On the null geometry of quasi generalized CR-submanifolds of indefinite nearly α-Sasakian manifolds

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To the family of Mr and Mrs E. B. Kabenga,
my sister Justine B. Nasejje,
brother T. Mutambuze,
and my fiancée Teddy Nasaazi.

Declaration

I, Samuel SSEKAJJA, declare that

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Abstract

Generalized CR (GCR)-lightlike submanifolds of indefinite almost contact manifolds were introduced by K. L. Duggal and B. Sahin, with the assumption that they are tangent to the structure vector field ξ of the almost contact structure $(\overline{\phi}, \eta, \xi)$. Contrary to the above assumption, we have introduced and studied a new class of CR-lightlike submanifold of an indefinite nearly α-Sasakian manifold, called *quasi generalized CR (QGCR)*-lightlike submanifold. We have showed that QGCR-lightlike submanifold include; ascreen QGCR, co-screen QGCR and the well known GCR-lightlike submanifolds. We have proved some existence (or non-existence) theorems and provided a thorough study of geometry of their distributions. Also, we have constructed many examples, where necessary, to illustrate the main ideas.

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Chapter 0

Introduction and statement of main results

0.1 Introduction

Cauchy-Riemannian (CR)-submanifolds of Riemannian manifolds were introduced by A. Bejancu [2] as submanifolds of complex manifolds. CR-submanifolds is an important class of submanifolds and many researchers investigated their properties, especially embedding properties. Later on, the study was extended to CR-submanifolds of almost contact manifolds, for example see [25], [27] and [28]. In [7] and [9], the authors introduced the notions of contact CR-lightlike submanifolds of semi-Riemannian manifolds, and later studied by other researchers, for instance, see [19] and [21] and other references therein.

Null or lightlike subspaces exist naturally in semi-Riemannian manifolds and they are of great concern to modern differential geometry. In fact, lightlike geometry is widely applied in mathematical physics, particularly, in general relativity and electromagnetism [7]. A thorough study of geometry of lightlike submanifolds was first presented in 1996 by Duggal and Bejancu [7] and later by Duggal and Sahin [9]. Since the tangent bundle of a lightlike subspace intersects its normal space, they considered a splitting of the ambient space into four non intersecting subspaces in which two of them are null and the other two are non-degenerate. Their approach was later adopted my many researchers, including among others; [8], [9], [10], [11], [19], [20] and [21].

In the two books [7] and [9], the authors initiated the study of generalized Cauchy-Riemann (GCR)-lightlike submanifolds of an indefinite Hermitian and Sasakian manifolds respectively. Later on, many papers appeared on these submanifolds in other spaces, like for example [13] and [14] on GCR-lightlike submanifolds of indefinite Kenmotsu and cosymplectic manifolds respectively. In all the above mentioned work, the structure vector field, ξ, of the almost contact structure carried by the ambient space was assumed to be tangent to the submanifold. Moreover, when ξ is tangent to the submanifold, Calin [6] proved that it belongs to the screen distribution. This assumption is widely accepted and it has been applied in many papers on contact lightlike geometry [9, 10, 19–21]. In a different view, D. H. Jin [16] introduced a new class of hypersurface, called, ascreen lightlike hypersurfaces which are not necessarily tangent to ξ and later extended the ideas to half-lightlike submanifolds of codimention 2 [15]. It is worthy mentioning that the position of ξ affects the geometry of the submanifold at hand and therefore, its restriction to the tangent bundle of the submanifold, especially, in its screen distribution is an over simplification which reduces computational work but narrowing the research to only those submanifolds which are tangent to ξ. Choosing ξ to be tangent to a CR-submanifold of Sasakian manifolds was first presented by Yano-Kon in their book [27], in which they showed that if ξ is normal then the resulting submanifold is not a CR-submanifold. In fact, one gets an anti-invariant submanifold. Their argument is based on the symmetry of the shape operator with respect to the induced Riemannian metric. It is important to notice that the shape operators of a lightlike submanifold are generally non-symmetric with respect to the induced lightlike metric and thus, we may not carry on with the assumption ξ is tangeent to the submanifold in case of a lightlike submanifold.

Contrary to the above mentioned assumption, we introduced a new class of CR-lightlike submanifold of a nearly Sasakian and nearly cosymplectic manifolds, known as *quasi generalized Cauchy-Riemann (QGCR)*-lightlike submanifold [22, 23] in which ξ was not necessarily tangent to the submanifold. We showed that QGCR-lightlike submanifolds includes GCR-lightlike submanifolds and two other spacial CR-lightlike submanifolds, called ascreen and co-screen QGCR-lightlike submanifolds. In this dissertation, we bring together all our results by generalizing the ambient space to an indefinite nearly α-Sasakian manifold, which includes nearly Sasakian and cosymplectic manifolds.

0.2 Main Results

In the recent work by Duggal and Sahin [10], they introduced the notions of generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Sasakian manifolds which are tangent to the structure vector field ξ. They proved that the above class of submanifold is an umbrella of a number of CR-submanifolds, for instance contact CR-lightlike and contact SCR-lightlike submanifolds. All these submanifolds can be found in their book [9].

This dissertation stands in contrast to Duggal and Sahin's original assumption that the structure vector field, ξ, is tangent to the submanifold and particularly, in the screen distribution. Thus, we have introduced a new class of CR-lightlike submanifold, called quasi generalized CR (QGCR)-lightlike submanifold of an indefinite nearly α-Sasakian manifolds, for which GCR-lightlike submanifold form part. In a QGCR-lightlike submanifold, the structure vector field, ξ, is defined generally on the ambient space, such that its restriction to the screen distribution reduces that QGCR-lightlike submanifold to the classical GCR-lightlike submanifold [9]. We have studied a number of special cases of QGCR-lightlike submanifold, like *ascreen* and *co-screen* QGCR-lightlike submanifolds.

In this dissertation, we establish some existence theorems for totally umbilical or totally geodesic QGCR-lightlike submanifolds of indefinite nearly α-Sasakian manifolds in accordance to the position of the structure vector field ξ. Some of the results are noted in the next theorems;

Theorem 0.2.1 ([22])**.** *Let $(M,g,S(TM), S(TM^{\perp}))$ be a totally umbilical or totally geodesic proper QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\eta,\xi,\overline{g})$, with the structure vector field ξ tangent, normal or transversal to M. Then $\alpha = 0$ i.e., $\overline{M}$ is an indefinite nearly cosymplectic manifold. Also, any QGCR-lightlike submanifold carrying the structure vector field ξ in its screen distribution is a GCR-lightlike submanifold.*

The above theorem offers a characterization of totally umbilical and totally geodesic QGCR-lightlike submanifolds of indefinite nearly α-Sasakian manifolds. Moreover, it is easy to see that the following corollary holds;

Corollary 0.2.2 ([22])**.** *There exist no totally umbilical or totally geodesic proper QGCR-lightlike submanifolds $(M,g,S(TM), S(TM^{\perp}))$ of an indefinite nearly Sasakian manifold $(\overline{M},\overline{\phi},\eta,\xi,\overline{g})$ with the structure vector field ξ tangent, normal or transversal to M.*

Also, we prove that the restriction of the $\overline{\phi}$-constant sectional curvature $\overline{c}$ to non-degenerate subbundles of a QGCR-lightlike submanifolds in an indefinite nearly cosymplectic manifold is actually a generalization of the results one gets if the indefinite nearly cosymplectic manifold is replaced with an indefinite cosymplectic manifold. Some of the findings are Theorems 0.2.3 and 0.2.4 below;

Theorem 0.2.3 ([23])**.** *Let $(M,g,S(TM), S(TM^{\perp}))$ be a totally umbilical or totally geodesic ascreen QGCR-lightlike submanifold of an indefinite nearly cosymplectic space form $\overline{M}(\overline{c})$, of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$, such that D_0 and $\overline{\phi}\mathscr{S}$ are spacelike and parallel distributions with respect to ∇. Then, $\overline{c} \geq 0$. Equality occurs when $\overline{M}(\overline{c})$ is an indefinite cosymplectic space form.*

When $(M,g,S(TM), S(TM^{\perp}))$ is irrotational, we have the following;

Theorem 0.2.4 ([23]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an irrotational ascreen QGCR-lightlike submanifold of an indefinite nearly cosymplectic space form $\overline{M}(c)$ of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$. Then, $\overline{c} \leq 0$ or $\overline{c} \geq 0$. Equality holds when $\overline{M}(\overline{c})$ is an indefinite cosymplectic space form.*

The rest of the dissertation is arranged as follows; In Chapter 1, we introduce the basic preliminaries which we will refer to in the other parts of the work. Chapter 2 introduces the idea of QGCR-lightlike submanifolds, discusses some existence or non existance theorems and also the integrability of distributions. In Chapter 3, we discuss the geometry of ascreen QGCR-lightlike submanifolds. Totally umbilical and totally geodesic submanifolds are discussed as well as minimal submanifolds. In Chapter 4, we discuss co-screen QGCR-lightlike submanifolds and lastly, Chapter 5 winds up the dissertation by summarizing the study and giving insights into future work.

Chapter 1

Preliminaries

1.1 Semi-Riemannian manifolds

In this brief section, we present the basic notions on semi-Riemannian manifolds from [7] and [9], necessary for this dissertation.

Let $\mathscr{V}$ be an m-dimensional vector space endowed with a symmetric bilinear map $\overline{g}$: $\mathscr{V} \times \mathscr{V} \longrightarrow \mathbb{R}$. One says that $\overline{g}$ is *degenerate* [7] on $\mathscr{V}$ if there exist a vector $E \neq 0$, of $\mathscr{V}$, such that

$$\overline{g}(E,v) = 0, \quad \forall v \in \mathscr{V},$$

else $\overline{g}$ is said to be *non-degenerate*.

Definition 1.1.1 ([7])**.** The *radical (or null)* space of $\mathscr{V}$ with respect to $\overline{g}$, is a subspace of $\mathscr{V}$ denoted by $\mathrm{Rad}\,\mathscr{V}$ and defined by

$$\mathrm{Rad}\,\mathscr{V} = \{E \in \mathscr{V} : \ \overline{g}(E,v) = 0, \ v \in \mathscr{V}\}$$

The dimension of $\mathrm{Rad}\,\mathscr{V}$ is known as the *nullity degree*, denoted by $null\,\mathscr{V}$. It is easy to see that $\overline{g}$ is degenerate or non-degerate if and only if $null\,\mathscr{V} > 0$ and $null\,\mathscr{V} = 0$ respectively. Further, the pair $(\mathscr{V},\overline{g})$ is called a *semi-Euclidean space* [7] if $\overline{g}$ is of index q, where $0 < q \leq m$, and defines a scalar product on $\mathscr{V}$.

Suppose that $\mathscr{W}$ is a subspace of $(\mathscr{V},\overline{g})$, then the restriction $\overline{g}|_{\mathscr{W}}$ is either degenerate or non-degenerate and in the first case, $\mathscr{W}$ is called a *lightlike* subspace of $\mathscr{V}$; otherwise, $\mathscr{W}$ is called a non-degenerate subspace of $\mathscr{V}$.

Let $\overline{M}$ be a real m-dimensional smooth manifold and $\overline{g}$ be a symmetric tensor of type $(0,2)$ on $\overline{M}$. If $\overline{g}$ is non-degenerate on $\overline{M}$, then one can say that $\overline{g}$ is *semi-Riemannian metric* [7].

Definition 1.1.2 ([7])**.** A *semi-Riemannian manifold* is a smooth manifold $\overline{M}$ endowed with a semi-Riemannian metric $\overline{g}$.

The tangent space $T_p\overline{M}$ of $\overline{M}$, where $p \in \overline{M}$, is a semi-Euclidean space and any vector $u \in T_p\overline{M}$ can fall in one of the following classes;

- *spacelike* if $\overline{g}(u,u) > 0$ or $u = 0$,
- *timelike* if $\overline{g}(u,u) < 0$, and
- *lightlike* (or *null*) if $\overline{g}(u,u) = 0$ and $u \neq 0$.

The above classification of vector u is known as its *casual character* [7].

Definition 1.1.3 ([7])**.** A linear connection $\overline{\nabla}$ on a semi-Riemannian manifold $(\overline{M},\overline{g})$ is called a *metric connection* if the metric tensor $\overline{g}$ is parallel with respect to $\overline{\nabla}$, that is to say, $\overline{\nabla}\overline{g} = 0$.

From now on, we denote by $\Gamma(\Xi)$ the set of all smooth sections of a vector bundle Ξ over $\overline{M}$ and $\mathscr{X}(\overline{M})$ to be a set of smooth functions on $\overline{M}$. The metric connection above is also known as the *Levi-Civita connection* and it satisfies the well known *Koszul's* formula

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y},\overline{Z}) &= \overline{X}(\overline{g}(\overline{Y},\overline{Z})) + \overline{Y}(\overline{g}(\overline{X},\overline{Z})) - \overline{Z}(\overline{g}(\overline{X},\overline{Y})) \\ &\quad - \overline{g}([\overline{X},\overline{Y}],\overline{Z}) + \overline{g}([\overline{Z},\overline{X}],\overline{Y}) - \overline{g}([\overline{Y},\overline{Z}],\overline{X}), \end{aligned} \tag{1.1.1}$$

for all $\overline{X},\overline{Y},\overline{Z} \in \Gamma(T\overline{M})$, where $[\,,\,]$ denotes the *Lie bracket* (see [7] and [9] for details).

Definition 1.1.4 ([7])**.** The *Riemannian curvature tensor* $\overline{R}$ is a type $(0,4)$ tensor defined as

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{U}) = \overline{g}(\overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{U}), \quad \forall \overline{X},\overline{Y},\overline{Z},\overline{U} \in \Gamma(T\overline{M}), \tag{1.1.2}$$

where the *curvature tensor field* $\overline{R}$ of type $(1,3)$ on the right hand side of (1.1.2) is given by

$$\overline{R}(\overline{X},\overline{Y})\overline{Z} = \overline{\nabla}_{\overline{X}}(\overline{\nabla}_{\overline{Y}}\overline{Z}) - \overline{\nabla}_{\overline{Y}}(\overline{\nabla}_{\overline{X}}\overline{Z}) - \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z}, \quad \forall \overline{X},\overline{Y},\overline{Z} \in \Gamma(T\overline{M}). \tag{1.1.3}$$

The following properties of the Riemannian curvature tensor $\overline{R}$ in (1.1.2) are also well known

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{U}) + \overline{R}(\overline{Y},\overline{X},\overline{Z},\overline{U}) = 0, \quad \overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{U}) + \overline{R}(\overline{X},\overline{Y},\overline{U},\overline{Z}) = 0, \tag{1.1.4}$$

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{U}) - \overline{R}(\overline{Z},\overline{U},\overline{X},\overline{Y}) = 0, \qquad \forall \overline{X},\overline{Y},\overline{Z},\overline{U} \in \Gamma(T\overline{M}). \tag{1.1.5}$$

Definition 1.1.5 ([7])**.** A *distribution* D of rank r on a smooth manifold $\overline{M}$ is a mapping defined on $\overline{M}$, which assigns to each point $p \in \overline{M}$ an r dimensional linear subspace D_p of $T_p\overline{M}$.

