

UNIVERSITY OF KWAZULU-NATAL

MASTERS DISSERTATION

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**Open-Loop Nash and Stackelberg  
Equilibrium for Non-Cooperative  
Differential Games with Pure State  
Constraints**

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*A dissertation submitted in fulfilment of the requirements  
for the degree of Master of Science (Applied Mathematics)*

*in the*

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Kristina JOSEPH

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Date

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Dr Jean Medard T NGNOTCHOUYE

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Date

*“To acquire knowledge, one must study; but to acquire wisdom, one must observe.”*

Marilyn vos Savant

UNIVERSITY OF KWAZULU-NATAL  
College of Agriculture, Engineering and Science  
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**Open-Loop Nash and Stackelberg Equilibrium  
for Non-Cooperative Differential Games with  
Pure State Constraints**

by Kristina JOSEPH

*Abstract*

This dissertation deals with the study of non-cooperative differential games with pure state constraints. We propose a first order optimality result for the open-loop Nash and Stackelberg equilibria solutions of an  $N$ -player differential game with pure state constraints. A numerical method to solve for an open-loop Nash equilibrium is discussed. Numerical examples are included to illustrate this method.

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# Abbreviations

<b>BVP</b>	<b>B</b> oundary <b>V</b> alue <b>P</b> roblem
<b>COP</b>	<b>C</b> onstrained <b>O</b> ptimization <b>P</b> roblem
<b>KKT</b>	<b>K</b> arush - <b>K</b> uhn - <b>T</b> ucker
<b>OCP</b>	<b>O</b> ptimal <b>C</b> ontrol <b>P</b> roblem
<b>ODE</b>	<b>O</b> rdinary <b>D</b> ifferential <b>E</b> quation
<b>RK4</b>	<b>R</b> unge- <b>K</b> utta 4

# Physical Constants

# Symbols

# Chapter 1

## Introduction

This dissertation deals with the study of differential games with pure state constraints. The theory of a differential game is concerned with multiple player decision making. A differential game comprises of, a dynamic decision process changing continuously in time, the players, their respective payoff (cost) functions and the information that the players may have access to.

Differential game theory can be considered as an extension of optimal control theory [1]. Optimal control theory is concerned with the optimization of a single objective function for one control variable only, whereas in an  $N$ -player differential game each player optimizes his own payoff function subject to the dynamics of the game which is given by a differential equation.

The study of differential games was first introduced by Isaacs [2] in 1954. In the early development of the theory, applications of pursuit-evasion type problems were popular. In the 1970's, an interest in the applications of differential games in economics developed. Differential game theory has since been applied to a large array of topics in economics, such as macroeconomics, microeconomics, industrial organization, oligopoly theory, resource and environmental economics, labour economics, marketing, production, and operations management, finance, and much more [3–6].

Primarily, the importance of differential games lies in military type applications such as, missile guidance, aircraft controls and aerial tactics [7]. A lot of research has been done on the application of differential games in warfare and pursuit [2, 8, 9].

Noteworthy pioneers in the development of differential game theory include Friedman [10], Başar and Olsder [11], Fudenberg and Tirole [12]. Most of this literature shows us that differential games started off as an extension of optimal control theory [13]. The connection between differential game theory and optimal control theory was clarified by the study of the Maximum Principle [1].

Recently, most research on the topic of differential games has focused on the case of  $N$ -player differential games with cooperation [14] and stochastic differential games [3]. Advances in mathematical computational tools have led to the study of numerical methods of differential games [15, 16]. The aim of this dissertation is to solve for the Nash and Stackelberg equilibriums solutions of a differential game with pure state constraints by using an application of the Maximum Principle.

A brief overview of the Chapters is as follows.

In Chapter 2, we review the concepts of constrained optimal control and optimization problems. In Section 2.1 we introduce the Maximum Principle for an optimal control problem with general inequality constraints. We then discuss the Maximum Principle for current value problems. We then move on to Section 2.2, here we discuss sufficiency conditions for optimal control problems. In Section 2.3 we describe the numerical method known as the Forward-Backward Sweep to solve optimal control problems. We conclude the section with some examples. Finally in Section 2.4 we discuss the Karush-Kuhn-Tucker conditions for optimization problems with constraints.

In Chapter 3, we investigate the theories of  $N$ -player differential games. We begin with a brief review of game theory and then move on to describe the theories of an  $N$ -player differential game with pure state constraints. Here we illustrate how we can use the maximum principle to obtain necessary conditions for the open-loop Nash and Stackelberg equilibriums. The section is concluded with some examples to illustrate these notions.

In Chapter 4, we investigate numerical methods to solve for an open-loop Nash equilibrium. An algorithm for solving  $N$ -player differential games with the Hamiltonian being a convex function to the control is described. The section is concluded with some examples to illustrate this method. The chapter is concluded with some examples.

In Chapter 5, we present the conclusion of the dissertation. Here we discuss the shortcomings of the N- player differential game problem. We then discuss how we may improve the problem and the approach to solving it. We end the chapter by discussing the agenda for future research of the problem.

## Chapter 2

# Overview of Constrained Optimal Control Theory and Optimization

This chapter deals with the theory and the numerical simulation of constrained optimal control and optimization problems. These concepts play an important role in the study of differential games which is the main topic of this dissertation.

### 2.1 Optimal Control Problems with General Inequality Constraints

We consider the constrained optimal control problem formulated as

$$\max \left\{ J = S(x(T)) + \int_0^T F(t, x(t), u(t)) dt \right\}, \quad (2.1)$$

$$\text{subject to } \dot{x} = f(t, x(t), u(t)), \quad x(0) = x_0, \quad (2.2)$$

$$g(t, x(t), u(t)) \geq 0, \quad t \in [0, T], \quad (2.3)$$

$$h(t, x(t)) \geq 0, \quad t \in [0, T], \quad (2.4)$$

where,  $x(t) : [0, T] \rightarrow \mathbb{R}^n$  is called the state variable,  $u(t) : [0, T] \rightarrow \mathbb{R}^m$  is called the control variable, the function  $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$  in (2.2) describes the dynamical system and the integrand of the integral,  $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ , in (2.1) is called the running payoff. The function  $S : \mathbb{R}^n \mapsto \mathbb{R}$  in (2.1) is called terminal payoff. The functions  $g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^q$  is a mixed inequality constraint and

$h : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^r$  is a pure state constraint. We assume that the functions  $g$ ,  $h$ ,  $f$ ,  $F$  and  $S$  are continuously differentiable in all their arguments.

**Definition 2.1.1** (Admissible Control). A control  $u$  which satisfies the equations (2.2) and (2.3) is called an admissible control.

For simplicity, we assume that the function  $h$  has only one component, that is  $r = 1$ . At any point where  $h(t, x(t)) > 0$ , the corresponding constraint  $h(t, x(t)) \geq 0$  is not binding and can be ignored. Otherwise at any interval where  $h(t, x(t)) = 0$ , we require the total derivative of the function  $h(t, x(t))$  with respect to time, denoted as  $\dot{h}(t, x(t))$  to be positive in order for  $h(t, x(t))$  to remain positive. The total time derivatives of the function  $h(t, x(t))$  can be worked out as follows

$$\begin{aligned} \dot{h}(t, x, u) &= \frac{d}{dt}h(t, x) = h_t + h_x \dot{x} \\ &= h_t + h_x f, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \ddot{h}(t, x, u) &= \frac{d^2}{dt^2}h(t, x) = h_x (\dot{u}f_u + \dot{x}f_x + f_t) + f (\dot{x}h_{xx} + h_{xt}) + \dot{x}h_{xt} + h_{tt} \\ &= h_x (\dot{u}f_u + f f_x + f_t) + f^2 h_{xx} + 2f h_{xt} + h_{tt}. \end{aligned}$$

**Definition 2.1.2** (Order of State Constraint). Let  $h(t, x(t)) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^r$  be  $r \in \mathbb{N}$  times continuously differentiable. The state constraint  $h(t, x(t)) \geq 0$  is said to be of  $j^{\text{th}}$  order if the first time that the control  $u$  appears explicitly in the total derivative of  $h(t, x(t))$  with respect to  $t$  is at the  $j^{\text{th}}$  derivative.

For simplicity, we will consider in the rest of this chapter only pure state constraints of order one.

**Definition 2.1.3** (Interior Interval, Boundary Interval, Junction Times). With regards to the constraint  $h(t, x(t)) \geq 0$  we have the following,

1. We call the interval  $(\theta_1, \theta_2) \subset [0, T]$  an *interior interval* if for the constraint  $h(t, x(t)) \geq 0$  we have  $h(t, x(t)) > 0$  for all  $t \in (\theta_1, \theta_2)$  with  $\theta_1 < \theta_2$ .
2. We call the interval  $[\tau_1, \tau_2]$  the *boundary interval* if the optimal trajectory satisfies  $h(t, x(t)) = 0$  for  $\tau_1 < t < \tau_2$ .

3. Collectively (i) - (iii) given below are known as the *junction times* of the interval.
- (i) If at  $t = \tau_1$  we have the end of interior interval and the start of a boundary interval, then  $\tau_1$  is called an *entry time*.
  - (ii) If at  $\tau_2$  we have the end of boundary interval and the start of a interior interval, then  $\tau_2$  is called an *exit time*.
  - (iii) If  $h(\tau, x(\tau)) = 0$  and if the trajectory is in the interior just before and after  $\tau$ , then  $\tau$  is known as a *contact time*.

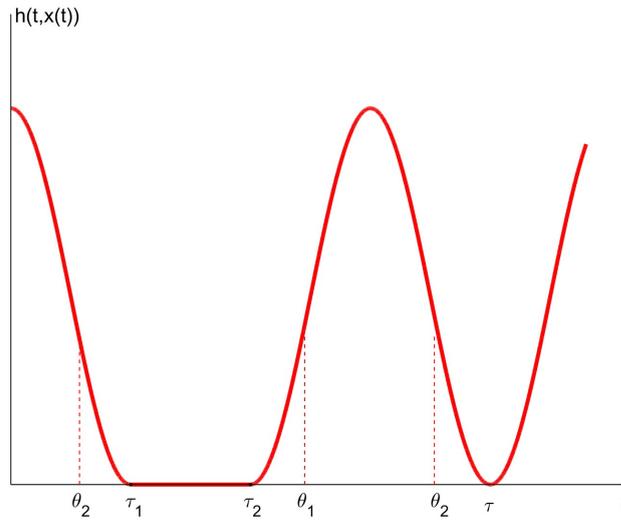


FIGURE 2.1: Illustration of Interior Interval, Boundary Interval, Junction Times

Figure 2.1 presents an illustration of the interior interval, boundary interval, and junction times as described in Definition 2.1.3.

For the rest of this chapter, we require that the mixed inequality constraints  $g$  satisfy the constraint qualifications

- (i) For all arguments of  $t, x(t), u(t)$ ,

$$\text{rank} \left[ \frac{\partial g}{\partial u}, \text{diag}(g) \right] = q, \tag{2.6}$$

- (ii) For any boundary interval  $[\tau_1, \tau_2]$ ,

$$\text{rank} \left[ \frac{\partial \dot{h}}{\partial u} \right] \neq 0, \tag{2.7}$$

for  $\dot{h}$  as defined in equation (2.5).

The condition (2.6) indicates that the gradients of all the constraints  $g$  with respect to  $u$  has to be linearly independent.

A first order optimality condition for the OCP (2.1)-(2.4) is now presented. The Hamiltonian allows us to handle the constraints (2.2) through a multiplier  $p$ . The Lagrangian allows us to handle the mixed inequality constraints (2.3) through a multiplier  $\eta$  and the pure state constraints (2.4) through a multiplier  $\mu$ . In practice there are two main ways to handle pure state constraints namely, the *direct* and *indirect adjoining methods*. In the direct adjoining method, the pure state constraints (2.4) is used directly to form the Lagrangian via a multiplier whereas in the indirect adjoining method, the total derivative of the pure state constraints (2.5) is used to form the Lagrangian when the constraints (2.4) are binding.

The Hamiltonian function  $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$  is given by,

$$H[t, x, u, p] = F(t, x, u) + p \cdot f(t, x, u), \quad (2.8)$$

where  $p$  is a row vector called the adjoint or co-state variable.

The Lagrangian function  $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^r \rightarrow \mathbb{R}$  using the indirect adjoining method is given by,

$$L[t, x, u, p, \eta, \mu] := H(t, x, u, p) + \eta \cdot g(t, x, u) + \mu \cdot \dot{h}(t, x, u). \quad (2.9)$$

We are now ready to state the necessary conditions for the maximum principle with general inequality constraints for problem (2.1)-(2.4).

**Theorem 2.1.4** (Maximum Principle with General Inequality Constraints). *Let the control  $u^*$  with corresponding state trajectory  $x^*$  be an optimal solution for the problem (2.1)-(2.4) satisfying the constraint qualifications (2.6)-(2.7). Then there exists adjoint variable  $p$ , multipliers  $\eta, \mu, \gamma$  and the jump parameter  $\zeta$  satisfying the following conditions (i)-(vi) at every time where  $u^*$  is continuous,*

(i) *The state equation*

$$\dot{x}^* = f(t, x^*(t), u^*(t)), \quad x^*(0) = x_0, \quad (2.10)$$

is satisfied together with the constraints

$$g(t, x^*(t), u^*(t)) \geq 0 \quad \text{and} \quad h(t, x^*(t)) \geq 0. \quad (2.11)$$

(ii) The adjoint equation given by,

$$\dot{p} = -\frac{\partial L}{\partial x} \quad (2.12)$$

with the transversality conditions

$$\begin{aligned} p(T^-) &= S_x(T, x^*(T)) + \gamma h_x(T, x^*(T)), \\ \gamma &\geq 0, \quad \gamma h(T, x^*(T)) = 0. \end{aligned} \quad (2.13)$$

(iii) The Hamiltonian maximizing condition given by,

$$H(t, x^*(t), u^*(t), p(t)) \geq H(t, x^*(t), u, p(t)), \quad (2.14)$$

for each  $t \in [0, T]$  for all  $u$  which satisfy

$$\begin{aligned} g[t, x^*(t), u(t)] &\geq 0, \quad \text{and} \\ \dot{h}[t, x^*(t), u(t)] &\geq 0 \quad \text{whenever} \quad h(t, x^*(t)) = 0. \end{aligned} \quad (2.15)$$

(iv) The jump conditions at any entry/contact time  $\tau$  are

$$\begin{aligned} p(\tau^-) &= p(\tau^+) + \zeta(\tau) h_x(\tau, x^*(\tau)) \quad \text{and} \\ H(\tau, x^*(\tau), u^*(\tau^-), p(\tau^-)) &= H(\tau, x^*(\tau), u^*(\tau^+), p(\tau^+)) - \zeta(\tau) h_t(\tau, x^*(\tau)) \end{aligned} \quad (2.16)$$

(v) The Lagrange multipliers are such that

$$\frac{\partial L}{\partial u} \Big|_{u=u^*(t)} = 0, \quad \frac{dH}{dt} = \frac{dL}{dt} = \frac{\partial L}{\partial t}. \quad (2.17)$$

(vi) The complementary slackness conditions

$$\begin{aligned} \eta(t) &\geq 0, \quad \eta(t) g(t, x^*(t), u^*(t)) = 0, \\ \mu(t) &\geq 0, \quad \mu(t) h(t, x^*(t)) = 0, \quad \dot{\mu}(t) \leq 0, \\ \zeta(\tau) &\geq 0, \quad \zeta(\tau) h(\tau, x^*(\tau)) = 0. \end{aligned} \quad (2.18)$$

hold.

In the case where the terminal time  $T$  is unspecified, we have an extra necessary transversality condition for  $T^*$  to be optimal which is,

$$H(T^*, x^*(T^*), u^*(T^*), p(T^*)) + S_T(T^*, x^*(T^*)) = 0, \quad (2.19)$$

as long as  $T^* \in (0, \infty)$ . It is important to note that if  $T$  is confined to an interval  $[\tau_a, \tau_b]$ , where  $\tau_b > \tau_a \geq 0$ , then we still use the additional transversality condition (2.19) given that  $T^* \in (\tau_a, \tau_b)$ . Now when we have the case where  $T^* = \tau_a$ , we will simply substitute the equality sign of the additional transversality condition (2.19) with  $\leq$  and for the case where  $T^* = \tau_b$ , we substitute the equality sign of the additional transversality condition (2.19) with  $\geq$ .

**Example 2.1.1.** *Consider an economy consisting of two sectors where, sector 1 produces investment goods and sector 2 produces consumption goods. Let*

$$x_i(t) = \text{the production in sector } i \text{ per unit of time, } i = 1, 2,$$

and let

$$u(t) = \text{the proportion of investments allocated to sector 1}$$

where

$$0 \leq u(t) \leq 1$$

The increase in production per unit of time in each sector is assumed to be proportional to investment allocated to the sector hence, we have the following system with dynamics

$$\begin{aligned} \dot{x}_1(t) &= \alpha u x_1 & x_1(0) &= x_{10}, \\ \dot{x}_2(t) &= \alpha(1-u)x_1, & x_2(0) &= x_{20}, \end{aligned} \quad t \in [0, T]. \quad (2.20)$$

where  $\alpha$  is a positive constant.

The payoff functional is given as

$$J = \int_0^T x_2 dt, \quad (2.21)$$

We aim to maximize the total consumption in the given planning period  $[0, T]$ .

**Solution.** The Hamiltonian for the problem is given as

$$\begin{aligned} H[t, x_1, x_2, u, p_1, p_2] &= F(t, x_1, x_2, u) + p_1 \cdot f_1(t, x_1, x_2, u) + p_2 \cdot f_2(t, x_1, x_2, u) \\ &= x_2 + p_1 \cdot (\alpha u x_1) + p_2 \cdot (\alpha(1 - u)x_1), \end{aligned} \quad (2.22)$$

Applying the Hamiltonian maximization condition (iii) of Theorem (2.1.4) we get

$$\begin{aligned} u^\# &= \underset{u \in U}{\operatorname{argmax}} \{H[t, x_1, x_2, u, p_1, p_2]\} \\ &= \underset{u \in U}{\operatorname{argmax}} \{x_2 + p_1 \cdot (\alpha u x_1) + p_2 \cdot (\alpha(1 - u)x_1)\} \\ &= \underset{u \in U}{\operatorname{argmax}} \{\alpha u x_1 (p_1 - p_2)\} \\ &= \begin{cases} 1 & \text{if } p_1 - p_2 > 0, \\ 0 & \text{if } p_1 - p_2 < 0. \end{cases} \end{aligned} \quad (2.23)$$

From condition (ii) of Theorem (2.1.4) we find that the adjoint equation is given by,

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial x_1} \\ &= -\frac{\partial}{\partial x_1} \{x_2 + p_1 \cdot (\alpha u x_1) + p_2 \cdot (\alpha(1 - u)x_1)\} \\ &= -p_1 \alpha u - p_2 \alpha(1 - u), \end{aligned}$$

and

$$\begin{aligned} \dot{p}_2 &= -\frac{\partial H}{\partial x_2} \\ &= -\frac{\partial}{\partial x_2} \{x_2 + p_1 \cdot (\alpha u x_1) + p_2 \cdot (\alpha(1 - u)x_1)\} \\ &= -1, \end{aligned}$$

with the transversality conditions

$$p_1(T) = S_{x_1}(T, x_1(T)) = 0,$$

and

$$p_2(T) = S_{x_2}(T, x_2(T)) = 0.$$

Putting all together gives

$$\left\{ \begin{array}{ll} u^\# = \begin{cases} 1 & \text{if } p_1 - p_2 > 0, \\ 0 & \text{if } p_1 - p_2 < 0. \end{cases} \\ x_1 = \alpha u^\# x_1, & x_1(0) = x_{10}, \\ x_2 = \alpha(1 - u^\#) x_1, & x_2(0) = x_{20}, \\ \dot{p}_1 = -p_1 \alpha u - p_2 \alpha(1 - u), & p_1(T) = 0, \\ \dot{p}_2 = -1, & p_2(T) = 0. \end{array} \right. \quad (2.24)$$

Solving (2.24) yields

$$u^* = \begin{cases} 1 & \text{if } 0 \leq t \leq T - 2, \\ 0 & \text{if } T - 2 < t \leq T, \end{cases}$$

$$x_1^* = \begin{cases} x_{10} e^{\alpha t} & \text{if } 0 \leq t \leq T - 2, \\ x_{10} e^{\alpha(T-2)} & \text{if } T - 2 < t \leq T, \end{cases}$$

$$x_2^* = \begin{cases} x_{20} & \text{if } 0 \leq t \leq T - 2, \\ x_{20} + \alpha x_{10} (t - T + \frac{2}{\alpha}) e^{\alpha(T-2)} & \text{if } T - 2 < t \leq T, \end{cases}$$

$$p_1^* = \begin{cases} \frac{2}{\alpha} e^{(-\alpha t + \alpha(T-2))} & \text{if } 0 \leq t \leq T - 2, \\ \frac{\alpha}{2} (t - T)^2 & \text{if } T - 2 < t \leq T, \end{cases}$$

$$p_2^* = T - t.$$

■

**Example 2.1.2** (Price of Goods). *We consider a firm manufacturing a good where,*

$$x(t) = \text{the sale price of a good at time } t,$$

and for the controls, at each time  $t$ , a good is produced by the firm. So the controls for our problem are defined as:

$$u(t) = \text{rate of production of a good at time } t,$$

and

$$d = \text{rate of consumption of a good at time } t,$$

$$c(s) = \text{cost function},$$

we assume for the sake of definiteness that

$$c(s) = \frac{s^2}{2}.$$

The price increases when the consumption of goods is larger than the production of goods, and decreases otherwise. That is we have the following system with dynamics

$$\dot{x}(t) = x(t)(d - u(t)), \quad x(0) = x_0, \quad t \in [0, T],$$

We have constraints given by

$$x(t) \geq 0. \tag{2.25}$$

The payoff functional is measured by the profit generated by sales, minus the cost  $c(u(t))$  of producing the good at rate  $u(t)$ . that is

$$J = \int_0^T [d \cdot x(t) - c(u(t))] dt.$$

We aim to find an optimal solution for the problem.

