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EXPLICIT MODELS FOR SPHERICAL STATIC FLUIDS IN EINSTEIN-GAUSS-BONNET GRAVITY

Muhammad Al-Zafar Khan

Explicit models for spherical static fluids in Einstein-Gauss-Bonnet gravity

Muhammad Al-Zafar Khan

Submitted in fulfilment of the academic requirements for the degree of Master in Science to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal.

As the candidate's supervisors, we have approved this dissertation for submission.

Signed:	Dr Riven Narain	December 2016
Signed:	Prof. Sunil Maharaj	December 2016

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Abstract

In this research, we explore modifications to the Einstein-Hilbert action by examining exact barotropic distributions in Einstein-Gauss-Bonnet gravity (EGB). We consider exact solutions of interior models in stars in five-dimensional EGB theory with spherical symmetry. We start by giving a brief introduction to the theory of general relativity and thereafter give a review of EGB gravity. From basic astrophysical modeling using the static five-dimensional metric, we obtain classical differential geometric quantities and thereafter produce the EGB field equations. These equations are a set of highly nonlinear partial differential equations and it is very difficult to solve exactly. By imposing a transformation proposed by Durgapal and Bannerji (1983), the field equations are written in equivalent form. Earlier EGB models are reviewed. New classes of exact solutions to the Einstein equations in five dimensions are found and their physical features are studied. In the EGB case we find two exact models with constant density. The first solution is the generalized Schwarzchild model. The second solution corresponds to a specific value of the Gauss-Bonnet constant.

Contents

Chapter 1

Introduction

The general theory of relativity has difficulty in explaining several inconsistencies such as taking into account the causes of the acceleration of the universe, its failure to be quantized - that is a quantum field theory cannot be generated for it, and its predictions of spacetime singularities. As a consequence several alternate theories of gravity have surfaced. An example is the Einstein-Gauss-Bonnet (EGB) theory which has been demonstrated to be promising in this regard, and has been studied rigorously. Consequently, in heterotic string theoretic models EGB gravity appears as a natural consideration of the effective action in the low energy limit. Therefore, EGB gravity generalizes the Einsteinian theory of gravity by adding an additional term to the Einstein-Hilbert action. This additional term is quadratic in the Riemann tensor and thus the variation of this term with respect to the metric allows for the attainment of a system of second order equations of motion which shares many nice properties with traditional general relativity. Additionally, EGB gravity is ghost-free about exact backgrounds, i.e. the negative norm state does not break unitarity and thus probabilities are strictly positive (Boulware and Deser - 1985).

In this thesis, we seek to find new exact solutions to the five-dimensional Einstein and EGB field equations for spherically symmetric, static, uncharged fluids. Using differential geometric quantities, we derive the field equations and thereafter explore solution strategies using a variety of ad hoc techniques. In the past, solutions have been found in terms of elementary functions see for example the works of Chilambwe *et al* (2015), Hansraj *et al* (2015) and Maharaj *et al* (2015). Solutions in this thesis have been found in terms of elementary functions and special functions. For a comprehensive review of generating exact solutions to the Einstein field equations, the reader is encouraged to see Stephani *et al* (2003).

The importance of traditional four-dimensional general relativity is quintessential to the understanding of gravitational phenomena such as stellar formation and gravitational collapse. However the importance of a higher dimensional theory of gravity cannot be overstated, and is pivotal to an improved understanding of many physical phenomena. For example, it is of tantamount importance to have an improved theory of gravity in order to explain the large scale structure of the universe, the expansion and acceleration of the universe and a concrete explanation for dark energy. Further, in order to attain a unification of the fundamental forces the Strong Nuclear Force, the Weak Nuclear Force, the Electromagnetic force and the Gravitational Force, i.e. a Grand Unified Theory (GUT), we require an improvement to traditional general relativity. In this regard, EGB gravity has proved to be a worthy successor to the Einsteinian theory of gravity with higher order curvature terms.

There exists several well known exact solutions to the five-dimensional Einstein and EGB field equations for spherically symmetric spacetimes. Of these proposed solutions, very few are physically viable. In the domain of neutral and charged isotropic spheres, many excellent solutions have been obtained by Durgapal and Bannerji (1983), Finch and Skea (1989) and Hansraj and Maharaj (2006). For charged, anisotropic matter, a recent paper has been published by Mafa Takisa and Maharaj (2013). Some results have been found by Hansraj *et al* (2015) and Maharaj *et al* (2015). Another interesting class of new solutions has been obtained by Chilambwe *et al* (2015). The reader is encouraged to seek out these papers to view a more modern approach to solution methodologies. The study of black hole solutions in EGB theory has been carried out by Wheeler (1986), Myers and Simon (1988) and Torii and Maeda (2005). The inhomogeneous collapse of dust in pressure-free fluids containing non-interacting particles in EGB theory was studied extensively by Maeda (2006), and solutions to this model were attained by Jhingan and Ghosh (2010).

In chapter 2, we give the mathematical preliminaries that are fundamental to the understanding of general relativity. We start with the assumption of a four-dimensional spacetime and thereafter extend the theory to five-dimensions. Firstly, we give the definition of the basic differential geometry quantities. Thereafter, we introduce the Einstein tensor and then give the Einstein field equations. We then describe the physical phenomenon of causality. The energy conditions are then stated, and it is pointed out that for a physical solution to be viable, they have to be satisfied. We conclude this chapter by introducing higher dimensional gravity in the form of the EGB field equations with the inclusion of the Lovelock term.

In chapter 3, using the five-dimensional, spherically symmetric, static line element, we derive the relevant differential geometry quantities. We combine the Einstein tensor and the Lanczos term linearly and equate to the matter term to form the EGB field equations. Using the pressure isotropy condition, we get the master gravitational equation. Thereafter, using a transformation proposed by Durgapal and Bannerji (1983), we convert the field equations into a form that is easier to work with. This transformation is also applied to the pressure isotropy equation and we get two representations of this equation, one in terms of the dependent variable Y and the other in terms of Z.

In chapter 4, we give a review of three known solutions that were found by

Chilambwe *et al* (2015), Hansraj *et al* (2015) and Maharaj *et al* (2015). We outline the strategy used and then give the solutions in terms of the matter variables for both the higher dimensional Einstein and EGB cases. The solutions indicate that the EGB equations provide a rich family of physically viable models.

In chapter 5, we examine the five-dimensional Einstein field equations and produce three new solutions. We thereafter examine the physical features of these solutions and perform a matching of the interior spacetime with the exterior Boulware-Deser metric at the surface of the star. Choosing specific values of the parameters, we produce graphical renditions of the solutions and discuss their physical viabilities. Lastly, we generalize the model to the higher dimensional Einstein case to include any arbitrary function.

In chapter 6, we consider the EGB equations with the Lovelock term present. We show that it is possible to integrate the field equations for a specific choice of one of the potentials. Two cases of exact solutions are identified.

In chapter 7, we conclude this thesis by discussing what has been accomplished through this work and the ramifications of finding a higher dimensional theory of gravity.

Chapter 2

Mathematical formalism of general relativity

2.1 Introduction

Mathematically speaking, the concept of spacetime is traditionally modelled as a four-dimensional, smooth, continuously differentiable (C^∞) manifold. A manifold is a topological space that is locally Euclidean because for every point in the manifold there is a neighbourhood that is topologically the same as the open unit ball in $\mathbb{R}^n$. From a point-set topological perspective, a spacetime is a Hausdorff space, because for any two non-identical points on the manifold, a continuous function exists that separates the two points, this is due to the condition of the separation axiom acting on the spacetime. What is meant here by smooth is that the manifold is defined everywhere, with no singularities, and functions have continuous partial derivatives. The differentiability of a manifold allows for the introduction of continuous coordinate systems, at least locally. This allows for the definition of curves, vector fields and tensor fields. From a local perspective, the spacetime manifold displays properties of a Euclidean space, in that orthogonality of the basis frames are present and the conditions of special relativity hold. Each point in the manifold has coordinates $(x^a, 0 \leq a \leq 3) = (x^0, x^1, x^2, x^3)$ where 1,2,3 denote the three spatial coordinates and $x^0 = ct$ (where c is the speed of light in a vacuum given exactly as 2.99792458×10^8 m/s) is the timelike coordinate. The spacetime structure forms the basis for the definition of invariant quantities in differential geometry, see for example the works of de Felice and Clarke (1990), Misner *et al* (1973), Wald (1984), Foster and Nightingale (2010) and Poisson (2004).

2.2 Differential geometry

The quantities T and T^* denote the space of all tangent and dual tangent vector spaces respectively on a curve in the manifold. The vectors $\{\mathbf{e}_a\}$ and $\{\mathbf{e}^a\}$ are basis vectors in T and T^* respectively. In order to consider metrical properties on the manifold we define a symmetric, nonsingular, covariant tensor $\mathbf{g}$ of rank two called the metric tensor field. Thus we have that $\mathbf{g} \in T \otimes T$ and $\mathbf{g} = g_{ab}\mathbf{e}^a \otimes \mathbf{e}^b$

where $\mathbf{g}$ is the bilinear functional

$$\mathbf{g} : (\mathbf{e}_a, \mathbf{e}_b) \to \mathbb{R},$$

relative to a basis $\{\mathbf{e}_a\}$. The metric tensor field $\mathbf{g}$ endows the manifold with the inner product

$$\begin{aligned}\langle \mathbf{X}, \mathbf{Y} \rangle &= \mathbf{g}(\mathbf{X}, \mathbf{Y}) \\ &= g_{ab} X^a Y^b,\end{aligned}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vector fields. The manifold in which the indefinite metric tensor $\mathbf{g}$ is defined, is called a pseudo-Riemannian manifold. The invariant quantity

$$s = \int_{t_1}^{t_2} |g_{ab} \dot{x}^a \dot{x}^b|^{1/2} \, \mathrm{d}t,$$

defines the length along a curve on the manifold between t_1 and t_2 which represents the values of the parameter t at the endpoints of the curve. This definition is independent of the coordinates used and does not depend on the way the curve is parametrised. The infinitesimal distance between neighbouring points with coordinates x^a and $x^a + \mathrm{d}x^a$ is defined by the invariant relativistic quantity

$$\mathrm{d}s^2 = g_{ab} \mathrm{d}x^a \mathrm{d}x^b, \tag{2.1}$$

called the line element or Riemannian fundamental form.

The metric connection Γ is defined in terms of the metric tensor field $\mathbf{g}$ and its derivatives. It is given by

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{bc,d}), \tag{2.2}$$

where the commas in (2.2) denote partial differentiation. Note that there exists a unique connection Γ that preserves inner products under parallel transport on the manifold (do Carmo - 1992).

We can now define a rank four tensor in terms of the Christoffel symbols and its related partial derivatives as follows

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}. \tag{2.3}$$

The quantity (2.3) is known as the Riemann or the curvature tensor which provides a measure of the amount of curvature of a manifold. The Riemann tensor measures how much a spacetime manifold deviates from flatness. A spacetime is Minkowski (flat space) if $R^d{}_{abc} = 0$ and for curved spacetimes $R^d{}_{abc} \neq 0$.

Performing a contraction on equation (2.3), we obtain the Ricci tensor

$$R_{ab} = R^c{}_{acb} = \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac}. \tag{2.4}$$

We form the Ricci scalar by further contracting equation (2.4) as follows

$$R = g^{ab}R_{ab}. \tag{2.5}$$

The equations (2.4) and (2.5) equip us with the machinery we need to form the Einstein tensor **G**. This is given by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}. \tag{2.6}$$

Since the Ricci and metric tensors are symmetric, it follows that the Einstein tensor is also symmetric. The Einstein tensor is divergence-free so that

$$G^{ab}{}_{;b} = 0. \tag{2.7}$$

Equation (2.7) is known famously as the Bianchi identity, from which conservation laws are generated. A proof of the result (2.7) can be found in Foster and Nightingale (2010).

The Weyl tensor is another tensor which can be obtained from the Riemann tensor, Ricci tensor and Ricci scalar. It is defined by

$$\begin{aligned} C_{abcd} = R_{abcd} &+ \frac{1}{2}\left(g_{ab}R_{bc} - g_{ac}R_{bd} + g_{bc}R_{ad} - g_{bd}R_{ac}\right) \\ &+ \frac{1}{6}\left(g_{ac}g_{bd} - g_{ad}g_{cb}\right)R, \end{aligned} \tag{2.8}$$

in four dimensions. The Weyl tensor measures the secondary effects of gravitational force that a particle experiences while traveling along a geodesic and represents tidal effects. The Weyl and Riemann tensors are different in that while the Riemann tensor precisely quantifies the change in volume of a particle, the Weyl tensor describes the distortion of the shape of the particle under the effect of the gravitational force. It has the same symmetries as the Riemann tensor but is trace-free; it is the Riemann tensor with the Ricci terms subtracted out. For an extensive treatment of differential geometry with applications to general relativity, the reader is referred to Bishop and Goldberg (1980), Borisenko and Tarapov (1968) and Wald (1984).

2.3 The matter tensor and the Einstein field equations

The matter content is described by the energy momentum tensor $\mathbf{T}$. It is defined as follows

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab}, \tag{2.9}$$

where ρ is the energy density and p is the isotropic pressure, q^a is the contravariant heat flow vector $(q^a u_a) = 0$ and π^{ab} is the pressure or stress tensor $(\pi^{ab} u_a = 0, \pi^a{}_a = 0)$. These quantities in equation (2.9) are measured with respect to a comoving fluid four velocity u^a, which is unit and timelike $(u^a u_a = -1)$. In the absence of heat flux and anisotropic stress $(q^a = 0,\ \pi^{ab} = 0)$ we have the simpler case

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}. \tag{2.10}$$

This is the form of a perfect fluid matter distribution. The distribution (2.10) is studied in this thesis.

In order to investigate how the mass of celestial bodies affects the curvature of spacetime, we let equation (2.6) equal to equation (2.9) in order to arrive at the famous Einstein field equations

$$G^{ab} = T^{ab}. \tag{2.11}$$

The Einstein field equations relate the gravitational field to the matter content. Equation (2.11) is a set of highly nonlinear partial differential equations, for which it is difficult to find exact solutions. We utilize geometric units in which the speed of light and the coupling constant are taken to be unity.