We shall assume that all distributions considered in this dissertation are smooth. Its obvious that any vector subbundle of $T\overline{M}$ defines a smooth distribution on $\overline{M}$.

Definition 1.1.6 ([9])**.** A distribution D is said to be *integrable* (or *involutive*) if for all $\overline{X},\overline{Y} \in \Gamma(D)$, then $[\overline{X},\overline{Y}] \in \Gamma(D)$.

1.2 Lightlike submanifolds

In this section, we quote a few aspects of lightlike submanifolds from [7] and [9] necessary for this dissertation. Extented details can be found in the above two books.

Let $(\overline{M},\overline{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold of constant index ν, $1 \leq \nu \leq m+n$ and M be a submanifold of $\overline{M}$ of codimension n. This means that $\overline{M}$ is never a Riemannian manifold. We assume that $m > 1$ and $n \geq 1$, which implies that M is not a curve of $\overline{M}$. For a point $p \in M$, we define the orthogonal complement $T_pM^{\perp}$ of the tangent space T_pM by

$$T_pM^{\perp} = \{X \in T_pM : \overline{g}(X,Y) = 0, \, \forall Y \in T_pM\}.$$

We put $\mathrm{Rad}\, T_pM = \mathrm{Rad}\, T_pM^{\perp} = T_pM \cap T_pM^{\perp}$. The submanifold M of $\overline{M}$ is said to be *r-lightlike submanifold* (one supposes that the index of $\overline{M}$ is $\nu \geq r$), if the mapping

$$\mathrm{Rad}\, TM : p \in M \longrightarrow \mathrm{Rad}\, T_pM$$

defines a *smooth distribution* on M of rank $r > 0$. We call $\mathrm{Rad}\, TM$ the radical distribution on M. In this case M is said to be r-lightlike submanifold of $\overline{M}$. In this dissertation, an r-lightlike submanifold will simply be called a *lightlike submanifold* and $g = \overline{g}|_M$ is *lightlike metric*, unless we need to specify r.

We say that a lightlike submanifold M of $\overline{M}$ is

1. r-lightlike if $1 \leq r < min\{m,n\}$;
2. co-isotropic if $1 \leq r = n < m$, $S(TM^{\perp}) = \{0\}$;
3. isotropic if $1 \leq r = m < n$, $S(TM) = \{0\}$;
4. totally lightlike if $r = n = m$, $S(TM) = \{0\} = S(TM^{\perp})$.

Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $\mathrm{Rad}\, TM$ in TM, that is,

$$TM = \mathrm{Rad}\, TM \perp S(TM). \tag{1.2.1}$$

Since M is assumed to be paracompact, then the screen distribution always exist on M. Clearly, the distribution $S(TM)$ is not unique, however, it is canonically isomorphic to the factor vector bundle $TM/\mathrm{Rad}\,TM$ [10].

On a lightlike submanifold M, the vector bundle $TM^{\perp}$ is not complementary to TM in $T\overline{M}|_M$, due to the fact that $\mathrm{Rad}\,TM = TM \cap TM^{\perp}$ is a distribution on M of rank $r > 0$. This poses a challenge since a vector in $T\overline{M}$ can not be uniquely decomposed into a component in TM and a component in $TM^{\perp}$. Thus, the standard book definition of the second fundamental forms and Gauss-Weingarten formulas do not generally apply in case of lightlike submanifolds. One possible way of dealing with the above problem is to split the ambient tangent bundle $T\overline{M}$ in four non-intersecting subbundles for which the screen distribution $S(TM)$ forms part. Now, the remaining three subbundles to make up $T\overline{M}$ are constructed as follows;

Choose a *screen transversal* bundle $S(TM^{\perp})$, which is semi-Riemannian and complementary to $\mathrm{Rad}\,TM$ in $TM^{\perp}$ such that $TM^{\perp}$ can be decomposed as follows;

$$TM^{\perp} = \mathrm{Rad}\,TM \perp S(TM^{\perp}). \tag{1.2.2}$$

Since $S(TM)$ is non-degerate then there is a complementary vector subbundle $S(TM)^{\perp}$ to $S(TM)$ in $T\overline{M}$ such that

$$TM^{\perp} = S(TM) \perp S(TM)^{\perp}. \tag{1.2.3}$$

It is clearly visible that $S(TM)^{\perp}$ is a submanifold of $STM)^{\perp}$ and the fact that both of them are non degenerate, one has the following orthogonal decomposition of $S(TM)^{\perp}$,

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}. \tag{1.2.4}$$

Let $\mathrm{tr}(TM)$ and $l\mathrm{tr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and to $\mathrm{Rad}\,TM$ in $\mathrm{tr}(TM)$ respectively. Then,

$$\mathrm{tr}(TM) = l\mathrm{tr}(TM) \perp S(TM^{\perp}), \tag{1.2.5}$$

$$T\overline{M} = S(TM) \perp S(TM^{\perp}) \perp \{\mathrm{Rad}\,TM \oplus l\mathrm{tr}(TM)\}. \tag{1.2.6}$$

Theory of lightlike submanifolds depends on $S(TM)$ and $S(TM^{\perp})$, therefore we shall also adopt the notation $(M, g, S(TM), S(TM^{\perp})$ for a lightlike submanifold. The following result is well known [7].

Theorem 1.2.1. *Let* $(M, g, S(TM)), S(TM^{\perp})$ *be an r-lightlike submanifold of* $(\overline{M}, \overline{g})$ *with* $r > 1$. *Suppose that* $\mathscr{U}$ *is a coordinate neighborhood of* M *and* $\{E_i\}$, *for* $i \in \{1, \cdots, r\}$, *be a* *basis of* $\Gamma(\mathrm{Rad}\,TM|_{\mathscr{U}})$. *Then, there exist smooth sections* $\{N_i\}$ *of* $S(TM^{\perp})^{\perp}|_{\mathscr{U}}$ *such that*

$$\overline{g}(E_i,N_j)=\delta_{ij}, \quad \textit{and} \quad \overline{g}(N_i,N_j)=0, \quad \forall i,j\in\{1,\cdots,r\}. \tag{1.2.7}$$

Proof. The proof can be found in [7] or [9]. □

Next, consider a local quasi-orthonormal field of frames of $\overline{M}$ along M, on $\mathscr{U}$ as

$$\{E_1,\cdots,E_r,N_1,\cdots,N_r,X_{r+1},\cdots,X_m,W_{1+r},\cdots,W_n\},$$

where $\{X_{r+1},\cdots,X_m\}$ and $\{W_{1+r},\ldots,W_n\}$ are respectively orthonormal bases of $\Gamma(S(TM)|_{\mathscr{U}})$ and $\Gamma(S(TM^{\perp})|_{\mathscr{U}})$ and that $\varepsilon_\beta=\overline{g}(W_\beta,W_\beta)$ be the signatures of W_β.

The induced geometrical objects

Let $(\overline{M},\overline{g})$ be a semi-Riemannian and denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$. Suppose that (M,g) is a lightlike submanifold of $\overline{M}$. Let $S(TM)$ and $l\mathrm{tr}(TM)$ be the screen distribution and the corresponding lightlike transversal bundle of M respectively. Then, the Gauss and Weingarten formulas for an r-lightlike submanifold are given by

$$\overline{\nabla}_X Y=\nabla_X Y+h(X,Y), \ \forall X,Y\in\Gamma(TM), \tag{1.2.8}$$

$$\overline{\nabla}_X V=-A_V X+\nabla^t_X V, \ \forall X\in\Gamma(TM), \ V\in\Gamma(\mathrm{tr}TM), \tag{1.2.9}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y),\nabla^t_X V\}$ belongs to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$ respectively. Further, ∇ and ∇^t are linear connections on M and $\mathrm{tr}\,TM$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(\mathrm{tr}(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$.

Let P be the projection morphism of TM onto $S(TM)$. Then, by straightforward calculations (see [7] for details), the above two equations gives

$$\overline{\nabla}_X Y=\nabla_X Y+h^l(X,Y)+h^s(X,Y) \tag{1.2.10}$$

$$\overline{\nabla}_X N=-A_N X+\nabla^l_X N+D^s(X,N) \tag{1.2.11}$$

$$\overline{\nabla}_X W=-A_W X+D^l(X,W)+\nabla^s_X W, \tag{1.2.12}$$

$$\nabla_X PY=\nabla^*_X PY+h^*(X,PY) \tag{1.2.13}$$

$$\nabla_X E=-A^*_E X+\nabla^{*t}_X E, \tag{1.2.14}$$

for any $E\in\Gamma(\mathrm{Rad}\,TM)$, $X,Y\in\Gamma(TM)$, $N\in\Gamma(\mathrm{tr}(TM))$ and $W\in\Gamma(S(TM^{\perp}))$. It is obvious that h^l and h^s are respectively $\Gamma(\mathrm{tr}(TM))$ and $\Gamma(S(TM^{\perp}))$ valued and they are called the *lightlike second fundamental form* and *screen second fundamental form* of M respectively. Notice that D^l and D^s are not metric connection. In fact, they are Otsuki connections [7]. Further, ∇^* and ∇^{*t} are liner connections on $S(TM)$ and $\mathrm{Rad}\,TM)$ respectively and they are in fact metric connections [9]. A_N and A_W are called the *shape operators* of M. Also, h^* and A^* are respectively $\Gamma(\mathrm{Rad}\,TM)$ and $\Gamma(S(TM))$ valued bilinear forms, called the second fundamental form and shape operator of $\mathrm{Rad}\,TM$ and $S(TM)$ respectively.

Next, we shall use the following set of indices

$$i,j,k \in \{1,\cdots,r\}, \quad a,b,c \in \{1+r,\cdots,m\} \quad \text{and} \quad \beta,\gamma,\iota \in \{1+r,\cdots,n\}.$$

Consider a coordinate neighborhood $\mathscr{U}$ of M and let $\{N_i, W_\beta\}$ be the basis of $\mathrm{tr}(TM)|_M$, where $N_i \in \Gamma(l\mathrm{tr}(TM)|_M)$, $i \in \{1,\cdots,r\}$ and $W_\beta \in \Gamma(S(TM^\perp)|_{\mathscr{U}})$, $\beta \in \{1+r,\cdots,n\}$. Then, the set of equations (1.2.10)-(1.2.14) reduces to

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^l(X,Y)N_i + \sum_{\beta=r+1}^{n} h_\beta^s(X,Y)W_\beta, \tag{1.2.15}$$

$$\overline{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{\beta=r+1}^{n} \rho_{i\beta}(X)W_\beta, \tag{1.2.16}$$

$$\overline{\nabla}_X W_\beta = -A_{W_\beta}X + \sum_{i=1}^{r} \varphi_{\beta i}(X)N_i + \sum_{\beta=r+1}^{n} \sigma_{\beta\gamma}(X)W_\gamma, \tag{1.2.17}$$

$$\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^{r} h_i^*(X,PY)E_i, \tag{1.2.18}$$

$$\nabla_X E_i = -A_{E_i}^* X - \sum_{j=1}^{r} \tau_{ji}(X)E_j, \quad \forall X,Y \in \Gamma(TM), \tag{1.2.19}$$

where h_i^l and h_β^s are symmetric bilinear forms known as *local lightlike* and *screen fundamental* forms of TM respectively. Also, A_{N_i}, $A_{E_i}^*$ and A_{W_β} are linear operators on TM while τ_{ij}, $\rho_{i\beta}$, $\varphi_{\beta i}$ and $\sigma_{\beta\gamma}$ are differential 1-forms on TM given by

$$\begin{aligned} &\tau_{ij}(X) = \overline{g}(\overline{\nabla}_X N_i, E_j), \quad \varepsilon_\beta \rho_{i\beta}(X) = \overline{g}(\overline{\nabla}_X N_i, W_\beta), \\ &\varphi_{\beta i}(X) = \overline{g}(\overline{\nabla}_X W_\beta, E_i) \quad \text{and} \quad \varepsilon_\gamma \sigma_{\beta\gamma}(X) = \overline{g}(\overline{\nabla}_X W_\beta, W_\gamma), \quad \forall X \in \Gamma(TM). \end{aligned} \tag{1.2.20}$$

It is well known [7] that

$$h_i^l(X,Y) = \overline{g}(\overline{\nabla}_X Y, E_i), \quad \forall X,Y \in \Gamma(TM), \tag{1.2.21}$$

from which we deduce the independence of h_i^l on the choice of $S(TM)$.

Next, it is easy to see that the second fundamental tensor h is given by

$$h(X,Y)=\sum_{i=1}^{r}h_i^l(X,Y)N_i+\sum_{\beta=r+1}^{n}h_\beta^s(X,Y)W_\beta,\quad X,Y\in\Gamma(TM). \tag{1.2.22}$$

Further, we stress that ∇ is generally not a metric connection and is given by

$$(\nabla_X g)(Y,Z)=\sum_{i=1}^{r}\{h_i^l(X,Y)\lambda_i(Z)+h_i^l(X,Z)\lambda_i(Y)\}, \tag{1.2.23}$$

for any $X,Y\in\Gamma(TM)$ and λ_i are differential 1-forms given by

$$\lambda_i(X)=\overline{g}(X,N_i),\quad \forall X\in\Gamma(TM). \tag{1.2.24}$$

The above three local second fundamental forms are related to their shape operators by the following set of equations

$$g(A^*_{E_i}X,Y)=h_i^l(X,Y)+\sum_{j=1}^{r}h_j^l(X,E_i)\lambda_j(Y),\quad \overline{g}(A^*_{E_i}X,N_j)=0, \tag{1.2.25}$$

$$g(A_{W_\beta}X,Y)=\varepsilon_\beta h_\beta^s(X,Y)+\sum_{i=1}^{r}\varphi_{\beta i}(X)\lambda_i(Y), \tag{1.2.26}$$

$$\overline{g}(A_{W_\beta}X,N_i)=\varepsilon_\beta\rho_{i\beta}(X),\quad \lambda_j(A_{N_i}X)+\lambda_i(A_{N_j}X)=0, \tag{1.2.27}$$

$$g(A_{N_i}X,Y)=h_i^*(X,PY),\quad \forall X,Y\in\Gamma(TM). \tag{1.2.28}$$

Replacing Y by E_j in the first equation of (1.2.25) we get

$$h_i^l(X,E_j)+h_j^l(X,E_i)=0,\quad h_i^l(X,E_i)=0 \quad\text{and}\quad h_i^l(E_j,E_k)=0, \tag{1.2.29}$$

for all $X\in\Gamma(TM)$. Further, replacing X with E_j in the first equation of (1.2.25) and then applying (1.2.29) one gets

$$h_i^l(X,E_j)=g(X,A^*_{E_i}E_j),\quad A^*_{E_i}E_j+A^*_{E_j}E_i=0 \quad\text{and}\quad A^*_{E_i}E_i=0,\quad \forall X\in\Gamma(TM). \tag{1.2.30}$$

For any r-lightlike submanifold, replacing Y by E_i in (1.2.26), we get

$$h_\beta^s(X,E_i)=-\varepsilon_\beta\varphi_{\beta i}(X),\quad \forall X\in\Gamma(TM). \tag{1.2.31}$$

Definition 1.2.2 ([9]). A lightlike submanifold $(M,g,S(TM),S(TM^\perp)$, of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is said to be totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $\mathscr{H} \in \Gamma(\mathrm{tr}(TM))$, called the transversal curvature vector of M such that

$$h(X,Y) = \mathscr{H} g(X,Y), \tag{1.2.32}$$

for all $X,Y \in \Gamma(TM)$. Moreover, it is easy to see that M is totally umbilical in $\overline{M}$, if and only if on each coordinate neighborhood $\mathscr{U}$ there exist smooth vector fields $\mathcal{H}^l \in \Gamma(l\mathrm{tr}(TM))$ and $\mathcal{H}^s \in \Gamma(S(TM^\perp))$ and smooth functions $\mathcal{H}_i^l \in F(l\mathrm{tr}(TM))$ and $\mathcal{H}_\beta^s \in F(S(TM^\perp))$ such that

$$\begin{aligned} h^l(X,Y) &= \mathcal{H}^l g(X,Y), \quad h^s(X,Y) = \mathcal{H}^s g(X,Y), \\ h_i^l(X,Y) &= \mathcal{H}_i^l g(X,Y), \quad h_\beta^s(X,Y) = \mathcal{H}_\beta^s g(X,Y), \end{aligned} \tag{1.2.33}$$

for all $X,Y \in \Gamma(TM)$.