**Solution.** The Hamiltonian for the problem is given as

$$\begin{aligned} H[t, x, u, p] &= F(t, x, u) + p \cdot f(t, x, u) \\ &= d \cdot x - \frac{u^2}{2} + p \cdot x(d - u), \end{aligned} \tag{2.26}$$

and the Lagrangian for the problem is given as

$$\begin{aligned} L[t, x, u, p, \mu] &= H[t, x, u, p] + \mu \cdot f(t, x, u) \\ &= d \cdot x - \frac{u^2}{2} + p \cdot x(d - u) + \mu \cdot x(d - u). \end{aligned}$$

Applying the Hamiltonian maximization condition (iii) of Theorem (2.1.4) we get

$$\begin{aligned} u^\# &= \underset{u \in U}{\operatorname{argmax}} \{H[t, x, u, p]\} \\ &= \underset{u \in U}{\operatorname{argmax}} \left\{ d \cdot x - \frac{u^2}{2} + p \cdot x(d - u) \right\} \\ &= -p \cdot x, \end{aligned} \tag{2.27}$$

for each  $t \in [0, T]$  and for all  $u$  which satisfy

$$x(d - u) \geq 0.$$

From condition (ii) of Theorem (2.1.4) we find that the adjoint equation is given by

$$\begin{aligned} \dot{p} &= -\frac{\partial L}{\partial x} \\ &= -\frac{\partial}{\partial x} \left\{ d \cdot x - \frac{u^2}{2} + p \cdot x(d - u) + \mu \cdot x(d - u) \right\} \\ &= -d - (p + \mu) \cdot (d - u), \end{aligned}$$

with the transversality conditions

$$\begin{aligned} p(T) &= S_x(T, x(T)) + \gamma h_x(T, x(T)) \\ &= \gamma \end{aligned}$$

$$\gamma \geq 0, \quad \gamma x(T) = 0.$$

From condition (vi) of Theorem (2.1.4) we find that the complementary slackness conditions,

$$\mu \geq 0, \quad \mu x = 0, \quad \dot{\mu} \leq 0.$$

Putting all together gives

$$\left\{ \begin{array}{ll} \dot{x} &= x(d - u^\sharp) \\ &= x(d + px), & x(0) = x_0, \\ \\ \dot{p} &= -d - (p + \mu) \cdot (d - u^\sharp), & p(T) = \gamma, \\ &= -d - (p + \mu) \cdot (d + px), & \gamma \geq 0, \quad \gamma x(T) = 0. \end{array} \right. \quad (2.28)$$

which satisfy the constraints

$$x(d - u) \geq 0, \quad (2.29)$$

and the slackness conditions

$$\mu \geq 0, \quad \mu x = 0, \quad \dot{\mu} \leq 0. \quad (2.30)$$

to determine the state  $x$  and the adjoint variable  $p$ .

■

**Example 2.1.3** (Commodity Trading). *Let*

$x_1(t)$  = *the money on hand at time  $t$ ,*

$x_2(t)$  = *the amount of wheat owned at time  $t$ ,*

*and for the controls, at each time  $t$ , we have the possibility to buy or to sell some wheat.*

*So the controls for our problem are defined as:*

$u(t)$  = *amount of wheat bought/sold at time  $t$ ,*

*where  $u(t) > 0$  means we are buying wheat  $u(t) < 0$  means we are selling wheat. Also*

$s$  = *cost of storing a unit of amount of wheat for a unit of time,*

$q$  = *price of wheat at time  $t$ ,*

*where  $s > 0$ .*

We have the following system with dynamics

$$\begin{aligned} \dot{x}_1(t) &= -s \cdot x_2(t) - q(t) \cdot u(t), & x_1(0) &= x_{10}, \\ \dot{x}_2(t) &= u(t), & x_2(0) &= x_{20}, \end{aligned} \quad t \in [0, T]. \quad (2.31)$$

We have constraints given by

$$\begin{aligned} x_1(t) &\geq 0, \\ x_2(t) &\geq 0, \\ -4 &\leq u(t) \leq 4. \end{aligned} \quad (2.32)$$

The payoff functional for the firm is given as

$$J = [x_1(T) + q(T) \cdot x_2(T)],$$

We aim to find an optimal solution for the problem.

**Solution.** In order to solve the problem, let us consider a particular situation, i.e

$$s = 3, \quad \text{and} \quad q(t) = t^2 + 1,$$

The Hamiltonian for the problem is given as

$$\begin{aligned} H[t, x_1, x_2, u, p_1, p_2] &= F(t, x_1, x_2, u) + p_1 \cdot f_1(t, x_1, x_2, u) + p_2 \cdot f_2(t, x_1, x_2, u) \\ &= p_1 \cdot (u - t^2 u - 3x_2) + p_2 \cdot u, \end{aligned} \quad (2.33)$$

and the Lagrangian for the problem is given as

$$\begin{aligned} L[t, x_1, x_2, u, p_1, p_2, \mu_1, \mu_2] &= H[t, x_1, x_2, u, p_1, p_2] + \mu_1 \cdot f_1(t, x_1, x_2, u) + \mu_2 \cdot f_2(t, x_1, x_2, u) \\ &= (p_1 + \mu_1) \cdot (u - t^2 u - 3x_2) + (p_2 + \mu_2) \cdot u. \end{aligned}$$

Applying the Hamiltonian maximization condition (iii) of Theorem (2.1.4) we get

$$\begin{aligned} u^\# &= \underset{u \in U}{\operatorname{argmax}} \{H[t, x_1, x_2, u, p_1, p_2]\} \\ &= \underset{u \in U}{\operatorname{argmax}} \{p_1 \cdot (u - t^2 u - 3x_2) + p_2 \cdot u\} \\ &= \begin{cases} 4 & \text{if } p_2 + p_1 - p_1 t^2 > 0, \\ 0 & \text{if } p_2 + p_1 - p_1 t^2 = 0, \\ -4 & \text{if } p_2 + p_1 - p_1 t^2 < 0. \end{cases} \end{aligned} \quad (2.34)$$

for each  $t \in [0, T]$  and for all  $u$  which satisfy

$$u - t^2u - 3x_2 \geq 0, \quad u \geq 0.$$

From condition (ii) of Theorem (2.1.4) we find that the adjoint equation is given by

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial L}{\partial x_1} \\ &= -\frac{\partial}{\partial x_1} \{(p_1 + \mu_1) \cdot (u - t^2u - 3x_2) + (p_2 + \mu_2) \cdot u\} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \dot{p}_2 &= -\frac{\partial L}{\partial x_2} \\ &= -\frac{\partial}{\partial x_2} \{(p_1 + \mu_1) \cdot (u - t^2u - 3x_2) + (p_2 + \mu_2) \cdot u\} \\ &= 3(p_1 + \mu_1), \end{aligned}$$

with the transversality conditions

$$\begin{aligned} p_1(T) &= S_{x_1}(T, x_1(T)) + \gamma_1 h_{1x_1}(T, x_1(T)) \\ &= 1 + \gamma_1, \end{aligned}$$

$$\gamma_1 \geq 0, \quad \gamma_1 x_1(T) = 0,$$

and

$$\begin{aligned} p_2(T) &= S_{x_2}(T, x_2(T)) + \gamma_2 h_{2x_2}(T, x_2(T)) \\ &= T^2 + 1 + \gamma_2, \end{aligned}$$

$$\gamma_2 \geq 0, \quad \gamma_2 x_2(T) = 0.$$

From condition (v) of Theorem (2.1.4) we find that the Lagrange multipliers are such that,

$$\begin{aligned}
 u^\sharp &= \underset{u \in U}{\operatorname{argmax}} \{L[t, x_1, x_2, u, p_1, p_2, \mu_1, \mu_2]\} \\
 &= \underset{u \in U}{\operatorname{argmax}} \{(p_1 + \mu_1) \cdot (u - t^2 u - 3x_2) + (p_2 + \mu_2) \cdot u\} \\
 &= \begin{cases} 4 & \text{if } (p_1 + \mu_1) \cdot (1 - t^2) + (p_2 + \mu_2) > 0, \\ 0 & \text{if } (p_1 + \mu_1) \cdot (1 - t^2) + (p_2 + \mu_2) = 0, \\ -4 & \text{if } (p_1 + \mu_1) \cdot (1 - t^2) + (p_2 + \mu_2) < 0. \end{cases}
 \end{aligned} \tag{2.35}$$

From condition (vi) of Theorem (2.1.4) we find that the complementary slackness conditions,

$$\begin{aligned}
 \mu_1 &\geq 0, & \mu_1 x_1 &= 0, & \dot{\mu}_1 &\leq 0, \\
 \mu_2 &\geq 0, & \mu_2 x_2 &= 0, & \dot{\mu}_2 &\leq 0.
 \end{aligned}$$

Putting all together gives

$$\left\{ \begin{array}{ll}
 \dot{x}_1 = -3x_2 - (t^2 + 1) \cdot u^\sharp, & x_1(0) = x_{10}, \\
 \dot{x}_2 = u^\sharp, & x_2(0) = x_{20} \\
 \dot{p}_1 = 0, & p_1(T) = 1 + \gamma_1, \\
 & \gamma_1 \geq 0, \quad \gamma_1 x_1(T) = 0, \\
 \dot{p}_2 = 3(p_1 + \mu_1), & p_2(T) = T^2 + 1 + \gamma_2. \\
 & \gamma_2 \geq 0, \quad \gamma_2 x_2(T) = 0.
 \end{array} \right. \tag{2.36}$$

which satisfy the constraints

$$u - t^2 u - 3x_2 \geq 0, \quad u \geq 0, \tag{2.37}$$

and the slackness conditions

$$\begin{aligned}
 \mu_1 &\geq 0, & \mu_1 x_1 &= 0, & \dot{\mu}_1 &\leq 0, \\
 \mu_2 &\geq 0, & \mu_2 x_2 &= 0, & \dot{\mu}_2 &\leq 0,
 \end{aligned} \tag{2.38}$$

to determine the states  $x_1, x_2$  and the adjoint variables  $p_1, p_2$ .

■

### 2.1.1 Current-Value Formulation

When the running payoff is discounted, we can still write the maximum principle in the usual form using the current value formulation of the problem.

$$\text{maximize } \left\{ J = e^{-rT} S(x(T)) + \int_0^T e^{-rt} F(t, x(t), u(t)) dt \right\}, \quad (2.39)$$

$$\text{subject to } \dot{x} = f(t, x(t), u(t)), \quad x(0) = x_0, \quad t \in [0, T],$$

$$g(t, x(t), u(t)) \geq 0, \quad t \in [0, T], \quad (2.40)$$

$$h(t, x(t)) \geq 0, \quad t \in [0, T], \quad (2.41)$$

The term  $e^{-rt}$  is the discount factor and  $r > 0$  is called the discount rate.

The standard Lagrangian function for problem (2.39)-(2.41) is given as,

$$\begin{aligned} L(t, x, u, p, \mu) &= H(t, x, u, p) + \eta \cdot g(t, x, u) + \mu \cdot \dot{h}(t, x, u), \\ &= e^{-rt} F(t, x, u) + p \cdot f(t, x, u) + \eta \cdot g(t, x, u) + \mu \cdot \dot{h}(t, x, u), \end{aligned}$$

where  $\dot{h}(t, x, u)$  is defined as in (2.5). If we multiply this Lagrangian by  $e^{rt}$  we will obtain the *current value Lagrangian* function which is given as,

$$\begin{aligned} L^c = Le^{rt} &= F(t, x, u) + p \cdot e^{rt} f(t, x, u) + \eta \cdot e^{rt} g(t, x, u) + \mu \cdot e^{rt} \dot{h}(t, x, u) \\ &= F(t, x, u) + \lambda f(t, x, u) + \varphi \cdot g(t, x, u) + \nu \cdot \dot{h}(t, x, u) \\ &= H^c(t, x, u, \lambda) + \varphi \cdot g(t, x, u) + \nu \cdot \dot{h}(t, x, u) \end{aligned}$$

where  $\lambda = p \cdot e^{rt}$ ,  $\varphi = e^{rt} \eta$ ,  $\nu = e^{rt} \mu$  and the

$$H^c(t, x, u, \lambda) = F(t, x, u) + p \cdot e^{rt} f(t, x, u)$$

is the *current value Hamiltonian* function.

From  $\lambda = p \cdot e^{rt}$  it follows that,

$$\dot{\lambda} = r\lambda + e^{rt} \dot{p}. \quad (2.42)$$

From  $L^c = Le^{rt}$ , it follows that

$$\frac{\partial L^c}{\partial x} = e^{rt} \frac{\partial L}{\partial x}. \quad (2.43)$$

From (2.42) and (2.43) we see that the adjoint equation will take on the form,

$$\dot{\lambda} - r\lambda = -\frac{\partial L^c}{\partial x}$$

with the corresponding *current value transversality condition* given by

$$\begin{aligned} \lambda(T) &= e^{rT} p(T) \\ &= S_x(T, x^*(T)) + e^{rT} \gamma h_x(T, x^*(T)) \\ &= S_x(T, x^*(T)) + \delta h_x(T, x^*(T)), \end{aligned}$$

where  $\delta = e^{rT} \gamma$ .

We are now ready to state the maximum principle for problem (2.39)-(2.41).

**Theorem 2.1.5** (Maximum Principle with General Inequality Constraints). *Let the control  $u^*$  with corresponding state trajectory  $x^*$  be an optimal solution for the problem (2.39)-(2.41) satisfying the constraint qualifications (2.6)-(2.7). Then there exists an adjoint variable  $\lambda$ , multipliers  $\varphi, \nu, \delta$  and the jump parameter  $\zeta$  satisfying the following conditions (i)-(vi) at every time where  $u^*$  is continuous,*

(i) *The state equation*

$$\dot{x}^* = f(t, x^*(t), u^*(t)), \quad x^*(0) = x_0, \quad (2.44)$$

*is satisfied together with the constraints*

$$g(t, x^*(t), u^*(t)) \geq 0 \quad \text{and} \quad h(t, x^*(t)) \geq 0. \quad (2.45)$$

(ii) *The adjoint equation given by*

$$\dot{\lambda} - r\lambda = -\frac{\partial L^c}{\partial x} \quad (2.46)$$

with the transversality conditions

$$\lambda(T^-) = S_x(T, x^*(T)) + \delta h_x(T, x^*(T)), \quad (2.47)$$

$$\delta \geq 0, \quad \delta h(T, x^*(T)) = 0.$$

(iii) The Hamiltonian maximizing condition given by

$$H^c(t, x^*(t), u^*(t), \lambda(t)) \geq H^c(t, x^*(t), u, \lambda(t)), \quad (2.48)$$

for each  $t \in [0, T]$  for all  $u$  which satisfy

$$\begin{aligned} g[t, x^*(t), u(t)] &\geq 0, \text{ and} \\ \dot{h}[t, x^*(t), u(t)] &\geq 0 \text{ whenever } h(t, x^*(t)) = 0. \end{aligned} \quad (2.49)$$

(iv) The jump conditions at any entry/contact time  $\tau$  are

$$\begin{aligned} \lambda(\tau^-) &= \lambda(\tau^+) + \zeta(\tau) h_x(\tau, x^*(\tau)) \text{ and} \\ H^c(\tau, x^*(\tau), u^*(\tau^-), \lambda(\tau^-)) &= H^c(\tau, x^*(\tau), u^*(\tau^+), \lambda(\tau^+)) - \zeta(\tau) h_t(\tau, x^*(\tau)). \end{aligned} \quad (2.50)$$

(v) The Lagrange multipliers are such that

$$\frac{\partial L^c}{\partial u} \Big|_{u=u^*(t)} = 0, \quad \frac{dH^c}{dt} = \frac{dL^c}{dt} = \frac{\partial L^c}{\partial t}. \quad (2.51)$$

(vi) The complementary slackness conditions

$$\begin{aligned} \varphi(t) &\geq 0, \quad \varphi(t) g(t, x^*(t), u^*(t)) = 0, \\ \nu(t) &\geq 0, \quad \nu(t) h(t, x^*(t)) = 0, \quad \dot{\nu}(t) \leq r\nu(t), \\ \zeta(\tau) &\geq 0, \quad \zeta(\tau) h(\tau, x^*(\tau)) = 0. \end{aligned} \quad (2.52)$$

hold.

**Example 2.1.4** (Vidale-Wolfe Advertising Model). We consider a firm wanting to increase their total number customers through advertising. Let

$x(t)$  = market share of the firm at time  $t$ ,

and for the controls, at each time  $t$ , an advertising strategy is implemented by the firm. So the controls for our problem are defined as:

$$u(t) = \text{advertising effort of the firm at time } t,$$

and

$$r = \text{interest rate},$$

$$\phi = \text{Firm's fractional revenue potential},$$

$$c(s) = \text{advertising cost function}$$

We assume for the sake of definiteness that

$$c(s) = k \frac{s^2}{2},$$

where  $k$  is a positive constant.

The number of customers at the firm is the difference between the number of customers attracted to the firm and the number of customers repelled from the firm, hence we have the following system with dynamics

$$\dot{x}(t) = u(t)(1 - x(t)) - x(t), \quad x(0) = x_0, \quad t \in [0, T]. \quad (2.53)$$

We have constraints given by

$$0 \leq x(t) \leq 1. \quad (2.54)$$

The payoff functional for the firm is given as

$$J = \int_0^T e^{-rt} [\phi \cdot x(t) - c(u(t))] dt,$$

We aim to find an optimal solution for the problem.

**Solution.** The Hamiltonian for the problem is given as

$$\begin{aligned} H^c[t, x, u, \lambda] &= F(t, x, u) + \lambda \cdot f(t, x, u) \\ &= \phi \cdot x - k \frac{u^2}{2} + \lambda \cdot (u(1 - x) - x), \end{aligned} \quad (2.55)$$

and the Lagrangian for the problem is given as

$$\begin{aligned} L^c[t, x, u, \lambda, \nu] &= H^c[t, x, u, \lambda] + \nu_1 \cdot f(t, x, u) - \nu_2 \cdot f(t, x, u) \\ &= \phi \cdot x - k \frac{u^2}{2} + (\lambda + \nu) \cdot (u(1 - x) - x), \end{aligned}$$

where  $\nu = \nu_1 - \nu_2$  and  $H^c[t, x, u, \lambda]$  is defined as in (2.55).

Applying the Hamiltonian maximization condition (iii) of Theorem (2.1.5) we get

$$\begin{aligned} u^\# &= \underset{u \in U}{\operatorname{argmax}} \{H^c[t, x, u, \lambda]\} \\ &= \underset{u \in U}{\operatorname{argmax}} \left\{ \phi \cdot x - k \frac{u^2}{2} + \lambda \cdot (u(1 - x) - x) \right\} \\ &= \frac{\lambda \cdot (1 - x)}{k}, \end{aligned}$$

for each  $t \in [0, T]$  for all  $u$  which satisfy

$$\begin{aligned} u(1 - x) - x &\geq 0, \\ x - u(1 - x) &\geq 0. \end{aligned}$$

From condition (ii) of Theorem (2.1.5) we find that the adjoint equation is given by,

$$\begin{aligned} \dot{\lambda} - r\lambda &= -\frac{\partial L^c}{\partial x} \\ &= -\frac{\partial}{\partial x} \left\{ \phi \cdot x - k \frac{u^2}{2} + (\lambda + \nu) \cdot (u(1 - x) - x) \right\} \\ &= -\phi + (\lambda + \nu)(u + 1), \end{aligned}$$

along with the transversality conditions

$$\begin{aligned} \lambda(T) &= S_x(T, x(T)) + \delta_1 \nabla_x h_1(T, x(T)) + \delta_2 \nabla_x h_2(T, x(T)), \\ &= \delta_1 - \delta_2 \end{aligned}$$

$$\begin{aligned} \delta_1 &\geq 0, \quad \delta_1 x(T) = 0 \\ \delta_2 &\geq 0, \quad \delta_2 (1 - x(T)) = 0. \end{aligned}$$

From condition (v) of Theorem (2.1.5) we find that the Lagrange multipliers are such that,

$$\begin{aligned} u^\sharp &= \underset{u \in U}{\operatorname{argmax}} \{L^c[t, x, u, \lambda, \nu]\} \\ &= \underset{u \in U}{\operatorname{argmax}} \left\{ \phi \cdot x - k \frac{u^2}{2} + (\lambda + \nu) \cdot (u(1-x) - x) \right\} \\ &= \frac{(\lambda + \nu) \cdot (1-x)}{k} \end{aligned}$$

From condition (vi) of Theorem (2.1.5) we find that the complementary slackness conditions,

$$\begin{aligned} \nu_1(t) &\geq 0, \quad \nu_1(t) x(t) = 0, \quad \dot{\nu}_1(t) \leq 0, \\ \nu_2(t) &\geq 0, \quad \nu_2(t) (1-x(t)) = 0, \quad \dot{\nu}_2(t) \leq 0. \end{aligned}$$

Putting all together gives

$$\left\{ \begin{array}{l} \dot{x} = u^\sharp(1-x) - x, \quad x(0) = x_0, \\ \quad = \frac{\lambda \cdot (1-x)^2}{k} - x \\ \dot{\lambda} - r\lambda = -\phi + (\lambda + \nu)(u^\sharp + 1) \\ \quad = -\phi + (\lambda + \nu) \left( \frac{\lambda \cdot (1-x)}{k} + 1 \right), \quad \lambda(T) = \delta_1 - \delta_2, \\ \delta_1 x(T) = 0, \quad \delta_1 \geq 0, \\ \delta_2 (1-x(T)) = 0, \quad \delta_2 \geq 0. \end{array} \right. \quad (2.56)$$

where,  $\nu = \nu_1 - \nu_2$ , satisfying the constraints

$$\begin{aligned} u(1-x) - x &\geq 0, \\ x - u(1-x) &\geq 0. \end{aligned} \quad (2.57)$$

and the slackness conditions

$$\left\{ \begin{array}{l} \nu_1(t) \geq 0, \quad \nu_1(t) x(t) = 0, \quad \dot{\nu}_1(t) \leq 0, \\ \nu_2(t) \geq 0, \quad \nu_2(t) (1-x(t)) = 0, \quad \dot{\nu}_2(t) \leq 0. \end{array} \right. \quad (2.58)$$

to determine the state  $x$  and the adjoint variable  $\lambda$ .

■

*Remark 2.1.6.* The system of ODE (2.56) - (2.58) is strongly coupled and therefore difficult to solve analytically. In Section 2.3 we will propose a numerical method for its solution.

## 2.2 Sufficiency Conditions for Optimal Control Problems

The maximum principle only provides necessary conditions for optimality, but it does not ensure that the given solution is optimal and whether an optimal solution exists. With some concavity conditions imposed on the Hamiltonian, the extremal provided by the maximum principle can be proven to be an optimal solution of the problem.

Optimal control theory has two main types of sufficiency theorems which are known as the Mangasarian and the Arrow theorems. Both of these theorems are important results since many applications rely on them [20].