2.4 Energy conditions and causality

Consider two events say A and B. For event A to be the cause of event B, it is only natural to assume that A occurs before B. But if some observer thinks that A occurs before B and another thinks that B occurred before A, then a contradiction occurs. More formally, " an event cannot occur from a cause which is not in the past light cone of that event. " This is known as the law of causality. In order to prevent contradictory circumstances and to ensure that the law of causality is not violated, a set of mathematical criteria has been established to eliminate unphysical solutions to the Einstein field equations. These criteria apply generally in the theory of general relativity and are called energy conditions. The energy conditions impose restrictions on the eigenvalues and eigenvectors of the energy momentum tensor. In a four-dimensional manifold, this would require us to solve a quartic polynomial.

Thus for relativistic fluids to be rendered physically viable, they should obey the following energy conditions:

The weak energy condition: For every timelike vector $\mathbf{A}$, the density of matter observed is nonnegative. This gives the condition $\rho = T_{ab}A^aA^b \geq 0$.

The strong energy condition: For every future pointing timelike vector **A**, the observed trace of the energy momentum tensor is always nonnegative. We obtain the restriction $\left(T_{ab} - \frac{1}{2}Tg_{ab}\right) A^a A^b \geq 0$.

The dominant energy condition: Provided that the weak energy condition holds, mass-energy can never be observed to be moving faster than the speed of light. More formally, for every vector **A** (both null and timelike), the contracted vector $-T^a{}_b A^b$ must be a future pointing vector.

In the special case of the perfect fluid energy momentum tensor (2.10), these general conditions take the form

(a) the weak energy condition: $\rho - p \geq 0$,

(b) the strong energy condition: $\rho + p \geq 0$,

(c) the dominant energy condition: $\rho + 3p \geq 0$.

For a mathematical treatment and an in depth exploration into the origins of the energy conditions, the reader is referred to Hawking and Ellis (1973) and Kolassis *et al* (1988) .

2.5 Einstein-Gauss-Bonnet gravity

The Einstein theory of gravity is highly successful in explaining many physical observations. However it has shortcomings in describing particular situations such as the late time expansion of the universe. It is therefore necessary to consider modified theories of gravity to study a wide variety of gravitational phenomena. One particular modified theory is the EGB gravity which is widely studied. The EGB theory has been extensively applied to many cosmological and astrophysical scenarios because its geometrical features are consistent with an acceptable covariant theory of gravity. The higher order curvature terms make a nonzero addition to the dynamical behaviour of the model. The EGB action in five dimensions is of the form

$$S = \int \sqrt{-g} \left[\frac{1}{2} \left(R - 2\Lambda + \alpha \mathcal{L}_{\mathrm{GB}} \right) \right] \mathrm{d}^5 x + S_{\mathrm{matter}}, \tag{2.12}$$

where α is the Gauss-Bonnet coupling constant and S_{matter} is the matter contribution to the action integral. The Lovelock term has the form

$$\mathcal{L}_{\mathrm{GB}} = R^2 + R_{abcd} R^{abcd} - 4 R_{cd} R^{cd}. \tag{2.13}$$

With the Gauss-Bonnet modification of gravity, we redefine the interaction between geometry and matter in terms of EGB field equations as

$$G_{ab} + \alpha H_{ab} = T_{ab}, \tag{2.14}$$

The tensor H_{ab} is a special term called the Lanczos tensor which plays a similar role in general relativity to that of the vector potential in electromagnetic theory (Lovelock - 1971). In the setting of this research, we define the Lanczos tensor as

$$H_{ab} = 2(R R_{ab} - 2 R_{ac} R^{c}{}_{b} - 2 R^{cd} R_{acbd} + R^{cde}{}_{a} R_{bcde}) - \frac{1}{2} g_{ab} \mathcal{L}_{\mathrm{GB}}. \tag{2.15}$$

Traditionally, the Lanczos tensor is defined as a rank three tensor, which is used to generate the Weyl tensor. However, we will not consider it's rank three formalism here.

Chapter 3

Einstein-Gauss-Bonnet theory

3.1 Introduction

In this chapter, we derive the master equations in EGB gravity that are the foremost objective of our study. In order to achieve this, we derive the relevant differential geometric quantities needed. We then find forms for the matter variables, that is, the density and the radial pressure. We thereafter generate the pressure isotropy condition in order to attain our master equation. This equation, in it's original form, is difficult to work with and cannot be solved exactly. Therefore we apply a transformation proposed by Durgapal and Bannerji (1983) and convert this equation into two other forms in terms of the new variables. We observe the presence of the Lovelock term increases the nonlinearity of the EGB field equations. In §3.2, using the five-dimensional spherically symmetric, static line element, we derive all relevant geometric quantities. In §3.3, we give the forms for the components of the Lanczos tensor explicitly. These forms are not easily obtainable in the prescribed literature, and thus we have endeavoured to categorically show them. We combine the Einstein tensor components from §3.2 with these Lanczos tensor components by introducing an arbitrary coupling constant. These expressions are then equated to the energy momentum tensor components to produce the EGB field equations. Lastly, we generate the pressure isotropy condition and transform it into two master equations.

3.2 Geometric quantities

The line element for a general five-dimensional static, spherically symmetric spacetime is given by

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + \mathrm{e}^{2\lambda}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2 + \sin^2\theta\ \sin^2\phi\ \mathrm{d}\psi^2), \tag{3.1}$$

in comoving coordinates $(x^0, x^1, x^2, x^3, x^4) = (t, r, \theta, \phi, \psi)$. The metric functions $\nu(r)$ and $\lambda(r)$ represent the gravitational potentials.

Thus, the metric tensor is

$$g_{ab} = \begin{bmatrix} -\mathrm{e}^{2\nu} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{2\lambda} & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & r^2 \sin^2\theta \sin^2\phi \end{bmatrix}, \tag{3.2}$$

which is diagonal.

The geometric quantities associated with the line element (3.1) are not well known and cannot be found easily in the prescribed literature. We have therefore calculated these quantities in full and present them here. These results have been checked with the software packages Maple and GRTensor.

The nonzero Christoffel symbols for the line element (3.1) are

$$\Gamma^0{}_{01} = \nu', \tag{3.3a}$$

$$\Gamma^1{}_{00} = \nu' \mathrm{e}^{2(\nu-\lambda)}, \tag{3.3b}$$

$$\Gamma^1{}_{11} = \lambda', \tag{3.3c}$$

$$\Gamma^1{}_{22} = -r\mathrm{e}^{-2\lambda}, \tag{3.3d}$$

$$\Gamma^1{}_{33} = -r\mathrm{e}^{-2\lambda}\sin^2\theta, \tag{3.3e}$$

$$\Gamma^1{}_{44} = \, -\, r\mathrm{e}^{-2\lambda}\sin^2\!\theta\,\sin^2\!\phi, \tag{3.3f}$$

$$\Gamma^2{}_{12} = \Gamma^3{}_{13} = \Gamma^4{}_{14} = \frac{1}{r}, \tag{3.3g}$$

$$\Gamma^2{}_{33} = \, -\, \frac{1}{2}\sin(2\theta), \tag{3.3h}$$

$$\Gamma^2{}_{44} = \, -\, \frac{1}{2}\sin(2\theta)\,\sin^2\!\phi, \tag{3.3i}$$

$$\Gamma^3{}_{23} = \Gamma^4{}_{24} = \cot\theta, \tag{3.3j}$$

$$\Gamma^3{}_{44} = \, -\, \frac{1}{2}\sin(2\phi), \tag{3.3k}$$

$$\Gamma^4{}_{34} = \cot\phi. \tag{3.3l}$$

The nonzero Riemann curvature tensor components are

$$R_{0101} = \mathrm{e}^{2\nu}[\nu'' + \nu'(\nu' - \lambda')], \tag{3.4a}$$

$$R_{0202} = r\nu'\mathrm{e}^{2(\nu-\lambda)}, \tag{3.4b}$$

$$R_{0303} = \sin^2\!\theta\; R_{0202}, \tag{3.4c}$$

$$R_{0404} = \sin^2\!\theta\,\sin^2\!\phi\; R_{0202}, \tag{3.4d}$$

$$R_{1212} = r\lambda', \tag{3.4e}$$

$$R_{1313} = \sin^2\!\theta\; R_{1212}, \tag{3.4f}$$

$$R_{1414} = \sin^2\theta \, \sin^2\phi \; R_{1212}, \tag{3.4g}$$

$$R_{2323} = \sin^2\theta \; r^2(1 - \mathrm{e}^{-2\lambda}), \tag{3.4h}$$

$$R_{2424} = \sin^2\phi \; R_{2323}, \tag{3.4i}$$

$$R_{3434} = \sin^2\theta \, \sin^2\phi \; (\cos^2\theta - 1) \; r^2 \; (\mathrm{e}^{-2\lambda - 1}). \tag{3.4j}$$

The nonzero Ricci tensor components are

$$R_{00} = \mathrm{e}^{2(\nu-\lambda)} \left[(\nu')^2 + \nu'' + \frac{3\nu'}{r} - \nu'\lambda' \right], \tag{3.5a}$$

$$R_{11} = \nu'\lambda' - (\nu')^2 - \nu'' + \frac{3\lambda'}{r}, \tag{3.5b}$$

$$R_{22} = r\mathrm{e}^{-2\lambda} \left[\lambda' + \frac{2\mathrm{e}^{2\lambda}}{r} - \nu' - \frac{2}{r} \right], \tag{3.5c}$$

$$R_{33} = \sin^2\theta \; R_{22}, \tag{3.5d}$$

$$R_{44} = \sin^2\theta \, \sin^2\phi \; R_{22}. \tag{3.5e}$$

The Ricci scalar becomes

$$R = \frac{6(1 - \mathrm{e}^{-2\lambda})}{r^2} + \frac{6\mathrm{e}^{-2\lambda}(\lambda' - \nu')}{r} - 2\mathrm{e}^{-2\lambda}\nu''$$

$$- 2\mathrm{e}^{-2\lambda}(\nu')^2 + 2\mathrm{e}^{-2\lambda}\lambda'\nu'. \tag{3.6}$$

The Einstein tensor components have the form

$$G_{00} = \frac{3}{r^2}\left[r\mathrm{e}^{2(\nu-\lambda)}\lambda' + \mathrm{e}^{2\nu} - \mathrm{e}^{2(\nu-\lambda)}\right], \tag{3.7a}$$

$$G_{11} = \frac{3}{r^2}\left[1 - \mathrm{e}^{2\lambda} + r\nu'\right], \tag{3.7b}$$

$$G_{22} = \mathrm{e}^{-2\lambda}[1 + 2r\nu' + r^2(\nu')^2 - r\lambda'(r\nu' + 2) + r^2\nu''] - 1, \tag{3.7c}$$

$$G_{33} = \sin^2\theta\, G_{22}, \tag{3.7d}$$

$$G_{44} = \sin^2\theta\, \sin^2\phi\, G_{22}. \tag{3.7e}$$

3.3 The Einstein-Gauss-Bonnet field equations

The energy momentum tensor components are

$$T_{00} = \rho\mathrm{e}^{2\nu}, \tag{3.8a}$$

$$T_{11} = p\mathrm{e}^{2\lambda}, \tag{3.8b}$$

$$T_{22} = pr^2, \tag{3.8c}$$

$$T_{33} = \sin^2\theta\, T_{22}, \tag{3.8d}$$

$$T_{44} = \sin^2\theta\, \sin^2\phi\, T_{22}. \tag{3.8e}$$

The components of the Lanczos tensor are

$$H_{00} = -\frac{12\mathrm{e}^{2(\nu-\lambda)(1+\mathrm{e}^{2\lambda})\lambda'}}{r^3}, \tag{3.9a}$$

$$H_{11} = \frac{3(4-3\mathrm{e}^{-2\lambda})\nu'}{r^3}, \tag{3.9b}$$

$$\begin{aligned} H_{22} &= 4\mathrm{e}^{-2\lambda}(3\mathrm{e}^{-2\lambda}-1)\lambda'\nu' + 4\mathrm{e}^{-2\lambda}(1-\mathrm{e}^{-2\lambda})(\nu')^2 \\ &\quad + 4\mathrm{e}^{-2\lambda}(1-\mathrm{e}^{-2\lambda})\nu'', \end{aligned} \tag{3.9c}$$

$$H_{33} = \sin^2\theta\ H_{22}, \tag{3.9d}$$

$$H_{44} = \sin^2\theta\ \sin^2\phi\ H_{22}. \tag{3.9e}$$

It remains to combine the components of the Einstein tensor G_{ab} and the Lanczos tensor H_{ab}. From equations (3.7) and (3.9), we can write

$$G_{00} + \alpha H_{00} = \frac{3\mathrm{e}^{2(\nu-\lambda)}(r\mathrm{e}^{2\lambda} + r^2\lambda' - 4\alpha\mathrm{e}^{2\lambda}\lambda' - 4\alpha\lambda' - r)}{r^3}, \tag{3.10a}$$

$$G_{11} + \alpha H_{11} = \frac{3\mathrm{e}^{-2\lambda}(r\mathrm{e}^{2\lambda} + r^2\mathrm{e}^{2\lambda}\nu' + 4\alpha\mathrm{e}^{2\lambda}\nu' - 3\alpha\nu' - r\mathrm{e}^{4\lambda})}{r^3}, \tag{3.10b}$$

$$\begin{aligned} G_{22} + \alpha H_{22} &= \mathrm{e}^{-2\lambda}[2r\nu' + r^2(\nu')^2 + 4\alpha(\nu')^2 + r^2\nu'' + 4\alpha\nu'' + 1] \\ &\quad + 4\alpha\mathrm{e}^{-4\lambda}[3\lambda'\nu' - (\nu')^2 - \nu''] - \mathrm{e}^{-2\lambda}[2r\lambda' + r^2\lambda'\nu'] \\ &\quad - 4\alpha\mathrm{e}^{-2\lambda}\lambda'\nu' - 1, \end{aligned} \tag{3.10c}$$

$$G_{33} + \alpha H_{33} = \sin^2\theta \; (G_{22} + \alpha H_{22}), \tag{3.10d}$$

$$G_{44} + \alpha H_{44} = \sin^2\theta \, \sin^2\phi \; (G_{22} + \alpha H_{22}). \tag{3.10e}$$