The above definition is independent of the choice of the screen distribution [9].

Definition 1.2.3 ([7])**.** A lightlike submanifold $(M, g, S(TM), S(TM^\perp)$, of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be totally geodesic if $h = 0$.

Next, by direct calculations using (1.2.10), (1.2.11) and (1.2.12) we obtain the following Gauss equation for the curvature tensors $\overline{R}$, R of $\overline{M}$ and M respectively.

$$\begin{aligned} \overline{R}(X,W,Z,Y) &= \overline{g}(R(X,W)Z,Y) + \overline{g}(A_{h^l(X,Z)}W,Y) \\ &- \overline{g}(A_{h^l(W,Z)}X,Y) + \overline{g}(A_{h^s(X,Z)}W,Y) \\ &- \overline{g}(A_{h^s(W,Z)}X,Y) + \overline{g}((\nabla_X h^l)(W,Z),Y) \\ &- \overline{g}((\nabla_W h^l)(X,Z),Y) + \overline{g}(D^l(X,h^s(W,Z)),Y) \\ &- \overline{g}(D^l(W,h^s(X,Z)),Y) + \overline{g}((\nabla_X h^s)(W,Z),Y) \\ &- \overline{g}((\nabla_W h^s)(X,Z),Y) + \overline{g}(D^s(X,h^l(W,Z)),Y) \\ &- \overline{g}(D^s(W,h^l(X,Z)),Y), \end{aligned} \tag{1.2.34}$$

for any $X,Y,Z,W \in \Gamma(TM)$.

1.3 Nearly α-Sasakian manifolds

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ \eta \circ \overline{\phi} = 0 \ \text{ and } \ \overline{\phi}(\xi) = 0. \tag{1.3.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an indefinite almost contact metric structure [4] on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that, for any vector field $\overline{X}$, $\overline{Y}$ on $\overline{M}$,

$$\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}). \tag{1.3.2}$$

It follows that, for any vector $\overline{X}$ on $\overline{M}$,

$$\eta(\overline{X}) = \overline{g}(\xi, \overline{X}). \tag{1.3.3}$$

If, moreover,

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} + (\overline{\nabla}_{\overline{Y}}\overline{\phi})\overline{X} = \alpha(2\overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X} - \eta(\overline{X})\overline{Y}), \tag{1.3.4}$$

for any vector fields $\overline{X}$, $\overline{Y}$ on $\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an *indefinite nearly α-Sasakian manifold* [17]. Notice that when $\alpha = 0$ (resp. $\alpha = 1$) then $\overline{M}$ reduces to the known nearly cosymplectic (resp. nearly Sasakian) manifold. Further, if $\alpha = 0$ and $\overline{H} = 0$, then $\overline{M}$ is the well known cosymplectic manifold with the following cosymplectic structure [14];

$$\begin{aligned}
&\eta = dz, \quad \xi = \partial z, \\
&\overline{g} = \eta \otimes \eta - \sum_{i=1}^{\frac{q}{2}} (dx^i \otimes dx^i + dy^i \otimes dy^i) + \sum_{i=q+1}^{m} (dx^i \otimes dx^i + dy^i \otimes dy^i), \\
&\phi_0\Big(\sum_{i=1}^{m} (X_i \partial x^i + Y_i \partial y^i) + Z\partial z\Big) = \sum_{i=1}^{m} (Y_i \partial x^i - X_i \partial y^i),
\end{aligned} \tag{1.3.5}$$

where (x^i, y^i, z) are Cartesian coordinates and $\partial t_k = \frac{\partial}{\partial t^k}$, for $t \in \overline{M}$. On the other hand, when $\alpha = 1$ and $\overline{H} = 0$, then $\overline{M}$ is the well known Sasakian manifold with the following Sasakian structure [10];

$$\begin{aligned}
&2\eta = dz - \sum_{i=1}^{m} y^i dx^i, \quad \xi = 2\partial z, \\
&4\overline{g} = 4\eta \otimes \eta - \sum_{i=1}^{\frac{q}{2}} (dx^i \otimes dx^i + dy^i \otimes dy^i) + \sum_{i=q+1}^{m} (dx^i \otimes dx^i + dy^i \otimes dy^i), \\
&\phi_0\Big(\sum_{i=1}^{m} (X_i \partial x^i + Y_i \partial y^i) + Z\partial z\Big) = \sum_{i=1}^{m} (Y_i \partial x^i - X_i \partial y^i) + \sum_{i=1}^{m} Y_i y^i \partial z,
\end{aligned} \tag{1.3.6}$$

where (x^i, y^i, z) are Cartesian coordinates and $\partial t_k = \frac{\partial}{\partial t^k}$, for $t \in \overline{M}$.

Let Ω be the fundamental 2-form of $\overline{M}$ defined by

$$\Omega(\overline{X}, \overline{Y}) = \overline{g}(\overline{X}, \overline{\phi}\,\overline{Y}), \quad \overline{X},\, \overline{Y} \in \Gamma(T\overline{M}). \tag{1.3.7}$$

Replacing $\overline{Y}$ by ξ in (1.3.4) we obtain

$$\overline{\nabla}_{\overline{X}}\xi - \overline{\phi}(\overline{\nabla}_{\xi}\overline{\phi})\overline{X} = -\alpha\overline{\phi}\,\overline{X}, \quad \forall\, \overline{X} \in \Gamma(T\overline{M}). \tag{1.3.8}$$

Introduce a (1,1)-tensor $\overline{H}$ on $\overline{M}$ taking

$$\overline{H}\,\overline{X} = -\overline{\phi}(\overline{\nabla}_{\xi}\overline{\phi})\overline{X}, \tag{1.3.9}$$

for any $\overline{X} \in \Gamma(T\overline{M})$, such that (1.3.8) reduces to

$$\overline{\nabla}_{\overline{X}}\xi = -\alpha\overline{\phi}\,\overline{X} - \overline{H}\,\overline{X}, \quad \forall\, \overline{X} \in \Gamma(T\overline{M}). \tag{1.3.10}$$

Lemma 1.3.1. *The linear operator $\overline{H}$ has the properties*

$$\begin{aligned} &\overline{H}\,\overline{\phi} + \overline{\phi}\,\overline{H} = 0, \;\; \overline{H}\xi = 0, \;\; \eta \circ \overline{H} = 0, \\ \textit{and}\;\; &\overline{g}(\overline{H}\,\overline{X}, \overline{Y}) = -\overline{g}(\overline{X}, \overline{H}\,\overline{Y}) \quad (\textit{i.e. } \overline{H} \textit{ is skew-symmetric}). \end{aligned} \tag{1.3.11}$$

Proof. The proof follows from a straightforward calculation. □

The fundamental 2-form Ω and the 1-form η are related as follows.

Lemma 1.3.2. *Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite nearly α-Sasakian manifold. Then,*

$$\alpha\Omega(\overline{X}, \overline{Y}) = d\eta(\overline{X}, \overline{Y}) + \overline{g}(\overline{H}\,\overline{X}, \overline{Y}), \tag{1.3.12}$$

for any $\overline{X}$, $\overline{Y} \in \Gamma(T\overline{M})$.

Moreover, $\overline{M}$ is α-Sasakian if and only if $\overline{H}$ vanishes identically on $\overline{M}$.

Proof. The relation (1.3.12) follows from a straightforward calculation. The second assertion follows from Theorem 3.2 in [3]. □

An almost contact metric manifold $\overline{M}$ is said to be normal [4] if the torsion tensor $N^{(1)}$ vanishes, that is

$$N^{(1)} = [\overline{\phi}, \overline{\phi}] + 2d\eta \otimes \xi, \tag{1.3.13}$$

where $[\overline{\phi}, \overline{\phi}]$ is the Nijenhuis tensor of $\overline{\phi}$. In [17] the authors showed that for any nearly α-Sasakian manifold, the relation below holds

$$\eta(N^{(1)}(\overline{X}, \overline{Y})) = 4d\eta(\overline{X}, \overline{Y}) - 4\alpha\overline{g}(\overline{X}, \overline{\phi}\,\overline{Y}), \tag{1.3.14}$$

for any $\overline{X}$, $\overline{Y} \in \Gamma(T\overline{M})$. From relation (1.3.14) and Lemma 4.3.2 we deduce that

$$4\overline{g}(\overline{X}, \overline{H}\,\overline{Y}) = \eta(N^{(1)}(\overline{X}, \overline{Y})), \quad \forall\, \overline{X}, \overline{Y} \in \Gamma(T\overline{M}). \tag{1.3.15}$$

Note that, for any $\overline{X}$, $\overline{Y}$, $\overline{Z} \in \Gamma(T\overline{M})$,

$$\overline{g}((\overline{\nabla}_{\overline{Z}}\overline{\phi})\overline{X}, \overline{Y}) = -\overline{g}(\overline{X}, (\overline{\nabla}_{\overline{Z}}\overline{\phi})\overline{Y}). \tag{1.3.16}$$

This means that the tensor $\overline{\nabla}\,\overline{\phi}$ is skew-symmetric.

Lemma 1.3.3. *Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be a nearly α-Sasakian manifold, then*

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{\phi}\,\overline{Y} = -\overline{\phi}(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} - \overline{g}(\overline{Y}, \overline{H}\,\overline{X})\xi - \eta(\overline{Y})\overline{H}\,\overline{X} - \alpha\overline{g}(\overline{Y}, \overline{\phi}\,\overline{X})\xi - \alpha\eta(\overline{Y})\overline{\phi}\,\overline{X},$$
$$(\overline{\nabla}_{\overline{\phi}\,\overline{X}}\overline{\phi})\overline{\phi}\,\overline{Y} = -(\overline{\nabla}_{\overline{Y}}\overline{\phi})\overline{X} - \eta(\overline{X})\overline{\phi}\,\overline{H}\,\overline{Y} + \eta(\overline{Y})\overline{\phi}\,\overline{H}\,\overline{X} - \alpha\eta(\overline{X})\overline{\phi}^2\overline{Y} + 2\alpha\eta(\overline{Y})\overline{\phi}^2\overline{X},$$

for all $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$.

Proof. By direct calculations we have

$$\begin{aligned}
(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{\phi}\,\overline{Y} &= \overline{\nabla}_{\overline{X}}\overline{\phi}^2\overline{Y} - \overline{\phi}\,\overline{\nabla}_{\overline{X}}\overline{\phi}\,\overline{Y} \\
&= \overline{\nabla}_{\overline{X}}(\eta(\overline{Y})\xi) - \overline{\phi}(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} - \eta(\overline{\nabla}_{\overline{X}}\overline{Y})\xi \\
&= -\overline{\phi}(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} + \overline{g}(\overline{Y}, \overline{\nabla}_{\overline{X}}\xi)\xi + \eta(\overline{Y})\overline{\nabla}_{\overline{X}}\xi.
\end{aligned} \tag{1.3.17}$$

Then, the first relation follows immediately from (1.3.17) by applying (1.3.10). The second relation follows from the first by replacing $\overline{X}$ with $\overline{\phi}\,\overline{X}$ and using the fact that $\overline{M}$ is nearly α-Sasakian manifold. □

By setting $\overline{Y} = \overline{X}$ in (1.3.4) and the second relation of Lemma 1.3.3 we respectively deduce

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{X} = \alpha\{\overline{g}(\overline{X}, \overline{X})\xi - \eta(\overline{X})\overline{X}\}, \tag{1.3.18}$$

$$\text{and} \quad (\overline{\nabla}_{\overline{\phi}\,\overline{X}}\overline{\phi})\overline{\phi}\,\overline{X} = -\alpha\{\overline{g}(\overline{X}, \overline{X})\xi - \eta(\overline{X})\overline{\phi}^2\overline{X}\}, \tag{1.3.19}$$

for any $\overline{X} \in \Gamma(T\overline{M})$.