Before we state the sufficiency conditions for problem (2.39) - (2.41), we must introduce the direct adjoining current value Hamiltonian function which is given by,

$$\begin{aligned} H^{c^d} &= H e^{rt} = F(t, x, u) + \lambda \cdot e^{rt} f(t, x, u) \\ &= F(t, x, u) + \lambda^d \cdot f(t, x, u), \end{aligned}$$

and the direct adjoining current value Lagrangian function which is given by,

$$\begin{aligned} L^{c^d} &= F(t, x, u) + p^d \cdot e^{rt} f(t, x, u) + \eta^d \cdot e^{rt} g(t, x, u) + \mu^d \cdot e^{rt} h(t, x) \\ &= H^{c^d} + \varphi^d \cdot g(t, x, u) + \nu^d \cdot h(t, x). \end{aligned}$$

where  $\lambda^d = e^{rt} \lambda$ ,  $\nu^d = e^{rt} \mu^d$  and  $\varphi^d = e^{rt} \eta^d$  are multipliers in the direct formulation which can be expressed in terms of the multipliers  $\lambda$ ,  $\nu$  and  $\varphi$  in the indirect formulation.

In order to formulate the sufficient conditions we need the following definition

$$\lambda^d(t) = \lambda(t) + \nu(t) h_x(t, x^*(t)) \quad \text{and} \quad \nu^d(t) = \nu(t).$$

We will now state the Mangasarian sufficiency theorem for OCP (2.39) - (2.41) following closely [21].

**Theorem 2.2.1** (Mangasarian Sufficiency Theorem). *Let the control  $u^*$  with corresponding state trajectory  $x^*$ , the adjoint function  $\lambda^d$ , the function  $\nu^d, \varphi^d$ , and the constants  $\delta$  and  $\zeta$  satisfy all conditions (i)-(vi) of Theorem (2.1.5) and let  $\lambda^d(t) = \lambda(t) + \nu(t)h_x(t, x^*(t))$ . Suppose that the set  $U$  is convex for every  $x \in \mathbb{R}^n, t \in [0, T]$ , the partial derivatives of the functions  $F(t, x, u)$  and  $f(t, x, u)$  are continuous with respect to their arguments. If the Hamiltonian function  $H^{c^d}(t, x, u, \lambda^d)$  exists and is concave in  $(x, u)$  for all  $t$ , the terminal payoff  $S(T, x)$  is concave in  $x$ ,  $g(t, x, u)$  is quasiconcave in  $(x, u)$  and the function  $h(t, x)$  is quasiconcave in  $x$ , then  $(x^*, u^*)$  is an optimal solution.*

*Furthermore if the Hamiltonian function  $H^{c^d}(t, x, u, \lambda^d)$  is strictly concave in  $(x, u)$  for all  $t$ , then the optimal solution  $(x^*, u^*)$  is unique.*

For most problems in optimal control theory sufficiency can be shown using the Mangasarian but there are many important problems in economics where we may find that the Hamiltonian function is not concave, hence we need to weaken this concavity condition as suggested by Arrow [17]. In order to do this we first need to define the *maximized Hamiltonian* function  $H^\sharp : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$  given as,

$$H^\sharp(t, x, u, \lambda^d) = \underset{u|g(t,x,u) \geq 0}{\text{maximize}} H^{c^d}(t, x, u, \lambda^d). \quad (2.59)$$

Let us assume that the maximisation problem in equation (2.59), has a unique solution denoted as  $u = u^\sharp(t, x, \lambda^d)$ . We can then write

$$H^\sharp(t, x, u, \lambda^d) = H^{c^d}(t, x, u^\sharp, \lambda^d), \quad (2.60)$$

Taking the partial derivatives with respect to  $x$  in equation (2.60) yields

$$H_x^\sharp(t, x, \lambda^d) = H_x^{c^d}(t, x, u^\sharp, \lambda^d) + H_u^{c^d}(t, x, u^\sharp, \lambda^d) \frac{\partial u^\sharp}{\partial x}. \quad (2.61)$$

To prove that

$$H_x^\sharp(t, x, u, \lambda^d) = H_x^{c^d}(t, x, u^\sharp, \lambda^d), \quad (2.62)$$

it is enough to prove that

$$H_u^{c^d}(t, x, u^\sharp, \lambda^d) \frac{\partial u^\sharp}{\partial x} = 0 \quad (2.63)$$

for all  $x$ . To do this we must consider the two cases of: (i) the unconstrained global maximum of  $H^{c^d}$  occurring in the interior of  $U(t)$ . So for this case we have  $H_u^{c^d}(t, x, u^\sharp, p) = 0$ . (ii) the unconstrained global maximum of  $H^{c^d}$  occurring outside of  $U(t)$ . So for this case we have  $\frac{\partial u^\sharp}{\partial x} = 0$ , since changing the state  $x$  does not effect the optimal value of the control  $u$ . We have then proven (2.62).

**Definition 2.2.2** (Quasi-concave). Consider the function  $f(x)$  defined on a convex subset  $S \in \mathbb{R}^n$  is said to be quasi-concave if for all real  $\alpha$  in  $\mathbb{R}$ , the set  $x \in S : f(x) \geq \alpha$  is convex.

We will now state the Arrow sufficiency theorem for OCP (2.39) - (2.41) following closely [21, 22].

**Theorem 2.2.3** (Arrow Sufficiency Theorem). *Let the control  $u^*$  with corresponding state trajectory  $x^*$ , the adjoint function  $\lambda^d$ , the function  $\nu^d, \varphi^d$ , and the constants  $\delta$  and  $\zeta$  satisfy all conditions (i)-(vi) of Theorem (2.1.5) and let  $\lambda^d(t) = \lambda(t) + \nu(t)h_x(t, x^*(t))$ . Suppose that the set  $U$  is convex for every  $x \in \mathbb{R}^n, t \in [0, T]$ , the partial derivatives of the functions  $F(t, x, u)$  and  $f(t, x, u)$  are continuous with respect to their arguments. If the Hamiltonian function  $H^\sharp(t, x, u, \lambda^d)$  defined in (2.59) exists and is concave in  $x$  for all  $t$ , the terminal payoff  $S(T, x)$  is concave in  $x$ ,  $g(t, x, u)$  is quasi-concave in  $(x, u)$  and the function  $h(t, x)$  is quasi-concave in  $x$ , then  $(x^*, u^*)$  is an optimal solution.*

*Furthermore if the maximized Hamiltonian function  $H^\sharp$  is strictly concave in  $x$  for all  $t$ , then the optimal state  $x^*$  (but not necessarily  $u^*$ ) is unique.*

To summarize, the Mangasarian sufficiency theorem basically says that the maximum principle is also sufficient if the Hamiltonian is concave with respect to both the state  $x$  and the control  $u$ , whereas with the Arrow sufficiency theorem we need the maximized Hamiltonian to be concave with respect to only the state  $x$  [20].

## 2.3 Numerical Solutions of Optimal Control Problems

This section is devoted to the presentation of the Forward-Backward sweep algorithm for the solution of optimal control problems. The algorithm alternates the numerical solution of the state equations forward in time, the adjoint equation backward in time and the update of the optimal control using the maximisation principle.

We present the algorithm for a basic optimal control problems without constraints. It can be extended in a straightforward way to a problem with constraints.

Consider the controlled dynamics below

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \quad t \in [0, T], \quad (2.64)$$

with payoff functionals given as

$$J = S(x(T)) + \int_0^T F(t, x(t), u(t)) dt. \quad (2.65)$$

An application of the maximum principle gives the following necessary conditions for optimality

$$\dot{x}^* = \nabla_p H(t, x^*, u^*, p), \quad x^*(0) = x_0, \quad (2.66)$$

$$\dot{p}^* = -\nabla_x H(t, x^*, u^*, p), \quad p(T) = \nabla S(x(T)), \quad (2.67)$$

$$H(t, x^*, u^*, p) \geq H(t, x^*, u, p), \quad (2.68)$$

for the Hamiltonian function

$$H(t, x, u, p) = p \cdot f(t, x, u) + F(t, x, u),$$

### 2.3.1 Forward-Backward Sweep Algorithm

Consider the following algorithm from [23].

**Algorithm 2.3.1** (Forward-Backward Sweep). *The iterative algorithm to solve (2.64)-(2.65) when  $H$  is a concave function of  $u$  is given by*

1. *Discretise the time variable  $t \in [0, T]$  as  $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \dots, t_k = k\Delta t, \dots, t_N = N\Delta t$ , where  $\Delta t$  is the time step and consider a piece-wise constant guess of the control variable*

$$u(t) = u_k, \quad t \in [t_k, \dots, t_N],$$

2. *Using the initial guess for  $u$  from step 1 and the initial condition  $x(0) = x_0$ , we solve the state equation (2.66) forward in time.*

3. Using the transversality condition  $p(T) = \nabla_x S(x(T))$  the values for  $u$  and  $x$ , we solve the adjoint equation (2.67) backwards in time for the adjoint  $p$ .
4. Update the control  $u$  using the maximality condition (2.68).
5. Check the convergence that is, check that for  $v \in x, u, p$ ,

$$\epsilon \sum_{i=1}^{N+1} |v(i)| - \sum_{i=1}^{N+1} |v(i) - v_{old}(i)| \geq 0, \quad (2.69)$$

where  $\epsilon$  is the tolerance,  $v(i)$  is the solution at the current iteration and  $v_{old}(i)$  is the solution at the last iteration.

- (a) If (2.69) is satisfied, then end the iterative procedure and output the optimal solution
- (b) If (2.69) is not satisfied, then go back to step 2.

In steps 2 and 3 any good ODE solver may be used. We will use the Runge-Kutta 4 (RK4) algorithm. Hence in order to solve the state given by the differential equation

$$\dot{x} = f(t, x(t), u(t)), \quad x(0) = x_0,$$

forward in time using RK4, we have

For  $i = 1, 2, \dots, N$ ,

$$x(i+1) = x(i) + \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4),$$

where  $K_1 = f(t(i), x(i), u(i))$ ,

$$K_2 = f(t(i) + \frac{h}{2}, x(i) + \frac{h}{2}K_1, \frac{1}{2}(u(i) + u_{i+1})),$$

$$K_3 = f(t(i) + \frac{h}{2}, x(i) + \frac{h}{2}K_2, \frac{1}{2}(u(i) + u(i+1))),$$

$$K_4 = f(t(i) + h, x(i) + hK_3, u(i+1)),$$

$$h = \frac{T}{N},$$

and solving the adjoint given by the differential equation

$$\dot{p} = -\nabla_x H(t, x, u, p), \quad p(T) = \nabla S(x(T)),$$

backward in time using RK4 gives us

For  $i = N + 1, N, N - 1, \dots, 2$

$$p(i - 1) = p(i) - \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4),$$

where  $K_1 = -\nabla_x H(t(i), x(i), u(i), p(i))$ ,

$$K_2 = -\nabla_x H(t(i) - \frac{h}{2}, \frac{1}{2}(x(i) + x(i - 1)), \frac{1}{2}(u(i) + u(i - 1)), p(i) - \frac{h}{2}K_1),$$

$$K_3 = -\nabla_x H(t(i) - \frac{h}{2}, \frac{1}{2}(x(i) + x(i - 1)), \frac{1}{2}(u(i) + u(i - 1)), p(i) - \frac{h}{2}K_2),$$

$$K_4 = -\nabla_x H(t(i) - h, x(i - 1), u(i - 1), p(i) - hK_3),$$

$$h = \frac{T}{N}.$$

We will demonstrate below that the RK4 is accurate for the solution of optimal control problems. For this we will display for some problems, the norm of the difference between the exact solution and the approximate solution.

There are different norms that can be used to measure the error in the solution of an optimal control problem. If there are discontinuities or singularities in the solution then the  $L_1$  norm is found to be the most suitable to use. The  $L_2$  norm is used when there is a uniform mesh. The max (or infinity) norm is the most sensitive measure of error and returns the value of the maximum absolute error over the entire domain of solutions [24].

We recall below the definitions of the  $L_1, L_2$  and  $L_\infty$  error norms of any vector  $y = (y_1, \dots, y_N)$ .

$$L_1 = || \hat{y} - y ||_1 = \sum_{i=1}^N | \hat{y}_i - y_i |,$$

$$L_2 = || \hat{y} - y ||_2 = \left( \sum_{i=1}^N | \hat{y}_i - y_i |^2 \right)^{1/2},$$

$$L_\infty = || \hat{y} - y ||_\infty = \max | \hat{y}_i - y_i |, \quad i = 1, 2, \dots, N.$$

where  $\hat{y}$  and  $y$  denotes the respective vectors of size  $N$  of the approximate and exact solutions.

Consider the following example from [17].

**Example 2.3.1.** Find the optimal control which maximizes the payoff

$$P = \int_0^T -(x(t)^2 + c \cdot u(t)^2) dt,$$

subject to the system with dynamics

$$\dot{x}(t) = u(t), \quad x(0) = x_0,$$

where  $c > 0$ .

The exact solutions were found analytically by Sydsæter et al. [17] to be

$$u^*(t) = x_0 \left( \frac{e^{-r(T-t)} - e^{r(T-t)}}{r^3 (e^{rT} + e^{-rT})} \right),$$

$$x^*(t) = x_0 \left( \frac{e^{r(T-t)} + e^{-r(T-t)}}{e^{rT} + e^{-rT}} \right),$$

$$p^*(t) = 2x_0 \left( \frac{e^{-r(T-t)} - e^{r(T-t)}}{r (e^{rT} + e^{-rT})} \right).$$

with  $r = \frac{1}{\sqrt{c}}$

**Solution.** The Hamiltonian for the problem is given by

$$\begin{aligned} H(t, x, u, p) &= p \cdot f(t, x, u) + F(t, x, u) \\ &= p \cdot u - x^2 - c \cdot u^2. \end{aligned}$$

Applying the maximum principle, we obtain the following necessary conditions which will assist us in solving the problem numerically

$$\begin{cases} u = \frac{p}{2c}, \\ \dot{x} = u, & x(0) = x_0, \\ \dot{p} = 2x, & p(T) = 0. \end{cases}$$

We will use the values of  $x(0) = 1$  for the initial condition,  $c = 1$  for the constant  $c$ ,  $T = 2$  for the terminal time,  $N = 1000$  for the number of subintervals and  $\epsilon = 0.001$  for the tolerance to solve the problem numerically using Algorithm 2.3.1.

Hence using Algorithm 2.3.1 we obtain the following graphical results given by Figures 2.2 - 2.3

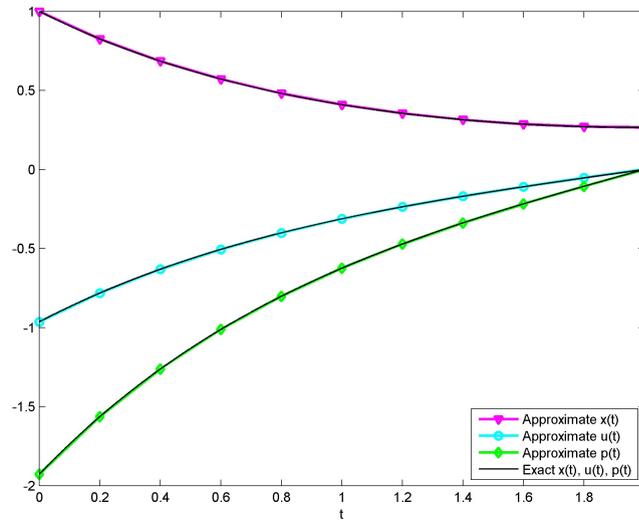


FIGURE 2.2

In Figure 2.2 we present the approximated optimal solutions of the state, control and adjoint plotted against their respective exact solutions for Example 2.3.1. The fact that the approximated optimal solution of the control, state and adjoint lies on the curve of its respective exact solution verifies that the results obtained from Algorithm 2.3.1 are indeed correct.

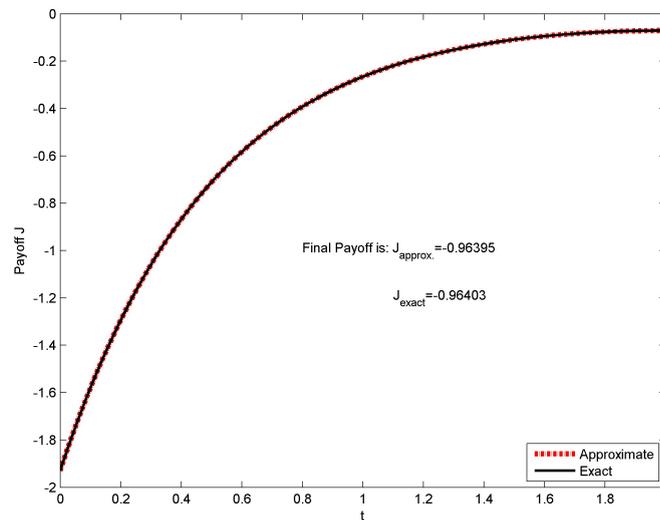


FIGURE 2.3: Optimal Payoff  $J^*(t)$  for Example 2.1.2

In Figure 2.3 we present the approximated optimal solution plotted against the exact solution of the payoff  $J^*(t)$  for Example 2.3.1. The value of the final payoff is given for the approximate solution as  $J_{\text{approx}}^*(t) = 0.96395$  and for the exact solution as  $J_{\text{exact}}^*(t) = 0.96403$ .

In order to verify our solution we will need to check the efficiency and accuracy of the results by finding the error norms.

$N$	$\ x_{\text{approx.}} - x_{\text{exact}}\ _{\infty}$	$\ u_{\text{approx.}} - u_{\text{exact}}\ _{\infty}$	$\ p_{\text{approx.}} - p_{\text{exact}}\ _{\infty}$
10	$1.8381 \times 10^{-3}$	$5.9773 \times 10^{-4}$	$1.4647 \times 10^{-3}$
50	$2.0976 \times 10^{-4}$	$8.4515 \times 10^{-4}$	$2.1960 \times 10^{-4}$
100	$2.2935 \times 10^{-4}$	$9.0854 \times 10^{-4}$	$1.8572 \times 10^{-4}$
500	$2.7569 \times 10^{-4}$	$9.6262 \times 10^{-4}$	$1.7581 \times 10^{-4}$
1000	$2.8006 \times 10^{-4}$	$9.6957 \times 10^{-4}$	$1.7552 \times 10^{-4}$

TABLE 2.1: Error norms for Example 2.3.1 for various values of  $N$ .

■

**Example 2.3.2.** Recall that previously the solution of Example 2.1.2 was found to satisfy

$$\left\{ \begin{array}{l} u^\# = -px \\ \dot{x} = x(d - u^\#) \\ \dot{p} = -d - (p + \mu) \cdot (d - u^\#), \end{array} \right. \quad \begin{array}{l} x(0) = x_0, \\ p(T) = \gamma, \\ \gamma \geq 0, \quad \gamma x(T) = 0. \end{array} \quad (2.70)$$

which satisfy the constraints

$$x(d - u) \geq 0, \quad (2.71)$$

and the slackness conditions

$$\mu \geq 0, \quad \mu x = 0, \quad \dot{\mu} \leq 0. \quad (2.72)$$

**Solution.** We will use the values of  $x(0) = 1$  for the initial condition,  $d = 0.8$  for the consumption rate  $d$ ,  $T = 1$  for the terminal time,  $N = 1000$  for the number of

subintervals and  $\epsilon = 0.001$  for the tolerance to solve the problem numerically using Algorithm 2.3.1.

Hence using Algorithm 2.3.1 we obtain the following graphical results given by Figure 2.4 - 2.5

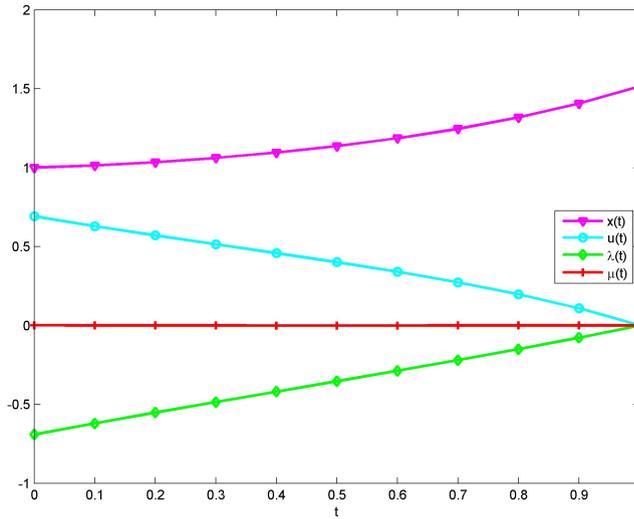


FIGURE 2.4: Optimal State  $x^*(t)$ , Control  $u^*(t)$ , Adjoint  $\lambda^*(t)$  and Multiplier  $\mu^*(t)$  for Example 2.1.2

In Figure 2.4 we present the approximated optimal solutions of the state, control, adjoint and Lagrange multiplier for Example 2.1.2.

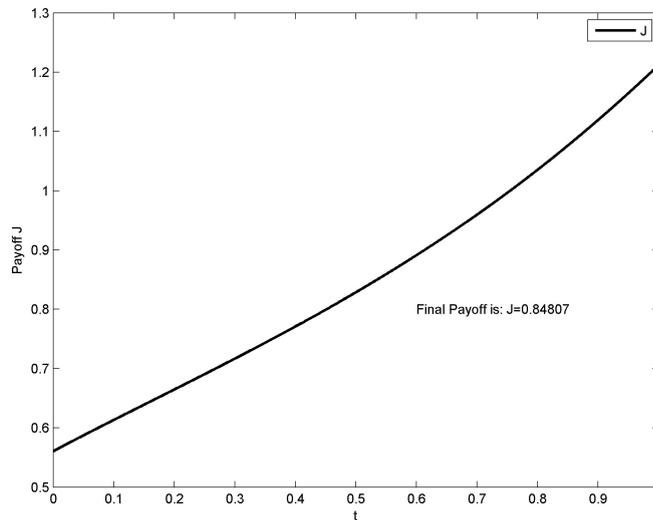


FIGURE 2.5: Payoff  $J$  for Example 2.1.2

In Figure 2.5 we present the approximated optimal solution of the payoff  $J^*(t)$  for Example 2.1.2. The value of the final payoff is given for the approximate solution as  $J^*(t) = 0.84807$ .