Equating (3.8) and (3.10), we arrive at the EGB field equations

$$\rho = \frac{3}{\mathrm{e}^{4\lambda} r^3} \left[r\mathrm{e}^{4\lambda} - r\mathrm{e}^{2\lambda} - 4\alpha\lambda' + r^2 \mathrm{e}^{2\lambda}\lambda' - 4\alpha\mathrm{e}^{2\lambda}\lambda' \right], \tag{3.11a}$$

$$p = \frac{3}{\mathrm{e}^{4\lambda} r^3} \left[(r^2\nu' + 4\alpha\nu' + r)\mathrm{e}^{2\lambda} - r\mathrm{e}^{4\lambda} - 3\alpha\nu' \right], \tag{3.11b}$$

$$\begin{aligned} p = {} & \frac{1}{e^{4\lambda} r^2} \left[12\alpha\lambda'\nu' - 4\alpha(\nu')^2 - 4\alpha\nu'' - \mathrm{e}^{4\lambda} \right] \\ & + \frac{1}{\mathrm{e}^{2\lambda} r^2} \left[2r\nu' + r^2(\nu')^2 - r^2\lambda'\nu' - 2r\lambda' + 1 \right] \\ & + \frac{1}{\mathrm{e}^{2\lambda} r^2} \left[4\alpha(\nu')^2 + (r^2 + 4\alpha)\nu'' - 4\alpha\lambda'\nu' \right]. \end{aligned} \tag{3.11c}$$

The pressure isotropy condition requires that the radial and tangential components of the pressure are equal. Equating (3.11b) and (3.11c), we get

$$\begin{aligned} & \mathrm{e}^{-2\lambda}[2r^2\lambda' + r^2\nu' + 12\alpha\nu' + r^3\lambda'\nu' + 4\alpha \; r\lambda'\nu' + 2r - r^3(\nu')^2] \\ & + \alpha\mathrm{e}^{-4\lambda}[4r(\nu')^2 + 4r\nu'' - 12r\lambda'\nu' - 9\nu'] + r^3\mathrm{e}^{-2\lambda}\nu'' \\ & - \mathrm{e}^{-2\lambda}[4\alpha r(\nu')^2 - 4\alpha r\nu''] - 2r = 0. \end{aligned} \tag{3.12}$$

Equation (3.12) is the condition that governs the behaviour of the model. It is a highly nonlinear and difficult equation to analyse. It is possible that new variables may reduce (3.12) to simpler form. Durgapal and Bannerji (1983) proposed the transformation for general relativity equations

$$e^{2\nu} = Y^2(x), \tag{3.13a}$$

$$e^{-2\lambda} = Z(x) \tag{3.13b}$$

$$x = Cr^2, \tag{3.13c}$$

where C in (3.13c) is an arbitrary constant.

Finch and Skea (1989), Hansraj and Maharaj (2006), Chilambwe *et al* (2015). Hansraj *et al* (2015) and Maharaj *et al* (2015) and others have utilized this transformation with great success and hence we employ its use in this thesis to simplify equation (3.12). Substituting (3.13) into (3.11), we get

$$\frac{\rho}{C} = 3\dot{Z} + \frac{3(Z-1)(1-4\alpha C\dot{Z})}{x}, \tag{3.14a}$$

$$\frac{p}{C} = \frac{3(Z-1)}{x} + \frac{6Z\dot{Y}}{Y} - \frac{24\alpha C(Z-1)Z\dot{Y}}{xY}. \tag{3.14b}$$

Substituting (3.13) into (3.12), we get the pressure isotropy condition

$$\begin{aligned} 2xZ\left[4\alpha C(Z-1) - x\right]\ddot{Y} &- \left[x^2\dot{Z} + 4\alpha C\left(x\dot{Z} - 2Z + 2Z^2 - 3xZ\dot{Z}\right)\right]\dot{Y} \\ &- \left(1 + x\dot{Z} - Z\right)Y = 0. \end{aligned} \tag{3.15}$$

We are treating (3.15) as a differential equation in Y when Z is specified.

Rearranging equation (3.15) in terms of Z and its derivatives, we get

$$[(-x^2 - 4\alpha Cx)\dot{Y} - xY]\dot{Z} + (12\alpha Cx\dot{Y})Z\dot{Z} + [8\alpha C(x\ddot{Y} - \dot{Y})]Z^2$$

$$+ [(-2x^2 - 8\alpha Cx)\ddot{Y} + 8\alpha C\dot{Y} + Y]Z - Y = 0. \tag{3.16}$$

We are treating (3.16) as a differential equation in Z when Y is specified.

Equations (3.15) and (3.16) become our master equations and our focus will be on solving them.

Chapter 4

A review of known solutions

4.1 Introduction

In this chapter, we briefly provide three known solutions to the five-dimensional Einstein and Einstein-Gauss-Bonnet field equations that were found by Chilambwe *et al* (2015), Hansraj *et al* (2015) and Maharaj *et al* (2015). Using the solution generating techniques, we present these solutions and forms for the matter variables ρ and p. For a comprehensive study and a discussion of the physical characteristics of these solutions, the reader is encouraged to refer to these three papers. In §4.2, we discuss the solutions found by Chilambwe *et al* (2015). By making some simplification assumptions to the master equation (3.16), we select a linear form for the dependent variable Y and then solve the resulting equation for Z. Setting the coupling constant $\alpha = 0$, we provide the solution for the five-dimensional Einstein field equations and thereafter considering a nonzero coupling constant, we provide a generalized EGB solution. In §4.3, we discuss the solutions found by Hansraj *et al* (2015). In this case, a coefficient of the master equation (3.16), arranged in terms of Z and its derivatives is made to vanish. In this way, a form for Y is attained. This form of Y and its first and second derivatives are back substituted into the master equation in order to attain an equation with one dependent variable Z. Solutions to this equation are then attained in both the five-dimensional Einstein and EGB cases. In §4.4, we discuss the solutions found by Maharaj *et al* (2015). Using the method of Frobenius, a solution to both the five-dimensional Einstein and EGB cases are carried out. We provide an equaton of state for this equation.

4.2 Chilambwe *et al* (2015)

Letting $\beta = 4\alpha C$ in (3.16), we get

$$(x^2\dot{Y} + xY + \beta xY + \beta x\dot{Y} - 3\beta x\dot{Y}Z)\dot{Z} + 2\beta(\dot{Y} - x\ddot{Y})Z^2$$
$$+ (2x^2\ddot{Y} + 2\beta x\ddot{Y} - 2\beta\dot{Y} - Y)Z + Y = 0. \tag{4.1}$$

On setting $Y = a + bx$, where a and b are arbitrary constants, equation (4.1) reduces to

$$[x(a + bx) + bx^2 b\beta x - 3b\beta xZ]\dot{Z} + 2b\beta Z^2 - (bx + 2b\beta + a)Z$$
$$+ (a + bx) = 0. \tag{4.2}$$

4.2.1 The higher dimensional Einstein case

Setting $\alpha = 0$ in equation (4.2) gives $\beta = 0$. Thus equation (4.2) reduces to

$$x(a + 2bx)\dot{Z} - (a + bx)Z + (a + bx) = 0. \tag{4.3}$$

Integration of (4.3) yields

$$Z = 1 + \frac{c_1 x}{\sqrt{a + 2bx}}, \tag{4.4}$$

where c_1 is the constant of integration. Substituting these forms for Y and Z into the EGB field equations (3.14), we get

$$\frac{\rho}{C} = \frac{3c_1(2a + 3bx)}{(a + 2bx)^{3/2}}, \tag{4.5a}$$

$$\frac{p}{C} = \frac{3c_1(a + 3bx) + 6b\sqrt{a + 2bx}}{(a + bx)\sqrt{a + 2bx}}. \tag{4.5b}$$

4.2.2 The Einstein-Gauss-Bonnet case

When $\alpha \neq 0$, equation (4.2) is a nonlinear modified Abel equation, and the solution is not elementary. However, due to the simplifying assumption that $\beta = 4\alpha C$, the solution of equation (4.2) can be expressed in terms of elementary functions as

$$Z = \frac{(1 + 2A) \pm (a + 2bx)(1 - A)}{3b\beta}, \tag{4.6}$$

where

$$A = \frac{(80c_1^2x^2 - B)^{1/2}[(80c_1^2 - B)^{1/2} - \sqrt{80}c_1x]^{1/3}}{B^{1/3}\{[(80c_1^2x^2 - B)^{1/2} - \sqrt{80}c_1x]^{2/3} - B^{1/3}\}}, \tag{4.7a}$$

$$B = a^3 - 6a^2b\beta + 12a(b\beta)^2 - 8(b\beta)^3 + 6b[a^2 + 4(b\beta)^2 - 4ab\beta]x$$

$$+ 12b^2(a - 2b\beta)x^2 + 8b^2x^3, \tag{4.7b}$$

and c_1 is the constant of integration. Substituting these forms for Y and Z into the field equations, we get

$$\frac{\rho}{C} = \frac{b(2b\beta - a - 8bx)}{3b^2\beta x} + \frac{Ab(2b\beta + 4bx - a)}{3b^2\beta x}$$

$$- \frac{2A^2b(2b\beta - a - 2bx)}{3b^2\beta x} + \frac{}{3b^2\beta x} + \frac{A\dot{A}(2b\beta - a - 2bx)^2}{3b^2\beta x}$$

$$+ \frac{\dot{A}(2b\beta - a - 2bx)(a - 2b\beta - bx)}{3b^2\beta x}, \tag{4.8a}$$

$$\frac{p}{C} = \frac{10b^2x^2 + b(7 + 4b\beta)x + (2b\beta - a)^2}{3b\beta x(a + bx)} - \frac{2A^2(2b\beta - a - 2b\beta x)^2}{3b\beta x(a + bx)}$$

$$+ \frac{A(2b\beta - a - 2bx)(2b\beta + bx - a)}{3b\beta x(a + bx)}, \tag{4.8b}$$

where $\dot{A} = \frac{\mathrm{d}A}{\mathrm{d}x}$, $\ddot{A} = \frac{\mathrm{d}^2A}{\mathrm{d}x^2}$.

4.3 Hansraj *et al* (2015)

4.3.1 The Einstein-Gauss-Bonnet case

Making the coefficient of Z^2 vanish in equation (3.16), we get

$$8\alpha C(\dot{Y} - x\ddot{Y}) = 0. \tag{4.9}$$

Integrating (4.9), we obtain

$$Y = \frac{1}{2}ax^2 + b, \tag{4.10}$$

where a and b are the constants of integration. Letting $\beta = C\alpha$ and $\epsilon = \frac{a}{b}$ and substituting the equation (4.10) into equation (3.16), we get

$$(3\epsilon x^3 + 8\beta\epsilon x^2 + 2x - 24\beta\epsilon x^2 Z)\dot{Z} + (3\epsilon x^2 - 2)Z + 2 = 0. \tag{4.11}$$

Integrating (4.11), we get

$$Z = \frac{3\epsilon x^2 + 8\beta\epsilon x \pm M}{24\beta\epsilon x}, \tag{4.12}$$

where

$$\begin{aligned} M = {} & [4(1 - 16\beta\epsilon x) + 4\epsilon(16\beta^2\epsilon + 144\beta^2\epsilon c_1 + 3)x^2 \\ & + 3\epsilon^2(3x^4 + 32\beta x^3)]^{1/2}, \end{aligned} \tag{4.13}$$

and c_1 is the constant of integration. Substituting (4.10) and (4.12) and their respective derivatives into the EGB field equations (3.14), we get

$$\frac{\rho}{C} = \frac{27\epsilon^2x^4 - 48\beta\epsilon^2x^3 - 32\beta^2\epsilon x}{48\beta\epsilon^2x^4} + \frac{4 - 4M(1-4\beta\epsilon x)}{48\beta\epsilon^2x^4}$$

$$- \frac{\dot{M}x(3\epsilon x^2 + 16\beta\epsilon x - 2) - M(M - x\dot{M})}{48\beta\epsilon^2x^4}, \tag{4.14a}$$

$$\frac{p}{C} = \frac{27\epsilon^2x^4 + 96\beta\epsilon^2x^3 + 8\epsilon(3+32\epsilon\beta^2)x^2}{24\epsilon\beta x^2(\epsilon x^2+2)}$$

$$- \frac{64\beta x - 4 + Mx(3\epsilon x^2 + 16\epsilon\beta x + 2M - 2)}{24\epsilon\beta x^2(\epsilon x^2+2)}, \tag{4.14b}$$

where $\dot{M} = \frac{\mathrm{d}M}{\mathrm{d}x}$, $\ddot{M} = \frac{\mathrm{d}^2M}{\mathrm{d}x^2}$.

4.3.2 The higher dimensional Einstein case

Setting $\alpha = 0$ in equation (3.16), we get

$$(2x^2Z)\ddot{Y} + (x^2\dot{Z})\dot{Y} + (1 - Z + x\dot{Z})Y = 0. \tag{4.15}$$

Substituting the form for Y in equation (4.10) into (4.15), we obtain

$$x(3\epsilon x^2+1)\dot{Z} + (3\epsilon x^2 - 1)Z + (\epsilon x^2+1) = 0. \tag{4.16}$$

Integrating (4.16), we get

$$Z = \frac{1 + c_1x - \bar{\epsilon}x^2}{3\bar{\epsilon}x^2+1}, \tag{4.17}$$

where $\bar{\epsilon} = \frac{1}{2}\epsilon$ and c_1 is the constant of integration. Substituting equations (4.10) and (4.10) into the EGB field equations (3.14), we have

$$\frac{\rho}{C} = \frac{6(c_1 - 6\bar{\epsilon}x - \bar{\epsilon}^2 x^3)}{(1 + 3\bar{\epsilon}x^2)^2}, \tag{4.18a}$$

$$\frac{p}{C} = \frac{3(c_1 + 5c_1\bar{\epsilon}x^2 - 8\bar{\epsilon}^2 x^3)}{(1 + \bar{\epsilon}x^2)(1 + 3\bar{\epsilon}x^2)}. \tag{4.18b}$$

4.4 Maharaj *et al* (2015)

4.4.1 The higher-dimensional Einstein case

Setting $Z = a$ in equation (4.15), where a is some arbitrary constant, we get

$$(2x^2)\ddot{Y} + (1 - a)Y = 0. \tag{4.19}$$

Case 1: $a = 1$

Setting $a = 1$ in (4.19), we get

$$2x^2\ddot{Y} = 0. \tag{4.20}$$

Integration of (4.20) yields

$$Y = a + bx, \tag{4.21}$$

where a and b are constants of integration. However, these forms for Y and Z are inadmissible because upon substitution into the field equations we obtain a zero density.