A plane section Π in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\overline{\phi}$-section Π is called a $\overline{\phi}$-sectional curvature [19]. A nearly cosymplectic manifold $\overline{M}$ (i.e, $\alpha = 0$) with constant pointwise $\overline{\phi}$-sectional curvature $\overline{c}$ is said to be a nearly cosymplectic space form and is denoted by $\overline{M}(\overline{c})$. The curvature tensor $\overline{R}$ of a nearly cosymplectic space form $\overline{M}(\overline{c})$ is given by [12] as

$$\begin{aligned}
4\overline{R}(\overline{X},\overline{W},\overline{Z},\overline{Y}) &= \overline{g}((\overline{\nabla}_{\overline{W}}\overline{\phi})\overline{Z},(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y}) - \overline{g}((\overline{\nabla}_{\overline{W}}\overline{\phi})\overline{Y},(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Z})\\
&\quad - 2\overline{g}((\overline{\nabla}_{\overline{W}}\overline{\phi})\overline{X},(\overline{\nabla}_{\overline{Y}}\overline{\phi})\overline{Z}) + \overline{g}(\overline{H}\,\overline{W},\overline{Z})\overline{g}(\overline{H}\,\overline{X},\overline{Y})\\
&\quad - \overline{g}(\overline{H}\,\overline{W},\overline{Y})\overline{g}(\overline{H}\,\overline{X},\overline{Z}) - 2\overline{g}(\overline{H}\,\overline{W},\overline{X})\overline{g}(\overline{H}\,\overline{Y},\overline{Z})\\
&\quad - \eta(\overline{W})\eta(\overline{Y})\overline{g}(\overline{H}\,\overline{X},\overline{H}\,\overline{Z}) + \eta(\overline{W})\eta(\overline{Z})\overline{g}(\overline{H}\,\overline{X},\overline{H}\,\overline{Y})\\
&\quad + \eta(\overline{X})\eta(\overline{Y})\overline{g}(\overline{H}\,\overline{W},\overline{H}\,\overline{Z}) - \eta(\overline{X})\eta(\overline{Z})\overline{g}(\overline{H}\,\overline{W},\overline{H}\,\overline{Y})\\
&\quad + \overline{c}\{\overline{g}(\overline{X},\overline{Y})\overline{g}(\overline{Z},\overline{W}) - \overline{g}(\overline{Z},\overline{X})\overline{g}(\overline{Y},\overline{W})\\
&\quad + \eta(\overline{Z})\eta(\overline{X})\overline{g}(\overline{Y},\overline{W}) - \eta(\overline{Y})\eta(\overline{X})\overline{g}(\overline{Z},\overline{W})\\
&\quad + \eta(\overline{Y})\eta(\overline{W})\overline{g}(\overline{Z},\overline{X}) - \eta(\overline{Z})\eta(\overline{W})\overline{g}(\overline{Y},\overline{X})\\
&\quad + \overline{g}(\overline{\phi}\,\overline{Y},\overline{X})\overline{g}(\overline{\phi}\,\overline{Z},\overline{W}) - \overline{g}(\overline{\phi}\,\overline{Z},\overline{X})\overline{g}(\overline{\phi}\,\overline{Y},\overline{W})\\
&\quad - 2\overline{g}(\overline{\phi}\,\overline{Z},\overline{Y})\overline{g}(\overline{\phi}\,\overline{X},\overline{W})\},
\end{aligned} \tag{1.3.20}$$

for all $\overline{X},\overline{Y},\overline{Z},\overline{W} \in \Gamma(T\overline{M})$. For extended reading and details on nearly α-Sasakian manifolds we refer the reader to [1], [3], [4], [5], [12], [18], [25] and [17].

1.4 CR and GCR-lightlike submanifolds

Here, we introduce the concepts of CR-submanifolds and GCR-lightlike submanifolds of indefinite almost contact manifolds. The discussion is mainly based on the three books [7], [9] and [27].

Definition 1.4.1 ([27])**.** Let $(\overline{M},\overline{g})$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi},\xi,\eta)$. A real submanifold M of $\overline{M}$ is called a *CR-submanifold* if there exist a differentiable distribution D on M such that

- D is invariant, that is, $\overline{\phi}D_p = D_p$, for each $p \in M$,
- the complimentary orthogonal distribution $D^{\perp}$ to D in TM is anti-invariant, that is, $\overline{\phi}D_p^{\perp} \subset T_pM^{\perp}$, for any $p \in M$.

Next, we introduce GCR-lightlike submanifolds of indefinite almost contact manifold $(\overline{M},\overline{g})$. Calin [6] proposed that if the structure vector field ξ of the almost contact structure

$(\overline{\phi}, \xi, \eta)$ is tangent to a lightlike submanifold, then, it belongs to its screen distribution $S(TM)$. Using the above assumption, Dugal and Sahin [9] (also see [10]) introduced a class of contact CR-lightlike submanifold, called generalized CR (GCR)-lightlike submanifold as follows;

Definition 1.4.2 ([9])**.** Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ such that ξ is tangent to M. Then, M is called generalized CR (GCR)-lightlike submanifold if the following conditions are satisfied;

1. There exist two subbundles D_1 and D_2 of $\mathrm{Rad}\,TM$ such that

$$\mathrm{Rad}\,TM = D_1 \oplus D_2, \quad \overline{\phi}D_1 = D_1, \quad \overline{\phi}D_2 \subset S(TM). \tag{1.4.1}$$

2. There exist two subbundle D_0 and $\overline{D}$ of $S(TM)$ such that

$$S(TM) = \{\overline{\phi}D_2 \oplus \overline{D}\} \perp D_0 \perp \{\xi\}, \quad \overline{\phi}\,\overline{D} = \mathscr{L} \perp \mathscr{S}, \tag{1.4.2}$$

where D_0 is an invariant non-degenerate distribution on M, $\{\xi\}$ is a line bundle spanned by ξ, $\mathscr{L}$ and $\mathscr{S}$ are respectively vector subbundles of $l\mathrm{tr}(TM)$ and $S(TM^{\perp})$.

Then, from the definition above, the tangent bundle TM of M decomposes as follows

$$TM = D \oplus \overline{D} \perp \{\xi\}, \quad D = \mathrm{Rad}\,TM \oplus D_0 \oplus \overline{\phi}D_2. \tag{1.4.3}$$

A GCR-lightlike submanifold $(M, g, S(TM))$ is said to be proper if $D_2 \neq \{0\}$, $D_1 \neq \{0\}$, $D_0 \neq \{0\}$ and $\mathscr{S} \neq \{0\}$.

The following two results are well known.

Proposition 1.4.3 ([9])**.** *A GCR-lightlike submanifold of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$, is a contact CR (respectively, contact SCR-lightlike submanifold) if and only if $D_1 = \{0\}$ (respectively, $D_2 = \{0\}$).*

Details on the two submanifolds in the above proposition can be found in [9].

Proposition 1.4.4 ([9])**.** *There exist no coisotropic, isotropic or totally lightlike proper GCR-lightlike submanifolds of an indefinite Sasakian manifold.*

We also note that any proper 9-dimensional contact GCR-lightlike submanifold is 3-lightlike. Further, the authors [7], [9] and [10] emphasizes that GCR-lightlike submanifolds are umbrella of real hypersurfaces, invariant, screen real and contact CR-lightlike submanifolds.

The books [7] and [9] have extended details of the above class of submanifold and its subcases.

Chapter 2

QGCR-lightlike submanifolds of nearly α-Sasakian manifod

2.1 Introduction

In this chapter, we introduce the notions of quasi generalised CR (QGCR)-lightlike submanifolds of indefinite nearly α-Sasakian manifolds. We establish some existence (or non-existence) theorems and also discuss the integrability of distributions.

Generalised CR (GCR)-lightlike submanifolds of indefinite Sasakian manifolds were introduced by [9, p. 334], in which the structure vector field ξ was assumed to be tangent to the submanifold. Later, their ideas were adopted by [13] and [14] for indefinite Kenmotsu and cosymplectic manifolds respectively.

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be a $(2n+1)$-dimensional manifold. Generally, the structure vector field ξ belongs to $T\overline{M}$. Therefore, the case $\xi \in \Gamma(S(TM))$ considered by [9], [13] and [14] for GCR-lightlike submanifolds of indefinite Sasakian, Kenmotsu and cosymplectic manifolds, respectively, is only a particular case of $\xi \in \Gamma(T\overline{M})$. It is important to note that restricting ξ to the screen distribution $S(TM)$, minimizes algebraic computations and at the same time narrowing the research by only looking at those CR-lightlike submanifolds which are tangent to ξ, and yet there are other classes of CR-lightlike submanifolds which are not necessarily carrying ξ in their screen distributions.

Contrary to the well-known assumption that $\xi \in \Gamma(S(TM))$ used in GCR-lightlike submanifolds of almost contact manifolds, we introduce a new class of CR-lightlike submanifold, called *quasi generalised CR (QGCR) lightlike submanifold*, in which $\xi \in \Gamma(T\overline{M})$ and we also show that it contains the known GCR-lightlike submanifolds.

We define ξ according to decomposition (1.2.6) as follows;

$$\xi = \xi_S + \xi_{S^\perp} + \xi_R + \xi_l, \tag{2.1.1}$$

where ξ_S is a smooth vector field of $S(TM)$ and $\xi_{S^\perp}$, ξ_R, ξ_l are defined as follows

$$\xi_R = \sum_{i=1}^{r} a_i E_i, \quad \xi_l = \sum_{i=1}^{r} b_i N_i, \quad \xi_{S^\perp} = \sum_{\beta=r+1}^{n} c_\beta W_\beta \tag{2.1.2}$$

with $a_i = \eta(N_i)$, $b_i = \eta(E_i)$ and $c_\beta = \varepsilon_\beta \eta(W_\beta)$ all smooth functions in $\mathscr{X}(\overline{M})$. In this dissertation, we shall assume, without loss of generality, that ξ is a unit spacelike vector field (see [26, p. 272] for details on this choice).

2.2 QGCR-lightlike submanifolds

In this section, we introduce a new class of CR-lightlike submanifold in which ξ belongs to $T\overline{M}$, called *quasi generalized CR (QGCR) lightlike submanifold* as follows;

Definition 2.2.1. Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g}, \overline{\phi}, \xi, \eta)$. We say that M is quasi generalized CR (QGCR)-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) there exist two distributions D_1 and D_2 of $\mathrm{Rad}\,TM$ such that

$$\mathrm{Rad}\,TM = D_1 \oplus D_2, \quad \overline{\phi} D_1 = D_1, \quad \overline{\phi} D_2 \subset S(TM), \tag{2.2.1}$$

(ii) there exist vector bundles D_0 and $\overline{D}$ of $S(TM)$ such that

$$S(TM) = \{\overline{\phi} D_2 \oplus \overline{D}\} \perp D_0, \tag{2.2.2}$$

$$\text{with} \quad \overline{\phi} D_0 \subseteq D_0, \quad \overline{D} = \overline{\phi}\mathscr{S} \oplus \overline{\phi}\mathscr{L}, \tag{2.2.3}$$

where D_0 is a non-degenerate and invariant distribution on M, $\mathscr{L}$ and $\mathscr{S}$ are respectively vector subbundles of $l\mathrm{tr}(TM)$ and $S(TM^\perp)$.

If $D_1 \neq \{0\}$, $D_0 \neq \{0\}$, $D_2 \neq \{0\}$ and $\mathscr{S} \neq \{0\}$, then M is called a *proper QGCR lightlike submanifold*.

Let M be a proper QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $\overline{M}$. If the structure vector field ξ is tangent to M, then, $\xi \in \Gamma(S(TM))$. The proof of this is similar to one given by Calin in Sasakian case [6]. In this case, if $X \in \Gamma(\mathscr{S})$ and $Y \in \Gamma(\mathscr{L})$, then $\eta(X) = \eta(Y) = 0$ and

$$\overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X,Y) - \eta(X)\eta(Y) = 0,$$

which reduces the direct sum $\overline{D}$ in (2.2.3) to the orthogonal direct sum $\overline{D} = \overline{\phi}\,\mathscr{S} \perp \overline{\phi}\,\mathscr{L}$, and thus $\overline{\phi}\,\overline{D} = \mathscr{S} \perp \mathscr{L}$. Since $\xi \in \Gamma(S(TM))$ and ξ is neither a vector field in $\overline{\phi}D_2$ nor in $\overline{D}$, ξ is in D_0. By $\overline{\phi}D_0 \subseteq D_0$, there exist a distribution D_0' of rank $(rank(D_0) - 1)$ and satisfying $\overline{\phi}D_0' = D_0'$ such that $D_0 = D_0' \perp \langle\xi\rangle$, where $\langle\xi\rangle$ is the 1-dimensional distribution spanned by ξ. Therefore, the QGCR-lightlike submanifold tangent to ξ reverts to a GCR-lightlike submanifold [10].

Proposition 2.2.2. *A QGCR-lightlike submanifold M of an indefinite nearly α-Sasakian manifold $\overline{M}$ tangent to the structure vector field ξ is a GCR-lightlike submanifold.*

Next, we follow Yano-Kon [28, p. 353] definition of contact CR-submanifolds and state the following definition for a quasi contact CR-lightlike submanifold.

Definition 2.2.3. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. We say that M is quasi contact CR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied;

(i) $\mathrm{Rad}\,TM$ is a distribution on M such that $\mathrm{Rad}\,TM \cap \overline{\phi}(\mathrm{Rad}\,TM) = \{0\}$;

(ii) there exist vector bundles D_0 and D' over $S(TM)$ such that

$$S(TM) = \{\overline{\phi}(\mathrm{Rad}\,TM) \oplus D'\} \perp D_0, \tag{2.2.4}$$

$$\text{with} \quad \overline{\phi}D_0 \subseteq D_0, \;\; D' = \overline{\phi}L_1 \oplus \overline{\phi}l\mathrm{tr}(TM), \tag{2.2.5}$$

where D_0 is a non-degenerate, L_1 is vector subbundle of $S(TM^{\perp})$.

It is easy to see that when the structure vector field ξ is tangent to the quasi contact CR-lightlike submanifold M, then M is a contact CR-lighlike submanifold.

Proposition 2.2.4. *A QGCR lightlike submanifold of an indefinite nearly α-Sasakian manifold $\overline{M}$, is a quasi contact CR (resp. quasi SCR)if and only if $D_1 = \{0\}$ (resp. $D_2 = \{0\}$).*

Proof. Let M be a quasi contact CR-lightlike submanifold. Then $\overline{\phi}(\mathrm{Rad}\,TM)$ is a distribution on M such that $\overline{\phi}(\mathrm{Rad}\,TM) \cap \mathrm{Rad}\,TM = \{0\}$. Therefore, $D_2 = \mathrm{Rad}\,TM$ and $D_1 = \{0\}$. Hence, $\overline{\phi}(l\mathrm{tr}(TM)) \cap l\mathrm{tr}(TM) = \{0\}$. Then it follows that $\overline{\phi}(l\mathrm{tr}(TM)) \subset S(TM)$. The converse is obvious. The second assertation also follows in the same way. □

From (1.2.1), the tangent bundle, TM, of any QGCR-lightlike submanifold can be rewritten as

$$TM = D \oplus \widehat{D}, \text{ where } D = D_0 \perp D_1 \text{ and } \widehat{D} = \{D_2 \perp \overline{\phi} D_2\} \oplus \overline{D}.$$

Notice that D is invariant with respect to $\overline{\phi}$ while $\widehat{D}$ is not generally anti-invariant with respect to $\overline{\phi}$.

Note the following for a proper QGCR-lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of an indefinite almost contact metric manifolds $\overline{M}$ according to Definition 2.2.1:

1. Condition (i) implies that $\dim(\mathrm{Rad}\, TM) = s \geq 3$.
2. Condition (ii) implies that $\dim(D) \geq 4l \geq 4$ and $\dim(D_2) = \dim(\mathscr{L})$.

