} + \tfrac{1}{R^2 \sin^2\theta}\left(e^{-2\psi} R \sin^2\theta \tfrac{\partial R}{\partial r}\right) e^{\psi} \\
&= \tfrac{1}{R} e^{-\psi} \tfrac{\partial R}{\partial r} + \tfrac{1}{R} e^{-\psi} \tfrac{\partial R}{\partial r} \\
&= \tfrac{2}{R} e^{-\psi} \tfrac{\partial R}{\partial r} = \tfrac{2R'}{R} e^{-\psi}. \qquad (7.20)
\end{aligned}$$

At the boundary, noting that $\mathcal{R} \equiv \mathcal{R}(\mathcal{T})$, (7.1) and (7.2) become

$$ds_-^2 = -e^{2\nu} dt^2 + R^2 d\Omega^2, \qquad (7.21)$$

and

$$ds_+^2 = -dT^2 \left( \left[1 - \tfrac{2M}{\mathcal{R}(T)}\right] - \tfrac{(\mathcal{R}_{,T})^2}{1 - \frac{2M}{\mathcal{R}(T)}} \right) + \mathcal{R}(T)^2 d\Omega^2, \qquad (7.22)$$

respectively. From (7.21) and (7.22), we find that

$$\frac{dT}{dt} = \frac{e^{\nu}}{\sqrt{\left[1 - \frac{2M}{\mathcal{R}(T)}\right] - \frac{(\mathcal{R}_{,T})^2}{1 - \frac{2M}{\mathcal{R}(T)}}}}. \qquad (7.23)$$

At the boundary

$$\mathcal{R}(T) = R(t),$$

implying that

$$\mathcal{R}_{,t} = R_{,t} = \dot{R}. \qquad (7.24)$$

Note that $\mathcal{R}_{,T} = \mathcal{R}_{,t}\left(\frac{dt}{dT}\right)$, which after using (7.24), becomes

$$\mathcal{R}_{,T} = \dot{R}\left(\frac{dt}{dT}\right). \tag{7.25}$$

Letting $\mathcal{D} = 1 - \frac{2M}{\mathcal{R}(T)}$ and using (7.25), we obtain from (7.23)

$$\frac{dT}{dt} = \frac{\sqrt{e^{2\nu}D + \dot{R}^2}}{\mathcal{D}} = e^{\nu}\frac{\sqrt{D + e^{-2\nu}\dot{R}^2}}{\mathcal{D}}. \tag{7.26}$$

The normal to the boundary hypersurface for the exterior spacetime is given by

$$n_r^+ = \left(0, \tfrac{dT}{dt}e^{-\nu}, 0, 0\right),$$

which we can write as

$$n^r = \left(0, \left(1 - \tfrac{2M}{\mathcal{R}}\right)\tfrac{dT}{dt}e^{-\nu}, 0, 0\right). \tag{7.27}$$

For the exterior spacetime

$$e^r = \left(0, \sqrt{1 - \tfrac{2M}{\mathcal{R}}}, 0, 0\right). \tag{7.28}$$

Since

$$n^r = \beta e^r, \tag{7.29}$$

for the exterior spacetime, we have from (7.27) and (7.28) that

$$\widetilde{\beta} = \sqrt{1 - \tfrac{2M}{\mathcal{R}}}\tfrac{dT}{dt}e^{-\nu}. \tag{7.30}$$

Thus

$$\widetilde{\beta}\widetilde{\phi} = \sqrt{1 - \tfrac{2M}{\mathcal{R}}}\tfrac{dT}{dt}e^{-\nu}\tfrac{2}{\mathcal{R}}\sqrt{1 - \tfrac{2M}{\mathcal{R}}} \tag{7.31}$$

$$= \mathcal{D}\tfrac{2}{\mathcal{R}}\tfrac{dT}{dt}e^{-\nu}, \tag{7.32}$$

where $\widetilde{\phi} = \frac{2}{\mathcal{R}}\sqrt{1 - \frac{2M}{\mathcal{R}}}$ is calculated in a similar manner as $\phi$. From (6.29), our matching condition for this example is

$$\phi = \widetilde{\beta}\widetilde{\phi}. \tag{7.33}$$

Applying (7.33), using (7.20) and (7.32), gives

$$\mathcal{D} + e^{-2\nu}\dot{R}^2 = R'^2 e^{-2\psi}, \tag{7.34}$$

which implies that

$$1 - \tfrac{2M}{R} = R'^2 e^{-2\psi} - \dot{R}^2 e^{-2\nu}. \tag{7.35}$$

But from Madhav, Goswami and Joshi (2005), $R'^2 e^{-2\psi} - \dot{R}^2 e^{-2\nu} = 1 - \frac{2\mathcal{M}}{R}$. Therefore (7.35) becomes

$$1 - \tfrac{2M}{R} = 1 - \tfrac{2\mathcal{M}}{R},$$

which implies that

$$\mathcal{M} = M = constant, \tag{7.36}$$

which further implies that

$$\dot{\mathcal{M}} = 0. \tag{7.37}$$

Thus (7.36) shows that the Misner-Sharp and Schwarzschild masses are exactly the same at the boundary. Equation (7.37) has physical significance as well. Using the field equations it can be shown that

$$\dot{\mathcal{M}} = -p\frac{\dot{K}}{K^{\frac{5}{2}}}. \tag{7.38}$$

Now (7.37) and (7.38) imply that

$$p = 0. \tag{7.39}$$

Hence the pressure on the boundary must be zero.