Case 2: $a \neq 1$

Using the method of Frobenius, we seek solutions of the form

$$Y_1(x) = \sum_{k \geq 0} a_k x^{k+c}, \tag{4.22}$$

where a_k is the coefficient of the power series and c is some constant introduced to satisfy the indicial equation. Since (4.19), and more generally (3.16) are hypergeometric differential equations, we can express $Y_1(x)$ as

$$Y_1(x) = 2 \sum_{k \geq 1} \frac{(-1)^k a_0}{A^k} \prod_{i=1}^{k} [i(i+1) + E] x^{k+2}, \tag{4.23}$$

where a_0, A and E are constants. Specifically $A = 4\alpha(1-a)$ and $E = \frac{1-a}{2a}$. Since the roots of the indicial equation for Y_1 differ integrally, i.e. by an integer value, we can express the second solution as

$$Y_2(x) = mY_1(x) \ln x + \sum_{k \geq 0} b_k x^k, \tag{4.5}$$

where m is some arbitrary constant and b_k is the coefficient of the power series. Using the principle of superposition, we express the solution for Y as a linear combination of the forms for Y_1 and Y_2 as

$$Y = c_1 Y_1 + c_2 Y_2, \tag{4.24}$$

where c_1 and c_2 are arbitrary constants. When $\alpha = 0$, Y takes the form

$$Y = c_1 x^{\frac{1-\sqrt{\frac{3a-2}{a}}}{2}} + c_2 x^{\frac{1+\sqrt{\frac{3a-2}{a}}}{2}}. \tag{4.25}$$

Substituting this form for Y in equation (4.25) and the form for Z into the EGB field equations (3.14), we get

$$\frac{\rho}{C} = \frac{3(1-a)}{x}, \tag{4.26a}$$

$$\frac{p}{C} = \frac{3a\left[c_1\left(1-\sqrt{\frac{3a-2}{a}}\right)x^{-\left(\frac{1+\sqrt{\frac{3a-2}{a}}}{2}\right)} + c_2\left(1-\sqrt{\frac{3a-2}{a}}\right)x^{-\left(\frac{1-\sqrt{\frac{3a-2}{a}}}{2}\right)}\right]}{c_1 x^{\frac{1-\sqrt{\frac{3a-2}{a}}}{2}} + c_2 x^{\frac{1+\sqrt{\frac{3a-2}{a}}}{2}}}. \tag{4.26b}$$

Setting the constant $c_1 = 0$, we get a linear barotropic equation of state

$$p = \left[\frac{a}{1-a}\left(1+\sqrt{\frac{3a-2}{a}}\right) - 1\right]\rho. \tag{4.27}$$

Thus, from this model it is evident that $\rho \propto \frac{1}{r^2}$. Therefore, this model is valid for spherically symmetric bodies with isothermal temperature profiles. Hence, we have found a five-dimensional Einstein relation, with a linear equation of state, from the EGB equations.

Chapter 5

The higher dimensional Einstein case

5.1 Introduction

In this chapter we present three new solutions to the Einstein field equations for spherically symmetric fluid bodies in five dimensions. By making the coupling constant vanish in the general master equations, we attain new master equations that become the object of our study. These solutions are attained through astute selections of forms for the dependent variables. Without loss of generality, when a form for the dependent variable Y is chosen, the master equation is then solved to produce a form for Z. Unlike previous studies conducted, where solutions were attained in terms of elementary functions, see the works of Chilambwe *et al* (2015), Hansraj *et al* (2015) and Maharaj *et al* (2015), some of the solutions attained here are in terms of special functions. We thereafter match the interior solutions attained to the exterior spacetime which we take to be the Boulware-Deser metric at the boundary of the celestial object. Lastly, we examine the physical features of these models by selecting specific forms for the parameters and producing graphical renditions of the solutions and discussing their physical viabilities. In §5.2, we analyze the master equation by selecting a form for the dependent variable Z and then solve the master equation for Y. It is shown that selecting a linear form for Z is a redundant avenue of pursuit, as the form for the matter variables lies in the complex field $\mathbb{C}$. By choosing the natural logarithmic form for Z, we attain a form for Y in terms of hypergeometric functions and thereafter we perform a matching of the solutions with exterior spacetime at the boundary. We then plot the matter variables, the energy conditions and show that the speed of sound is less than the speed of light throughout the star. In §5.3, firstly we choose a form for Y that encapsulates all polynomials and transcendentals of degree n, with some restrictions to the value of n. A form for Z is then found explicitly and matching is thereafter carried out. The physical features of this model is then studied by plotting the various equations for specific values of the parameters. It is shown graphically that for the region $x \in (0, 3.5461)$ that the weak energy condition is defied. Thereafter, we choose the scaled exponential function and we attain a form for Z in terms of the exponential integral function. As with the other models, the matching is carried out and physical features is studied. Lastly, in §5.4, we select an arbitrary functional for Y and then using the method of integrating factors, find an arbitrary functional form for Z. This method generalizes all solutions to the five-dimensional Einstein equations. We then express the matter variables ρ and p in terms if these arbitrary functions.

5.2 Analysis in terms of Y

Equation (4.15) when $\alpha = 0$ is quintessential to the analysis of this section, we restate it here as

$$(2x^2 Z)\ddot{Y} + (x^2 \dot{Z})\dot{Y} + (x\dot{Z} - Z + 1)Y = 0. \tag{5.1}$$

Equation (5.1) becomes our master equation in this section and solution methodologies will be explored based on this equation.

5.2.1 Choosing a general linear functional form for Z

Selecting $Z = ax + b$, where a and b are arbitrary constants, and substituting this form into equation (5.1), we get

$$(2ax^3 + 2bx^2)\ddot{Y} + (ax^2)\dot{Y} + (1 - b)Y = 0. \tag{5.2}$$

Equation (5.2) is a second order linear differential equation with variable coefficients; upon integration we get

$$Y = c_1\sqrt{ax}\,\exp\left[-\sqrt{\frac{3b-2}{b}}\,\operatorname{arctanh}\left(\sqrt{\frac{ax+b}{b}}\right)\right]$$
$$- c_2\sqrt{\frac{x}{a(3b-2)}}\,\exp\left[\sqrt{\frac{3b-2}{b}}\,\operatorname{arctanh}\left(\sqrt{\frac{ax+b}{b}}\right)\right], \tag{5.3}$$

where c_1 and c_2 are the constants of integration. From equation (5.3), we see that the constants a and b are restricted by $a \in (0, \infty)$ and $b \in (\frac{2}{3}, \infty)$. Substituting these forms for Y and Z into the equation (3.14b) for pressure, we get a function whose domain is $x \in (-\infty, 0)$. This is unrealistic and we do not pursue this case any further.

5.2.2 Solution I

Selecting $Z = \ln x$ and substituting this form into (5.1), we get

$$(2x^2 \ln x)\ddot{Y} + x\dot{Y} + (2 - \ln x)Y = 0. \tag{5.4}$$

Equation (5.4) is a second order nonlinear differential equation with varying coefficients. Using the solution strategy proposed by Polyanin and Zaitsev (2002), we find the solution of (5.4) to be

$$Y = \frac{c_1\, x^{(1-\sqrt{3})/2}\, \mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ln x\right) \sqrt{\ln x}}{2^{1/4}}$$

$$+ \frac{c_2\, x^{(1-\sqrt{3})/2}\, \mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ln x\right) \sqrt{\ln x}}{2^{1/4}}, \tag{5.5}$$

where c_1 and c_2 are the constants of integration. The quantities $\mathcal{U}$ and $\mathcal{M}$ are the *Tricomi* and *Kummer confluent hypergeometric functions* respectively, and are defined by

$$\mathcal{U}(a,b,x) := \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-xt} t^{a-1} (1-t)^{b-a-1}\, \mathrm{d}t, \tag{5.6a}$$

$$\mathcal{M}(a,b,x) := \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-1)} \int_0^1 \mathrm{e}^{xt} t^{a-1} (1-t)^{b-a-1}\, \mathrm{d}t, \tag{5.6b}$$

and $\Gamma(\zeta)$ is the gamma function defined as

$$\Gamma(\zeta) := \int_0^\infty x^{\zeta-1} \mathrm{e}^{-x}\, \mathrm{d}x. \tag{5.6c}$$

For a comprehensive treatment of these special functions, the reader should see the excellent texts of Andrews *et al* (1999), Andrews (1985), Bell (2004) and Polyanin and Mazhirov (2007). Substituting these forms of Y and Z into the field equations (3.14), we get

$$\frac{\rho}{C} = \frac{3\ \ln x}{x}, \tag{5.7a}$$

$$\frac{p}{C} = \frac{3\ \ln x}{x} - \frac{3}{x} + (3 + 3\ \ln x - 3\sqrt{3}\ \ln x)(\Phi + \Psi)$$

$$+ (3\sqrt{3}\ \ln x - 5\ \ln x)\Lambda + (15\ \ln x - 9\sqrt{3})\Upsilon, \tag{5.7b}$$

where we have introduced

$$\Phi = \Phi(x) = \frac{c_2\mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right)}{x\left[c_1\mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right) + c_2\mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right)\right]}, \tag{5.8a}$$

$$\Psi = \Psi(x) = \frac{c_1\mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right)}{x\left[c_1\mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right) + c_2\mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right)\right]}, \tag{5.8b}$$

$$\Lambda = \Lambda(x) = \frac{c_2\mathcal{M}\left(\frac{21-5\sqrt{3}}{12}, \frac{5}{2}, \sqrt{3}\ \ln x\right)}{x\left[c_1\mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right) + c_2\mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right)\right]}, \tag{5.8c}$$

$$\Upsilon = \Upsilon(x) = \frac{c_1\mathcal{U}\left(\frac{21-5\sqrt{3}}{12}, \frac{5}{2}, \sqrt{3}\ \ln x\right)}{2x\left[c_1\mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right) + c_2\mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln x\right)\right]}. \tag{5.8d}$$

Physical Features

We study the physical features related to our proposed exact solutions. We produce graphical renditions for the parameter values $c_1 = 56$, $c_2 = 0$ and $C = \frac{1}{1000}$. From the graphs, we see that the energy density and pressure look very similar for this choice of the parameter values. The energy density in Figure 5.1, which is plotted for the range $x \in [1.05, 100]$, has a singularity at the centre of the star. Further, the energy density is a monotonically decreasing function and approaches zero as we move away from the centre of the star. The pressure in Figure 5.2 is also a monotonically decreasing function and eventually tends toward zero as we move to the boundary. This defines the pressure-free surface for the star, thus guaranteeing that the stellar surface does in fact exist, and there exists a realistic star candidate. The pressure was plotted for the range $x \in [1.41151, 100]$, and also has a singularity at the origin.

In Figure 5.3 the speed of sound parameter converges to a numerical value smaller than one. Mathematically,

$$\lim_{x\to\infty} \frac{\mathrm{d}p}{\mathrm{d}\rho} = 2 - \sqrt{3} < 1.$$

Thus, throughout the star the speed of sound is less than the speed of light and causality is maintained. i.e. for this region the the sound speed is subluminal. We also observe in Figure 5.4 that $\rho - p \geq 0$, $\rho + p \geq 0$ and $\rho + 3p \geq 0$. This implies that the weak, strong and dominant energy conditions are satisfied throughout the star. The speed of sound parameter was plotted for the range $x \in [21.6, 100]$.

In order to illustrate the physical viability of this stellar model, we match the interior solution found above to the exterior spacetime. We take the exterior spacetime to be the Boulware-Deser metric given by

$$\mathrm{d}s^2 = -F(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2 + \sin^2\theta \sin^2\phi\, \mathrm{d}\psi^2), \tag{5.9a}$$

where

$$F(r) = 1 + \frac{r^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8M\alpha}{r^4}}\right), \tag{5.9b}$$

and M is the mass of the gravitating hypersphere.

On the inside of the star, we require that

$$e^{2\lambda} = \frac{1}{\ln(Cr^2)}, \tag{5.10a}$$

$$e^{2\nu} = \frac{\left[c_1\mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln(Cr^2)\right) + c_2\mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln(Cr^2)\right)\right]^2 \ln(Cr^2)\ (Cr^2)^{1-\sqrt{3}}}{\sqrt{2}}. \tag{5.10b}$$

At the boundary of the gravitating hypersphere $r = R$, we require that the interior solution generated matches the metric (5.9a). Thus

$$\frac{1}{\ln(CR^2)} = \frac{4\alpha}{4\alpha + R^2\left(1 - \sqrt{1 + \frac{8M\alpha}{R^4}}\right)}, \tag{5.11a}$$

$$\frac{[\tilde{a}c_1 + \tilde{b}c_2]^2\ \ln(CR^2)\ (CR^2)^{1-\sqrt{3}}}{\sqrt{2}} = \frac{R^2}{4\alpha}\left(1 - \sqrt{\frac{R^4 + 8M\alpha}{R^4}}\right) + 1. \tag{5.11b}$$

The third junction condition that we require is that of a pressure-free hypersurface at the boundary. Therefore, we obtain

$$\begin{aligned} &\frac{3\ln(CR^2)}{CR^2} - \frac{3}{CR^2} + \frac{3 + 3\ln(CR^2) - 3\sqrt{3}\ \ln(CR^2)}{CR^2} \\ &+ \left[3\sqrt{3}\ln(CR^2 - 5\ln(CR^2))\right]\left[\frac{c_2\tilde{d}}{CR^2(c_1\tilde{a} + c_2\tilde{b})}\right] \\ &+ [15\ln(CR^2) - 9\sqrt{3}]\left[\frac{c_1\tilde{c}}{2CR^2(c_1\tilde{a} + c_2\tilde{b})}\right] = 0. \end{aligned} \tag{5.11c}$$

In equations (5.11b) and (5.11c),

$$\tilde{a} = \mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln(CR^2)\right), \tag{5.12a}$$

$$\tilde{b} = \mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\ \ln(CR^2)\right), \tag{5.12b}$$

$$\tilde{c} = \mathcal{U}\left(\frac{21-5\sqrt{3}}{12}, \frac{5}{2}, \sqrt{3}\ \ln(CR^2)\right), \tag{5.12c}$$

$$\tilde{d} = \mathcal{M}\left(\frac{21-5\sqrt{3}}{12}, \frac{5}{2}, \sqrt{3}\ \ln(CR^2)\right). \tag{5.12d}$$

In principle, one can solve (5.11b) and (5.11c) for c_1 and c_2 simultaneously and thus uniquely fix these arbitrary constants in terms of the stellar radius R and the mass of the spherical hypersphere M.