2.3 Characterization theorems

In this section, we discuss some existence (or non-existence) theorems for proper QGCR-lightlike submanifolds of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$.

Theorem 2.3.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical proper QGCR-lighlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$ with the structure vector field ξ tangent to M. Then, $\alpha = 0$.*

Proof. Since $\xi \in \Gamma(TM)$ then $\xi = \xi_R + \xi_S$ and $b_i = c_\beta = 0$. Using (1.3.10) and (1.2.15), we get

$$-\alpha \overline{\phi} X = \overline{H} X + \nabla_X \xi + \sum_{i=1}^{r} h_i^l(X, \xi) N_i + \sum_{\beta=r+1}^{n} h_\beta^s(X, \xi) W_\beta, \tag{2.3.1}$$

for all $X \in \Gamma(TM)$. Taking the $\overline{g}$-product of (2.3.1) with respect to $W_\beta \in \Gamma(\mathscr{S})$ we get

$$\alpha g(X, \overline{\phi} W_\beta) = \overline{g}(\overline{H} X, W_\beta) + \varepsilon_\beta h_\beta^s(X, \xi), \ \forall X \in \Gamma(TM). \tag{2.3.2}$$

Now, letting $X = \overline{\phi} W_\beta$ in (2.3.2) we obtain

$$\alpha g(\overline{\phi} W_\beta, \overline{\phi} W_\beta) = \overline{g}(\overline{H}\, \overline{\phi} W_\beta, W_\beta) + \varepsilon_\beta h_\beta^s(\overline{\phi} W_\beta, \xi). \tag{2.3.3}$$

Since $c_\beta = \varepsilon_\beta \eta(W_\beta) = 0$, then $-\overline{H}\,\overline{\phi}W_\beta = (\overline{\nabla}_{W_\beta}\overline{\phi})\xi + \alpha W_\beta$ and the first term on the right hand side of (2.3.3) therefore simplifies as follows using (1.3.4)

$$\begin{aligned}
-\overline{g}(\overline{H}\,\overline{\phi}W_\beta, W_\beta) &= \overline{g}((\overline{\nabla}_{W_\beta}\overline{\phi})\xi, W_\beta) + \alpha\overline{g}(W_\beta, W_\beta) \\
&= -\overline{g}(\xi, (\overline{\nabla}_{W_\beta}\overline{\phi})W_\beta) + \alpha\overline{g}(W_\beta, W_\beta) \\
&= -\alpha\overline{g}(\xi, \bar{g}(W_\beta, W_\beta)\xi) + \alpha\overline{g}(W_\beta, W_\beta) \\
&= -\alpha\overline{g}(W_\beta, W_\beta) + \alpha\overline{g}(W_\beta, W_\beta) = 0. \qquad (2.3.4)
\end{aligned}$$

Then substituting $\overline{g}(\overline{H}\,\overline{\phi}W_\beta, W_\beta) = 0$ in (2.3.3) we obtain

$$\alpha g(\overline{\phi}W_\beta, \overline{\phi}W_\beta) = \varepsilon_\beta h^s_\beta(\overline{\phi}W_\beta, \xi). \qquad (2.3.5)$$

By virtue of the fact that M is totally umbilical in $\overline{M}$, (2.3.5) yields

$$\alpha g(\overline{\phi}W_\beta, \overline{\phi}W_\beta) = \varepsilon_\beta \mathcal{H}^s_\beta g(\overline{\phi}W_\beta, \xi) = 0. \qquad (2.3.6)$$

Then, simplifying (2.3.6) while considering $\eta(W_\beta) = 0$, we get

$$\alpha g(\overline{\phi}W_\beta, \overline{\phi}W_\beta) = \alpha g(W_\beta, W_\beta) = \alpha\varepsilon_\beta = 0,$$

from which we can see that $\alpha = 0$. □

Corollary 2.3.2. *There exist no totally umbilical proper QGCR-lightlike submanifolds $(M, g, S(TM), S(TM^\perp))$ of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$ with the structure vector field ξ tangent to M.*

Theorem 2.3.3. *Let $(M, g, S(TM), S(TM^\perp))$ be a totally geodesic QGCR-lighlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$ with the structure vector field ξ tangent to M. Then, $\alpha = 0$.*

Proof. The proof is similar to that in Theorem 2.3.1. Hence, we omit it here. □

Using Theorem 2.3.1 and Corollary 2.3.3 above we get the following theorem;

Theorem 2.3.4. *Let $(M, g, S(TM), S(TM^\perp))$ be a totally umbilical or totally geodesic geodesic proper QGCR-lighlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$ with the structure vector field ξ tangent to M. Then, $\alpha = 0$.*

When the structure vector field ξ is normal, we have the following.

Theorem 2.3.5. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a proper QGCR-lighlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$ with the structure vector field ξ normal to M. Then, $\alpha = 0$.*

Proof. Suppose by contradiction that $\alpha \neq 0$ and $\xi \in \Gamma(TM^{\perp})$, then

$$\xi = \xi_R + \xi_{S^{\perp}}, \quad \xi_l = \xi_S = 0, \quad b_i = 0, \quad a_i \neq 0 \quad \text{and} \quad c_\beta \neq 0. \tag{2.3.7}$$

Differentiating the first equation of (2.3.7) with respect to X and using (1.3.8), (1.2.15) and (1.2.17), we get

$$\begin{aligned} -\alpha\overline{\phi}X = & \sum_{i=1}^{r} X(a_i)E_i + \sum_{\beta=r+1}^{n} X(c_\beta)W_\beta \\ & + \sum_{i=1}^{r} a_i \left\{ \nabla_X E_i + \sum_{j=1}^{r} h_j^l(X, E_i)N_j + \sum_{\gamma=r+1}^{n} h_\gamma^s(X, E_i)W_\gamma \right\} \\ & + \sum_{\beta=r+1}^{n} c_\beta \left\{ -A_{W_\beta}X + \sum_{i=1}^{r} \varphi_{\beta i}(X)N_i + \sum_{\gamma=r+1}^{n} \sigma_{\beta\gamma}(X)W_\gamma \right\} + \overline{H}X, \end{aligned} \tag{2.3.8}$$

for all $X \in \Gamma(TM)$. Taking the $\overline{g}$-product of (2.3.8) with respect to E_k and $\overline{\phi}N_k \in \Gamma(S(TM))$ in turn, where $N_k \in \Gamma(\mathscr{L})$, we get

$$\alpha\overline{g}(X, \overline{\phi}E_k) = -\sum_{i=1}^{r} a_i h_i^l(X, E_k) - \sum_{\beta=r+1}^{n} c_\beta h_\beta^s(X, E_k) + \overline{g}(\overline{H}X, E_k). \tag{2.3.9}$$

Replacing X with $\overline{\phi}N_k$ in (2.3.9) we obtain

$$\alpha\overline{g}(N_k, E_k) = -\sum_{i=1}^{r} a_i h_i^l(\overline{\phi}N_k, E_k) - \sum_{\beta=r+1}^{n} c_\beta h_\beta^s(\overline{\phi}N_k, E_k) + \overline{g}(\overline{H}\,\overline{\phi}N_k, E_k). \tag{2.3.10}$$

The $\overline{g}$-product with $\overline{\phi}N_k$ yields

$$\begin{aligned} -\alpha\overline{g}(\overline{\phi}X, \overline{\phi}N_k) = & -\sum_{i=1}^{r} a_i \overline{g}(A_{E_i}^* X, \overline{\phi}N_k) + \sum_{i=1}^{r} a_i \sum_{j=1}^{r} h_j^l(X, E_i)\lambda_j(\overline{\phi}N_k) \\ & - \sum_{\beta=r+1}^{n} c_\beta \overline{g}(A_{W_\beta}X, \overline{\phi}N_k) + \sum_{\beta=r+1}^{n} c_\beta \sum_{j=1}^{r} \varphi_{\beta j}(X)\lambda_j(\overline{\phi}N_k) \\ & + \overline{g}(\overline{H}X, \overline{\phi}N_k). \end{aligned} \tag{2.3.11}$$

Now, using (1.2.24), (1.2.25) and (1.2.26) in (2.3.11), we obtain

$$\alpha \overline{g}(\overline{\phi}X,\overline{\phi}N_k) = \sum_{i=1}^{r} a_i \overline{g}(A^*_{E_i}X,\overline{\phi}N_k) + \sum_{\beta=r+1}^{n} c_\beta \overline{g}(A_{W_\beta}X,\overline{\phi}N_k) - \overline{g}(\overline{H}X,\overline{\phi}N_k),$$

which on replacing X with E_k and simplifying gives

$$\alpha \overline{g}(E_k,N_k) = \alpha b_k a_k + \sum_{i=1}^{r} a_i h^l_i(E_k,\overline{\phi}N_k) + \sum_{\beta=r+1}^{n} c_\beta h^s_\beta(E_k,\overline{\phi}N_k) - \overline{g}(\overline{H}E_k,\overline{\phi}N_k). \quad (2.3.12)$$

Adding (2.3.10) to (2.3.12) yields

$$2\alpha \overline{g}(E_k,N_k) = \overline{g}(\overline{H}\,\overline{\phi}N_k,E_k) - \overline{g}(\overline{H}E_k,\overline{\phi}N_k). \quad (2.3.13)$$

But $\overline{H}$ is skew-symmetric and thus (2.3.13) becomes

$$\overline{g}(\overline{H}\,\overline{\phi}N_k,E_k) = \alpha. \quad (2.3.14)$$

By virtue of (2.3.14) and the fact that $\alpha \neq 0$, it easy to see that $\frac{1}{\alpha}\overline{H}\,\overline{\phi}N_k \in \Gamma(l\mathrm{tr}(TM))$ when $\alpha > 0$ or $\frac{1}{\alpha}\overline{\phi}\,\overline{H}N_k \in \Gamma(l\mathrm{tr}(TM))$ when $\alpha < 0$. Hence, there exist non vanishing smooth functions e_k such that $\overline{H}\,\overline{\phi}N_k = \alpha e_k N_k$ or $\overline{\phi}\,\overline{H}N_k = \alpha e_k N_k$. Taking the $\overline{g}$-product of the first equation with respect to ξ, we get $0 = \overline{g}(\overline{H}\,\overline{\phi}N_k,\xi) = \overline{g}(\alpha e_k N_k,\xi) = \alpha e_k \overline{g}(N_k,\xi) = \alpha e_k a_k$, from which $a_k = 0$, a contradiction. Thus, $\alpha = 0$. The second also yields similar results since $\eta \circ \overline{\phi} = 0$. □

From the above theorem, we have the following corollaries;

Corollary 2.3.6. *There is no proper QGCR-lightlike submanifolds* $(M,g,S(TM),\,S(TM^\perp))$ *of an indefinite nearly Sasakian manifold* $(\overline{M},\overline{\phi},\eta,\xi,\overline{g})$ *with the structure vector field* ξ *normal to* M.

Corollary 2.3.7. *There is no totally umbilical or totally geodesic proper QGCR-lightlike submanifolds* $(M,g,S(TM),S(TM^\perp))$ *of an indefinite nearly Sasakian manifold* $(\overline{M},\overline{\phi},\eta,\xi,\overline{g})$ *with the structure vector field* ξ *normal to* M.

Note from Theorem 2.3.5 and its corollaries that the normality of ξ considered is the special case when $a_i \neq 0$ and $c_\beta \neq 0$, such that a contradiction of any of the two makes the assertion invalid.

Theorem 2.3.8. *Let* $(M,g,S(TM),S(TM^\perp))$ *be a totally umbilical proper QGCR-lightlike submanifold of an indefinite nearly* α*-Sasakian manifold* $(\overline{M},\overline{\phi},\eta,\xi,\overline{g})$ *with the structure* *vector field ξ transversal to M. If the screen distribution $S(TM)$ is totally umbilical, then $\alpha = 0$.*

Proof. Suppose by contradiction that $\alpha \neq 0$ and $\xi \in \Gamma(\mathrm{tr}(TM))$, then,

$$\xi = \xi_l + \xi_{S^\perp}, \quad \xi_R = \xi_S = 0, \quad a_i = 0, \quad b_i \neq 0 \quad \text{and} \quad c_\beta \neq 0. \tag{2.3.15}$$

Differentiating the first equation of (2.3.15) with respect to X and using (1.3.8), (1.2.16) and (1.2.17), we get

$$\begin{aligned}
-\alpha\overline{\phi}X = &\sum_{i=1}^{r} X(b_i)N_i + \sum_{\beta=r+1}^{n} X(c_\beta)W_\beta \\
&+ \sum_{i=1}^{r} b_i \left\{ -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{\beta=r+1}^{n} \rho_{i\beta}(X)W_\beta \right\} \\
&+ \sum_{\beta=r+1}^{n} c_\beta \left\{ -A_{W_\beta}X + \sum_{i=1}^{r} \varphi_{\beta i}(X)N_i + \sum_{\gamma=r+1}^{n} \sigma_{\beta\gamma}(X)W_\gamma \right\} + \overline{H}X,
\end{aligned}$$

for all $X \in \Gamma(TM)$. Now, taking the $\overline{g}$-product of the above equation with respect to $\overline{\phi}N_k \in \Gamma(S(TM))$ where $N_k \in \Gamma(\mathscr{L})$, we get

$$-\alpha\overline{g}(\overline{\phi}X, \overline{\phi}N_k) = -\sum_{i=1}^{r} b_i g(A_{N_i}X, \overline{\phi}N_k) - \sum_{\beta=r+1}^{n} c_\beta g(A_{W_\beta}X, \overline{\phi}N_k) + \overline{g}(\overline{H}X, \overline{\phi}N_k). \tag{2.3.16}$$

Replacing X with $E_k \in \Gamma(D_2)$ in (2.3.16), we obtain

$$\begin{aligned}
-\alpha\overline{g}(\overline{\phi}E_k, \overline{\phi}N_k) = &-\sum_{i=1}^{r} b_i g(A_{N_i}E_k, \overline{\phi}N_k) - \sum_{\beta=r+1}^{n} c_\beta g(A_{W_\beta}E_k, \overline{\phi}N_k) \\
&+ \overline{g}(\overline{H}E_k, \overline{\phi}N_k).
\end{aligned} \tag{2.3.17}$$

Substituting (1.2.25) and the first equation of (1.2.27) in (2.3.17) gives

$$-\alpha\overline{g}(\overline{\phi}E_k, \overline{\phi}N_k) = -\sum_{i=1}^{r} b_i h_i^*(E_k, \overline{\phi}N_k) - \sum_{\beta=r+1}^{n} c_\beta h_\beta^s(E_k, \overline{\phi}N_k) + \overline{g}(\overline{H}E_k, \overline{\phi}N_k). \tag{2.3.18}$$

Since M is totally umbilical in $\overline{M}$, with a totally umbilical screen, then (2.3.18) yields

$$-\alpha\overline{g}(\overline{\phi}E_k, \overline{\phi}N_k) = \overline{g}(\overline{H}E_k, \overline{\phi}N_k), \tag{2.3.19}$$

which reduces to $\overline{g}(\overline{\phi}\,\overline{H}E_k, N_k) = \alpha$. It is easy to see from this equation that $\frac{1}{\alpha}\overline{\phi}\,\overline{H}E_k \in \Gamma(\mathrm{Rad}\,TM)$ when $\alpha > 0$ or $\frac{1}{\alpha}\overline{H}\,\overline{\phi}E_k \in \Gamma(\mathrm{Rad}\,TM)$ when $\alpha < 0$. Hence, there exist non vanishing smooth functions w_k such that $\overline{\phi}\,\overline{H}E_k = \alpha w_k E_k$ or $\overline{H}\,\overline{\phi}E_k = \alpha w_k E_k$. Taking the $\overline{g}$-product of the first equation with respect to ξ, we obtain $0 = \overline{g}(\overline{\phi}\,\overline{H}E_k, \xi) = \alpha w_k \overline{g}(E_k, \xi) = \alpha w_k b_k$. Hence, $b_k = 0$ which is contradiction. Thus, $\alpha = 0$. The second gives similar results since $\eta \circ \overline{H} = 0$. □

Corollary 2.3.9. *There exist no totally umbilical proper QGCR-lightlike submanifolds $(M, g, S(TM), S(TM^{\perp}))$, with totally geodesic screen distributions, of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$ with the structure vector field ξ transversal to M.*

Theorem 2.3.10. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally geodesic proper QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$ with the structure vector field ξ transversal to M. If the screen distribution $S(TM)$ is totally geodesic, then $\alpha = 0$.*

Proof. The proof is similar to that in Theorem 2.3.8. Hence, we ommit it. □

Note from Theorem 2.3.8 and its corollaries that the transversality of ξ considered is the special case when $b_i \neq 0$ and $c_\beta \neq 0$, such that a contradiction of any of the two makes the assertion invalid.