■

**Example 2.3.3.** *Recall that previously the solution of Example 2.1.1 was found to satisfy*

$$\left\{ \begin{array}{ll} u^\# = \begin{cases} 1 & \text{if } p_1 - p_2 > 0, \\ 0 & \text{if } p_1 - p_2 < 0. \end{cases} \\ x_1 = \alpha u^\# x_1, & x_1(0) = x_{10}, \\ x_2 = \alpha(1 - u^\#)x_1, & x_2(0) = x_{20}, \\ \dot{p}_1 = -p_1\alpha u - p_2\alpha(1 - u), & p_1(T) = 0, \\ \dot{p}_2 = -1, & p_2(T) = 0, \end{array} \right. \quad (2.73)$$

with the exact solutions given as

$$u^* = \begin{cases} 1 & \text{if } 0 \leq t \leq T - 2, \\ 0 & \text{if } T - 2 < t \leq T, \end{cases}$$

$$x_1^* = \begin{cases} x_{10}e^{\alpha t} & \text{if } 0 \leq t \leq T - 2, \\ x_{10}e^{\alpha T - 2} & \text{if } T - 2 < t \leq T, \end{cases}$$

$$x_2^* = \begin{cases} x_{20} & \text{if } 0 \leq t \leq T - 2, \\ x_{20} + \alpha x_{10}(t - T + \frac{2}{\alpha})e^{\alpha T - 2} & \text{if } T - 2 < t \leq T, \end{cases}$$

$$p_1^* = \begin{cases} \frac{2}{\alpha}e^{(-\alpha t + \alpha T - 2)} & \text{if } 0 \leq t \leq T - 2, \\ \frac{\alpha}{2}(t - T)^2 & \text{if } T - 2 < t \leq T, \end{cases}$$

$$p_2^* = T - t.$$

**Solution.** We will use the values of  $x_1(0) = 0.6$  and  $x_2(0) = 0.4$  for the initial conditions,  $\alpha = 1$ , for the constant,  $T = 3$  for the terminal time,  $N = 1000$  for the number of subintervals and  $\epsilon = 0.001$  for the tolerance to solve the problem numerically using Algorithm 2.3.1.

By using Algorithm 2.3.1 we obtain the following graphical results given by Figure 2.6 - 2.9

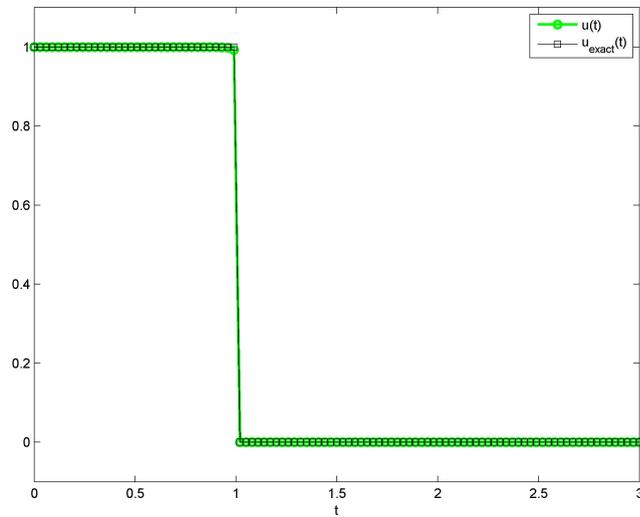


FIGURE 2.6: Optimal Control  $u^*(t)$  for Example 2.1.1

In Figure 2.6 we present the approximated optimal solutions plotted against the exact solutions of the control for Example 2.1.1.

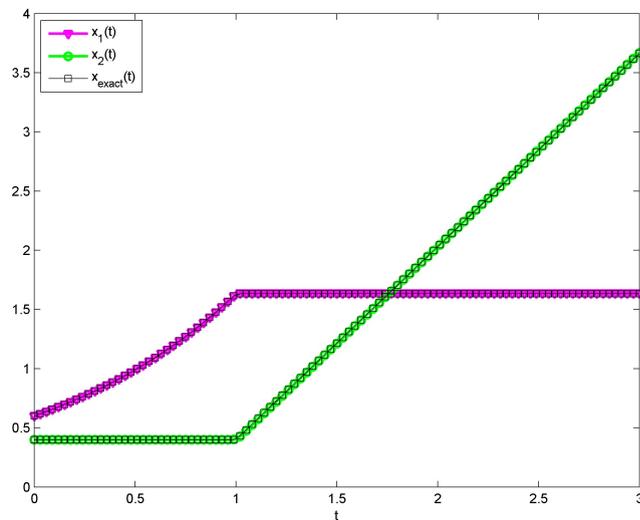


FIGURE 2.7: Optimal States  $x_1^*(t), x_2^*(t)$  for Example 2.1.1

In Figure 2.7 we present the approximated optimal solutions plotted against the exact solutions of the state for Example 2.1.1.

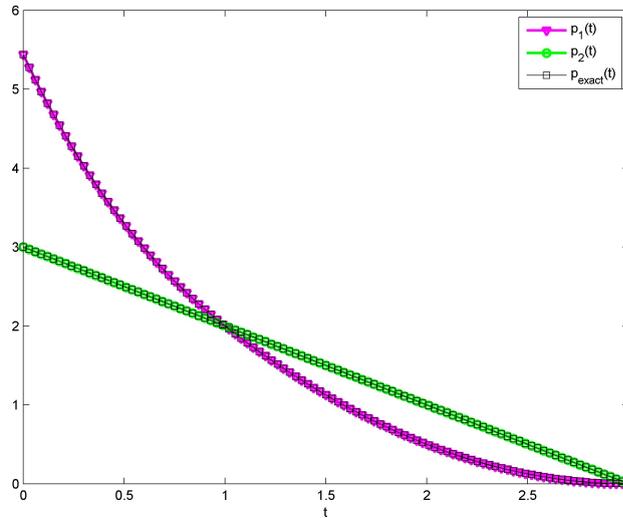


FIGURE 2.8: Optimal Ajoins  $p_1^*(t), p_2^*(t)$  for Example 2.1.1

In Figure 2.8 we present the approximated optimal solutions plotted against the exact solutions of the adjoint for Example 2.1.1.

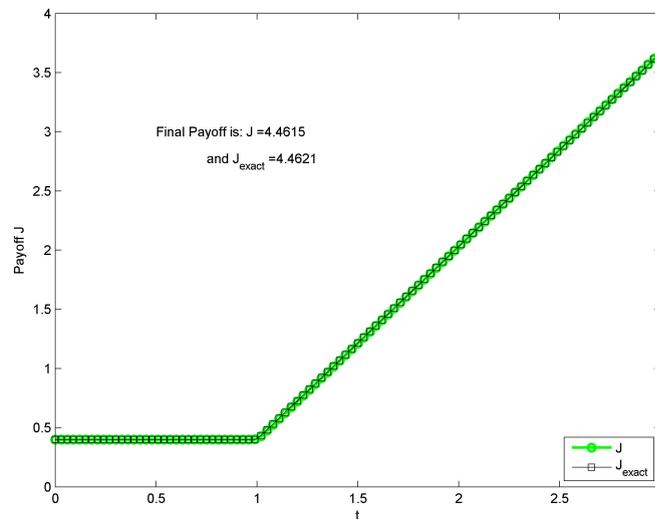


FIGURE 2.9: Payoff  $P$  for Example 2.1.1

In Figure 2.9 we present the approximated optimal solution of the payoff  $J^*(t)$  for Example 2.1.1. The value of the final payoff is given for the approximate solution as  $J^*(t) = 4.4615$  and for the exact solution as  $J_{\text{exact}}^*(t) = 4.4621$ .

The fact that the approximated optimal solution of the control, state, adjoint and payoff lies on the curve of its respective exact solution verifies that the results obtained from Algorithm 2.3.1 are indeed correct.

The following table verifies the efficiency and accuracy of the results by displaying the error norms.

$N$	<i>Iterations</i>	$\ x_1 - \tilde{x}_1\ _\infty$	$\ x_2 - \tilde{x}_2\ _\infty$	$\ u - \tilde{u}\ _\infty$	$\ p_1 - \tilde{p}_1\ _\infty$	$\ p_2 - \tilde{p}_2\ _\infty$
100	14	$7.0831 \times 10^{-3}$	$6.8497 \times 10^{-3}$	$7.7515 \times 10^{-3}$	$6.5239 \times 10^{-4}$	$6.5239 \times 10^{-4}$
250	14	$2.1739 \times 10^{-3}$	$1.9344 \times 10^{-3}$	$1.5564 \times 10^{-2}$	$5.3563 \times 10^{-4}$	$5.3563 \times 10^{-4}$
500	14	$2.8851 \times 10^{-3}$	$3.0193 \times 10^{-3}$	$1.5564 \times 10^{-3}$	$5.1461 \times 10^{-3}$	$5.1461 \times 10^{-3}$
1000	14	$1.4409 \times 10^{-3}$	$1.2975 \times 10^{-3}$	$6.2439 \times 10^{-2}$	$5.1200 \times 10^{-4}$	$5.1200 \times 10^{-4}$

TABLE 2.2: Error norms for Example 2.1.1 for various values of  $N$ .

■

**Example 2.3.4.** Recall that previously the solution of Example 2.1.4 was found to satisfy

$$\left\{ \begin{array}{l} u^\# = \frac{\lambda \cdot (1 - x)}{k}, \\ \dot{x} = u^\#(1 - x) - x, \quad x(0) = x_0, \\ \dot{\lambda} - r\lambda = -\phi + (\lambda + \nu)(u^\# + 1), \quad \lambda(T) = \delta_1 - \delta_2, \\ \delta_1 x(T) = 0, \quad \delta_1 \geq 0, \\ \delta_2 (1 - x(T)) = 0, \quad \delta_2 \geq 0. \end{array} \right.$$

where  $\nu = \nu_1 - \nu_2$ , satisfying the constraints

$$\begin{aligned} u(1 - x) - x &\geq 0, \\ x - u(1 - x) &\geq 0, \end{aligned} \tag{2.74}$$

and the slackness conditions

$$\left\{ \begin{array}{l} \nu_1(t) \geq 0, \quad \nu_1(t) x(t) = 0, \quad \dot{\nu}_1(t) \leq 0, \\ \nu_2(t) \geq 0, \quad \nu_2(t) (1 - x(t)) = 0, \quad \dot{\nu}_2(t) \leq 0. \end{array} \right. \tag{2.75}$$

to determine the state  $x$  and the adjoint variable  $\lambda$ .

**Solution.** We will use the values of

$x(0) = 0.45$  for the initial condition,

$k = 1$  for the constants,

$r = 0.09$  for the interest rates,

$\phi = 0.5$  for the fractional revenues,

$T = 1$  for the terminal time,

$N = 1000$  for the number of subintervals and

$\epsilon = 0.001$  for the tolerance

to solve the problem numerically using Algorithm 2.3.1.

Hence using Algorithm 2.3.1 we obtain the following graphical results given by Figure 2.10 - 2.11

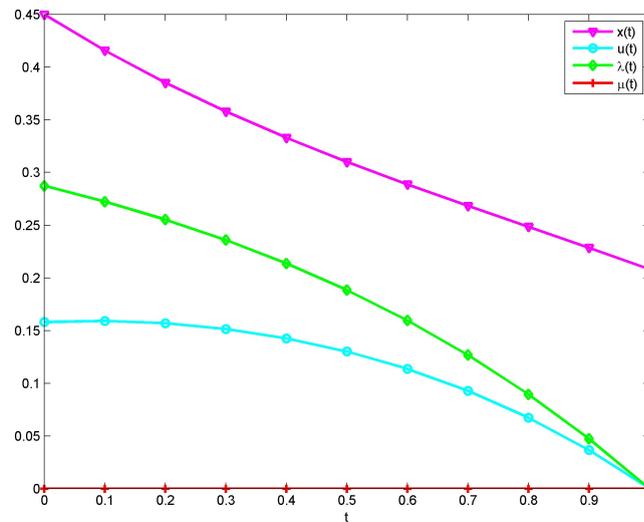
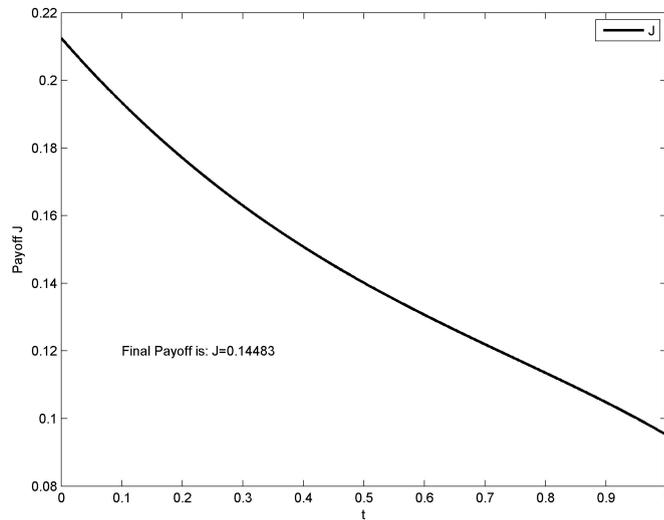


FIGURE 2.10: Optimal State  $x^*(t)$ , Control  $u^*(t)$ , Adjoint  $\lambda^*(t)$  and Multiplier  $\mu^*(t)$  for Example 2.1.4

In Figure 2.10 we present the approximated optimal solutions of the state, control, adjoint and Lagrange multiplier for Example 2.1.4.

FIGURE 2.11: Payoff  $P$  for Example 2.1.4

In Figure 2.11 we present the approximated optimal solution of the payoff  $J^*(t)$  for Example 2.1.4. The value of the final payoff is given for the approximate solution as  $J^*(t) = 0.14483$ .

■

## 2.4 Constrained Optimization

The general form of a constrained optimization problem (COP) is

$$\begin{aligned}
 & \underset{x,u}{\text{maximize}} && f(x) \\
 & \text{subject to} && g_i(x) = 0, \quad i = 1, 2, \dots, m, \\
 & && h_j(x) \geq 0, \quad j = 1, 2, \dots, p
 \end{aligned} \tag{2.76}$$

where  $f$  and all the  $g_i$ 's and  $h_j$ 's are differentiable.

By use of the Karush-Kuhn-Tucker (KKT) conditions the idea of Lagrange multipliers can be extended to handle inequality constraints as well as equality constraints. These conditions provide a first order optimality condition for the problem (2.76) known as

the KKT conditions. The full set of KKT conditions are given by,

$$\begin{aligned}
\nabla f(x) + \sum_{i=1}^m \eta_i \nabla g_i(x) + \sum_{j=1}^p \mu_j \nabla h_j(x) &= 0, \\
g_i(x) &= 0, \quad \text{for } i = 1, 2, \dots, m \\
\mu_j h_j(x) &= 0, \quad \text{for } j = 1, 2, \dots, p \\
h_j(x) &\geq 0, \quad \text{for } j = 1, 2, \dots, p \\
\mu_j &\geq 0, \quad \text{for } j = 1, 2, \dots, p
\end{aligned} \tag{2.77}$$

where  $\eta_1, \eta_2, \dots, \eta_m$  are the Lagrange multipliers and  $\mu_1, \mu_2, \dots, \mu_p$  are the KKT multipliers.

Let us consider the utility maximization problem. This problem is studied in various areas of economics and finance [25–27]. For the basic problem a utility function determines the satisfaction or utility experienced by a consumer after receiving a certain amount of a specific good (or service) and a different amount of another good (or service). The problem the consumer faces is to calculate the amount of each good (or service) they need to buy in order to maximize their utility while abiding by a fixed budget.

Consider the simple modified utility maximization problem from [28].

**Example 2.4.1.** *Consider a firm selling two products in a competitive market where,*

$$\begin{aligned}
x_1 &= \text{a unit of the first product,} \\
x_2 &= \text{a unit of the second product,}
\end{aligned}$$

*A unit of the first product  $x_1$  costs  $p_1$  and a unit of the second product  $x_2$  costs  $p_2$ . The consumer wishes to spend a complete total of  $B$ . The consumer can buy no more than  $d_1$  units of the first product and  $d_2$  units of the second product. The consumer aims to maximize their utility which is given by*

$$U(x_1, x_2) = a \ln x_1 + (1 - a) \ln x_2$$

**Solution.** Mathematically this problem is to be formulated as the following COP,

$$\begin{aligned} & \underset{x_1, x_2}{\text{maximize}} \quad a \ln x_1 + (1 - a) \ln x_2 \\ & \text{subject to} \quad p_1 x_1 + p_2 x_2 = B, \\ & \quad \quad \quad x_1 \leq d_1, \\ & \quad \quad \quad x_2 \leq d_2. \end{aligned}$$

Let  $a = 0.4$ ,  $p_1 = 20$ ,  $p_2 = 30$ ,  $B = 600$ ,  $d_1 = 20$ ,  $d_2 = 15$ . The Lagrangian for the problem is given as,

$$L(x_1, x_2, \eta, \mu_1, \mu_2) = 0.4 \ln x_1 + 0.6 \ln x_2 + \eta [600 - 20x_1 - 30x_2] + \mu_1 [20 - x_1] + \mu_2 [15 - x_2].$$

Applying the KKT conditions we get the following system of equations,

$$\left\{ \begin{array}{l} \frac{2}{5x_1} - 20\eta - \mu_1 = 0, \quad \frac{3}{5x_2} - 30\eta - \mu_2 = 0, \\ 600 - 20x_1 - 30x_2 = 0, \\ 20 - x_1 \geq 0, \quad 15 - x_2 \geq 0, \\ \mu_1 [20 - x_1] = 0, \quad \mu_2 [15 - x_2] = 0, \\ \mu_1 \geq 0, \quad \mu_2 \geq 0. \end{array} \right. \quad (2.78)$$

We will now consider the following four cases,

Case 1:  $\mu_1 = 0$ ,  $\mu_2 = 0$ ,  $20 - x_1 \neq 0$ ,  $15 - x_2 \neq 0$ , for this case solving the above system of equations given by (2.78) simultaneously we get the values of,

$$x_1 = 12, \quad x_2 = 12, \quad \eta = \frac{1}{600}, \quad \mu_1 = 0, \quad \mu_2 = 0.$$

Case 2:  $\mu_1 = 0$ ,  $\mu_2 \neq 0$ ,  $20 - x_1 \neq 0$ ,  $15 - x_2 = 0$ , for this case solving the above system of equations given by (2.78) simultaneously we get the values of,

$$x_1 = \frac{15}{2}, \quad x_2 = 15, \quad \eta = \frac{1}{375}, \quad \mu_1 = 0, \quad \mu_2 = -\frac{1}{25}.$$

Case 3:  $\mu_1 \neq 0$ ,  $\mu_2 = 0$ ,  $20 - x_1 = 0$ ,  $15 - x_2 \neq 0$ , for this case solving the above system of equations given by (2.78) simultaneously we get the values of,

$$x_1 = 20, \quad x_2 = \frac{20}{3}, \quad \eta = \frac{3}{1000}, \quad \mu_1 = -\frac{1}{25}, \quad \mu_2 = 0.$$

Case 4:  $\mu_1 \neq 0$ ,  $\mu_2 \neq 0$ ,  $20 - x_1 = 0$ ,  $15 - x_2 = 0$ , for this case solving the above system of equations given by (2.78) simultaneously we get the values of  $x_1$ , and  $x_2$  as 20 and 15 respectively. Notice that for these values of  $x_1$  and  $x_2$  the budget constraint  $20x_1 + 30x_2 = 600$  is not satisfied, hence we can disregard this case.

From the dual feasibility condition of the KKT we know that  $\mu_1 \geq 0$  and  $\mu_2 \geq 0$  hence we only consider those solutions which satisfy this condition. The optimal solution is identified as,

$$x_1 = 12, x_2 = 12, \eta = \frac{1}{600}, \mu_1 = 0, \mu_2 = 0.$$

with the maximum  $f(x) = 0.4 \ln x_1 + 0.6 \ln x_2 = \ln(12)$ .

■

## Chapter 3

# *N*-player Differential Games

Previously in Chapter 2 we dealt with the case where a single individual made a decision and optimized a payoff. In this chapter, we consider the case of an  $N$ -player differential game where there are many decision makers, which we will call players, aim to optimize their respective objective functions under the dynamics of the game.

Differential games are an extension of static game where players choose their strategy from a finite number of possibilities. These games are also called matrix game [29–33]. Unlike in optimal control or optimisation problems where it is very clear what optimality is, in game theory the concept of solution is not trivial. This is because what one player considers as optimality can be very disastrous for any of the other players.

Different concepts of solutions have been proposed [30]. One solution concept is that of Pareto optimality, where it is not possible to strictly increase the payoff of one player without strictly decreasing the payoff of the other. Another concept of solution in game theory, which we will focus on in this dissertation is the concept of Nash equilibrium. A set of strategies is called a Nash equilibrium if any player who deviates from the equilibrium strategy fails to improve his payoff. Other solution concepts can be found in [34–37].

A noteworthy classification in game theory is between cooperative and non-cooperative games. In a cooperative game, the players can communicate and negotiate with each other and arrive at a mutual decision for their actions [38]. In a non-cooperative game there is no form of communication between the players, therefore each player will aim to optimize their objective function whilst deducing the behaviour of the other players.

### 3.1 Formulation of the Problem

**Definition 3.1.1** (*N*-player differential game). In general a *N*-player differential game played on a time interval  $[0, T]$  has following elements:

- (i) A set of players  $N = \{1, 2, \dots, n\}$ ,
- (ii) For each Player  $i \in N$ , a vector of controls  $\mathbf{u}_i(t) \in U_i \subset \mathbb{R}^{n_i}$ , where  $U_i$  is the set of admissible control values for Player  $i$ ,
- (iii) A vector of state variables  $x \in X \subset \mathbb{R}^n$ , where  $X$  is the set of admissible states. The evolution of the state variables is governed by a system of differential equations, called the state equations:

$$\dot{x}(t) = f(t, x(t), \mathbf{u}(t)), \quad x(0) = x_0 \quad (3.1)$$

where  $\mathbf{u}(t) = (u_1(t), u_2(t), \dots, u_n(t))$ . Equation (3.1) is often called the state equation. It describes the state of the game at every time  $t$ .

- (iv) A payoff for Player  $i \in N$ ,

$$J_i(\mathbf{u}) \doteq S_i(x(T)) + \int_0^T F_i(t, x(t), \mathbf{u}(t)) dt, \quad (3.2)$$

where function  $F_i$  is Player  $i$ 's running payoff and function  $S_i$  is his terminal payoff,

- (v) An information structure, that is information available to Player  $i$  when he chooses strategy  $u_i(t)$  at time  $t$ ,
- (vi) A strategy set  $\Psi_i$ , where a strategy  $\psi_i \in \Psi_i$  is a decision rule that defines the control  $u_i(t) \in U_i$  as a function of the information available at time  $t$ .