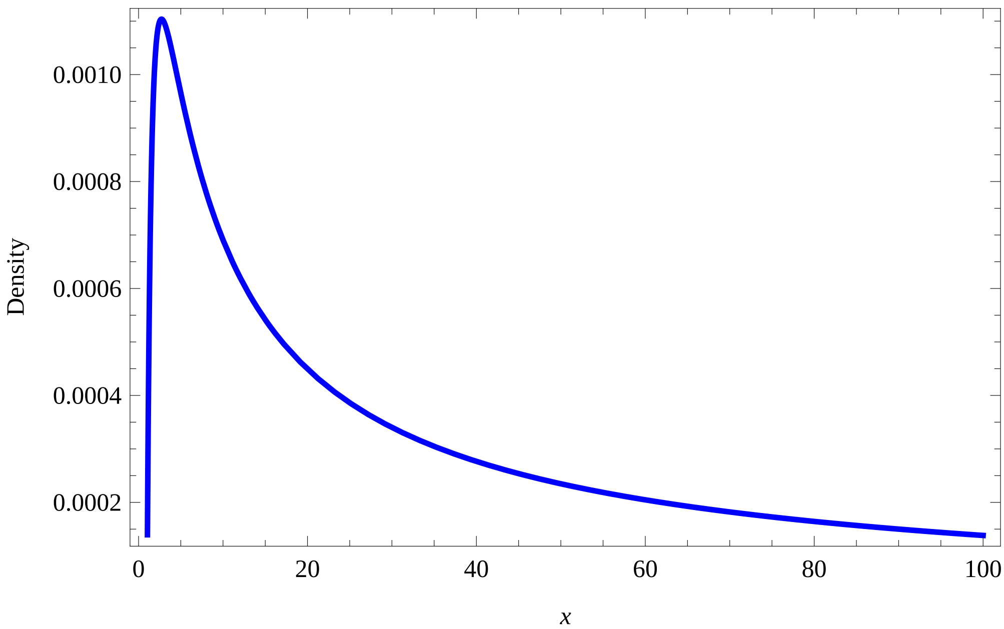

Figure 5.1: Plot of the energy density versus the radial coordinate x.

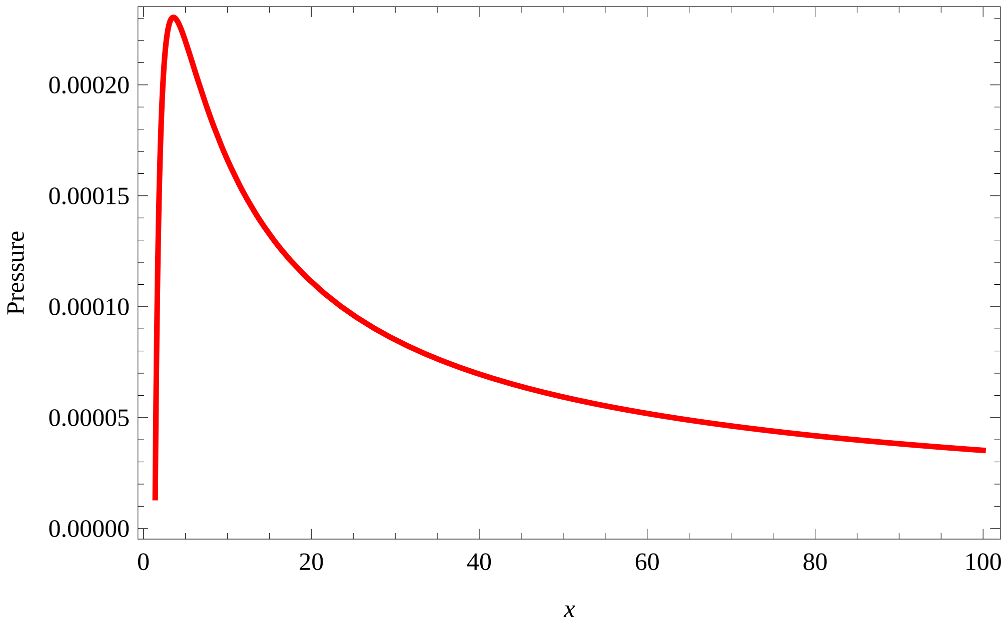

Figure 5.2: Plot of the pressure versus the radial coordinate x.

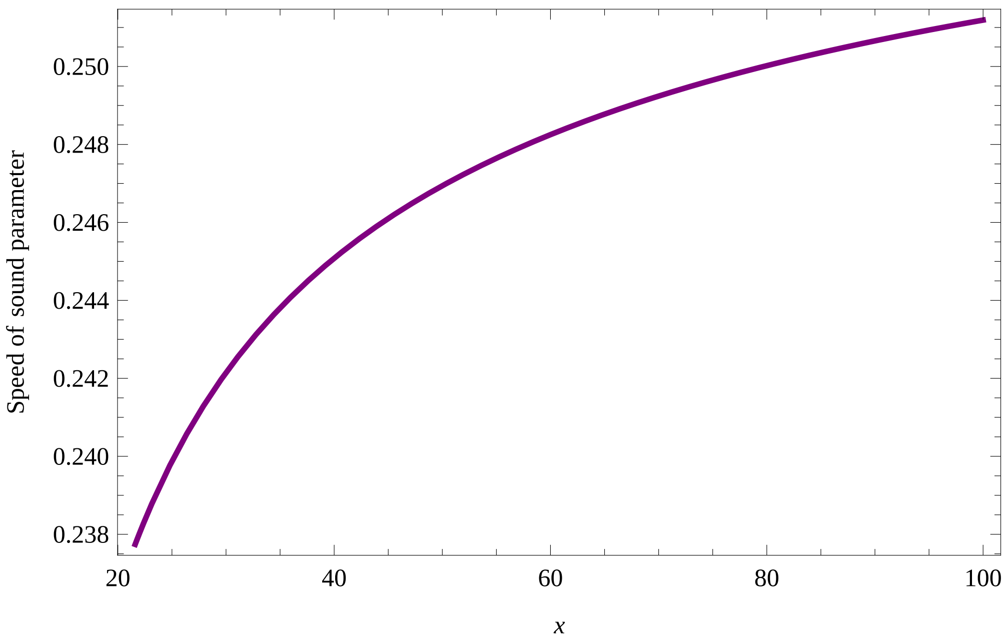

Figure 5.3: Plot of the speed of sound parameter versus the radial coordinate x.

The energy conditions are governed by the equations

$$\frac{\rho-p}{C}=\frac{3}{x}+(3\sqrt{3}\ \ln x-3\ \ln x-3)(\Phi+\Psi)$$

$$+(5\ \ln x-3\sqrt{3}\ \ln x)\Lambda+(9\sqrt{3}-15\ \ln x)\Upsilon, \tag{5.13a}$$

$$\frac{\rho+p}{C}=\frac{6\ \ln x}{x}-\frac{3}{x}+(3+3\ \ln x-3\ \sqrt{3}\ \ln x)(\Phi+\Psi)$$

$$+(3\sqrt{3}\ \ln x-5\ \ln x)\Lambda+(15\ \ln x-9\sqrt{3})\Upsilon, \tag{5.13b}$$

$$\frac{\rho+3p}{C}=\frac{12\ \ln x}{x}-\frac{9}{x}+(9+9\ \ln x-9\sqrt{3}\ \ln x)(\Phi+\Psi)$$

$$+(9\sqrt{3}\ \ln x-15\ \ln x)\Lambda+(45\ \ln x-27\sqrt{3}\ \ln x)\Upsilon. \tag{5.13c}$$

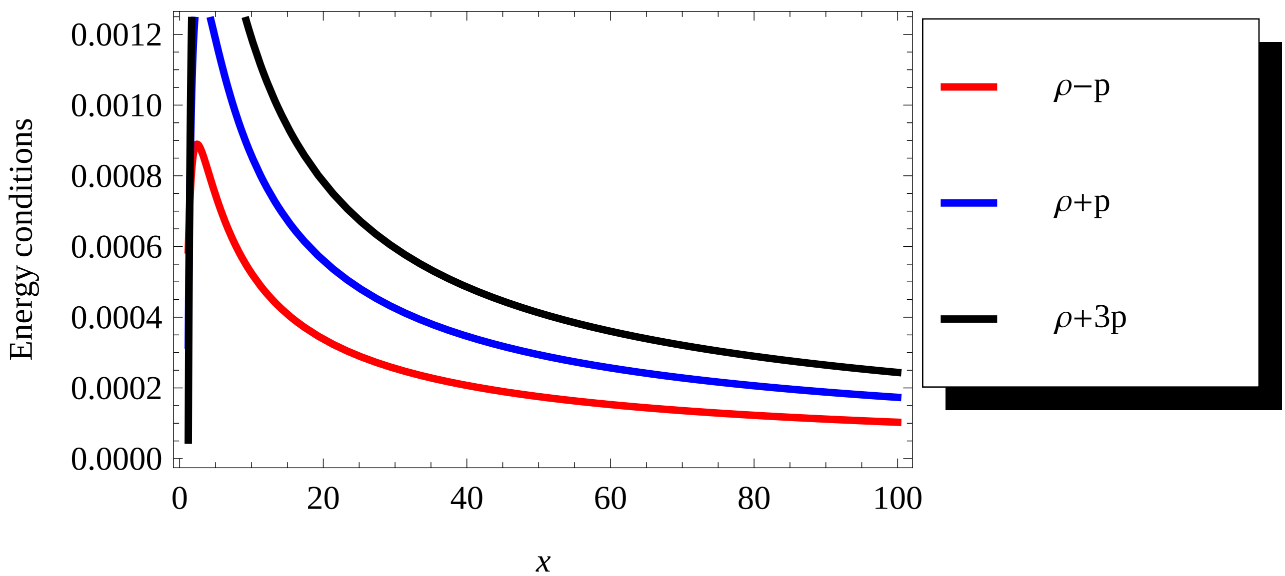

Figure 5.4: Plot of the energy conditions versus the radial coordinate x.

5.3 Analysis in terms of Z

Setting $\alpha = 0$ in equation (3.16), we get

$$x(x\dot{Y} + Y)\dot{Z} + (2x^2\ddot{Y} - Y)Z + Y = 0. \tag{5.14}$$

Equation (5.14) becomes our master equation in this section, and solution methodologies will be explored based on this equation.

5.3.1 Solution II

Selecting $Y = x^n$, where n is some arbitrary constant, and substituting this form for Y into equation (5.14), we get

$$(n+1)x\dot{Z} + (2n^2 - 2n - 1)Z + 1 = 0. \tag{5.15}$$

Equation (5.15) is a first order linear differential equation, direct integration of this equation yields

$$Z = \frac{1}{1 + 2n - 2n^2} + c_1[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}, \tag{5.16}$$

where c_1 is the constant of integration. From equation (5.16), we get the restriction that $n \neq -1, \frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}$. Substituting our forms for Y and Z into the field equations (3.14), we get

$$\frac{\rho}{C} = \frac{6n(n-1)}{(1-2n-2n^2)x} + \frac{3c_1(2+3n-2n^2)[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}}{(n+1)x}, \tag{5.17a}$$

$$\frac{p}{C} = \frac{6n^2}{(1+2n-2n^2)x} + \frac{3c_1(2n+1)[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}}{x}. \tag{5.17b}$$

Physical Features

We study the physical features related to our exact solutions. We produce graphical renditions for the parameter values $c_1 = -178$, $n = 92$, $C = -23$. From the graphs we see that the energy density and pressure look very similar for this choice of the parameter values. The energy density in Figure 5.5 is plotted for the range $x \in [0.1.100]$, we observe that the star has a singularity at the origin. Further, the energy density is a monotonically decreasing function and approaches zero as we move away from the centre of the star. The pressure in Figure 5.6 is plotted for the range $x \in [0, 100]$. The pressure is also a monotonically decreasing function and eventually tends toward zero as we move away from the centre of the star to the boundary. This defines a pressure-free surface for the star.