Next, we consider the special case of nearly α-Sasakian manifold with $\overline{H} = 0$. More precisely, the indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \eta, \xi, \overline{g})$ with $\overline{H} = 0$ becomes α-Sasakian manifold. We define the concept of generalised α-Sasakian space form as follows;

Definition 2.3.11. An indefinite α-Sasakian manifold $\overline{M}$ will be called a generalised α-Sasakian space form, denoted by $\overline{M}(f_1, f_2)$, if its curvature tensor $\overline{R}$ is given by

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y})\overline{Z} = {}& f_1\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}\} + f_2\{\eta(\overline{X})\eta(\overline{Z})\overline{Y} \\ & - \eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\xi \\ & + \overline{g}(\overline{\phi}\,\overline{Y},\overline{Z})\overline{\phi}\,\overline{X} + \overline{g}(\overline{\phi}\,\overline{Z},\overline{X})\overline{\phi}\,\overline{Y} - 2\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\overline{\phi}\,\overline{Z}\},\end{aligned} \tag{2.3.20}$$

for any $\overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M})$ and $f_1, f_2 \in \mathscr{X}(\overline{M})$.

If $f_1 = \frac{\overline{c}+3}{4}$ and $f_2 = \frac{\overline{c}-1}{4}$, then $\overline{M}(f_1, f_2)$ becomes a 1-Sasakian space form or simply Sasakian space form [19]. If $f_1 = f_2 = \frac{\overline{c}}{2}$, then $\overline{M}(f_1, f_2)$ becomes a cosymplectic space form [12]. Now, using (2.3.20) we have the following existence theorem.

Theorem 2.3.12. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite generalised α-Sasakian space form $\overline{M}(f_1, f_2)$ with $f_2 \neq 0$. Then, M is a QGCR-lightlike submanifold of $\overline{M}(f_1, f_2)$ if and only if*

(a) The maximal invariant subspaces of T_pM, $p \in M$ define a distribution

$$D = D_0 \perp D_1,$$

where $\mathrm{Rad}\,TM = D_1 \oplus D_2$ and D_0 is a non-degenerate invariant distribution.

(b) There exists a lightlike transversal vector bundle $l\mathrm{tr}(TM)$ such that

$$\overline{g}(\overline{R}(X,Y)E,N) = 0,\ \forall X,Y \in \Gamma(D_0),\ E \in \Gamma(\mathrm{Rad}\,TM),\ N \in \Gamma(l\mathrm{tr}(TM)).$$

(c) There exists a vector subbundle M_2 on M such that

$$\overline{g}(\overline{R}(X,Y)W,W') = 0,\ \ \forall W,W' \in \Gamma(M_2),$$

where M_2 is orthogonal to D and $\overline{R}$ is the curvature tensor of $\overline{M}(c)$.

Proof. Suppose M is a QGCR-lightlike submanifold of $\overline{M}(f_1, f_2)$ with $f_2 \neq 0$. Then, $D = D_0 \perp D_1$ is a maximal invariant subspace. Next, from (2.3.20), for $X,Y \in \Gamma(D_0)$, $E \in \Gamma(D_2)$ and $N \in \Gamma(l\mathrm{tr}(TM))$ we have

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)E,N) &= f_2\{\eta(X)\eta(E)\overline{g}(Y,N) - \eta(Y)\eta(E)\overline{g}(X,N) - 2g(\overline{\phi}X,Y)\overline{g}(\overline{\phi}E,N)\}\\ &= 2f_2 g(\overline{\phi}X,Y)\overline{g}(\overline{\phi}E,N).\end{aligned}$$

Since $g(\overline{\phi}X,Y) \neq 0$ and $\overline{g}(\overline{\phi}E,N) = 0$, we have $\overline{g}(\overline{R}(X,Y)E,N) = 0$. Similarly, from (2.3.20), one obtains

$$\overline{g}(\overline{R}(X,Y)W,W') = 2f_2 g(\overline{\phi}X,Y)\overline{g}(\overline{\phi}W,W'),$$

$\forall X,Y \in \Gamma(D_0)$ and $W,W' \in \Gamma(\overline{\phi}\mathscr{S})$. Let $W = \overline{\phi}W_1$ and $W' = \overline{\phi}W_2$ with $W_1, W_2 \in \Gamma(\mathscr{S})$. Since $g(\overline{\phi}X,Y) \neq 0$ and $\overline{g}(\overline{\phi}W,W') = \overline{g}(\overline{\phi}^2 W_1, \overline{\phi}W_2) = \overline{g}(\overline{\phi}W_1, W_2) = 0$. Therefore, we have $\overline{g}(\overline{R}(X,Y)W,W') = 0$.

Conversely, assume that (a), (b) and (c) are satisfied. Then (a) implies that $D = D_0 \perp D_1$ is invariant. From (b) and (2.3.20) we get

$$\overline{g}(\overline{\phi}E,N) = 0, \tag{2.3.21}$$

which implies $\overline{\phi}E \in \Gamma(S(TM))$. Thus, some part of Rad$TM$, say D_2, belongs to $S(TM)$ under the action of $\overline{\phi}$. Further, (2.3.21) implies $\overline{g}(\overline{\phi}E,N) = \overline{g}(\overline{\phi}^2E,\overline{\phi}N) = \overline{g}(-E+\eta(E)\xi,\overline{\phi}N) = -\overline{g}(E,\overline{\phi}N) = 0$. Therefore, a part of $l$tr(TM), say $\mathscr{L}$, also belongs to $S(TM)$ under the action of $\overline{\phi}$. On the other hand, (c) and (2.3.20) imply $\overline{g}(\overline{\phi}W,W') = 0$. Hence we obtain $\overline{\phi}M_2 \perp M_2$. Also, $g(\overline{\phi}E,W) = -g(E,\overline{\phi}W) = -c_\beta\eta(E)$ implies that generally $\overline{\phi}M_2 \oplus$ RadTM or equivalently, $M_2 \oplus \overline{\phi}RadTM$. Now, from $M_2 \oplus \overline{\phi}RadTM$ and the fact that $\overline{\phi}D_1 = D_1$, then $M_2 \perp D_1$ and $M_2 \oplus \overline{\phi}D_2$. This also tells us that $\overline{\phi}M_2$ has a component along $l$tr(TM), essentially coming from ξ. On the other hand, invariant and non-degenerate D_0 implies $g(\overline{\phi}W,X) = 0$, for $X \in \Gamma(D_0)$. Thus, $M_2 \perp D_0$ and $\overline{\phi}M_2 \perp D_0$. Since $\xi \in \Gamma(T\overline{M})$, we sum up the above results and conclude that

$$S(TM) = \{\overline{\phi}D_2 \oplus M_1 \oplus M_2\} \perp D_0,$$

where $M_1 = \overline{\phi}\mathscr{L}$. Hence M is QGCR-lightlike submanifold of $\overline{M}(f_1,f_2)$ and the proof is completed. □

Note that conditions (b) and (c) are independent of the position of ξ and hence valid for GCR-lightlike submanifolds [10] and QGCR-lightlike submanifolds of indefinite Sasakian space form $\overline{M}\left(\frac{\overline{c}+3}{4},\frac{\overline{c}-1}{4}\right)$ and indefinite cosymplectic space form $\overline{M}\left(\frac{\overline{c}}{2},\frac{\overline{c}}{2}\right)$. When ξ is tangent to M, it is well known that $\xi \in \Gamma(S(TM))$ [6]. In this case, one has a GCR-lightlike submanifold, in which $D_2 \perp \overline{\phi}D_2$ is an invariant subbundle of TM, leading to $D = D_1 \perp D_2 \perp \overline{\phi}D_2 \perp D_0$ as the maximal invariant subspace of TM. On the other hand, when M is QGCR-lightlike submanifold then $\xi \in \Gamma(T\overline{M})$ and thus $D_2 \perp \overline{\phi}D_2$ is generally not an invariant subbundle of TM since the action of $\overline{\phi}$ on it gives a component along ξ. In particular, let $E \in \Gamma(D_2)$ then $E+\overline{\phi}E \in \Gamma(D_2 \perp \overline{\phi}D_2)$. But on applying $\overline{\phi}$ to this subbundle and considering the fact that $\eta(E) \neq 0$ we get $-E+\overline{\phi}E+\eta(E)\xi \notin \Gamma(D_2 \perp \overline{\phi}D_2)$. Hence, $D = D_0 \perp D_1$ is the maximal invariant subbundle of TM. Further, in the case of QGCR-lightlike submanifold, $\overline{\phi}D_2 \oplus M_2$. In fact, let $\overline{\phi}E \in \Gamma(\overline{\phi}D_2)$ and $W = \overline{\phi}W_1 \in \Gamma(M_2)$, where $W_1 \in \Gamma(\mathscr{S})$. Then, $\overline{g}(\overline{\phi}E,W) = \overline{g}(\overline{\phi}E,\overline{\phi}W_1) = -\eta(E)\eta(W_1) \neq 0$. This explains the second direct sum in decomposition $S(TM) = \{\overline{\phi}D_2 \oplus M_1 \oplus M_2\} \perp D_0$. For the case of GCR-lightlike submanifold, $\eta(E) = \eta(W_1) = 0$, hence $\overline{g}(\overline{\phi}E,W) = \overline{g}(\overline{\phi}E,\overline{\phi}W_1) = 0$. This implies that $\overline{\phi}D_2 \perp M_2$ and hence the first direct orthogonal sum in the decomposition $S(TM) = \{\overline{\phi}D_2 \oplus M_1\} \perp M_2 \perp D_0 \perp \langle\xi\rangle$.

2.4 Integrability of distributions

Let M be a proper QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. From (1.2.1), the tangent bundle of any QGCR-lightlike submanifold, TM, can be rewritten as

$$TM = D \oplus \widehat{D}, \tag{2.4.1}$$

where $D = D_0 \perp D_1$ and $\widehat{D} = \{D_2 \perp \overline{\phi} D_2\} \oplus \overline{D}$.

Notice that D is invariant with respect to $\overline{\phi}$ while $\widehat{D}$ is not generally anti-invariant with respect to $\overline{\phi}$.

Let π and $\widehat{\pi}$ be the projections of TM onto D and $\widehat{D}$ respectively. Then, using the first equation of (2.4.1) we can decompose X as

$$X = \pi X + \widehat{\pi} X, \ \ \forall X \in \Gamma(TM). \tag{2.4.2}$$

It is easy to see that $\overline{\phi}\pi X \in \Gamma(D)$. However, the action of $\overline{\phi}$ on $\widehat{\pi}X$ gives a tangential and transversal component due to a generalized ξ, i.e.,

$$\overline{\phi} X = P_1 X + P_2 X + QX, \ \ \forall X \in \Gamma(TM), \tag{2.4.3}$$

where $P_1 X = \overline{\phi}\pi X$ while $P_2 X$ is the tangential component of $\overline{\phi}\widehat{\pi}X$ and QX is the transversal component of $\overline{\phi}X$, essentially coming from $\overline{\phi}\widehat{\pi}X$ since $\overline{\phi}D = D$.

By grouping the tangential and transversal parts in (2.4.3), it is easy to see that

$$\overline{\phi} X = PX + QX, \ \ \forall X \in \Gamma(TM), \tag{2.4.4}$$

where $PX = P_1 X + P_2 X$.

Note that if $X \in \Gamma(D)$, then $P_2 X = QX = 0$, and $\overline{\phi}X = P_1 X$.

The equation (2.4.4) can be properly understood through the following specific case of vector field in $\overline{D} \subset \widehat{D}$. Let ξ_M and $\xi_{\mathrm{tr}M}$ be the tangential and transversal components of ξ. If $X \in \Gamma(\overline{D})$ and since $\overline{D} = \overline{\phi}\mathscr{S} \oplus \overline{\phi}\mathscr{L}$, then

$$\overline{\phi} X = SX + LX - \{\eta(SX) + \eta(LY)\}\xi_M - \{\eta(SX) + \eta(LY)\}\xi_{\mathrm{tr}M}.$$

Consequently, for $X \in \Gamma(\overline{D})$,

$$\begin{aligned}
P_1 X &= 0,\\
P_2 X &= -\{\eta(SX) + \eta(LY)\}\xi_M,\\
\text{and } \; QX &= SX + LX - \{\eta(SX) + \eta(LY)\}\xi_{\mathrm{tr}M}.
\end{aligned}$$

Similarly, for any $V \in \Gamma(\mathrm{tr}(TM))$, $V = SV + LV$, and

$$\overline{\phi}V = tV + fV, \tag{2.4.5}$$

where tV and fV are the tangential and transversal components of $\overline{\phi}V$, respectively.