We may have additional constraints on either the control set of the players  $U_i$  or on the state of the game  $X$ . This may lead to having a strategy set depending on the control set  $U_i$ . In this dissertation, we will consider the case for where the set of admissible states is defined via an inequality of the form,

$$h(t, x) \geq 0, \quad (3.3)$$

which is called a pure state constraint.

In order to have a well posed problem we need to specify what information is available to the players.

**Definition 3.1.2.** In the *N*-player differential game given by Definition 3.1.1 of a duration  $[0, T]$ , we say that Player *i*'s information structures is

- (i) **Open-loop** if at time  $t$  the only information available to Player *i* is the initial state of the game  $x_0$ , hence his strategy set can be written as  $\Psi_i(t) = \{x_0\}, t \in [0, T]$ .
- (ii) **Feedback** if Player *i* uses the state of the game  $(t, x(t))$  as information basis, hence his strategy set can be written as  $\Psi_i(t) = \{x(t)\}, t \in [0, T]$ .
- (iii) **Closed-loop perfect state** if Player *i* recalls the whole history of the state up to time  $t$ , hence his strategy set can be written as  $\Psi_i(t) = \{x(s), 0 \leq s \leq t\}, t \in [0, T]$ .
- (iv)  **$\epsilon$ -delayed closed-loop perfect state** if the information available to Player *i* is the state with a delay of  $\epsilon$  units, hence his strategy set can be written as  $\Psi_i(t) = \{x(s), 0 \leq s \leq t - \epsilon\}$ , where  $\epsilon > 0$  is fixed.

Note that we will be focusing on open-loop games only. Feedback equilibrium solutions are found using a dynamic programming approach. The theorem presented below, specifies conditions on the functions  $f, F, \psi$  for which the differential equation given by (3.1) has a unique solution.

**Theorem 3.1.3.** For the *N*-player differential game given by Definition 3.1.1 of a duration  $[0, T]$  which has any one of the information structures given above. If

- (i)  $f(t, x, u_1, u_2, \dots, u_n)$  is continuous in  $t \in [0, T]$  for each  $x \in \mathbb{R}^n, i \in N$ ,
- (ii)  $f(t, x, u_1, u_2, \dots, u_n)$  is uniformly Lipschitz in  $x, u_1, u_2, \dots, u_n$ . That is for some  $k > 0$ ,

$$|f(t, x, u_1, u_2, \dots, u_n) - f(t, \bar{x}, \bar{u}_1, \bar{u}_2, \dots, \bar{u}_n)| \leq k \max_{0 \leq t \leq T} \left\{ |x(t) - \bar{x}(t)| + \sum_{i \in N} |u_i - \bar{u}_i| \right\},$$

$$x(\cdot), \bar{x}(\cdot) \in \mathbf{C}^n[0, T],$$

$$u_i(\cdot), \bar{u}_i(\cdot) \in \mathbf{U}_i \quad (i \in N)$$

- (iii) for  $\psi_i \in \Psi_i, (i \in N)$ ,  $\psi_i(t, x)$  is continuous in  $t$  for each  $x(\cdot) \in \mathbf{C}^n[0, T]$  and uniformly Lipschitz in  $x(\cdot) \in \mathbf{C}^n[0, T]$ ,

differential equation given by (3.1) has a unique solution that is continuous.

### 3.1.1 Open-Loop Nash Equilibrium

The idea of a Nash equilibrium was proposed by John Forbes Nash, Jr [20, 39]. Game situations where cooperation between the players are not allowed normally concern Nash equilibrium solutions. The Nash equilibrium solution concept safe guards against any attempts by a single player of unilaterally altering his strategy, since a deviation from the Nash equilibrium may not be beneficial to him [34].

For brevity, we denote  $[u_i, u_{-i}^*]$  as the  $n$ -tuple of strategies for the players such that

$$[u_i, u_{-i}^*] = (u_1^*, u_2^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_n^*)$$

The Nash equilibrium can be defined as,

**Definition 3.1.4** (Nash Equilibrium). A Nash solution  $u_i^*$ ,  $i \in N$  is defined by

$$J_i(\mathbf{u}^*) \geq J_i([u_i, u_{-i}^*]), \quad (3.4)$$

for all admissible  $u_i$ ,  $i \in N$ , where  $J_i$  is the criterion which Player  $i$  wants to maximize.

and hence in terms of optimization and optimal control we have,

**Theorem 3.1.5** (Nash Equilibrium). *The  $n$ -tuple of control functions  $u^*$  is an open-loop Nash Equilibrium for the  $N$ -player differential game given by Definition 3.1.1 and constrained by (3.3) if for all  $i \in N$  the following holds*

- For Player  $i$ , the control  $u_i^*$  gives the solution to the OCP:

$$\max J_i([u_i, u_{-i}^*]) = S_i(x(T)) + \int_0^T F_i(t, x, [u_i, u_{-i}^*]) dt, \quad (3.5)$$

over all the controls  $u_i$ , for the system with dynamics

$$\dot{x} = f(t, x, [u_i, u_{-i}^*]), \quad x(0) = x_0 \in \mathbb{R}^n, \quad (3.6)$$

subject to the pure state constraints

$$h(t, x(t)) \geq 0. \quad (3.7)$$

From Definition 3.1.5, we can see that finding a Nash equilibrium amounts to simultaneously solving optimal control problems, one for each player. Thanks to the maximum principle, we can derive a set of necessary conditions for the Nash equilibrium of any differential game. For  $i = 1, 2, \dots, n$  we introduce the Hamiltonian of Player  $i$  as

$$H_i(t, x, \mathbf{u}, p_i) = p_i \cdot f(t, x, \mathbf{u}) + F_i(t, x, \mathbf{u}), \quad (3.8)$$

which allows us to handle the constraints (3.6) through a multiplier  $p$ . Using the indirect adjoining method [see, Section 2.1], we define the Lagrangian of Player  $i$  as,

$$L_i[t, x, \mathbf{u}, p_i, \mu_i] = H_i(t, x, \mathbf{u}, p_i) + \mu_i \cdot \dot{h}(t, x, \mathbf{u}), \quad (3.9)$$

which allows us to handle the constraints (3.7) through a multiplier  $\mu$ .

We require that the maximisation condition of the maximum principle [see, Theorem 2.1.4], always has a solution as presented in the following assumption.

**Assumption 1.** For all  $(t, x) \in [0, T] \times \mathbb{R}^N$  and any vectors  $p_i, \in \mathbb{R}^N$ , there exists a unique  $n$ -tuple  $u_i^\# \in U_i$  such that

$$u_i^\# = \underset{u_i \in U_i}{\operatorname{argmax}} \left\{ H_i(t, x, [u_i, u_{-i}^\#], p_i) \right\}. \quad (3.10)$$

for each  $t \in [0, T]$  for all  $u_i$  which satisfies

$$\dot{h}[t, x(t), u_i(t)] \geq 0 \text{ whenever } h[t, x(t)] = 0.$$

where  $H_i$  is the Hamiltonian function as defined by (3.8). The corresponding map will be denoted by

$$(t, x, p_i) \mapsto u_i^\#(t, x, p_i).$$

We can now propose a method for finding open loop Nash equilibrium for an  $N$ -player differential game.

**Proposition 3.1.6.** Let Assumption 1 hold. For  $i \in N$ , let the control  $u_i^*$  be an optimal solution for the  $N$ -player differential game given by Definition 3.1.1 and constrained by

(3.3). Then  $u_i^*$  solves the following two point boundary value problem (BVP)

$$\left\{ \begin{array}{l} \dot{x} = f(t, x, u_i^\sharp), \quad x(0) = x_0, \\ \dot{p}_i = -\frac{\partial L_i}{\partial x}(t, x, u_i^\sharp, \mu_i), \quad p_i(T) = \nabla_x S_i(x(T)) + \gamma \nabla h(T, x(T)), \\ \gamma \geq 0, \quad \gamma h(T, x^*(T)) = 0. \end{array} \right. \quad (3.11)$$

which satisfies the complementary slackness conditions,

$$\mu_i(t) \geq 0, \quad \mu_i(t) h(t, x^*(t)) = 0, \quad \dot{\mu}_i(t) \leq 0.$$

Consider the following example for the case where  $N = 2$  from Bressan [40].

**Example 3.1.1** (Producer-Consumer Game). *We consider a firm manufacturing a good where,*

$$x(t) = \text{the sale price of a good at time } t,$$

and for the controls, at each time  $t$ , a good is produced by the firm at rate  $u_1$  and consumed by the consumer at rate  $u_2$ . So the controls for our problem are defined as:

$$\begin{aligned} u_1(t) &= \text{rate of production of a good at time } t, \\ u_2(t) &= \text{rate of consumption of a good at time } t, \end{aligned}$$

and

$$c_i(s) = \text{cost function of Firm } i$$

we assume for the sake of definiteness that

$$\begin{aligned} c_1(s) &= \frac{s^2}{2}, \\ c_2(s) &= 2\sqrt{s}. \end{aligned}$$

The price increases when the consumption of goods is larger than the production of goods, and decreases otherwise, that is we have the following system with dynamics

$$\dot{x}(t) = x(t)(u_2(t) - u_1(t)), \quad x(0) = x_0, \quad t \in [0, T],$$

with constraints given by

$$x(t) \geq 0. \quad (3.12)$$

The payoff functional for Firm 1 is measured by the profit generated by sales, minus the cost  $c_1(u_1(t))$  of producing the good at rate  $u_1(t)$  and for Firm 2 is measured by the initial capital the consumer has, minus the price paid to buy the  $x(t)u_2(t)$  of producing the good at rate  $u_2(t)$ , that is we have

$$\begin{aligned} J_1 &= \int_0^T [x(t) \cdot u_2(t) - c_1(u_1(t))] dt, \\ J_2 &= \int_0^T [c_2(u_2(t)) - x(t) \cdot u_2(t)] dt, \end{aligned} \quad (3.13)$$

We aim to find the necessary conditions for an open-loop Nash Equilibrium for the problem.

**Solution.** The Hamiltonian functions for the players in this problem is given by

$$\begin{aligned} H_1(t, x, u_1, u_2, p_1) &= p_1 \cdot f(t, x, u_1, u_2) + F_1(t, x, u_1, u_2) \\ &= p_1 \cdot x(u_2 - u_1) + xu_2 - \frac{u_1^2}{2}, \end{aligned}$$

and

$$\begin{aligned} H_2(t, x, u_1, u_2, p_2) &= p_2 \cdot f(t, x, u_1, u_2) + F_2(t, x, u_1, u_2) \\ &= p_2 \cdot x(u_2 - u_1) + 2\sqrt{u_2} - xu_2. \end{aligned}$$

The Lagrangian functions for the players in this problem is given by

$$\begin{aligned} L_1[t, x, u_1, u_2, p_1, \mu_1] &= H_1[t, x, u_1, u_2, p_1] + \mu_1 \cdot \dot{h}(t, x, u_1, u_2) \\ &= H_1[t, x, u_1, u_2, p_1] + \mu_1 \cdot (h_t + h_x \cdot f) \\ &= xu_2 - \frac{u_1^2}{2} + x \cdot (p_1 + \mu_1)(u_2 - u_1), \end{aligned}$$

and

$$\begin{aligned} L_2[t, x, u_1, u_2, p_2, \mu_2] &= H_2[t, x, u_1, u_2, p_2] + \mu_2 \cdot \dot{h}(t, x, u_1, u_2) \\ &= H_2[t, x, u_1, u_2, p_2] + \mu_2 \cdot (h_t + h_x \cdot f) \\ &= 2\sqrt{u_2} - xu_2 + x \cdot (p_2 + \mu_2)(u_2 - u_1). \end{aligned}$$

From Assumption 1 and the Hamiltonian maximization condition (iii) of Theorem (2.1.4) we can write our pair of optimal controls  $(u_1^\sharp(t, x, \lambda_1), u_2^\sharp(t, x, \lambda_2))$  as

$$\begin{aligned} u_1^\sharp &= \underset{u_1 \in U_1}{\operatorname{argmax}} \{H_1(t, x, u_1, u_2, \lambda_1)\} \\ &= \underset{u_1 \in U_1}{\operatorname{argmax}} \left\{ p_1 \cdot x(u_2 - u_1) + xu_2 - \frac{u_1^2}{2} \right\} \\ &= -p_1 \cdot x, \end{aligned}$$

and

$$\begin{aligned} u_2^\sharp &= \underset{u_2 \in U_2}{\operatorname{argmax}} \{H_2(t, x, u_1, u_2, \lambda_2)\} \\ &= \underset{u_2 \in U_2}{\operatorname{argmax}} \{p_2 \cdot x(u_2 - u_1) + 2\sqrt{u_2} - xu_2\} \\ &= \frac{1}{(x \cdot (1 - p_2))^2} \end{aligned}$$

for each  $t \in [0, T]$  and for all  $u_1, u_2$  which satisfies

$$x(u_2 - u_1) \geq 0.$$

From condition (ii) of Theorem (2.1.4) we find that the adjoint equations are given by,

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial L_1}{\partial x} \\ &= -\frac{\partial}{\partial x} \left\{ xu_2 - \frac{u_1^2}{2} + x \cdot (p_1 + \mu_1)(u_2 - u_1) \right\} \\ &= (p_1 + \mu_1)(u_1 - u_2) - u_2, \end{aligned}$$

and

$$\begin{aligned} \dot{p}_2 &= -\frac{\partial L_2}{\partial x} \\ &= -\frac{\partial}{\partial x} \{2\sqrt{u_2} - xu_2 + x \cdot (p_2 + \mu_2)(u_2 - u_1)\} \\ &= (p_2 + \mu_2)(u_1 - u_2) + u_2, \end{aligned}$$

along with the transversality conditions

$$\begin{aligned} p_1(T) &= \nabla S_1(T, x(T)) + \delta_1 \nabla h(T, x(T)) \\ &= \delta_1, \\ \delta_1 x(T) &= 0, \quad \delta_1 \geq 0, \end{aligned}$$

and

$$\begin{aligned} p_2(T) &= \nabla S_2(T, x(T)) + \delta_2 \nabla h(T, x(T)) \\ &= \delta_2, \\ \delta_2 x(T) &= 0, \quad \delta_2 \geq 0. \end{aligned}$$

Putting all together gives

$$\left\{ \begin{array}{l} \dot{x} = x(u_2^\# - u_1^\#) \\ \quad x \cdot \left( \frac{1}{(x \cdot (1 - p_2))^2} + p_1 x \right), \quad x(0) = x_0, \\ \\ \dot{p}_1 = (p_1 + \mu_1)(u_1^\# - u_2^\#) - u_2^\#, \quad p_1(T) = \delta_1 \\ \quad = -(p_1 + \mu_1) \left( \frac{1}{(x \cdot (1 - p_2))^2} + p_1 x \right) - \frac{1}{(x \cdot (1 - p_2))^2}, \quad \delta_1 x(T) = 0, \\ \quad \delta_1 \geq 0, \\ \\ \dot{p}_2 = (p_2 + \mu_2)(u_1^\# - u_2^\#) + u_2^\#, \quad p_2(T) = \delta_2, \\ \quad = -(p_2 + \mu_2) \left( \frac{1}{(x \cdot (1 - p_2))^2} + p_1 x \right) + \frac{1}{(x \cdot (1 - p_2))^2}, \quad \delta_2 x(T) = 0, \\ \quad \delta_2 \geq 0. \end{array} \right. \quad (3.14)$$

which satisfy the constraints

$$x(u_2 - u_1) \geq 0, \quad (3.15)$$

and the slackness conditions

$$\left\{ \begin{array}{l} \mu_1 \geq 0, \quad \mu_1 x = 0, \quad \dot{\mu}_1 \leq 0, \\ \mu_2 \geq 0, \quad \mu_2 x = 0, \quad \dot{\mu}_2 \leq 0. \end{array} \right. \quad (3.16)$$

to determine the state  $x$  and the adjoint variables  $p_1$  and  $p_2$ .

■

*Remark 3.1.7.* In Section 4.1 we will solve this problem further using a numerical method.

Consider the following example for the case where  $N = 2$  from Torres and Esparza [1].

**Example 3.1.2** (Duopolistic Market). Consider two firms, Firm 1 and Firm 2, selling the same product in a competitive market where,

$$\begin{aligned} x(t) &= \text{Firm 1's market share at time } t, \\ 1 - x(t) &= \text{Firm 2's market share at time } t, \end{aligned}$$

and for the controls, at each time  $t$ , an advertising strategy is implemented by the firms. So the controls for our problem are defined as:

$$u_i(t) = \text{advertising effort of Firm } i \text{ at time } t,$$

and

$$\begin{aligned} r_i &= \text{Firm } i \text{'s interest rate,} \\ \phi_i &= \text{Firm } i \text{'s fractional revenue potential,} \\ c_i(s) &= \text{advertising cost function.} \end{aligned}$$

We assume for the sake of completeness that

$$c_i(s) = \frac{k_i s^2}{2},$$

where  $k_i$  is a positive constant.

The number of customers at Firm 1 increases by the advertising efforts of Firm 1 and the advertising efforts of Firm 2 draws customers away from Firm 1, hence we have the following system with dynamics

$$\dot{x}(t) = u_1(t)(1 - x(t)) - u_2(t)x(t), \quad x(0) = x_0, \quad t \in [0, T], \quad (3.17)$$

with constraints given by

$$0 \leq x(t) \leq 1, \quad (3.18)$$

The payoff functionals for the 2 firms are given as

$$\begin{aligned} P^{\text{Firm 1}} &= \int_0^T e^{-r_1 t} [\phi_1 x(t) - c_1(u_1(t))] dt, \\ P^{\text{Firm 2}} &= \int_0^T e^{-r_2 t} [\phi_2 (1 - x(t)) - c_2(u_2(t))] dt. \end{aligned}$$

Here, the action of one firm does not directly effect the other firms payoff, but it effects the payoff indirectly through the state dynamics. We aim to find the necessary conditions for an open-loop Nash Equilibrium for the problem.

**Solution.** The Hamiltonian functions for the players in this problem is given by

$$\begin{aligned} H_1^c(t, x, u_1, u_2, \lambda_1) &= \lambda_1 \cdot f(t, x, u_1, u_2) + F_1(t, x, u_1, u_2) \\ &= \lambda_1(u_1(1-x) - u_2x) + \phi_1x - \frac{k_1u_1^2}{2}, \end{aligned}$$

and

$$\begin{aligned} H_2^c(t, x, u_1, u_2, \lambda_2) &= \lambda_2 \cdot f(t, x, u_1, u_2) + F_2(t, x, u_1, u_2) \\ &= \lambda_2(u_1(1-x) - u_2x) + \phi_2(1-x) - \frac{k_2u_2^2}{2}. \end{aligned}$$

where  $\lambda_1 = e^{r_1t} \cdot p_1$  and  $\lambda_2 = e^{r_2t} \cdot p_2$ .

The Lagrangian functions for the players in this problem is given by

$$\begin{aligned} L_1^c[t, x, u_1, u_2, \lambda_1, \nu_{11}, \nu_{12}] &= H_1^c[t, x, u_1, u_2, \lambda_1] + \nu_{11} \cdot f - \nu_{12} \cdot f \\ &= H_1^c[t, x, u_1, u_2, \lambda_1] + (\nu_{11} - \nu_{12})(u_1(1-x) - u_2x) \\ &= (\lambda_1 + \nu_1)(u_1(1-x) - u_2x) + \phi_1x - \frac{k_1u_1^2}{2}, \end{aligned}$$

and

$$\begin{aligned} L_2^c[t, x, u_1, u_2, \lambda_2, \nu_{21}, \nu_{22}] &= H_2^c[t, x, u_1, u_2, \lambda_2] + \nu_{21} \cdot f - \nu_{22} \cdot f \\ &= H_2^c[t, x, u_1, u_2, \lambda_2] + (\nu_{21} - \nu_{22})(u_1(1-x) - u_2x) \\ &= (\lambda_2 + \nu_2)(u_1(1-x) - u_2x) + \phi_2(1-x) - \frac{k_2u_2^2}{2}. \end{aligned}$$

where  $\nu_1 = \nu_{11} - \nu_{12}$  and  $\nu_2 = \nu_{21} - \nu_{22}$ .