From Figure 5.7 we observe that $\frac{\mathrm{d}p}{\mathrm{d}\rho} = 1$ throughout the star, so the speed of sound equals the speed of light throughout the star. We plot the energy conditions, $\rho - p$, $\rho + p$ and $\rho + 3p$ for the range $x \in [13, 100]$ in Figure 5.8. For this range, we see that all three energy conditions are strictly positive and satisfied. We match the interior solution found above with the exterior spacetime defined by the metric

(5.9a). For the inside of the star, we require

$$e^{2\lambda} = \frac{1}{\frac{a}{1+2n-2n^2} + c_1[(n+1)(Cr^2)]^{\frac{1+2n-2n^2}{n+1}}}, \tag{5.18a}$$

$$e^{2\nu} = (Cr^2)^{2n}. \tag{5.18b}$$

At the boundary $r = R$, we require that the interior solution generated matches the exterior spacetime (5.9a). Thus

$$\frac{1}{\frac{a}{1+2n-2n^2} + c_1[(n+1)(CR^2)]^{\frac{1+2n-2n^2}{n+1}}} = \frac{4\alpha}{4\alpha + R^2\left(1 - \sqrt{1 + \frac{8M\alpha}{R^4}}\right)}, \tag{5.19a}$$

$$(CR^2)^{2n} = \frac{R^2}{4\alpha}\left(1 - \sqrt{\frac{R^4 + 8M\alpha}{R^4}}\right) + 1. \tag{5.19b}$$

Further, the pressure at the boundary must be zero. Therefore

$$\frac{6n^2}{(1+2n-2n^2)CR^2} + \frac{3c_1(2n+1)[(n+1)CR^2]^{\frac{1+2n-2n^2}{n+1}}}{(n+1)CR^2} = 0. \tag{5.19c}$$

Using equation (5.19c), we find c_1 to be

$$c_1 = \frac{2n^2(n+1)[CR^2(n+1)]^{\frac{2n^2-2n-1}{n+1}}}{(2n+1)(2n^2-2n-1)}. \tag{5.20}$$

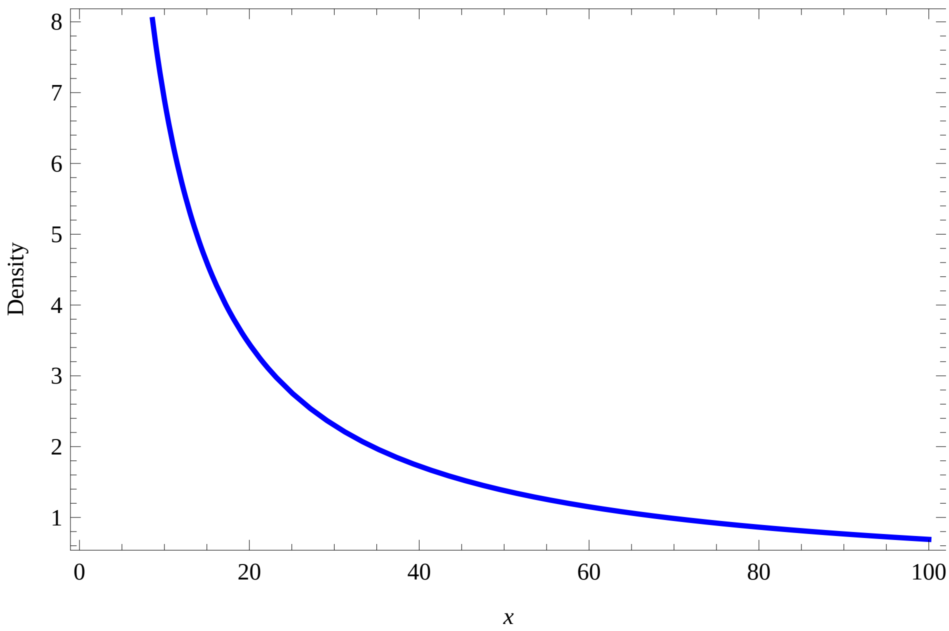

Figure 5.5: Plot of the energy versus the radial coordinate x.

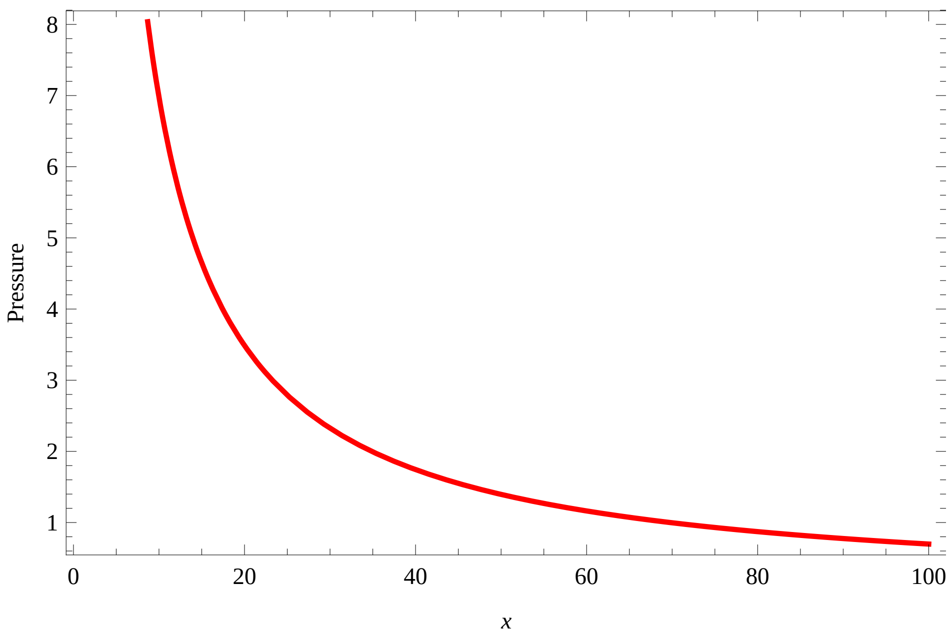

Figure 5.6: Plot of the pressure versus the radial coordinate x.

The speed of sound parameter has the form

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{\mathcal{A}}{\mathcal{B}}, \tag{5.21}$$

where

$$\mathcal{A} = \frac{2n^2}{2n^2-2n-1} - c_1(2n+1)(2n^2-2n-1)x[(n+1)x]^{\frac{n(1-2n)}{n+1}} - c_1(2n+1)[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}, \tag{5.22a}$$

$$\mathcal{B} = \frac{2n(n-1)}{2n^2-2n-1} + c_1(2n^2-3n-2)x[(n+1)x]^{\frac{n(1-2n)}{n+1}} + \frac{c_1(2n^2-3n-2)(2n^2-2n-1)[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}}{(n+1)^2}. \tag{5.22b}$$

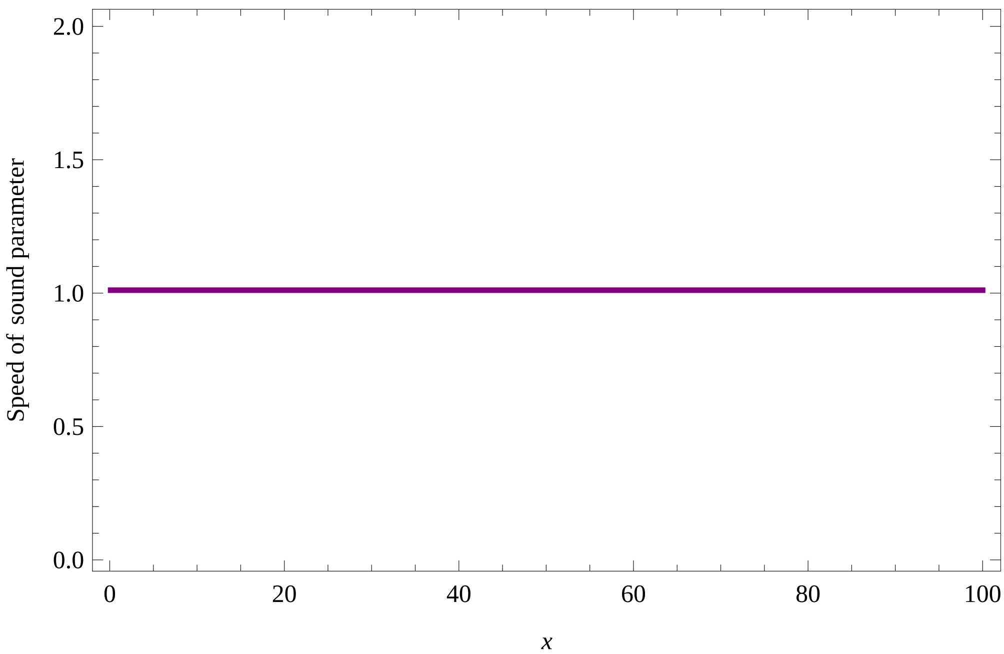

Figure 5.7: Plot of the speed of sound parameter versus the radial coordinate x.

The energy conditions have the form

$$\frac{\rho - p}{C} = \frac{6n}{(2n^2 - 2n - 1)x} + \frac{3c_1(1-4n^2)[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}}{x}, \tag{5.23a}$$

$$\frac{\rho + p}{C} = \frac{6n(2n-1)}{(1+2n-2n^2)x} + \frac{9c_1(2n+1)[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}}{(n+1)x}, \tag{5.23b}$$

$$\frac{\rho + 3p}{C} = \frac{6n(4n-1)}{(1+2n-2n^2)x} + \frac{3c_1(4n^2+12n+5)[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}}{(n+1)x}. \tag{5.23c}$$

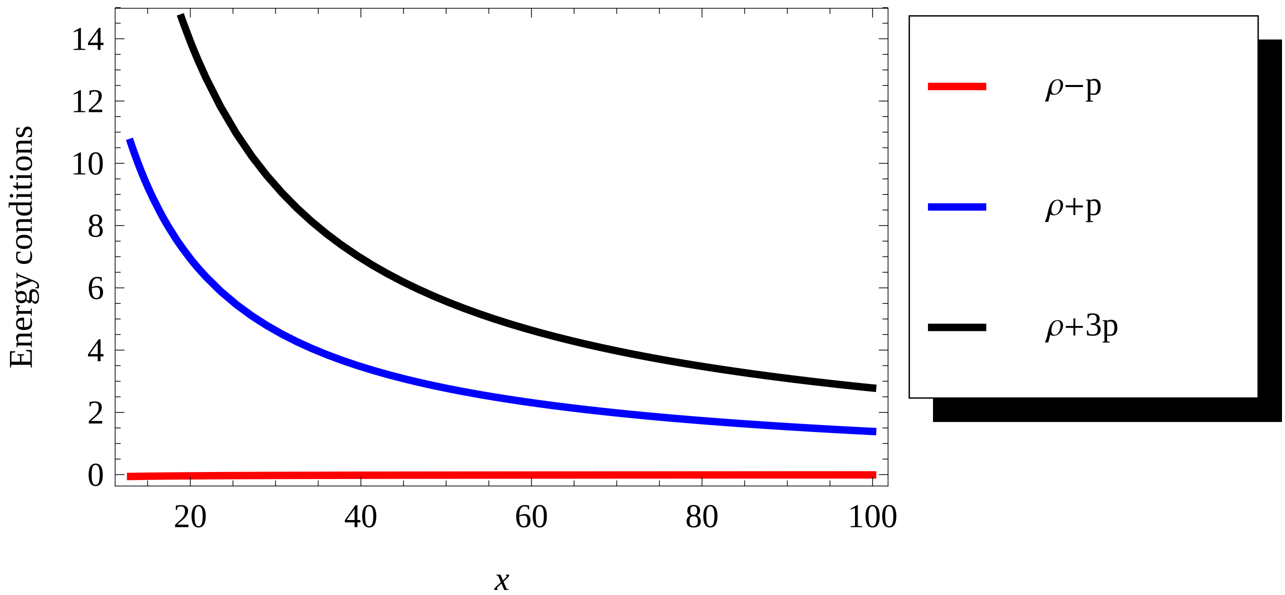

Figure 5.8: Plot of the energy conditions versus the radial coordinate x.

5.3.2 Solution III

Selecting $Y = \mathrm{e}^{\beta x}$, where β is some arbitrary constant and substituting into equation (5.14), we get

$$x(1+\beta x)\dot{Z} + (2\beta^2 x^2 - 1)Z + 1 = 0. \tag{5.24}$$

Equation (5.24) admits solutions in terms of elementary and special functions. We have

$$Z = 1 + 2\beta x + \mathrm{e}^{-2(1+\beta x)} x(1+\beta x)[\mathrm{e}^2 c_1 - 4\beta\, \mathcal{E}(2+2\beta x)], \tag{5.25}$$

where c_1 is the constant of integration and $\mathcal{E}(\tau)$ is the exponential integral function. Polyanin and Manzhirov (2007) have defined the exponential integral function as

$$\mathcal{E}(\tau) = \int_{-\tau}^{\infty} \frac{\mathrm{e}^{-t}}{t}\, \mathrm{d}t. \tag{5.26}$$

Then substituting our forms for Y and Z into the field equations, we get

$$\frac{\rho}{C} = 12\beta(1-\beta x) - 3\mathrm{e}^{-2(1+\beta x)}(2\beta^2 x^2 - \beta x - 2)[c_1 \mathrm{e}^2 - 4\beta\mathcal{E}(2+2\beta x)], \tag{5.27a}$$

$$\frac{p}{C} = 12\beta(1+\beta x) + 3\mathrm{e}^{-2(1+\beta x)}(2x^2\beta^2 + 3x\beta + 1)[c_1 \mathrm{e}^2 - 4\beta\mathcal{E}(2+2\beta x)]. \tag{5.27b}$$

Physical Features

We study the physical features to our proposed exact solutions. We produce graphical renditions for the parameter values $c_1 = 5$, $\beta = 5$ and $C = -5$. From the graphs we see that the energy density and pressure look very similar for this choice of the parameter values. The energy density in Figure 5.9 which is plotted for the range $x \in [1, 100]$, has a singularity at the centre of the star. Further, the energy density is a monotonically decreasing function and approaches zero as we move away from the centre of the star. The pressure in Figure 5.10 is a monotonically decreasing function and eventually tends towards zero as we move to the boundary. This defines the pressure free surface for the star, thus guaranteeing that the stellar surface does in fact exist, and there is a realistic star candidate. The pressure was plotted for the range $x \in [1.2, 100]$ and also has a singularity at the origin.