## 7.3 Discussion

In this chapter we have tested our matching conditions for LRS-II spacetimes with a simple example: the matching of a general spherically symmetric spacetime and the exterior Schwarzschild spacetime, which represented a spherically symmetric star immersed in vacuum. We found in this example that the Misner-Sharp and Schwarzschild masses are exactly the same at the boundary, and that the pressure at the boundary vanishes. This is pleasing as these are known results.

# Chapter 8

# Discussion

In this thesis we have outined the main results regarding general matching conditions in general relativity. We have briefly explained the 1+3 and 1+1+2 covariant formalisms and given a description of LRS-II spacetimes. We have employed the Israel-Darmois matching conditions to match two general spacetime regions in LRS-II across a timelike hypersurface using the semi-tetrad 1+1+2 covariant formalism, which gave us the new and unique matching conditions for LRS-II spacetimes. We thereafter applied our new result to a simple example: the matching of a general spherically symmetric spacetime and the exterior Schwarzschild spacetime, which represented a spherically symmetric star immersed in vacuum. This particular example leads to known results: the Misner-Sharp and Schwarzschild masses are exactly the same at the boundary (Misner and Sharp, 1964), and the pressure at the boundary is zero.

The matching conditions found for LRS-II spacetimes are

$$\beta\phi - \alpha\left(\Sigma - \tfrac{2}{3}\Theta\right) = \widetilde{\beta}\widetilde{\phi} - \widetilde{\alpha}\left(\widetilde{\Sigma} - \tfrac{2}{3}\widetilde{\Theta}\right), \tag{8.1}$$

$$f_1\dot{\alpha} - \alpha\beta f_2\dot{\beta} + \alpha\beta f_2\hat{\alpha} + f_3\hat{\beta} = \widetilde{f}_1\mathring{\widetilde{\alpha}} - \widetilde{\alpha}\widetilde{\beta}\widetilde{f}_2\mathring{\widetilde{\beta}} + \widetilde{\alpha}\widetilde{\beta}\widetilde{f}_2\bar{\widetilde{\alpha}} + \widetilde{f}_3\bar{\widetilde{\beta}}, \tag{8.2}$$

where

$$f_1 = \left(1+\alpha^2\right)^3 - \alpha^2\beta^2\left(1-\beta^2\right) - 2\alpha^2\beta^2\left(1+\alpha^2\right), \tag{8.3}$$

$$f_2 = \left(1+\alpha^2\right)^2 + \left(1-\beta^2\right)^2 + \left(1+\alpha^2\right)\left(1-\beta^2\right) - \alpha^2\beta^2, \tag{8.4}$$

$$f_3 = \left(1-\beta^2\right)^3 - \alpha^2\beta^2\left(1+\alpha^2\right) - 2\alpha^2\beta^2\left(1-\beta^2\right), \tag{8.5}$$

$$\widetilde{f}_1 = \left(1+\widetilde{\alpha}^2\right)^3 - \widetilde{\alpha}^2\widetilde{\beta}^2\left(1-\widetilde{\beta}^2\right) - 2\widetilde{\alpha}^2\widetilde{\beta}^2\left(1+\widetilde{\alpha}^2\right), \tag{8.6}$$

$$\widetilde{f}_2 = \left(1+\widetilde{\alpha}^2\right)^2 + \left(1-\widetilde{\beta}^2\right)^2 + \left(1+\widetilde{\alpha}^2\right)\left(1-\widetilde{\beta}^2\right) - \widetilde{\alpha}^2\widetilde{\beta}^2, \tag{8.7}$$

$$\widetilde{f}_3 = \left(1-\widetilde{\beta}^2\right)^3 - \widetilde{\alpha}^2\widetilde{\beta}^2\left(1+\widetilde{\alpha}^2\right) - 2\widetilde{\alpha}^2\widetilde{\beta}^2\left(1-\widetilde{\beta}^2\right). \tag{8.8}$$

We emphasize that these matching conditions have been derived for the first time in this research.

The first matching condition (8.1), is a relation which contains the gravitational potential, shear and expansion terms. It thus has physical meaning as it describes the geometry of the spacetimes concerned. The second matching condition, (8.2), only has terms involving $\alpha$ and $\beta$, as such it describes the geometry of the hypersurface. In cases where $\alpha$, $\beta$, $\widetilde{\alpha}$ and $\widetilde{\beta}$ are constant, (8.1) should be used, as (8.2) reduces to zero. The example in chapter 6 validated the matching condition (8.1). For situations where $\alpha$, $\beta$, $\widetilde{\alpha}$ and $\widetilde{\beta}$ are not constant, the hypersurface geometry must be considered and thus (8.2) needs to be taken into account. This has not been studied in this thesis, but has been left open for further research.

Our example in chapter 6 represented a spherically symmetric star in vacuum. This is a very idealised situation, and in reality the star would be emitting radiation and matter and could be absorbing matter from the nonempty external spacetime which is no longer a perfect vacuum. In such a radiative case the Schwarschild spacetime would have to be replaced with the Vaidya spacetime. This is a possible next step to extend the matching conditions we've found. Fayos, Senovilla and Torres (1996) matched the Friedmann-Lemaître-Robertson-Walker and Vaidya spacetimes. This could be an interesting case to consider for our matching conditons. Another very interesting problem is that of gravitational collapse. The idealised problem of gravitational collapse was studied by Oppenheimer and Snyder (1939) but other more realistic gravitational collapse models such as that done by Glass (1981) could be studied as well. The remarkable LRS-II matching conditions found in this thesis could possibly also be applied to brane-world cosmology. There are numerous applications for the LRS-II matching conditions, (8.1) and (8.2), which it is hoped will be employed to examine more complex physical problems.

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