From Assumption 1 and the Hamiltonian maximization condition (iii) of Theorem (2.1.5) we can write our pair of optimal controls  $(u_1^\sharp(t, x, \lambda_1), u_2^\sharp(t, x, \lambda_2))$  as

$$\begin{aligned} u_1^\sharp &= \underset{u_1 \in U_1}{\operatorname{argmax}} \{H_1^c(t, x, u_1, u_2, \lambda_1)\} \\ &= \underset{u_1 \in U_1}{\operatorname{argmax}} \left\{ \lambda_1(u_1(1-x) - u_2x) + \phi_1x - \frac{k_1u_1^2}{2} \right\} \\ &= \frac{\lambda_1(1-x)}{k_1}, \end{aligned}$$

and

$$\begin{aligned} u_2^\# &= \underset{u_2 \in U_2}{\operatorname{argmax}} \{H_2^c(t, x, u_1, u_2, \lambda_2)\} \\ &= \underset{u_2 \in U_2}{\operatorname{argmax}} \left\{ \lambda_2(u_1(1-x) - u_2x) + \phi_2(1-x) - \frac{k_2 u_2^2}{2} \right\} \\ &= \frac{-\lambda_2 x}{k_2}. \end{aligned}$$

for each  $t \in [0, T]$  and for all  $u_1, u_2$  which satisfy

$$\begin{aligned} u_1(1-x) - u_2x &\geq 0, \\ u_2x - u_1(1-x) &\geq 0. \end{aligned}$$

From condition (ii) of Theorem (2.1.5) we find that the adjoint equations are given by,

$$\begin{aligned} \dot{\lambda}_1 - r_1 \lambda_1 &= -\frac{\partial L_1^c}{\partial x} \\ &= -\frac{\partial}{\partial x} \left\{ (\lambda_1 + \nu_1)(u_1(1-x) - u_2x) + \phi_1x - \frac{k_1 u_1^2}{2} \right\} \\ &= (\lambda_1 + \nu_1)(u_1 + u_2) - \phi_1, \end{aligned}$$

and

$$\begin{aligned} \dot{\lambda}_2 - r_2 \lambda_2 &= -\frac{\partial L_2^c}{\partial x} \\ &= -\frac{\partial}{\partial x} \left\{ (\lambda_2 + \nu_2)(u_1(1-x) - u_2x) + \phi_2(1-x) - \frac{k_2 u_2^2}{2} \right\} \\ &= (\lambda_2 + \nu_2)(u_1 + u_2) + \phi_2, \end{aligned}$$

along with the transversality conditions

$$\begin{aligned} \lambda_1(T) &= \nabla S_1(T, x(T)) + \delta_{11} \nabla h_1(T, x(T)) + \delta_{12} \nabla h_2(T, x(T)), \\ &= \delta_{11} - \delta_{12}, \end{aligned}$$

$$\begin{aligned} \lambda_2(T) &= \nabla S_2(T, x(T)) + \delta_{21} \nabla h_1(T, x(T)) + \delta_{22} \nabla h_2(T, x(T)), \\ &= \delta_{21} - \delta_{22}, \end{aligned}$$

$$\begin{aligned} \delta_{11}x(T) &= 0, \quad \delta_{12}(1-x(T)) = 0, \quad \delta_{21}x(T) = 0, \quad \delta_{22}(1-x(T)) = 0, \\ \delta_{11} &\geq 0, \quad \delta_{12} \geq 0, \quad \delta_{21} \geq 0, \quad \delta_{22} \geq 0. \end{aligned}$$

Putting all together gives

$$\left\{ \begin{array}{l} \dot{x} = u_1^\#(1-x) - u_2^\#x \\ \quad = \frac{\lambda_1(1-x)^2}{k_1} + \frac{\lambda_2 x^2}{k_2}, \\ \\ \dot{\lambda}_1 = r_1 \lambda_1 + (\lambda_1 + \nu_1)(u_1^\# + u_2^\#) - \phi_1 \\ \quad = r_1 \lambda_1 + (\lambda_1 + \nu_1) \left( \frac{\lambda_1(1-x)}{k_1} - \frac{\lambda_2 x}{k_2} \right) - \phi_1, \\ \\ \dot{\lambda}_2 = r_2 \lambda_2 + (\lambda_2 + \nu_2)(u_1^\# + u_2^\#) + \phi_2, \\ \quad = r_2 \lambda_2 + (\lambda_2 + \nu_2) \left( \frac{\lambda_1(1-x)}{k_1} - \frac{\lambda_2 x}{k_2} \right) + \phi_2, \end{array} \right. \quad \begin{array}{l} x(0) = x_0, \\ \\ \lambda_1(T) = \delta_{11} - \delta_{12} \\ \delta_{11} x(T) = 0, \quad \delta_{12} (1-x(T)) = 0, \\ \delta_{11} \geq 0, \quad \delta_{12} \geq 0, \\ \\ \lambda_2(T) = \delta_{21} - \delta_{22}, \\ \delta_{21} x(T) = 0, \quad \delta_{22} (1-x(T)) = 0, \\ \delta_{21} \geq 0, \quad \delta_{22} \geq 0. \end{array} \quad (3.19)$$

which satisfy the constraints

$$\begin{array}{l} u_1(1-x) - u_2 x \geq 0, \\ u_2 x - u_1(1-x) \geq 0, \end{array} \quad (3.20)$$

and the slackness conditions

$$\left\{ \begin{array}{l} \nu_{11} \geq 0, \quad \nu_{11} x = 0, \quad \nu_{11} \leq 0, \\ \nu_{12} \geq 0, \quad \nu_{12} (1-x) = 0, \quad \nu_{12} \leq 0, \\ \nu_{21} \geq 0, \quad \nu_{21} x = 0, \quad \nu_{21} \leq 0, \\ \nu_{22} \geq 0, \quad \nu_{22} (1-x) = 0, \quad \nu_{22} \leq 0. \end{array} \right. \quad (3.21)$$

to determine the state  $x$  and the adjoint variables  $\lambda_1$  and  $\lambda_2$ .

■

*Remark 3.1.8.* In order to ensure that the open-loop Nash equilibrium for the  $N$ -player differential is indeed optimal, we will need to check for sufficiency.

### Sufficiency Conditions of open-loop Nash equilibrium solutions

Using the maximum principle to solve the  $N$ -player differential game problem only provides us with necessary conditions for optimality, but it does not assure us that the

$n$ -tuple of controls  $(u_1^*, u_2^*, \dots, u_n^*)$  is indeed an open-loop Nash equilibrium since this  $n$ -tuple may not be optimal. We will use Theorem (2.2.1) to derive a set of sufficiency conditions for the problem.

**Theorem 3.1.9** (Sufficient Conditions for open-loop Nash equilibrium). *For the  $N$ -player differential game given by Definition 3.1.1 and constrained by (3.3), consider the measurable function  $t \mapsto u_i^* \in U_i$  and the continuous functions  $x^*, p_i, \mu_i$  satisfying*

$$\begin{cases} \dot{x} = f(t, x, u_i^*), & x(0) = x_0, \\ \dot{p}_i = -\frac{\partial L_i}{\partial x}(t, x, u_i^*, p_i, \mu_i), & p_i(T) = \nabla_x S_i(x(T)) + \gamma \nabla_x h(T, x(T)), \\ \gamma \geq 0, & \gamma h(T, x(T)) = 0. \end{cases}$$

together with

$$H_i(t, x^*, u_i^*, p_i^*) = \max_{u_i \in U_i} \{H_i(t, x^*, u_i, p_i^*)\},$$

for all  $u_i$  which satisfy

$$\dot{h}[t, x(t), u_i(t)] \geq 0, \text{ whenever } h[t, x(t)] = 0.$$

and also the complementary slackness conditions,

$$\mu_i(t) \geq 0, \quad \mu_i(t) h(t, x^*(t)) = 0, \quad \dot{\mu}_i(t) \leq 0.$$

Suppose that the set  $U_i$  is convex for every  $x \in \mathbb{R}^n, t \in [0, T]$ , the partial derivatives of the functions  $F_i(t, x, u_i)$  and  $f(t, x, u_i)$  are continuous with respect to their arguments.

If the Hamiltonian function  $H_i$  exists and is concave in  $x, u_i$  for all  $t$ , the terminal payoff  $S_i(T, x)$  is concave in  $x$ , and  $h(t, x)$  is quasiconcave in  $(x, u_i)$ , then  $u_i^*$  is a Nash equilibrium solution and  $x^*$  is the corresponding trajectory.

If the Hamiltonian function  $H_i$  is strictly concave in  $(x, u_i)$  for all  $t$ , then  $u_i^*$  is the unique Nash equilibrium solution and  $x^*$  is the corresponding trajectory.

### 3.1.2 Open-Loop Stackelberg Equilibrium

Here we have an asymmetric game situation where the players can communicate, and therefore they can swap ideas on their strategies. For this equilibrium solution there is a hierarchy in the decision making process. The concept of solutions for hierarchical game situations where one player has domination over the other player was introduced by von Stackelberg in 1934. Hierarchical game situation results in players taking on either the role of a leader or of a follower [41]. A Stackelberg solution is obtained when the leaders have achieved their maximum payoff, based on and considering the optimal strategies of the followers [42].

We will consider a 2-player differential game where Player 1 takes on the role of the leader and Player 2 takes on the role of the follower.

Let us consider the following definition.

**Definition 3.1.10** (Best Response). The strategy  $u_2 \in U_2$  is Player 2's best response to Player 1's strategy  $u_1 \in U_1$  if

$$J_2(u_1, u_2) \geq J_2(u_1, u_2^*) \quad \forall u_2^* \in U_2 \quad u_2^* \neq u_2$$

i.e Player 2's strategy  $u_2$  will give the highest possible payoff given that Player 1 plays strategy  $u_1$ .

We are now ready to define the Stackelberg equilibrium.

The set of best responses for Player 2, given Player 1 chooses the strategy  $u_1 \in U_1$  is denoted as

$$\mathcal{R}_2(u_1) = \{u_2 \in U_2; J_2(u_1, u_2) \geq J_2(u_1, u_2^*) \quad \forall u_2^* \in U_2 \quad u_2 \neq u_2^*\},$$

and similarly the set of best responses for Player 1, given Player 2 chooses the strategy  $u_2 \in U_2$  is denoted as

$$\mathcal{R}_1(u_2) = \{u_1 \in U_1; J_1(u_1, u_2) \geq J_1(u_1^*, u_2) \quad \forall u_1^* \in U_1 \quad u_1 \neq u_1^*\}.$$

Hence assuming Player 1 chooses any admissible control  $u_1^* : [0, T] \mapsto U_1$  then the set of best responses for Player 2 is  $\mathcal{R}_2(u_1^*)$ . This simply means that  $\mathcal{R}_2(u_1^*)$  is the set of

all admissible control functions  $u_2 : [0, T] \mapsto U_2$  in relation with  $u_1^*$  that maximizes the payoff for Player 2. That is, the set of best responses for Player 2 solves the OCP

$$\text{maximize } J_2(u_1^*, u_2) = S_2(x(T)) + \int_0^T F_2(t, x(t), u_1^*(t), u_2(t)) dt,$$

over all the controls  $u_1, u_2$ , for the system with dynamics

$$\begin{aligned} \dot{x}(t) &= f(t, x, u_1^*, u_2), & x(0) &= x_0 \in \mathbb{R}^n, \\ h(t, x) &\geq 0 & u_2(t) &\in U_2. \end{aligned}$$

**Definition 3.1.11** (Open-Loop Stackelberg Equilibrium). A pair of control functions  $t \mapsto (u_1^*(t), u_2^*(t))$  is an open-loop Stackelberg equilibrium for the 2-player differential game within the framework of Definition 3.1.1 and constrained by (3.3) if the following conditions hold:

- (i)  $u_2^* \in \mathcal{R}_2(u_1^*)$
- (ii) Given any admissible control  $u_1$  for Player 1 and every best response  $u_2 \in \mathcal{R}_2(u_1)$  for Player 2, we have

$$J_1(u_1^*, u_2^*) = \max_{u_2 \in \mathcal{R}_2(u_1)} J_1(u_1, u_2)$$

subject to the dynamics

$$\begin{aligned} \dot{x}(t) &= f(t, x, u_1^*, u_2^*), & x(0) &= x_0 \in \mathbb{R}^n, \\ h(t, x) &\geq 0. \end{aligned}$$

From Definition 3.1.11, we can see that for us to get an open-loop Stackelberg equilibrium, Player 1 will have to find the best response of Player 2 for each and every one of his controls  $u_1$ , and he will have to choose the control function  $u_1^*$  such that his own payoff will be maximized.

Due to the hierarchy between the leader and follower a set of necessary conditions for an open-loop Stackelberg equilibrium will first be found for the follower and then for the leader [43]. To find these necessary conditions, we use a combination of variational analysis [see, 44–46] along with the maximum principle.

Let  $(u_1^*, u_2^*)$  be the controls for Players 1 and 2 respectively and  $x^*$  be the corresponding optimal trajectory of the dynamic system.

**For Player 2:** We know that  $u_2^*$  is the optimal response for Player 2, thus by the maximum principle there exists an adjoint vector  $p_2^*$  such that

$$\begin{aligned} \dot{x}^*(t) &= f(t, x^*, u_1^*, u_2^*), \quad x^*(0) = x_0 \\ \dot{p}_2^*(t) &= -\frac{\partial L_2}{\partial x}(t, x^*, u_1^*, u_2^*, p_2^*, \mu_2^*), \\ p_2^*(T) &= \nabla S_2(x^*(T)) + \gamma \nabla h(T, x^*(T)) \\ \gamma &\geq 0, \quad \gamma h(T, x^*(T)) = 0. \end{aligned} \tag{3.22}$$

Furthermore, the subsequent conditions for optimality

$$u_2^* \in \underset{u_2 \in U_2}{\operatorname{argmax}} \{H_2(t, x^*, u_1^*, u_2, p_2^*)\} \tag{3.23}$$

for almost every  $t \in [0, T]$  for all  $u_1, u_2$  which satisfy

$$\dot{h}[t, x^*(t), u_1(t), u_2(t)] \geq 0, \quad \text{whenever } h(t, x^*(t)) = 0. \tag{3.24}$$

hold.

**For Player 1:** In order to obtain a set of necessary conditions for optimality, we will need to assume

**Assumption 2.** *For every  $(t, x, u_1, p_2) \in [0, T] \times \mathbb{R}^n \times U_1 \times \mathbb{R}^n$  there exists a unique optimal choice  $u_2^\sharp \in U_2$  for Player 2, to be specific*

$$u_2^\sharp \doteq \underset{u_2 \in U_2}{\operatorname{argmax}} \{H_2(t, x, u_1, u_2, p_2)\}$$

for all  $u_1, u_2$  which satisfy

$$\dot{h}[t, x(t), u_1(t), u_2(t)] \geq 0, \quad \text{whenever } h[t, x(t)] = 0.$$

We are now able to construct Player 1's optimization problem as an OCP. This OCP will have an extended state space with the state variables  $(x, p_2) \in \mathbb{R}^n \times \mathbb{R}^n$ . That is, the OCP for Player 1 is

$$\text{maximize } S_1(x(T)) + \int_0^T F_1(t, x(t), u_1(t), u_2^\sharp(t, x(t), u_1(t), p_2(t))) dt \tag{3.25}$$

for the system on  $\mathbb{R}^{2n}$  with dynamics

$$\begin{aligned}
\dot{x}(t) &= f(t, x, u_1, u_2^\sharp(t, x, u_1, p_2)), \quad x(0) = x_0, \\
\dot{p}_2(t) &= -\frac{\partial L_2}{\partial x}(t, x, u_1, u_2^\sharp(t, x, u_1, p_2), \mu_2), \\
p_2(T) &= \nabla S_2(x(T)) + \gamma \nabla h(T, x(T)), \\
\gamma &\geq 0, \quad \gamma h(T, x(T)) = 0.
\end{aligned} \tag{3.26}$$

Although this is clearly a standard problem in optimal control theory, notice that the of state variables  $(x, p_2)$  are not both given at time  $t = 0$ . Rather, at  $t = 0$  the initial constraint  $x = x_0$  is given while at  $t = T$  the end condition  $p_2 = \nabla_x S_2(x) + \gamma \nabla_x h(T, x)$  is provided.

To ensure the existence of solution to (3.26), we need the following assumption.

**Assumption 3.** For all fixed  $t \in [0, T]$  and  $u_1 \in U_1$ , the maps

$$\begin{aligned}
(x, p_2) &\mapsto \tilde{F}(t, x, u_1, p_2) \doteq F_1(t, x, u_1, u_2^\sharp(t, x, u_1, p_2)) \\
(x, p_2) &\mapsto \tilde{f}(t, x, u_1, p_2) \doteq f(t, x, u_1, u_2^\sharp(t, x, u_1, p_2)) \\
(x, p_2) &\mapsto \tilde{L}(t, x, u_1, p_2, \mu_2) \doteq -\nabla_x L_2(t, x, u_1, u_2^\sharp(t, x, u_1, p_2), \mu_2) \\
x &\mapsto \tilde{S}_1(T, x(T)) \doteq \nabla S_1(T, x) + \gamma \nabla h(T, x), \\
x &\mapsto \tilde{S}_2(T, x(T)) \doteq \nabla S_2(T, x) + \gamma \nabla h(T, x),
\end{aligned}$$

are continuously differentiable.

If we apply the maximum principle for constrained initial and end points to the OCP (3.25)-(3.26), then we will obtain a set of necessary condition for open-loop Stackelberg equilibrium solutions.

**Theorem 3.1.12** (Open-loop Stackelberg equilibrium necessary conditions). *Let Assumption 2 and 3 hold. Let  $t \mapsto (u_1^*, u_2^*)$  be the open-loop Stackelberg equilibrium for the 2-player differential game within the framework of Definition 3.1.1 with constraint (3.3), and  $x^*, p_2^*$  be the corresponding trajectory and adjoint vector for Player 2 that satisfy (3.22)-(3.24).*

Then there exists a constant  $\kappa_0 \geq 0$  and two absolutely continuous adjoint vectors  $\kappa_1, \kappa_2$  which are not all equal to zero, that satisfy the equations

$$\begin{cases} \dot{\kappa}_1 = -\kappa_0 \frac{\partial \tilde{F}}{\partial x} - \kappa_1 \frac{\partial \tilde{f}}{\partial x} - \kappa_2 \frac{\partial \tilde{L}}{\partial x}, \\ \dot{\kappa}_2 = -\kappa_0 \frac{\partial \tilde{F}}{\partial p_2} - \kappa_1 \frac{\partial \tilde{f}}{\partial p_2} - \kappa_2 \frac{\partial \tilde{L}}{\partial p_2}, \end{cases} \quad (3.27)$$

for almost every  $t \in [0, T]$  together with the boundary conditions

$$\begin{aligned} \kappa_2(0) &= 0, \\ \kappa_1(T) &= \kappa_0 \tilde{S}_1 - \kappa_2(T) D^2 \tilde{S}_2. \end{aligned} \quad (3.28)$$

Furthermore, for almost every  $t \in [0, T]$  we have

$$u_1^*(t) = \underset{u_1 \in U_1}{\operatorname{argmax}} \left\{ \begin{array}{l} \kappa_0 \tilde{F}(t, x^*(t), u_1, p_2^*(t)) + \kappa_1(t) \tilde{f}(t, x^*(t), u_1, p_2^*(t)) \\ + \kappa_2(t) \tilde{L}(t, x^*(t), u_1, p_2^*(t), \mu_2^*(t)) \end{array} \right\}. \quad (3.29)$$

for all  $u_1, u_2$  which satisfy

$$\dot{h}[t, x(t), u_1(t), u_2(t)] \geq 0, \quad \text{whenever } h[t, x(t)] = 0. \quad (3.30)$$

The right hand side of (3.27) is calculated at  $(t, x^*(t), p_2^*(t), \mu_2^*(t), u_1^*)$ , also the function  $D^2 \tilde{S}_2$  denotes the Hessian matrix of second derivatives of the function  $S_2(x) + \gamma h(T, x)$  at point  $x$ .

Consider the following example from Bressan [40].

**Example 3.1.3** (Growth of an Economy). *We consider the growth of an economy with respect to Player 1 (government) and Player 2 (capitalist) where,*

$$\begin{aligned} x(t) &= \text{the total wealth of the Capitalists at time } t, \\ a &= \text{constant growth rate of the wealth,} \end{aligned}$$

and for the controls, at each time  $t$ , the government will impose a capital tax rate which will effect the amount of consumption of the capitalists. So the controls for our problem are defined as:

$$\begin{aligned} u_1(t) &= \text{capital tax rate imposed by the government,} \\ u_2(t) &= \text{instantaneous amount of consumption,} \end{aligned}$$

also, we have

$$\phi_i = \text{a utility function,}$$

we assume for the sake of definiteness that

$$\phi_i(s) = \frac{c_i s^2}{2}.$$

We have the following system with dynamics

$$\dot{x}(t) = ax(t) - u_1(t)x(t) - u_2(t), \quad x(0) = x_0, \quad t \in [0, T]. \quad (3.31)$$

We have constraints given by

$$x(t) \geq 0.$$

The payoff functionals for the 2 players are given as

$$\begin{aligned} J_1 &= bx(T) + \int_0^T [\phi_1(u_1(t)x(t))] dt, \\ J_2 &= x(T) + \int_0^T [\phi_2(u_2(t))] dt. \end{aligned}$$

We aim to find an open-loop Stackelberg Equilibrium for the problem, where the government is the leader who announces in advance the tax rate  $u_1$  and the capitalists are the followers.