In Figure 5.11 we observe that the speed of sound parameter is greater than one. This implies that the speed of sound is greater than the speed of light and that causality is defied. This is an unfortunate negative feature of this class of solution. We speculate that this behaviour arises because of the specific choice of the parameters made. Possibly another choice may lead to causal features. Other features are physically reasonable such as the energy conditions. In Figure 5.12 we observe that in the range $x \in [13, 100]$, $\rho - p$, $\rho + p$ and $\rho + 3p$ are strictly positive and satisfied. Lastly, we match the interior solution generated with the exterior metric given by (5.9a). For the inside of the star we require

$$e^{2\lambda} = \frac{1}{1 + 2\beta Cr^2 + e^{-2(1+\beta Cr^2)} Cr^2(1+\beta Cr^2)[e^2 c_1 - 4\beta\mathcal{E}(2+2\beta Cr^2)]}, \qquad (5.28a)$$

$$e^{2\nu} = e^{2\beta Cr^2}. \qquad (5.28b)$$

At the boundary $r = R$, we require that the interior solution generated matches the exterior spacetime (5.9a). Thus

$$\frac{1}{1 + 2\beta CR^2 + e^{-2(1+\beta CR^2)} CR^2(1+\beta CR^2)[e^2 c_1 - 4\beta\mathcal{E}(2+2\beta CR^2)]}$$

$$= \frac{4\alpha}{4\alpha + R^2\left(1 - \sqrt{1 + \frac{8M\alpha}{R^4}}\right)}, \qquad (5.29a)$$

$$e^{2\beta CR^2} = \frac{R^2}{4\alpha}\left(\sqrt{1 - \frac{R^4 + 8M\alpha}{R^4}}\right) + 1. \qquad (5.29b)$$

Further, the pressure at the boundary must be zero. Therefore

$$3e^{-2(1+\beta CR^2)}(2C^2R^4\beta^2 + 3CR^2\beta + 1)[c_1 e^2 - 4\beta\mathcal{E}(2+2\beta CR^2)]$$

$$+ 12\beta(1+\beta CR^2) = 0. \qquad (5.30)$$

Solving (5.30) for c_1, we get

$$c_1 = \frac{4\beta[2\beta CR^2\mathcal{E}(2+2\beta CR^2) + \mathcal{E}(2+2\beta CR^2) - e^{2(1+\beta CR^2)}]}{e^2(1+2\beta CR^2)}. \qquad (5.31)$$

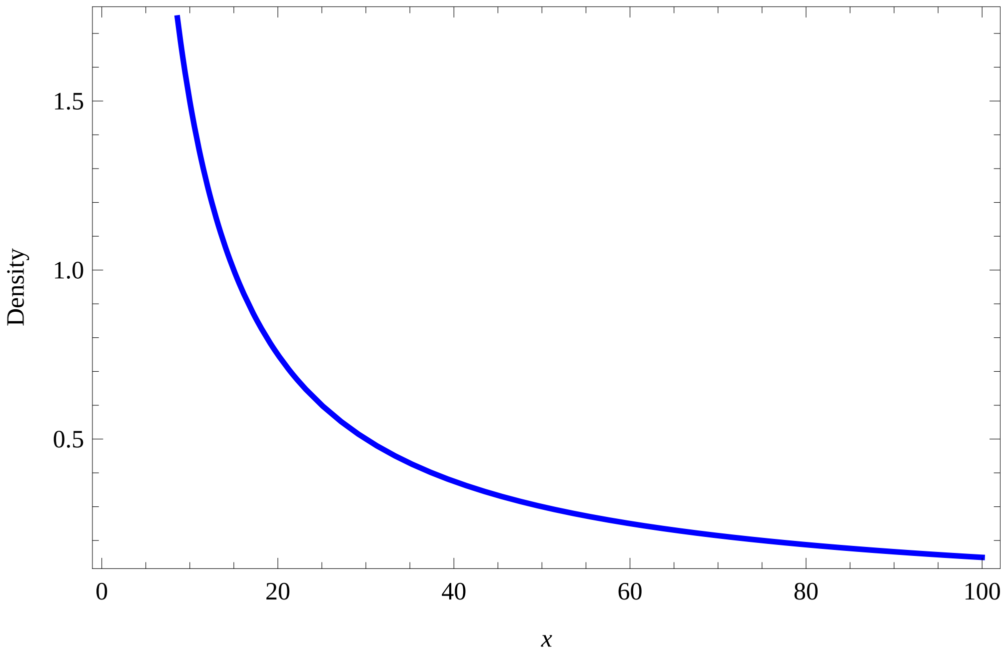

Figure 5.9: Plot of the energy density versus the radial coordinate x.

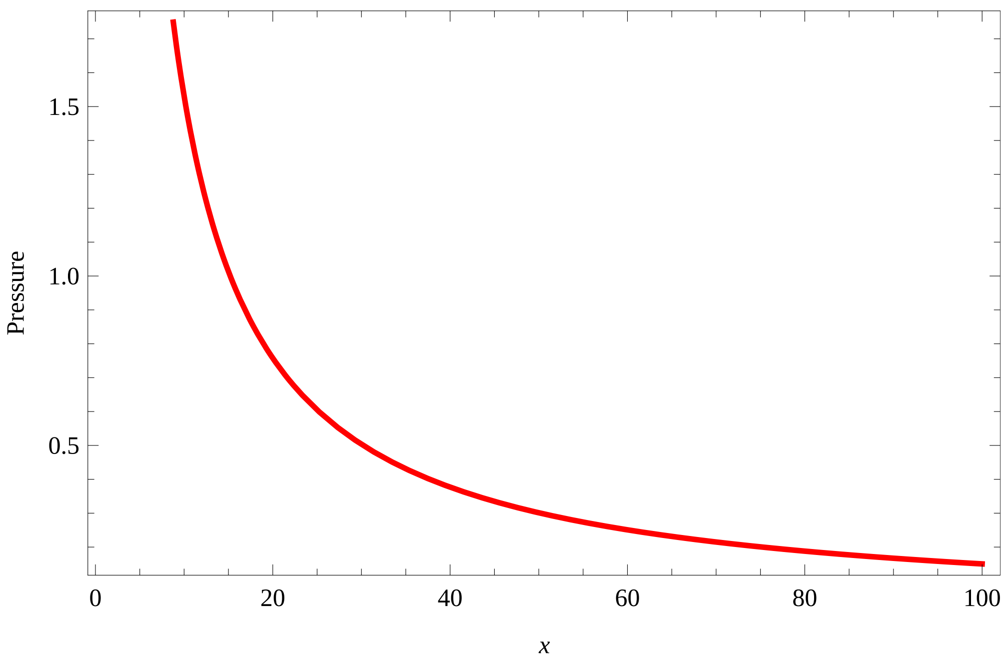

Figure 5.10: Plot of the pressure versus the radial coordinate x.

The speed of sound parameter has the form

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{\mathcal{C}}{\mathcal{D}}, \tag{5.32}$$

where

$$\begin{aligned}\mathcal{C} &= (1 - \beta x - 6\beta^2 x^2 - 4\beta^3 x^3)[c_1 \mathrm{e}^2 - 4\beta\mathcal{E}(2 + 2\beta x)] \\ &\quad - 8\beta^2 \mathrm{e}^{2(1+\beta x)} x(1 + \beta x),\end{aligned} \tag{5.33a}$$

$$\begin{aligned}\mathcal{D} &= (4\beta^3 x^3 - 2\beta^2 x^2 - 9\beta x - 3)[c_1 \mathrm{e}^2 - 4\beta\mathcal{E}(2 + 2\beta x)] \\ &\quad + 4\beta \mathrm{e}^{2(1+\beta x)}(2\beta^2 x^2 - 2\beta x - 3).\end{aligned} \tag{5.33b}$$

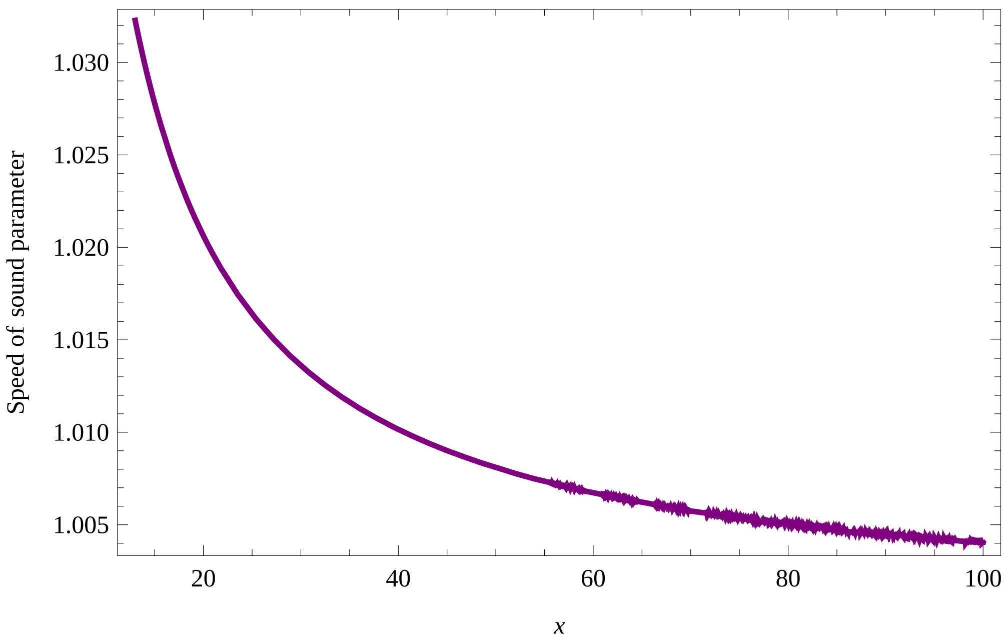

Figure 5.11: Plot of the speed of sound parameter versus the radial coordinate x.

The energy conditions have the form

$$\frac{\rho - p}{C} = 3\mathrm{e}^{-2(1+\beta x)}(1 - 2\beta x - 4\beta^2 x^2)[\mathrm{e}^2 c_1 - 4\beta + \mathcal{E}(2 + 2\beta x)]$$

$$- 24\beta^2 x, \tag{5.34a}$$

$$\frac{\rho + p}{C} = 3\mathrm{e}^{-2(1+\beta x)}(3 + 4\beta x)[\mathrm{e}^2 c_1 - 4\beta\mathcal{E}(2 + 2\beta x)] + 24\beta, \tag{5.34b}$$

$$\frac{\rho + 3p}{C} = 3\mathrm{e}^{-2(1+\beta x)}(5 + 10\beta x + 4\beta^2 x^2)[\mathrm{e}^2 c_1 - 4\beta\mathcal{E}(2 + 2\beta x)]$$

$$+ 24\beta(2 + \beta x). \tag{5.34c}$$

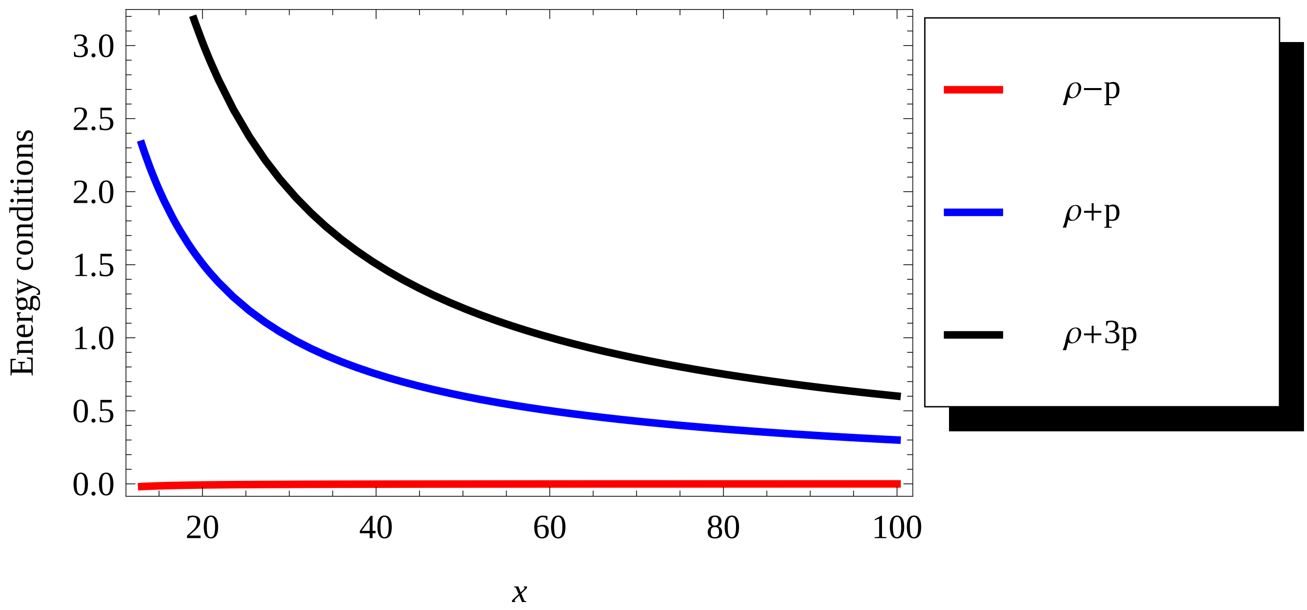

Figure 5.12: Plot of the energy density versus the radial coordinate x.

5.4 Solution in terms of arbitrary functions

We select $Y = f(x)$, where f is some general function of the variable x. Substituting this form of Y into (5.14), we get

$$(-xf - x^2 f')\dot{Z} + (f - 2x^2 f'')Z - f = 0. \tag{5.35}$$

Equation (5.35) is a first order linear equation, using the method of integrating factors, we find that (5.35) admits the solution

$$Z = \mathrm{e}^{F(x)} \left(\int \frac{G(x)}{\mathrm{e}^{F(x)}} \, \mathrm{d}x + \beta \right), \tag{5.36}$$

where β is the constant of integration and

$$F(x) = \int \frac{f - 2x^2 f''}{x(xf' + f)} \, \mathrm{d}x, \tag{5.37a}$$

$$G(x) = -\frac{f}{x(xf' + f)}. \tag{5.37b}$$

The implication of this result is that we can select any functional form for Y and get a corresponding functional form for Z. Substituting this result into the field equations, we get

$$\frac{\rho}{C} = \frac{3\mathrm{e}^{F(x)}(2x^2 f'' - xf' - 2f)\left(\int \frac{G(x)}{\mathrm{e}^{F(x)}} \, \mathrm{d}x - \beta\right)}{x(xf' + f)} - \frac{3(xf' + 2f)}{x(xf' + f)}, \tag{5.38a}$$

$$\frac{p}{C} = \frac{3\mathrm{e}^{F(x)}(2xf' + f)\left(\beta - \int \frac{G(x)}{\mathrm{e}^{F(x)}} \, \mathrm{d}x\right)}{xf} - \frac{3}{x}. \tag{5.38b}$$

Any choice of $f(x)$ allows us to complete the integration in (5.37), an produce an explicit solution. Clearly the physical conditions will restrict the functional forms of $f(x)$.

Chapter 6

The Einstein-Gauss-Bonnet case

6.1 Introduction

In this chapter we consider the form of the master equation derived in §3.3 in terms of the dependent variable Z. By making the coefficient of Y vanish, we generate a linear functional form for Z. Thereafter, the master equation is solved and we show that Y can be expressed in general surd form. These particular forms of the potentials Y and Z correspond to the classical Schwarzchild solution in terms of the matter variables. However, we obtain two cases from the master equation, and generate a new constant density exact solution to the EGB field equations. We point out that this solution results from integration and holds for any arbitrary form of the potential Y. In §6.3 we examine the case of the constant density solution that was found using the intuitive reasoning of Dadhich *et al* (2010) in five-dimensional EGB theory. We show that the related method we employed to this integration procedure is unique, and is a generalization of the traditional Schwarzchild solution. In §6.4 we fix the value of the integration constant c_1 in terms of the Gauss-Bonnet coupling constant α and the transformation constant C. In so doing, we obtain a new constant density solution in five-dimensional EGB gravity. Lastly, we show that this solution is distinct because the form for the isotropic pressure p is arbitrary.