Differentiating (2.4.4) with respect to Y we get

$$\overline{\nabla}_Y PX + \overline{\nabla}_Y QX = \overline{\nabla}_Y \overline{\phi} X. \tag{2.4.6}$$

Then using (1.2.15), (1.2.16), (1.2.17) and (1.3.4) we have

$$\overline{\nabla}_Y PX + \overline{\nabla}_Y QX = \nabla_Y PX + h(PX,Y) - A_{QX}Y + \nabla^t_Y QX, \tag{2.4.7}$$

and from (1.3.4), we have;

$$\begin{aligned}
\overline{\nabla}_Y \overline{\phi} X &= \overline{\phi}(\nabla_Y X) + \overline{\phi}(\nabla_X Y) + 2\overline{\phi} h(X,Y) - \overline{\nabla}_X \overline{\phi} Y\\
&\quad + 2\alpha\overline{g}(X,Y)\xi_M + 2\alpha\overline{g}(X,Y)\xi_{\mathrm{tr}M} - \alpha\eta(Y)X - \alpha\eta(X)Y\\
&= P(\nabla_Y X) + Q(\nabla_Y X) + P(\nabla_X Y) + Q(\nabla_X Y)\\
&\quad + 2th(X,Y) + 2fh(X,Y) - \nabla_X PY - \nabla^t_X QY\\
&\quad - h(X,PY) + A_{QY}X + 2\alpha\overline{g}(X,Y)\xi_M + 2\alpha\overline{g}(X,Y)\xi_{\mathrm{tr}M}\\
&\quad - \alpha\eta(Y)X - \alpha\eta(X)Y.
\end{aligned} \tag{2.4.8}$$

Finally putting (2.4.7) and (2.4.8) in (2.4.6) and then comparing the tangential and transversal components of the resulting equation, we obtain

$$\begin{aligned}
(\nabla_Y P)X + (\nabla_X P)Y &= A_{QX}Y + A_{QY}X + 2th(X,Y)\\
&\quad + 2\alpha\overline{g}(X,Y)\xi_M - \alpha\eta(X)Y - \alpha\eta(Y)X,
\end{aligned} \tag{2.4.9}$$

and

$$
\begin{aligned}
(\nabla^T_Y Q)X + (\nabla^T_X Q)Y = & -h(PX,Y) - h(X,PY) \\
& + 2fh(X,Y) + 2\alpha\overline{g}(X,Y)\xi_{\mathrm{tr}M},
\end{aligned} \tag{2.4.10}
$$

for all $X,Y \in \Gamma(TM)$, where

$$
(\nabla_Y P)X = \nabla_Y PX - P(\nabla_Y X) \quad \text{and} \quad (\nabla^T_Y Q)X = \nabla^t_Y QX - Q(\nabla_Y X). \tag{2.4.11}
$$

Proposition 2.4.1. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a proper QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\eta,\xi,\overline{g})$. Then,*

$$
\begin{aligned}
P[X,Y] = & -\nabla_Y PX - \nabla_X PY + 2P\nabla_X Y + A_{QX}Y + A_{QY}X \\
& + 2th(X,Y) + 2\alpha\overline{g}(X,Y)\xi_M - \alpha\eta(X)Y - \alpha\eta(Y)X,
\end{aligned} \tag{2.4.12}
$$

and

$$
\begin{aligned}
Q[X,Y] = & -\nabla^t_Y QX - \nabla^t_X QY + 2Q\nabla_X Y - h(PX,Y) - h(X,PY) \\
& + 2fh(X,Y) + 2\alpha\overline{g}(X,Y)\xi_{\mathrm{tr}M},
\end{aligned} \tag{2.4.13}
$$

for all $X,Y \in \Gamma(TM)$.

Proof. The proof follows from (2.4.9) and (2.4.10). □

Theorem 2.4.2. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a proper QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\eta,\xi,\overline{g})$. Then, the distribution D is integrable if and only if*

$$
\begin{aligned}
& h(P_1X,Y) + h(X,P_1Y) = 2(Q\nabla_X Y + fh(X,Y) + \alpha\overline{g}(X,Y)\xi_{\mathrm{tr}M}), \\
\textit{and} \quad & P_2[X,Y] = 0.
\end{aligned}
$$

for all $X,Y \in \Gamma(D)$.

Proof. The proof is a straightforward calculation. □

The integrability of $\widehat{D}$ is discussed as follows. Note that the distribution $\widehat{D}$ is integrable if and only if, for any X, $Y \in \Gamma(\widehat{D})$, $[X,Y] \in \Gamma(\widehat{D})$. The latter is equivalent to $P_1[X,Y] = 0$.

Theorem 2.4.3. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\eta,\xi,\overline{g})$. Then, the distribution $\widehat{D}$ is integrable if and* *only if*

$$\begin{aligned}&A_{QX}Y + A_{QY}X - \nabla_Y P_2 X - \nabla_X P_2 Y\\&+2(P_1(\nabla_X Y) + \alpha\overline{g}(X,Y)\xi_M + th(X,Y)) \in \Gamma(\widehat{D}),\end{aligned}$$

for all $X,Y \in \Gamma(\widehat{D})$.

Proof. Let $X,Y \in \Gamma(\widehat{D})$, then it is easy to see that $P_1X = P_1Y = 0$. Hence, $PX = P_2X$ and $PY = P_2Y$. Now using (2.4.12), we derive

$$\begin{aligned}\overline{\phi}[X,Y] &= P[X,Y] + Q[X,Y]\\&= -\nabla_Y PX - \nabla_X PY + 2P\nabla_X Y + A_{QX}Y\\&+A_{QY}X + 2th(X,Y) + 2\alpha\overline{g}(X,Y)\xi_M - \alpha\eta(X)Y\\&-\alpha\eta(Y)X + Q[X,Y]\\&= -\nabla_Y P_2 X - \nabla_X P_2 Y + 2P_1\nabla_X Y + A_{QX}Y\\&+A_{QY}X + 2th(X,Y) + 2\alpha\overline{g}(X,Y)\xi_M + 2P_2\nabla_X Y\\&-\alpha\eta(X)Y - \alpha\eta(Y)X + Q[X,Y]. \qquad (2.4.14)\end{aligned}$$

It is obvious from (2.4.14) that the last four terms belongs to $\widehat{D}$. Hence, the assertion follows from the remaining terms. □

Chapter 3

Ascreen QGCR-lightlike submanifolds of nearly α-Sasakian manifod

3.1 Introduction

In this chapter, we study a special QGCR-lightlike submanifold of indefinite nearly α-Sasakian manifolds, called, ascreen QGCR-lighlike submanifold. We discuss totally umblical, totally geodesic, mixed geodesic and minimal ascreen QGCR-lightlike submanifolds.

Definition 3.1.1 ([15]). A lightlike submanifold M of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be ascreen if the structural vector field, ξ, belongs to $\mathrm{Rad}\,TM \oplus l\mathrm{tr}(TM)$.

Note that, since $\mathscr{L}$ defined in Definition 2.2.1 is a subbundle of $l\mathrm{tr}(TM)$, there is a complementary subbundle ν of $l\mathrm{tr}(TM)$ such that

$$l\mathrm{tr}(TM) = \mathscr{L} \perp \nu.$$

It is easy to check that the complementary subbundle ν is invariant under $\overline{\phi}$, i.e. $\overline{\phi}\nu = \nu$.

Let M be an ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $\overline{M}$. Then by Definition 3.4.1, the structure vector field $\xi \in \mathrm{Rad}\,TM \oplus l\mathrm{tr}(TM)$. This means that ξ can possibly be in $\mathrm{Rad}\,TM$ or $l\mathrm{tr}(TM)$. If $\xi \in \Gamma(\mathrm{Rad}\,TM)$, then $\xi \in \Gamma(D_2)$ since $\overline{\phi}D_1 = D_1$ and $\overline{\phi}\xi = 0$. On the other hand, if $\xi \in \Gamma(l\mathrm{tr}(TM))$, then $\xi \in \Gamma(\mathscr{L})$ because of the fact that $\overline{\phi}\nu = \nu$ and $\overline{\phi}\xi = 0$. Since, ξ is a unit spacelike vector field, that is $\overline{g}(\xi, \xi) = 1$, then it can easily be seen that $\xi \notin \Gamma(D_2)$ or $\xi \notin \Gamma(\mathscr{L})$, since D_2 and $\mathscr{L}$ are both null subbundles. Therefore, we have the following definition for an ascreen QGCR-lightlike submanifold.

Definition 3.1.2. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g}, \overline{\phi}, \xi, \eta)$. We say that M is ascreen QGCR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) there exist two distributions D_1 and D_2 of $\operatorname{Rad} TM$ such that

$$\operatorname{Rad} TM = D_1 \oplus D_2, \quad \overline{\phi} D_1 = D_1, \quad \overline{\phi} D_2 \subset S(TM), \tag{3.1.1}$$

(ii) there exist vector bundles D_0 and $\overline{D}$ over $S(TM)$ such that

$$S(TM) = \{\overline{\phi} D_2 \oplus \overline{D}\} \perp D_0, \tag{3.1.2}$$

$$\text{with} \quad \overline{\phi} D_0 \subseteq D_0, \quad \overline{D} = \overline{\phi}\mathscr{S} \perp \overline{\phi}\mathscr{L}, \tag{3.1.3}$$

where D_0 is a non-degenerate and invariant distribution on M, $\mathscr{L}$ and $\mathscr{S}$ are respectively vector subbundles of $l\mathrm{tr}(TM)$ and $S(TM^{\perp})$.

If $D_1 \neq \{0\}$, $D_0 \neq \{0\}$, $D_2 \neq \{0\}$ and $\mathscr{S} \neq \{0\}$, then M is called a *proper ascreen QGCR lightlike submanifold.*

Lemma 3.1.3. *If $(M, g, S(TM), S(TM^{\perp}))$ is an ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$, then $\xi \in \Gamma(D_2 \oplus \mathscr{L})$.*

Theorem 3.1.4. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a 3-lightlike QGCR-submanifold of an indefinite almost contact manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. Then M is ascreen lightlike submanifold if and only if $\overline{\phi}\mathscr{L} = \overline{\phi} D_2$.*

Proof. Suppose that M is ascreen. Then, by Lemma 3.1.3, $\xi \in \Gamma(D_2 \oplus \mathscr{L})$. Since M is a 3-lightlike QGCR submanifold, and $\operatorname{Rad} TM = D_1 \oplus D_2$ with $\overline{\phi} D_1 = D_1$ and $l\mathrm{tr}(TM) = \mathscr{L} \perp \nu$ with $\overline{\phi}\nu = \nu$, the distributions D_2 and $\mathscr{L}$ are of rank 1. Consequently,

$$\xi = aE + bN, \tag{3.1.4}$$

where $E \in \Gamma(D_2)$ and $N \in \Gamma(\mathscr{L})$, and $a = \eta(N)$ and $b = \eta(E)$ are non-zero smooth functions. Applying $\overline{\phi}$ to (3.1.4) and using the fact that $\overline{\phi}\xi = 0$, we get

$$a\overline{\phi}E + b\overline{\phi}N = 0. \tag{3.1.5}$$

From (3.1.5), one gets $\overline{\phi}E = \omega\overline{\phi}N$, where $\omega = -\frac{b}{a} \neq 0$, a non vanishing smooth function. This implies that $\overline{\phi}\mathscr{L} \cap \overline{\phi} D_2 \neq \{0\}$. Since $rank(\overline{\phi} D_2) = rank(\overline{\phi}\mathscr{L}) = 1$, it follows that $\overline{\phi}\mathscr{L} = \overline{\phi} D_2$.

Conversely, suppose that $\overline{\phi}\mathscr{L} = \overline{\phi}D_2$. Then, there exist a non-vanishing smooth function ω such that

$$\overline{\phi}E = \omega\overline{\phi}N. \tag{3.1.6}$$

Taking the $\overline{g}$-product of (3.1.6) with respect to $\overline{\phi}E$ and $\overline{\phi}N$ in turn, we get

$$b^2 = \omega(ab-1) \text{ and } \omega a^2 = ab-1. \tag{3.1.7}$$

Since $\omega \neq 0$, by (3.1.7), we have $a \neq 0$, $b \neq 0$ and $b^2 = (\omega a)^2$. The latter gives $b = \pm\omega a$. The case $b = \omega a$ implies that $ab = \omega a^2 = ab - 1$, which is a contradiction. Thus $b = -\omega a$, from which $2ab = 1$. Since $\omega = -\frac{b}{a}$, $a \neq 0$ and $\overline{\phi}E = \omega\overline{\phi}N$, it is easy to see that $a\overline{\phi}E + b\overline{\phi}N = 0$. Applying $\overline{\phi}$ to this equation, and using the first relation in (1.3.1), together with $2ab = 1$, we get $\xi = aE + bN$. Therefore, M is ascreen lightlike submanifold of $\overline{M}$. □

In the ascreen QGCR-lightlike submanifold case, the item (ii) of Definition 2.2.1 implies that $\dim(D) \geq 4l \geq 4$ and $\dim(D_2) = \dim(\mathscr{L})$. Thus $\dim(M) \geq 7$ and $\dim(\overline{M}) \geq 11$, and any 7-dimensional ascreen QGCR-lightlike submanifold is 3-lightlike.

Next, we construct two examples of ascreen QGCR-lightlike submanifolds. First, when the ambient manifold is Sasakian (i.e., $\alpha = 1$ and $\overline{H} = 0$) and then when the ambient manifold is cosymplectic (i.e., $\alpha = 0$ and $\overline{H} = 0$).

Using structure 1.3.6 (i.e., $\overline{M}$ is Sasakian manifold), we have the following example;

Example 3.1.5. Let $\overline{M} = (\mathbb{R}^{11}_4, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,-,+,+,+,-,-,+,+,+,+)$ with respect to the canonical basis

$$(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z).$$

Let (M,g) be a submanifold of $\overline{M}$ given by

$$x^1 = y^4,\ \ y^1 = -x^4,\ \ z = x^2\sin\theta + y^2\cos\theta \ \text{ and } \ y^5 = (x^5)^{\frac{1}{2}},$$

where $\theta \in (0, \frac{\pi}{2})$. By direct calculations, we can easily check that the vector fields

$$\begin{aligned}
&E_1 = \partial x_4 + \partial y_1 + y^4\partial z, \quad E_2 = \partial x_1 - \partial y_4 + y^1\partial z,\\
&E_3 = \sin\theta\partial x_2 + \cos\theta\partial y_2 + \partial z, \quad X_1 = 2y^5\partial x_5 + \partial y_5 + 2(y^5)^2\partial z,\\
&X_2 = -\cos\theta\partial x_2 + \sin\theta\partial y_2 - y^2\cos\theta\partial z, \quad X_3 = 2\partial y_3,\ X_4 = 2(\partial x_3 + y^3\partial z),
\end{aligned}$$

form a local frame of TM. From this, we can see that $\mathrm{Rad}\,TM$ is spanned by $\{E_1, E_2, E_3\}$, and therefore, M is 3-lightlike. Further, $\overline{\phi}_0 E_1 = E_2$, therefore we set $D_1 = \mathrm{Span}\{E_1, E_2\}$.