**Solution.** The Hamiltonian function for the capitalists in this problem is given by

$$\begin{aligned} H_2(t, x, u_1, u_2, p_2) &= p_2 \cdot f(t, x, u_1, u_2) + F_2(t, x, u_1, u_2) \\ &= p_2 (ax - u_1x - u_2) + \frac{c_2(u_2)^2}{2} \end{aligned}$$

and the Lagrangian function for the capitalists in this problem is given by

$$\begin{aligned} L_2[t, x, u_1, u_2, p_2, \mu_2] &= H_2[t, x, u_1, u_2, p_2] + \mu_2 \cdot \dot{h}(t, x, u_1, u_2) \\ &= H_2[t, x, u_1, u_2, p_2] + \mu_2 \cdot (h_t + h_x \cdot f) \\ &= H_2[t, x, u_1, u_2, p_2] + \mu_2 \cdot f \\ &= (p_2 + \mu_2) (ax - u_1x - u_2) + \frac{c_2(u_2)^2}{2}. \end{aligned}$$

From Assumption 2 we have the unique optimal choice  $u_2^\# \in U_2$  for the capitalists given as

$$\begin{aligned} u_2^\# &= \underset{u_2 \in U_2}{\operatorname{argmax}} \{H_2(t, x, u_1, u_2, p_2)\} \\ &= \underset{u_2 \in U_2}{\operatorname{argmax}} \left\{ p_2 (ax - u_1x - u_2) + \frac{c_2(u_2)^2}{2} \right\} \\ &= \frac{p_2}{c_2} \end{aligned}$$

for all  $u_1, u_2$  which satisfy

$$ax - u_1x - u_2 \geq 0.$$

From Assumption 3 we have the following functions

$$\begin{aligned} \tilde{F}(t, x, u_1, p_2) &= F_1(t, x, u_1, u_2^\#(t, x, u_1, p_2)) \\ &= \frac{c_1(u_1x)^2}{2}, \end{aligned}$$

$$\begin{aligned} \tilde{f}(t, x, u_1, p_2) &= f(t, x, u_1, u_2^\#(t, x, u_1, p_2)) \\ &= ax - u_1x - u_2^\# \\ &= ax - u_1x - \frac{p_2}{c_2}, \end{aligned}$$

$$\begin{aligned} \tilde{L}(t, x, u_1, p_2, \mu_2) &= -\frac{\partial}{\partial x} L_2(t, x, u_1, u_2^\#(t, x, u_1, p_2), \mu_2) \\ &= -\frac{\partial}{\partial x} \left\{ (p_2 + \mu_2) (ax - u_1x - u_2) + \frac{c_2(u_2)^2}{2} \right\} \\ &= -(p_2 + \mu_2) (a - u_1), \end{aligned}$$

$$\begin{aligned} \tilde{S}_2(T, x(T)) &= \nabla S_2(x) + \nabla \gamma h(T, x) \\ &= 1 + \gamma. \end{aligned}$$

Hence the government who is our leader will have the following optimization problem to solve

$$\text{maximize } \left\{ bx(T) + \int_0^T \left[ \frac{c_1(u_1(t)x(t))^2}{2} \right] dt \right\}$$

for the system with dynamics

$$\begin{aligned}\dot{x}(t) &= ax - u_1x - \frac{p_2}{c_2}, \quad x(0) = x_0, \\ \dot{p}_2(t) &= -(p_2 + \mu_2)(a - u_1), \quad p_2(T) = 1 + \gamma \\ \gamma &\geq 0, \quad \gamma x(T) = 0.\end{aligned}$$

By applying Theorem (3.1.12) we obtain the optimal control for constants  $\kappa_0 \geq 0, \kappa_1, \kappa_2$  as

$$\begin{aligned}u_1^\# &= \underset{u_1 \geq 0}{\operatorname{argmax}} \left\{ \kappa_0 \tilde{F} + \kappa_1 \tilde{f} + \kappa_2 \tilde{L} \right\} \\ &= \underset{u_1 \geq 0}{\operatorname{argmax}} \left\{ \kappa_0 \frac{c_1(u_1x)^2}{2} + \kappa_1 \left( ax - u_1x - \frac{p_2}{c_2} \right) + \kappa_2 \left( -(p_2 + \mu_2)(a - u_1) \right) \right\} \\ &= \frac{\kappa_1x - \kappa_2(p_2 + \mu_2)}{\kappa_0c_1x^2}\end{aligned}$$

for all  $u_1, u_2$  which satisfy

$$ax - u_1x - u_2 \geq 0.$$

Putting all together gives

$$\left\{ \begin{aligned} \dot{x} &= (a - u_1^\#)x - \frac{p_2}{c_2} \\ &= \left( a - \frac{\kappa_1x + \kappa_2p_2}{\kappa_0c_1x^2} \right) x - \frac{p_2}{c_2}, \\ \dot{p}_2 &= -(p_2 + \mu_2)(a - u_1^\#) \\ &= (p_2 + \mu_2) \left( \frac{\kappa_1x + \kappa_2p_2}{\kappa_0c_1x^2} - a \right), \\ \dot{\kappa}_1 &= -\kappa_0c_1x(u_1^\#)^2 - \kappa_1(a - u_1^\#) \\ &= -\kappa_0c_1x \left( \frac{\kappa_1x - \kappa_2(p_2 + \mu_2)}{\kappa_0c_1x^2} \right)^2 + \kappa_1 \left( \frac{\kappa_1x - \kappa_2(p_2 + \mu_2)}{\kappa_0c_1x^2} - a \right), \\ \dot{\kappa}_2 &= \frac{\kappa_1}{c_2} + \kappa_2(a - u_1^\#) \\ &= \frac{\kappa_1}{c_2} - \kappa_2 \left( \frac{\kappa_1x - \kappa_2(p_2 + \mu_2)}{\kappa_0c_1x^2} - a \right). \end{aligned} \right. \quad (3.32)$$

with the initial and end conditions

$$\begin{cases} x(0) = x_0, & p_2(T) = 1 + \gamma, \\ \kappa_1(T) = \kappa_0 b, & \kappa_2(0) = 0, \\ \gamma x(T) = 0, & \gamma \geq 0. \end{cases} \quad (3.33)$$

to determine the state  $x$  and the adjoint variables  $p_1$  and  $p_2$

■

## Chapter 4

# Numerical Algorithms for Differential Games

In the previous section we have seen that analytically solving an  $N$ -player differential game problem completely is a challenging task. In this chapter, we propose numerical algorithms that allows us to approximately solve an  $N$ -player differential game.

This is done by adapting the Forward-Backward sweep algorithm presented in Section 2.3 for the solution of differential games. The algorithm is derived from the maximum principle of optimal control. As we have pointed out in Section 3.1.1, finding open loop Nash equilibrium for a differential game amounts to solving many optimal control problems, one for each player.

For  $i = 1, 2, \dots, n$ , consider the  $N$ -player differential game which has the following system with dynamics

$$\dot{x}(t) = f(t, x(t), \mathbf{u}(t)), \quad x(0) = x_0, \quad t \in [0, T], \quad (4.1)$$

and payoff functionals given as

$$J_i(\mathbf{u}(t)) = S_i(x(T)) + \int_0^T F_i(t, x(t), \mathbf{u}(t)) dt, \quad (4.2)$$

where  $\mathbf{u}(t) = (u_1(t), u_2(t), \dots, u_n(t))$ .

We denote  $[u_i, u_{-i}^*]$  as the  $n$ -tuple of strategies for the players such that

$$[u_i, u_{-i}^*] = (u_1^*, u_2^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_n^*).$$

Applying the maximum principle, we obtain the following necessary conditions for optimization.

$$\dot{x} = f(t, x, u_i^*), \quad x(0) = x_0, \quad (4.3)$$

$$\dot{p}_i = -\frac{\partial H_i}{\partial x}(t, x, u_i^*, p_i), \quad p_i(T) = \nabla_x S_i(x(T)), \quad (4.4)$$

$$u_i^* = \underset{u_i \in U_i}{\operatorname{argmax}} \{H_i(t, x^*, [u_i, u_{-i}^*], p_i^*)\}. \quad (4.5)$$

for the Hamiltonian functions

$$H_i(t, x, \mathbf{u}, p_i) = p_i \cdot f(t, x, \mathbf{u}) + F_i(t, x, \mathbf{u}),$$

When the strategy of the players are unbounded, the maximality conditions amounts to  $\frac{\partial H_i}{\partial u_i} = 0$ .

In order to solve (4.1)-(4.2) we will use an extension of the Forward-Backward Sweep algorithm.

## 4.1 Forward-Backward Sweep Algorithm

Consider the following algorithm which is an extension of Algorithm 2.3.1.

**Algorithm 4.1.1** (Forward-Backward Sweep). *The iterative algorithm to solve (4.1)-(4.2) when  $H_i$  is a concave function of  $u_i$  is given by*

1. Discretise the time variable  $t \in [0, T]$  as  $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \dots, t_k = k\Delta t, \dots, t_N = N\Delta t$ , where  $\Delta t$  is the time step and consider a piece-wise constant guess for the strategy of the players

$$\mathbf{u}(t) = \mathbf{u}(t_k), \quad t \in [t_k, \dots, t_N],$$

2. Using the initial guess for  $u_i$  from step 1 and the initial condition  $x(0) = x_0$ , we solve the state equation (4.3) forward in time.
3. Using the transversality condition  $p_i(T) = \nabla_x S_i(x)$  and the stored values for  $u_i$  and  $x$ , we solve the adjoint equation (4.4) backwards in time for the adjoint  $p_i$ .
4. Update the control  $u$  using the maximality condition (4.5).
5. Check the convergence that is, check that for  $v \in x, u_i, p_i$ ,

$$\epsilon \sum_{i=1}^{N+1} |v(i)| - \sum_{i=1}^{N+1} |v(i) - v^{old}(i)| \geq 0, \quad (4.6)$$

where  $\epsilon$  is the tolerance,  $v(i)$  is the solution at the current iteration and  $v_{old}(i)$  is the solution at the last iteration.

- (a) If (4.6) is satisfied, then end the iterative procedure and output the optimal solution
- (b) If (4.6) is not satisfied, then go back to step 2.

Lets now consider the following example for the case where  $N = 2$  from Sethi and Thompson [22].

**Example 4.1.1** (Bilinear Quadratic Advertising Model). We consider a simple advertising competition game between Firm  $i$  for  $i = 1, 2$  where,

$x_i(t)$  = the market share Firm  $i$  has at time  $t$

and for the controls, at each time  $t$ , an advertising rate is implemented by each of the firms. So the controls for our problem are defined as:

$u_i(t)$  = advertising rate Firm  $i$  at time  $t$

and also the following parameters are defined as:

- $a_i$  = ineffectiveness of Firm  $i$ 's advertising strategy
- $b_i$  = effectiveness of Firm  $i$ 's advertising strategy
- $c_i$  = cost of Firm  $i$ 's advertising
- $w_i$  = Firm  $i$ 's wealth

We have the following system with dynamics

$$\begin{aligned}\dot{x}_1(t) &= b_1 u_1(t)(1 - x_1(t) - x_2(t)) - a_1 x_1(t), \\ x_1(0) &= x_{10}, \quad t \in [0, T],\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}\dot{x}_2(t) &= b_2 u_2(t)(1 - x_1(t) - x_2(t)) - a_2 x_2(t), \\ x_2(0) &= x_{20}, \quad t \in [0, T].\end{aligned}\tag{4.8}$$

and payoff functional for Firm  $i$  is given as

$$P^{Firm\ i} = \int_0^T [c_i x_i(t) - (u_i(t))^2] dt + w_i x_i(T).\tag{4.9}$$

where  $b_i$ ,  $a_i$ ,  $w_i$ ,  $c_i$ , and  $r$  are positive constants. We aim to find the necessary conditions for the open-loop Nash Equilibrium.

**Solution.** The Hamiltonian for each of the firms are

$$\begin{aligned}H_1(t, x_1, x_2, u_1, u_2, p_{11}, p_{12}) &= p_{11}(b_1 u_1(1 - x_1 - x_2) - a_1 x_1) + c_1 x_1 - u_1^2 \\ &\quad + p_{12}(b_2 u_2(1 - x_1 - x_2) - a_2 x_2),\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}H_2(t, x_1, x_2, u_1, u_2, p_{21}, p_{22}) &= p_{21}(b_1 u_1(1 - x_1 - x_2) - a_1 x_1) + c_2 x_2 - u_2^2 \\ &\quad + p_{22}(b_2 u_2(1 - x_1 - x_2) - a_2 x_2),\end{aligned}\tag{4.11}$$

where the adjoint variables for each of firms are denoted as  $p_{11}$ ,  $p_{12}$ ,  $p_{21}$ ,  $p_{22}$ .

**Applying the maximum principle**

We obtain the following necessary conditions for optimization.

$$\begin{aligned} u_1^* &= \underset{u_1 \geq 0}{\operatorname{argmax}} \{H_1(t, x_1^*, x_2^*, u_1^*, u_2^*, p_{11}, p_{12})\} \\ &= \frac{p_{11}b_1(1 - x_1 - x_2)}{2}, \end{aligned}$$

$$\begin{aligned} u_2^* &= \underset{u_2 \geq 0}{\operatorname{argmax}} \{H_2(t, x_1^*, x_2^*, u_1^*, u_2^*, p_{21}, p_{22})\} \\ &= \frac{p_{22}b_2(1 - x_1 - x_2)}{2}, \end{aligned}$$

$$\begin{aligned} \dot{x}_1^* &= b_1u_1^*(1 - x_1^* - x_2^*) - a_1x_1^* \\ &= \frac{p_{11}b_1^2(1 - x_1^* - x_2^*)}{2} - a_1x_1^*, \end{aligned} \quad x_1(0) = x_{10},$$

$$\begin{aligned} \dot{x}_2^* &= b_2u_2^*(1 - x_1^* - x_2^*) - a_2x_2^* \\ &= \frac{p_{22}b_2^2(1 - x_1^* - x_2^*)}{2} - a_2x_2^*, \end{aligned} \quad x_2(0) = x_{20},$$

$$\begin{aligned} \dot{p}_{11}^* &= -\nabla_{x_1} H_1(t, x_1^*, x_2^*, u_1^*, u_2^*, p_{11}, p_{12}) \\ &= p_{11}b_1u_1^* + p_{11}a_1 + p_{12}b_2u_2^* - c_1 \\ &= \frac{(1 - x_1 - x_2)}{2} (p_{11}^2b_1^2 + p_{12}p_{22}b_2^2) + p_{11}a_1 - c_1, \quad p_{11}(T) = w_1, \end{aligned}$$

$$\begin{aligned} \dot{p}_{12}^* &= -\nabla_{x_2} H_1(t, x_1^*, x_2^*, u_1^*, u_2^*, p_{11}, p_{12}) \\ &= p_{11}b_1u_1^* + p_{12}b_2u_2^* - a_2p_{12} \\ &= \frac{(1 - x_1 - x_2)}{2} (p_{11}^2b_1^2 + p_{12}p_{22}b_2^2) - a_2p_{12}, \quad p_{12}(T) = 0, \end{aligned}$$

$$\begin{aligned} \dot{p}_{21}^* &= -\nabla_{x_1} H_2(t, x_1^*, x_2^*, u_1^*, u_2^*, p_{21}, p_{22}) \\ &= p_{21}b_1u_1^* + p_{22}b_2u_2^* - a_1p_{21} \\ &= \frac{(1 - x_1 - x_2)}{2} (p_{11}p_{21}b_1^2 + p_{22}^2b_2^2) + p_{21}a_1, \quad p_{21}(T) = 0, \end{aligned}$$

$$\begin{aligned} \dot{p}_{22}^* &= -\nabla_{x_2} H_2(t, x_1^*, x_2^*, u_1^*, u_2^*, p_{21}, p_{22}) \\ &= p_{21}b_1u_1^* + p_{22}b_2u_2^* + p_{22}a_2 - c_2 \\ &= \frac{(1 - x_1 - x_2)}{2} (p_{11}p_{21}b_1^2 + p_{22}^2b_2^2) + p_{22}a_2 - c_2, \quad p_{22}(T) = w_2. \end{aligned}$$

We will use the values of  $x_1(0) = 0.45$  and  $x_2(0) = 0.35$  for the initial condition,  $a_1 = 0.25$ ,  $a_2 = 0.65$ ,  $b_1 = 0.75$ ,  $b_2 = 0.35$ ,  $c_1 = c_2 = 1$ , for the constants,  $T = 1$  for the

terminal time,  $N = 1000$  for the number of subintervals and  $\epsilon = 0.001$  for the tolerance to solve the problem numerically.

Hence using Algorithm 4.1.1 we obtain the following graphical results given by Figure 4.1 - 4.4

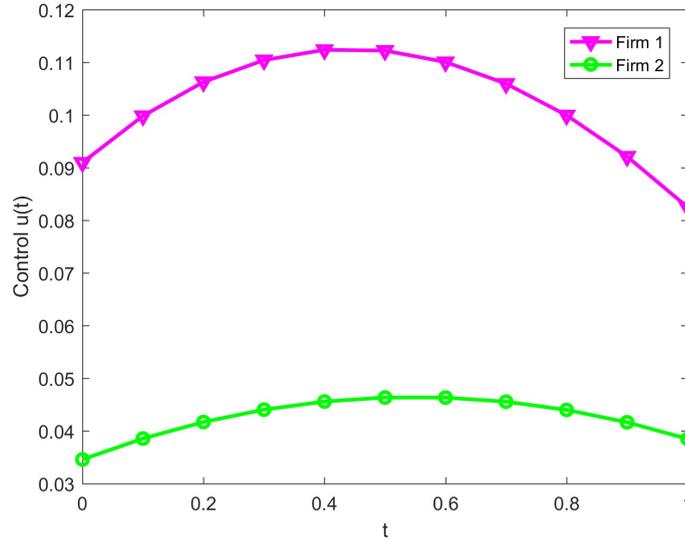
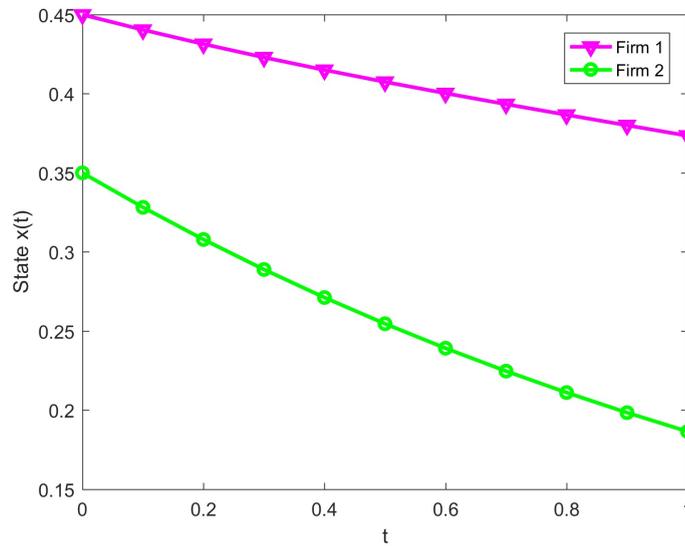
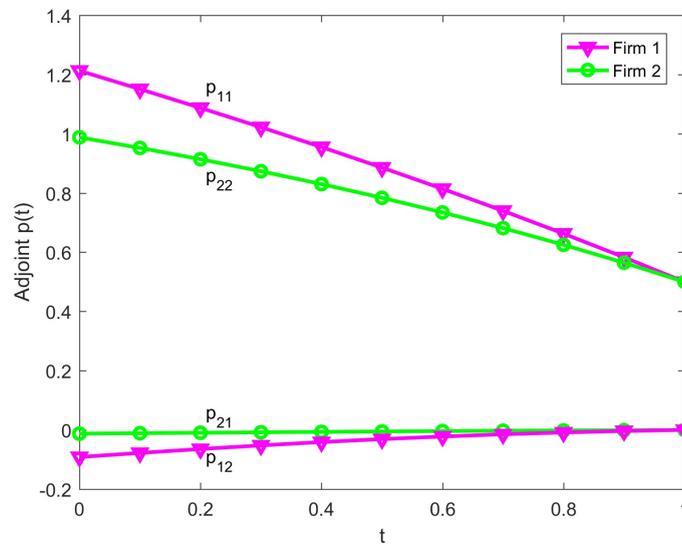


FIGURE 4.1: Optimal Controls  $u_1^*(t)$  and  $u_2^*(t)$  for Example 4.1.1

In Figure 4.1 we present the approximated optimal solutions of the control for Example 4.1.1. For this example the control  $u_i$  represented the advertising rate of its respective Firm at time  $t$ . From Figure 4.5 we see that the market share for both firms increases, reaches some turning point and then decreases. Also we see that Firm 1 has a greater advertising rate than Firm 2 on the interval  $t \in [0, 1]$ .

FIGURE 4.2: Optimal States  $x_1^*(t)$  and  $x_2^*(t)$  for Example 4.1.1

In Figure 4.2 we present the approximated optimal solutions of the state for Example 4.1.1. For this example the state  $x_i$  represented the market share of its respective Firm at time  $t$ . From Figure 4.2 we see that the market share for both of the firms is decreasing. Also we see that Firm 1 has a greater market share than Firm 2 on the interval  $t \in [0, 1]$ .

FIGURE 4.3: Optimal Adjoints  $p_{11}^*(t)$ ,  $p_{12}^*(t)$ ,  $p_{21}^*(t)$  and  $p_{22}^*(t)$  for Example 4.1.1

In Figure 4.3 we present the approximated optimal solutions of the adjoint for Example 4.1.1. The adjoint here represents the marginal value of the market share. In other words, the adjoint for this example is the rate of change in the payoff for small changes

in the market share. From Figure 4.3 we see that for Firm 1,  $p_{11}$  decreases whilst  $p_{12}$  increases and for Firm 2,  $p_{21}$  increases whilst  $p_{22}$  decreases on the interval  $t \in [0, 1]$ .

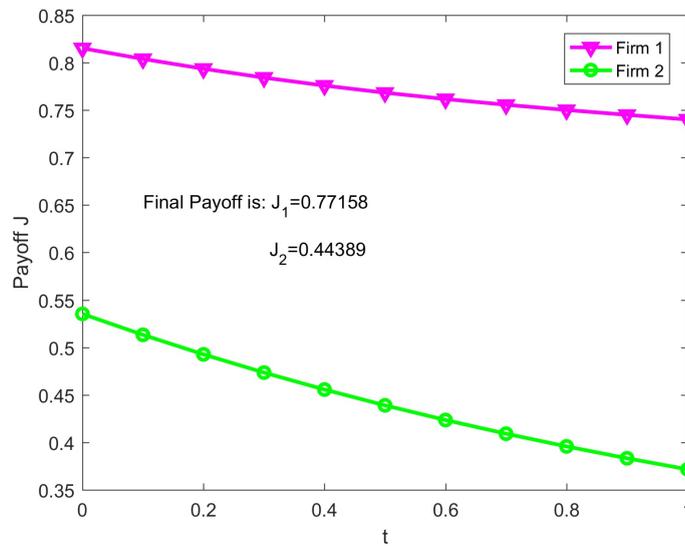


FIGURE 4.4: Optimal Payoffs  $J_1^*(t)$  and  $J_2^*(t)$  for Example 4.1.1

In Figure 4.4 we present the approximated optimal solution of the payoff functions  $J_1^*(t)$  and  $J_2^*(t)$  for Example 4.1.1. From Figure 4.4 we see that the value of the final payoffs are given as  $J_1^*(t) = 0.77158$  and  $J_2^*(t) = 0.44389$ . Also from Figure 4.4 we see that Firm 1 has a greater payoff than Firm 2 on the interval  $t \in [0, 1]$ .