6.2 The Schwarzchild solution extended

We show that it is possible to generate a simple class of exact solutions to the EGB equations. By making the coefficient of Y vanish in equation (3.15), we get

$$1 + x\dot{Z} - Z = 0. \tag{6.1}$$

Integrating (6.1), we get the form for Z

$$Z = 1 + c_1 x, \tag{6.2}$$

where c_1 is the constant of integration. Substituting this form for Z and its first derivative into equation (3.15), we get

$$\begin{aligned} &2x^2(1 + c_1 x)(4\alpha C c_1 - 1)\ddot{Y} - [c_1 x^2 - 12\alpha C c_1 x(c_1 x + 1)]\dot{Y} \\ &- 4\alpha C[c_1 x - 2(c_1 x + 1) + 2(1 + c_1 x)^2]\dot{Y} = 0. \end{aligned} \tag{6.3}$$

We can write then (6.3) in the simplified form

$$(4\alpha C c_1 - 1)[2(1 + c_1 x)\ddot{Y} + c_1 \dot{Y}] = 0. \tag{6.4}$$

From (6.4) we observe that two cases arise. We will consider each case in turn.

6.3 Case 1

In this case

$$4\alpha C c_1 \neq 0. \tag{6.5}$$

Then we get from (6.4) the condition

$$2(1 + c_1 x)\ddot{Y} + c_1 \dot{Y} = 0. \tag{6.6}$$

Integrating (6.6) we obtain

$$Y = \frac{c_2}{c_1}\,(1 + c_1 x)^{1/2} + c_3, \tag{6.7}$$

where c_2 and c_3 are constants of integration. Therefore for this case we obtain the line element

$$\begin{aligned} \mathrm{d}s^2 = &- \left[\left(\frac{c_2}{c_1} \right)^2 (1 + c_1 x) + \frac{2 c_2 c_3}{c_1} (1 + c_1 x)^{1/2} + c_3^2 \right] \mathrm{d}t^2 \\ &+ \frac{\mathrm{d}r^2}{1 + c_1 x} + r^2 (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 + \sin^2\theta \, \sin^2\phi \, \mathrm{d}\psi^2). \end{aligned} \tag{6.8}$$

The energy density and pressure are given by

$$\frac{\rho}{C} = 6c_1(1 - 2\alpha C c_1), \tag{6.9a}$$

$$\frac{p}{C} = \frac{3c_1[c_1c_3 - 2c_2(2\alpha C c_1 - 1)(1 + c_1x)^{1/2}]}{c_2(1 + c_1x)^{1/2} + c_1c_3}. \tag{6.9b}$$

Observe the solution (6.7) of equation (6.9) corresponds to the constant density model in five-dimensional EGB theory. We interpret this result as an EGB generalization of the conventional Schwarzschild solution in general relativity. Note that we have obtained this model by a direct integration of the field equations. Dadhich *et al* (2010) obtained a similar form of the generalized Schwarzschild solution using the principle of universality without any integration. We have shown that the five-dimensional EGB constant metric can be generated exactly by integrating the condition of pressure isotropy.

6.4 Case 2

In this case

$$2(1 + c_1x)\ddot{Y} + c_1Y \neq 0. \tag{6.10}$$

Then we have the condition that

$$c_1 = \frac{1}{4\alpha C}. \tag{6.11}$$

Thus the condition of pressure isotropy is always satisfied for this value of the integration constant c_1. Therefore in this case we generate the line element

$$\mathrm{d}s^2 = -Y^2(x)\mathrm{d}t^2+\left(\frac{4\alpha C}{x+4\alpha C}\right)\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\ \mathrm{d}\phi^2+\sin^2\theta\ \sin^2\phi\ \mathrm{d}\psi^2). \quad (6.12)$$

The energy density and pressure are given by

$$\frac{\rho}{C} = 6c_1(1-2\alpha C c_1), \quad (6.13a)$$

$$\frac{p}{C} = \frac{3[c_1 Y - 2(1+c_1 x)(4\alpha C c_1 - 1)\dot{Y}]}{Y}. \quad (6.13b)$$

Note that (6.12) and (6.13) correspond to a constant density solution in five-dimensional EGB gravity. It is important to observe (6.12) and (6.13) are different from (6.8) and (6.9) respectively in §6.3. Therefore we have generated a new constant density solution in EGB theory. It holds for the special value of $c_1 = \frac{1}{4\alpha C}$ and the pressure is arbitrary. We have not seen this particular solution in the literature. The special choice of α producing this solution may affect the dynamical evolution of the model. In this class of solutions the function Y is arbitrary and we can interpret the model as a cosmological solution. In the astrophysical setting the boundary conditions at the surface of the relativistic star may place additional conditions on Y. It is necessary to check if any types of constant density solutions are possible in EGB gravity by integrating the field equations for forms of the potential Z different from $1 + c_1 x$. This is an area for future research.

Chapter 7

Conclusion

In this thesis we have considered new exact solutions to the five-dimensional Einstein and EGB field equations for static, spherically symmetric spacetimes. Three new exact solutions to the five-dimensional Einstein equations are found in terms of both elementary and special functions. The master equations were analysed in terms of both dependent variables Y and Z. One solution was found when analysed in terms of Y, and two solutions were found when analysed in terms of Z. We carried out a matching of the interior solutions generated with the exterior Boulware-Deser metric at the respective boundaries. For the EGB case, the master equations were not easy to solve. By making the coefficient of Y vanish we were able to attain two cases which we examined in turn. By elementary factorization of the master equation in terms of the dependent variable Y, we were able to attain the constant density solution by direct integration of the field equations. Then by selection of a specific form of the integration constant c_1, we were able to generate a new exact constant density solution that is applicable in the cosmological setting.

Below we present an overview of the thesis:

In chapter 2, we introduced the relevant differential geometric quantities that were quintessential to the astrophysical modeling of EGB gravity. A brief description of the energy conditions and causality was presented in order to illustrate the physical viability of an astrophysical model. We concluded this chapter by introducing the action integral for five-dimensional EGB gravity and provided explanations of the Lovelock and Lanczos tensors. The EGB field equations were shown to be a linear combination of the Einstein tensor and Lanczos tensor by introducing the Gauss-Bonnet coupling constant. This expression is then equated to the energy momentum tensor, and thus the EGB field equations were formed.

In chapter 3, we derived all the relevant differential geometric quantities for the spherically symmetric, static, uncharged metric

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + \mathrm{e}^{2\lambda}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \sin^2\theta\,\sin^2\phi\,\mathrm{d}\psi^2). \tag{7.1}$$

It was pointed out that these quantities are not easily attainable from the literature and thus we have calculated them in full and presented them here. We thereafter form the EGB field equations and generate the pressure isotropy condition. The pressure isotropy condition is also made more comprehensible by the implementation of a transformation and we attained two forms for the master gravitational equations. The first being

$$2xZ\left[4\alpha C(Z-1)-x\right]\ddot{Y}-\left[x^2\dot{Z}+4\alpha C\left(x\dot{Z}-2Z+2Z^2-3xZ\dot{Z}\right)\right]\dot{Y}$$
$$-\left(1+x\dot{Z}-Z\right)Y=0, \tag{7.2a}$$

which is in terms of Y and when Z is specified, and

$$[(-x^2-4\alpha Cx)\dot{Y}-xY]\dot{Z}+(12\alpha Cx\dot{Y})Z\dot{Z}+[8\alpha C(x\ddot{Y}-\dot{Y})]Z^2$$
$$+[(-2x^2-8\alpha Cx)\ddot{Y}+8\alpha C\dot{Y}+Y]Z-Y=0, \tag{7.2b}$$

which is in terms of Z when Y is specified.

In chapter 4, we presented known solutions that were found by Chilambwe *et al* (2015), Hansraj *et al* (2015) and Maharaj *et al* (2015) for the five-dimensional Einstein and EGB cases and expressions for the matter variables ρ and p. In the research conducted by Chilambwe *et al* (2015), the simplification assumption $\beta = 4\alpha C$ was used in order simplify equation (7.2b). By choosing a general linear form for Y and setting $\alpha = 0$, a direct integration yielded a surd form for Z. In the more general EGB case when $\alpha \neq 0$, the equation was integrated directly to produce a form for Z in terms of elementary functions. In the investigation carried out by Hansraj *et al* (2015), the coefficient of the dependent variable Z^2 in equation (7.2b) was made to vanish. In so doing, the resulting equation could be integrated to produce a quadratic form for Y. In the analysis of Maharaj *et al* (2015), by selecting a constant form for Z an ordinary differential equation was attained. The resulting equation was then solved using the power series method of Frobenius to generate a form for Y. Two cases arose, when the constant was made unity, the constant density solution found by Dadhich *et al* (2010) was attained and when the constant was nonunity, a new class of exact solutions was generated.

In chapter 5, we examined the case of the five-dimensional Einstein field equations that are attained when the Gauss-Bonnet coupling constant α equals zero. Two types of analysis is carried out in terms of both equations (7.2a) and (7.2b). In the first analysis, we start by selecting a general linear form for Z and successfully solve the resulting ordinary differential equation to attain a form for Y.Thereafter, by selecting

$$Z = \ln x, \tag{7.3}$$

we attained the form

$$\begin{aligned} Y = {} & \frac{c_1 x^{1-\sqrt{3}} \mathcal{U}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\, \ln x\right)}{2^{1/4}} \\ & + \frac{c_2 x^{1-\sqrt{3}} \mathcal{M}\left(\frac{9-5\sqrt{3}}{12}, \frac{3}{2}, \sqrt{3}\, \ln x\right)}{2^{1/4}}. \end{aligned} \tag{7.4}$$

We performed an analysis of the physical features of this solution. It was shown that throughout the star causality is maintained and thus speed of sound is less than the speed of light. The density exhibited a monotonically decreasing profile and we infer that due to the decreasing nature of the pressure equation, a pressure-free boundary is defined for the star. The energy conditions are all strictly positive and decreasing. This is a new solution for the five-dimensional Einstein case. In the second analysis we chose

$$Y = x^n, \tag{7.5}$$

and obtained the form

$$Z = \frac{1}{1+2n-2n^2} + c_1[(n+1)x]^{\frac{1+2n-2n^2}{n+1}}. \tag{7.6}$$

As before, we performed an analysis of the physical features. The solution is well behaved. We believe that this is another new solution in the Einstein case. We also selected the form

$$Y = \mathrm{e}^{\beta x}, \tag{7.7}$$

and solved the resulting differential equation to obtain the form

$$Z = 1 + 2\beta x + \mathrm{e}^{-2(1+\beta x)} x(1+\beta x)[\mathrm{e}^2 c_1 - 4\beta\, \mathcal{E}(2+2\beta x)]. \tag{7.8}$$

Again we performed an analysis of the physical features. We observed that throughout the star the speed of sound is greater than the speed of light and thus this model describes a superluminous fluid in ther interior of the star. The remaining physical conditions were satisfied. We demonstrated a general algorithm to solve the field equations by selecting an arbitrary functional form

$$Y = f(x), \tag{7.9}$$

and then treating the resulting differential equation as first order. Therefore we could express the solution in terms of the matter variables

$$\frac{\rho}{C} = \frac{3\mathrm{e}^{F(x)}(2x^2f'' - xf' - 2f)\left(\int \frac{G(x)}{\mathrm{e}^{F(x)}}\,\mathrm{d}x - \beta\right)}{x(xf' + f)} - \frac{3(xf' + 2f)}{x(xf' + f)}, \tag{7.10a}$$

$$\frac{p}{C} = \frac{3\mathrm{e}^{F(x)}(2xf' + f)\left(\beta - \int \frac{G(x)}{\mathrm{e}^{F(x)}}\,\mathrm{d}x\right)}{xf} - \frac{3}{x}, \tag{7.10b}$$

where $F(x)$ and $G(x)$ are defined in equation (5.37).

In chapter 6, we analyzed equation (7.2a) by making the coefficient of Y vanish to obtain the form

$$Z = 1 + c_1 x. \tag{7.11}$$

Back substitution of (7.11) into (7.2a) yields the product

$$(4\alpha C c_1 - 1)[2(1 + c_1 x)\ddot{Y} + c_1 \dot{Y}] = 0, \tag{7.12}$$

when factorized. We obtain two cases from (7.12) and we examined each case in turn. For the first case, $4\alpha C c_1 - 1 \neq 0$. This produced the form

$$Y = \frac{c_2}{c_1}\,(1 + c_1 x)^{1/2} + c_3, \tag{7.13}$$

which is related to the Schwarzchild constant density model. For the second case, $2(1 + c_1 x)\ddot{Y} + c_1 Y \neq 0$. In this manner, we fixed the value of the integration constant to be

$$c_1 = \frac{1}{4\alpha C}, \tag{7.14}$$

and the metric function Y is arbitrary. This is another constant density solution that exists in EGB gravity. This is in contrast to the interior Schwarzchild solution which is unique.

The models generated in this thesis were as a result of selecting specific forms of the gravitating variables Y and Z and manipulations of the master equations by making coefficients vanish and fixing integration constants. The advocation of future research in this field and other modified theories of gravity will be to find exact solutions of the field equations by the application of Lie algebras to the system of equations. In so doing, simplification assumptions will not have to be made and the equations can be solved directly to obtain generalized solutions. In the astrophysical setting, Abebe *et al* (2013) were able to solve partial differential equations that resulted from modeling conformally flat radiating stars. Msomi *et al* (2010) were able to obtain exact models for spherically symmetric fluids in gravitating fields by finding the Lie symmetries of the underlying equations. The aggrandizement of this research can be accomplished by modeling spherically symmetric fluids in five-dimensional EGB theory with pressure anisotropy. In this regard, Abbas and Zubair (2015) have modeled gravitationally collapsing fluid spheres with unequal radial and tangential pressures. This research can also be enhanced by taking into consideration adiabatic and nonadiabatic effects. These analyses will be investigated in future work.

Chapter 8

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