■

**Example 4.1.2.** Recall that previously the solution of Example 3.1.2 was found to satisfy

$$\left\{ \begin{array}{l} u_1^\# = \frac{\lambda_1(1-x)}{k_1}, \\ u_2^\# = \frac{-\lambda_2 x}{k_2}, \\ \dot{x} = u_1^\#(1-x) - u_2^\#x, \quad x(0) = x_0, \\ \dot{\lambda}_1 = r_1\lambda_1 + (\lambda_1 + \nu_1)(u_1^\# + u_2^\#) - \phi_1 \quad \lambda_1(T) = \delta_{11} - \delta_{12} \\ \delta_{11} x(T) = 0, \quad \delta_{12} (1-x(T)) = 0, \\ \delta_{11} \geq 0, \quad \delta_{12} \geq 0, \\ \dot{\lambda}_2 = r_2\lambda_2 + (\lambda_2 + \nu_2)(u_1^\# + u_2^\#) + \phi_2, \quad \lambda_2(T) = \delta_{21} - \delta_{22}, \\ \delta_{21} x(T) = 0, \quad \delta_{22} (1-x(T)) = 0, \\ \delta_{21} \geq 0, \quad \delta_{22} \geq 0. \end{array} \right. \quad (4.12)$$

for all  $u_1, u_2$  which satisfy the constraints

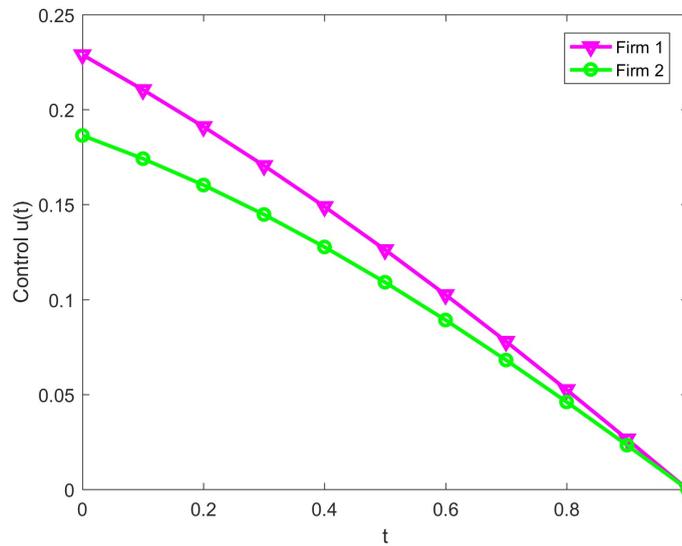
$$\begin{aligned} u_1(1-x) - u_2x &\geq 0, \\ u_2x - u_1(1-x) &\geq 0. \end{aligned} \quad (4.13)$$

and the slackness conditions

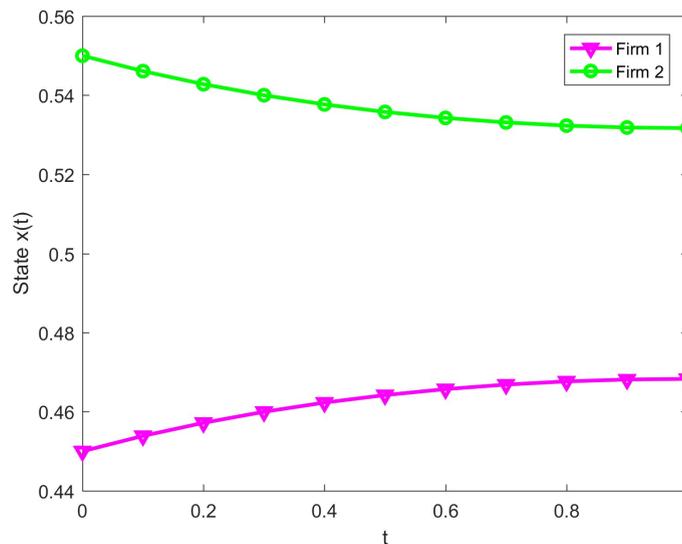
$$\left\{ \begin{array}{l} \nu_{11} x = 0, \quad \nu_{12} (1-x) = 0, \\ \nu_{21} x = 0, \quad \nu_{22} (1-x) = 0, \\ \nu_{11} \geq 0, \quad \nu_{12} \geq 0, \quad \nu_{21} \geq 0, \quad \nu_{22} \geq 0, \\ \nu_{11} \leq 0, \quad \nu_{12} \leq 0, \quad \nu_{21} \leq 0, \quad \nu_{22} \leq 0. \end{array} \right. \quad (4.14)$$

We will use the values of  $x(0) = 0.45$  for the initial condition,  $k_1 = k_2 = 1$ ,  $r_1 = 0.09$ ,  $r_2 = 0.08$ ,  $\phi_1 = \phi_2 = 0.5$ , for the constants, interest rates and fractional revenues,  $T = 1$  for the terminal time,  $N = 1000$  for the number of subintervals and  $\epsilon = 0.001$  for the tolerance to solve the problem numerically using Algorithm 4.1.1.

**Solution.** Hence using Algorithm 4.1.1 we obtain the following graphical results given by Figure 4.5 - 4.9

FIGURE 4.5: Optimal Controls  $u_1^*(t)$  and  $u_2^*(t)$  for Example 3.1.2

In Figure 4.5 we present the approximated optimal solutions of the control for Example 3.1.2. For this example the control  $u_i$  represented the advertising effort of its respective Firm at time  $t$ . From Figure 4.5 we see that the advertising effort of both firms is decreasing. Also we see that Firm 1 has a greater advertising effort than Firm 2 over the interval  $t \in [0, 1]$ .

FIGURE 4.6: Optimal State  $x^*(t)$  for Example 3.1.2

In Figure 4.6 we present the approximated optimal solutions of the state for Example 3.1.2. For this example the state  $x_i$  represented the market share of its respective Firm at time  $t$ . From Figure 4.6 we see that the market share for Firm 1 increases whilst the

market share for Firm 2 decreases. Also we see that Firm 2 has a greater market share than Firm 1 over the interval  $t \in [0, 1]$ .

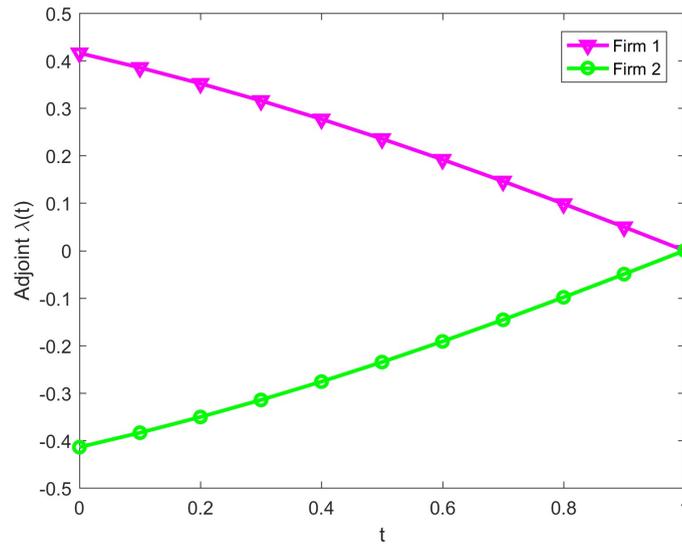


FIGURE 4.7: Optimal Adjoints  $\lambda_1^*(t)$  and  $\lambda_2^*(t)$  for Example 3.1.2

In Figure 4.7 we present the approximated optimal solutions of the adjoint for Example 3.1.2. The adjoint here represents the marginal value of the market share. In other words, the adjoint for this example is the rate of change in the payoff for small changes in the market share. From Figure 4.7 we see that for Firm 1,  $\lambda_1$  decreases whilst for Firm 2,  $\lambda_2$  increases on the interval  $t \in [0, 1]$ .

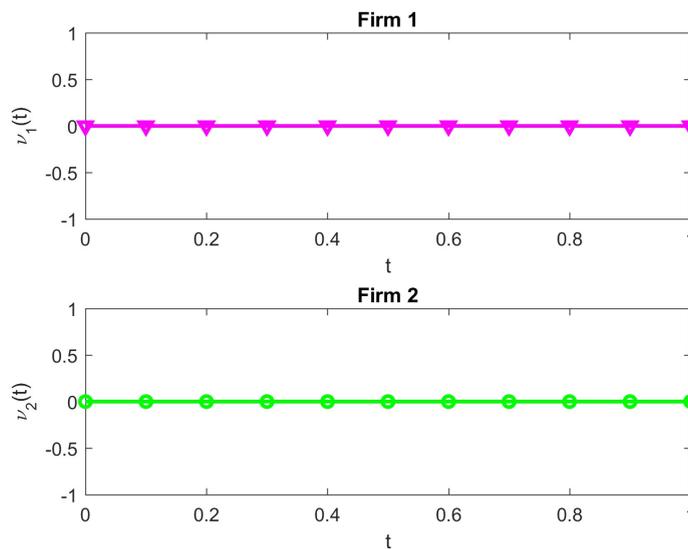


FIGURE 4.8: Optimal Lagrange Multipliers  $\nu_1^*(t)$  and  $\nu_2^*(t)$  for Example 3.1.2

In Figure 4.8 we present the approximated optimal solution of the Lagrange multipliers  $\nu_1^*(t)$  and  $\nu_2^*(t)$  for Example 3.1.2. For simplicity, when solving this example we let the multiplier  $\nu_1 = \nu_{11} - \nu_{12}$  and  $\nu_2 = \nu_{21} - \nu_{22}$ . The multiplier for this example is the rate of change in the payoff for small changes in the pure state constraint. From Figure 4.8 we see that  $\nu_1 = \nu_2 = 0$ .

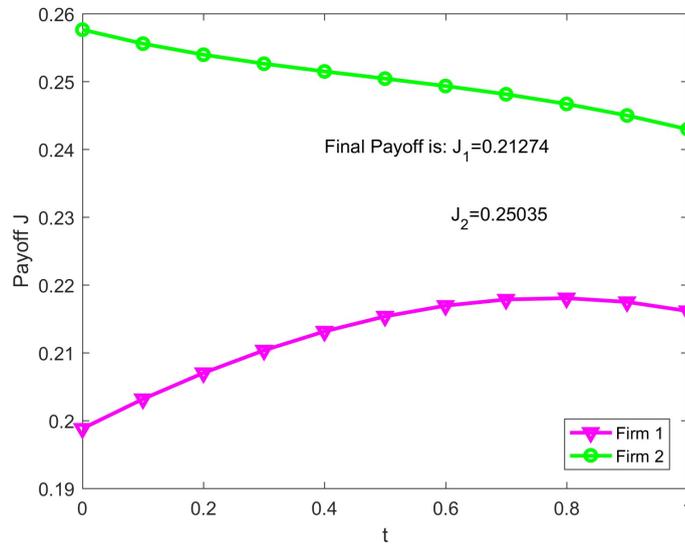


FIGURE 4.9: Optimal Payoffs  $J_1^*(t)$  and  $J_2^*(t)$  for Example 3.1.2

In Figure 4.9 we present the approximated optimal solution of the payoff functions  $J_1^*(t)$  and  $J_2^*(t)$  for Example 3.1.2. From Figure 4.9 we see that the value of the final payoffs are given as  $J_1^*(t) = 0.21274$  and  $J_2^*(t) = 0.25035$ . Also from Figure 4.9 we see that Firm 2 has a greater payoff than Firm 1 on the interval  $t \in [0, 1]$ .

■

**Example 4.1.3.** Recall that previously the solution of Example 3.1.1 was found to satisfy

$$\left\{ \begin{array}{l} u_1^\# = -p_1 \cdot x, \\ u_2^\# = \frac{1}{(x \cdot (1 - p_2))^2}, \\ \dot{x} = x(u_2^\# - u_1^\#), \quad x(0) = x_0, \\ \dot{p}_1 = (p_1 + \mu_1)(u_1^\# - u_2^\#) - u_2^\#, \quad p_1(T) = \delta_1 \\ \delta_1 x(T) = 0, \\ \delta_1 \geq 0, \\ \dot{p}_2 = (p_2 + \mu_2)(u_1^\# - u_2^\#) + u_2^\#, \quad p_2(T) = \delta_2, \\ \delta_2 x(T) = 0, \\ \delta_2 \geq 0. \end{array} \right. \quad (4.15)$$

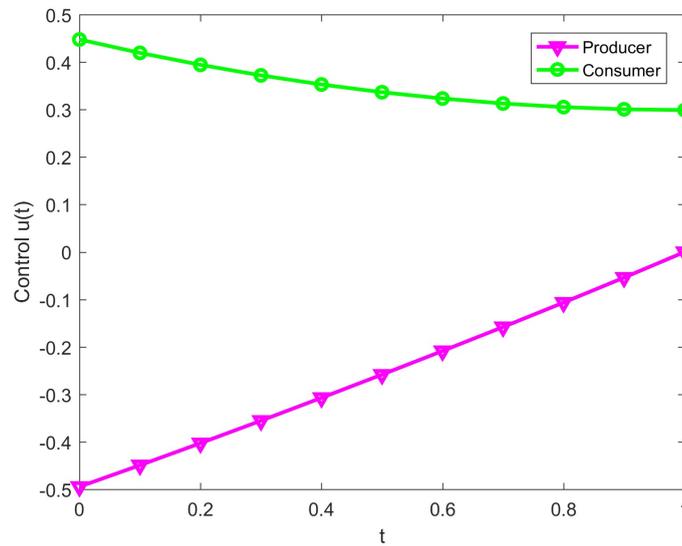
which satisfy the slackness conditions

$$\left\{ \begin{array}{l} \mu_1 \geq 0, \quad \mu_1 x = 0, \quad \mu_1 \leq 0, \\ \mu_2 \geq 0, \quad \mu_2 x = 0, \quad \mu_2 \leq 0. \end{array} \right. \quad (4.16)$$

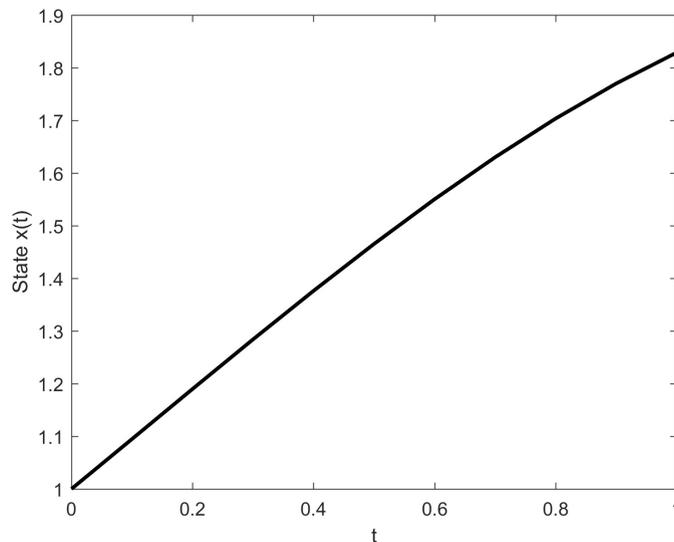
to determine the state  $x$  and the adjoint variables  $p_1$  and  $p_2$ .

We will use the values of  $x(0) = 1$ ,  $T = 1$ ,  $N = 1000$  and  $\epsilon = 0.001$  for the initial condition, terminal time, number of subintervals and tolerance respectively to solve the problem numerically using Algorithm 4.1.1.

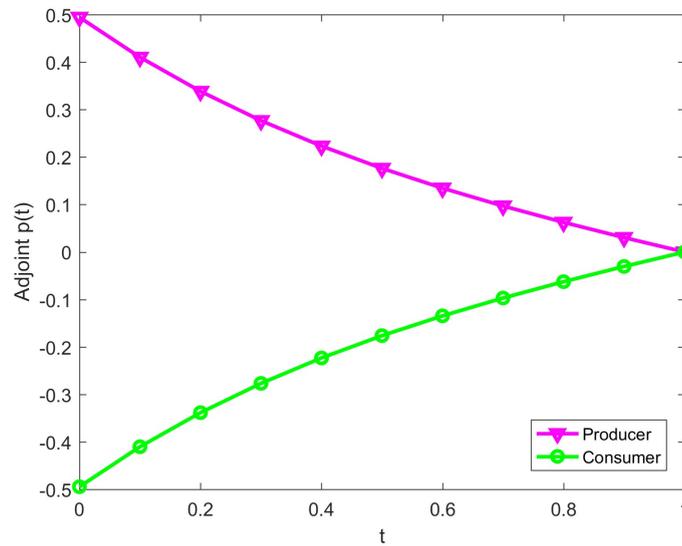
**Solution.** Hence using Algorithm 4.1.1 we obtain the following graphical results given by Figure 4.10 - 4.14

FIGURE 4.10: Optimal Controls  $u_1^*(t)$  and  $u_2^*(t)$  for Example 3.1.1

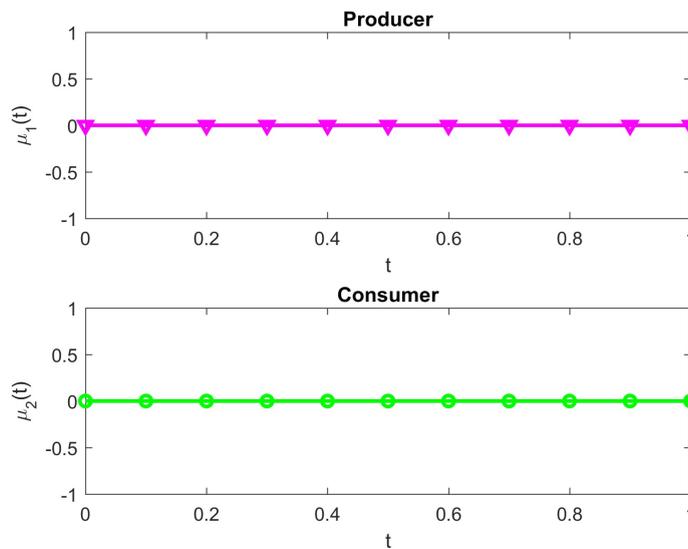
In Figure 4.10 we present the approximated optimal solutions of the control for Example 3.1.1. For this example the control  $u_1$  and  $u_2$  represented the production and consumption rates respectively. From Figure 4.10 we see that the rate of production is increasing whilst the rate of consumption is decreasing. Also we see that the rate of consumption is greater than the rate of production over the interval  $t \in [0, 1]$ .

FIGURE 4.11: Optimal State  $x^*(t)$  for Example 3.1.1

In Figure 4.11 we present the approximated optimal solutions of the state for Example 3.1.1. For this example the state  $x$  represented the sale price. From Figure 4.11 we see that the sale price is increasing on the interval  $t \in [0, 1]$ .

FIGURE 4.12: Optimal Adjoints  $\lambda_1^*(t)$  and  $\lambda_2^*(t)$  for Example 3.1.1

In Figure 4.12 we present the approximated optimal solutions of the adjoint for Example 3.1.1. The adjoint here represents the marginal value of the sale price. In other words, the adjoint for this example is the rate of change in the payoff for small changes in the sale price. From Figure 4.12 we see that for the producer,  $\lambda_1$  decreases whilst for the consumer,  $\lambda_2$  increases on the interval  $t \in [0, 1]$ .

FIGURE 4.13: Optimal Lagrange Multipliers  $\mu_1^*(t)$  and  $\mu_2^*(t)$  for Example 3.1.1

In Figure 4.13 we present the approximated optimal solution of the Lagrange multipliers  $\mu_1^*(t)$  and  $\mu_2^*(t)$  for Example 3.1.1. The multiplier for this example is the rate of change

in the payoff for small changes in the pure state constraint. From Figure 4.13 we see that  $\mu_1 = \mu_2 = 0$ .

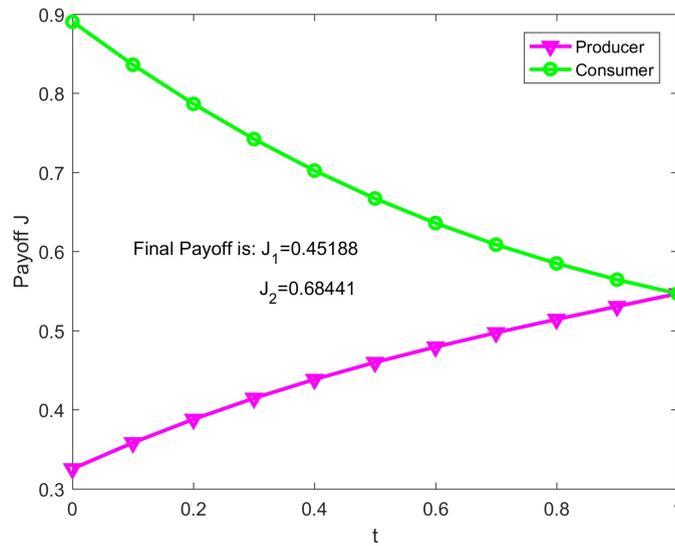


FIGURE 4.14: Optimal Payoffs  $J_1^*(t)$  and  $J_2^*(t)$  for Example 3.1.1

In Figure 4.14 we present the approximated optimal solution of the payoff functions  $J_1^*(t)$  and  $J_2^*(t)$  for Example 3.1.1. From Figure 4.14 we see that the value of the final payoffs are given as  $J_1^*(t) = 0.45188$  and  $J_2^*(t) = 0.68441$ . Also from Figure 4.4 we see that the consumer has a greater payoff than the producer on the interval  $t \in [0, 1]$ .

■

## Chapter 5

# Conclusion

This dissertation successfully obtained open-loop Nash and Stackelberg equilibrium solutions for an  $N$ -player differential game with pure state constraints by using an application of the maximum principle.

In solving for an open-loop Nash equilibrium, an application of the maximum principle yielded a set of necessary conditions for the solution. By assuming concavity and convexity of the appropriate functions and sets, we were able to get sufficient conditions for an existence of an open-loop Nash equilibrium solution. However, in practice there may be situations where these concavity and convexity assumptions fail, and hence the solutions may not be optimal.

Also for an open-loop Nash Equilibrium to exist, every player has to be able to deduce the equilibrium strategies for themselves and their opponents. In practice, these players can be people, animals, or computer programs. As a result there is a possibility of human error, mistakes or technical malfunctions while implementing a strategy, and hence the game will involve some uncertainty.

A possible solution to the above problems for the open-loop Nash equilibriums would be to introduce the notion of a randomized strategy to the two player differential game. By adding the element of probability to the strategies, we obtain a class of differential games known as stochastic differential games. For this class of differential games, we will obtain a more general existence of an open-loop Nash equilibrium. The theory of stochastic differential games provides us with the first possible area for future work in differential games.

Another shortcoming of an open-loop Nash equilibrium is that it's possible for a game to have multiple equilibrium solutions. Sufficient conditions guarantee that an equilibrium solution exists but the players might not know which solution they should focus on. A remedy to this problem would be to allow the players to communicate with each other so that they can choose an equilibrium through negotiation. This leads us to a class of games known as cooperative differential games, providing us with the second possible area for future work in differential games.

In solving for the necessary conditions of an open-loop Stackelberg equilibrium, we used a combination of variational calculus and the application of the maximum principle. Here, we first found a set of necessary conditions for the follower and then for the leader. In order to solve for this equilibrium solution, we had assumed that the roles of the players remained fixed at the beginning of the game. However, in practice this is not always the case. It is not always of a player's best interest to remain the leader, or alternatively the follower of the game. Also, there are situations in which being the leader may be of best interest to both of the players or alternatively to none of them. There are also situations that may arise in which the roles of the players are not clear. A possible solution to this leadership problem would be to introduce the notion of a mixed leadership differential game as in Başar et al [41]. The theory of mixed leadership differential games provides us with the third possible area for future work in differential games.

Also in solving for the necessary conditions of an open-loop Stackelberg equilibrium, we had focused on a game situation with a single leader and follower. In practice, there are many differential game situations where there are multiple leaders and followers. Hence an extension of the theory to solve for an open-loop Stackelberg equilibrium with multiple leaders and followers may be the fourth possible area for future work.

In solving an  $N$ -player differential game problem completely we had to adopt the use of computer techniques and numerical methods. We successfully described and illustrated a numerical algorithm for solving for an open-loop Nash equilibrium, but failed to do so for an open-loop Stackelberg equilibrium. This paves the way for the fifth area for future work in the sense that we can apply numerical methods to solve for an open-loop Stackelberg equilibrium of an  $N$ -player differential game.

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