Also $\overline{\phi}_0 E_3 = -X_2$ and thus $D_2 = \mathrm{Span}\{E_3\}$. It is easy to see that $\overline{\phi}_0 X_3 = X_4$, so we set $D_0 = \mathrm{Span}\{X_3, X_4\}$. On the other hand, following direct calculations, we have

$$N_1 = \frac{1}{2}(\partial x_4 - \partial y_1 + y^4 \partial z), \quad N_2 = \frac{1}{2}(-\partial x_1 - \partial y_4 + y^1 \partial z),$$
$$N_3 = \frac{1}{2}(-\sin\theta\partial x_2 - \cos\theta\partial y_2 + \partial z), \quad W = \partial x_5 - 2y^5 \partial y_5 + y^5 \partial z,$$

from which $l\mathrm{tr}(TM) = \mathrm{Span}\{N_1, N_2, N_3\}$ and $S(TM^\perp) = \mathrm{Span}\{W\}$. Clearly, $\overline{\phi}_0 N_2 = -N_1$. Further, $\overline{\phi}_0 N_3 = \frac{1}{2}X_2$ and thus $\mathscr{L} = \mathrm{Span}\{N_3\}$. Notice that $\overline{\phi}_0 N_3 = -\frac{1}{2}\overline{\phi}_0 E_3$ and therefore $\overline{\phi}_0 \mathscr{L} = \overline{\phi}_0 D_2$. Also, $\overline{\phi}_0 W = -X_1$ and therefore $\mathscr{S} = \mathrm{Span}\{W\}$. Finally, we calculate ξ as follows; From Theorem 3.4.2 we have $\xi = aE_3 + bN_3$. Applying $\overline{\phi}_0$ to this equation we obtain $a\overline{\phi}_0 E_3 + b\overline{\phi}_0 N_3 = 0$. Now, substituting for $\overline{\phi}_0 E_3$ and $\overline{\phi}_0 N_3$ in this equation we get $2a = b$, from which we get $\xi = \frac{1}{2}(E_3 + 2N_3)$. Since $\overline{\phi}_0 \xi = 0$ and $\overline{g}(\xi, \xi) = 1$, we conclude that (M, g) is an ascreen QGCR-lightlike submanifold of $\overline{M}$.

Next, considering structure 1.3.5 (i.e., $\overline{M}$ is cosymplectic manifold), we have the following example;

Example 3.1.6. Let $\overline{M} = (\mathbb{R}^{11}_4, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,-,+,+,+,-,-,+,+,+,+)$ with respect to the canonical basis

$$(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z).$$

Let (M, g) be a submanifold of $\overline{M}$ given by

$$x^1 = y^4, \; y^1 = -x^4, \; z = \frac{1}{\sqrt{2}}x^2 + \frac{1}{\sqrt{2}}y^2 \text{ and } y^5 = (x^5)^{\frac{1}{2}}.$$

By direct calculations, we can easily check that the vector fields

$$E_1 = \partial x_4 + \partial y_1, \quad E_2 = \partial x_1 - \partial y_4,$$
$$E_3 = \frac{1}{\sqrt{2}}\partial x_2 + \frac{1}{\sqrt{2}}\partial y_2 + \partial z, \quad X_1 = 2y^5 \partial x_5 + \partial y_5,$$
$$X_2 = -\frac{1}{\sqrt{2}}\partial x_2 + \frac{1}{\sqrt{2}}\partial y_2, \quad X_3 = \partial y_3, \, X_4 = \partial x_3,$$

form a local frame of TM. From this, we can see that $\mathrm{Rad}\,TM$ is spanned by $\{E_1, E_2, E_3\}$, and therefore, M is 3-lightlike. Further, $\overline{\phi}_0 E_1 = E_2$, therefore we set $D_1 = \mathrm{Span}\{E_1, E_2\}$. Also $\overline{\phi}_0 E_3 = -X_2$ and thus $D_2 = \mathrm{Span}\{E_3\}$. It is easy to see that $\overline{\phi}_0 X_3 = X_4$, so we set

$D_0 = \text{Span}\{X_3, X_4\}$. On the other hand, following direct calculations, we have

$$N_1 = \frac{1}{2}(\partial x_4 - \partial y_1), \quad N_2 = \frac{1}{2}(-\partial x_1 - \partial y_4),$$
$$N_3 = \frac{1}{2}(-\frac{1}{\sqrt{2}}\partial x_2 - \frac{1}{\sqrt{2}}\partial y_2 + \partial z), \quad W = \partial x_5 - 2y^5 \partial y_5,$$

from which $l\text{tr}(TM) = \text{Span}\{N_1, N_2, N_3\}$ and $S(TM^{\perp}) = \text{Span}\{W\}$. Clearly, $\overline{\phi}_0 N_2 = -N_1$. Further, $\overline{\phi}_0 N_3 = \frac{1}{2}X_2$ and thus $\mathscr{L} = \text{Span}\{N_3\}$. Notice that $\overline{\phi}_0 N_3 = -\frac{1}{2}\overline{\phi}_0 E_3$ and therefore $\overline{\phi}_0 \mathscr{L} = \overline{\phi}_0 D_2$. Also, $\overline{\phi}_0 W = -X_1$ and therefore $\mathscr{S} = \text{Span}\{W\}$. Finally, we calculate ξ as follows; Using Theorem 3.4.2 we have $\xi = aE_3 + bN_3$. Applying $\overline{\phi}_0$ to this equation we obtain $a\overline{\phi}_0 E_3 + b\overline{\phi}_0 N_3 = 0$. Now,substituting for $\overline{\phi}_0 E_3$ and $\overline{\phi}_0 N_3$ in this equation we get $2a = b$, from which we get $\xi = \frac{1}{2}(E_3 + 2N_3)$. Since $\overline{\phi}_0 \xi = 0$ and $\overline{g}(\xi, \xi) = 1$, we conclude that (M, g) is an ascreen QGCR-lightlike submanifold of $\overline{M}$.

3.2 Totally umbilical and totally geodesic ascreen QGCR-lightlike submanifolds

In this section, we prove two main theorems concerning totally umbilical, totally geodesic and irrotational ascreen QGCR-lightlike submanifolds of $\overline{M}$. Notice that D_0 and $\overline{\phi}\mathscr{S}$ are orthogonal and non-degenerate subbundles of TM and that when M is ascreen QGCR-lightlike submanifold, we observe that

$$\eta(X) = \eta(Z) = 0, \quad \forall X \in \Gamma(D_0), \quad Z \in \Gamma(\overline{\phi}\mathscr{S}). \tag{3.2.1}$$

Theorem 3.2.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical or totally geodesic proper ascreen QGCR-lightlike submanifold of an indefinite nearly cosymplectic space form $\overline{M}(\overline{c})$, of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$, such that D_0 and $\overline{\phi}\mathscr{S}$ are spacelike and parallel distributions with respect to ∇. Then, $\overline{c} \geq 0$. Equality occurs when $\overline{M}(\overline{c})$ is an indefinite cosymplectic space form.*

Proof. Let X and Z be vector fields in D_0 and $\overline{\phi}\mathscr{S}$, respectively. Replacing $\overline{W}$ with $\overline{\phi}X$ and $\overline{Y}$ with $\overline{\phi}Z$ in (1.3.20), we get

$$\begin{aligned}
4\overline{R}(X,\overline{\phi}X,Z,\overline{\phi}Z) &= \overline{g}((\overline{\nabla}_{\overline{\phi}X}\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})\overline{\phi}Z) \\
&- \overline{g}((\overline{\nabla}_{\overline{\phi}X}\overline{\phi})\overline{\phi}Z,(\overline{\nabla}_X\overline{\phi})Z) - 2\overline{g}((\overline{\nabla}_{\overline{\phi}X}\overline{\phi})X,(\overline{\nabla}_{\overline{\phi}Z}\overline{\phi})Z) \\
&+ \overline{g}(\overline{H}\,\overline{\phi}X,Z)\overline{g}(\overline{H}X,\overline{\phi}Z) - \overline{g}(\overline{H}\,\overline{\phi}X,\overline{\phi}Z)\overline{g}(\overline{H}X,Z) \\
&- 2\overline{g}(\overline{H}\,\overline{\phi}X,X)\overline{g}(\overline{H}\,\overline{\phi}Z,Z) - 2\overline{c}g(\overline{\phi}Z,\overline{\phi}Z)g(\overline{\phi}X,\overline{\phi}X).
\end{aligned} \tag{3.2.2}$$

Considering the first three terms on the right hand side of (3.2.2), we have

$$\overline{g}((\overline{\nabla}_{\overline{\phi}X}\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})\overline{\phi}Z) = -\,\overline{g}((\overline{\nabla}_Z\overline{\phi})\overline{\phi}X,(\overline{\nabla}_X\overline{\phi})\overline{\phi}Z). \tag{3.2.3}$$

Applying the first equation of Lemma 1.3.3 on (3.2.3) we derive

$$\begin{aligned}
&\overline{g}((\overline{\nabla}_{\overline{\phi}X}\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})\overline{\phi}Z) = -\overline{g}((\overline{\nabla}_Z\overline{\phi})\overline{\phi}X,(\overline{\nabla}_X\overline{\phi})\overline{\phi}Z) \\
&= -\overline{g}(\overline{\phi}(\overline{\nabla}_Z\overline{\phi})X,\overline{\phi}(\overline{\nabla}_X\overline{\phi})Z) - \overline{g}(X,\overline{\nabla}_Z\xi)\overline{g}(Z,\overline{\nabla}_X\xi) \\
&= -\overline{g}((\overline{\nabla}_Z\overline{\phi})X,(\overline{\nabla}_X\overline{\phi})Z) + \eta((\overline{\nabla}_Z\overline{\phi})X))\eta((\overline{\nabla}_X\overline{\phi})Z) + \overline{g}(Z,\overline{H}X)^2 \\
&= \overline{g}((\overline{\nabla}_X\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})Z) + \overline{g}(Z,(\overline{\nabla}_X\overline{\phi})\xi)\overline{g}(X,(\overline{\nabla}_Z\overline{\phi})\xi) + \overline{g}(Z,\overline{H}X)^2 \\
&= \overline{g}((\overline{\nabla}_X\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})Z) + \overline{g}(Z,\overline{\phi}\,\overline{H}X)\overline{g}(X,\overline{\phi}\,\overline{H}Z) + \overline{g}(Z,\overline{H}X)^2 \\
&= \overline{g}((\overline{\nabla}_X\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})Z) - \overline{g}(\overline{\phi}Z,\overline{H}X)^2 + \overline{g}(Z,\overline{H}X)^2.
\end{aligned} \tag{3.2.4}$$

In a similar way, using the second equation of Lemma 1.3.3, we get

$$-\,\overline{g}((\overline{\nabla}_{\overline{\phi}X}\overline{\phi})\overline{\phi}Z,(\overline{\nabla}_X\overline{\phi})Z) = \overline{g}((\overline{\nabla}_X\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})Z), \tag{3.2.5}$$

and

$$-\,2\overline{g}((\overline{\nabla}_{\overline{\phi}X}\overline{\phi})X,(\overline{\nabla}_{\overline{\phi}Z}\overline{\phi})Z) = 0. \tag{3.2.6}$$

Now substituting (3.2.4), (3.2.5) and (3.2.6) in (3.2.2), we get

$$\begin{aligned}
4\overline{R}(X,\overline{\phi}X,Z,\overline{\phi}Z) &= 2\overline{g}((\overline{\nabla}_X\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})Z) - \overline{g}(\overline{\phi}Z,\overline{H}X)^2 + \overline{g}(Z,\overline{H}X)^2 \\
&+ \overline{g}(\overline{H}\,\overline{\phi}X,Z)\overline{g}(\overline{H}X,\overline{\phi}Z) - \overline{g}(\overline{H}\,\overline{\phi}X,\overline{\phi}Z)\overline{g}(\overline{H}X,Z) \\
&- 2\overline{g}(\overline{H}\,\overline{\phi}X,X)\overline{g}(\overline{H}\,\overline{\phi}Z,Z) - 2\overline{c}g(\overline{\phi}Z,\overline{\phi}Z)g(\overline{\phi}X,\overline{\phi}X),
\end{aligned} \tag{3.2.7}$$

from which we obtain

$$2\overline{R}(X,\overline{\phi}X,Z,\overline{\phi}Z) = \overline{g}((\overline{\nabla}_X\overline{\phi})Z,(\overline{\nabla}_X\overline{\phi})Z) + \overline{g}(Z,\overline{H}X)^2 - \overline{c}g(Z,Z)g(X,X). \tag{3.2.8}$$

Then using the facts D_0 and $\overline{\phi}\mathscr{S}$ are spacelike and parallel with respect to ∇, we have

$$(\overline{\nabla}_Z\overline{\phi})X = (\nabla_Z\overline{\phi})X \in \Gamma(D_0),$$

and (3.2.8) reduces to

$$2\overline{R}(X,\overline{\phi}X,Z,\overline{\phi}Z) = ||(\nabla_Z\overline{\phi})X||^2 + \overline{g}(Z,\overline{H}X)^2 - \overline{c}||X||^2||Z||^2, \tag{3.2.9}$$

where $||.||$ denotes the norm on $D_0 \perp \overline{\phi}\mathscr{S}$ with respect to g.

On the other hand, when we replace $\overline{W}$ with $\overline{\phi}X$ and $\overline{Y}$ with $\overline{\phi}Z$ in (1.2.34), we have

$$\overline{R}(X,\overline{\phi}X,Z,\overline{\phi}Z) = \overline{g}((\nabla_X h^s)(\overline{\phi}X,Z),\overline{\phi}Z) - \overline{g}((\nabla_{\overline{\phi}X}h^s)(X,Z),\overline{\phi}Z), \tag{3.2.10}$$

where,

$$(\nabla_X h^s)(\overline{\phi}X,Z) = \nabla^s_X h^s(\overline{\phi}X,Z) - h^s(\nabla_X\overline{\phi}X,Z) - h^s(\overline{\phi}X,\nabla_X Z). \tag{3.2.11}$$

By the fact that M is totally umbilical in $\overline{M}$, we have $h^s(\overline{\phi}X,Z) = 0$. Thus, using Definition 1.2.2, equation (3.2.11) becomes

$$\begin{aligned}(\nabla_X h^s)(\overline{\phi}X,Z) &= -h^s(\nabla_X\overline{\phi}X,Z) - h^s(\overline{\phi}X,\nabla_X Z)\\ &= -g(\nabla_X\overline{\phi}X,Z)\mathcal{H}^s - g(\overline{\phi}X,\nabla_X Z)\mathcal{H}^s.\end{aligned} \tag{3.2.12}$$

Differentiating $\overline{g}(\overline{\phi}X,Z) = 0$ covariantly with respect to X and then applying (1.2.15), we obtain

$$g(\nabla_X\overline{\phi}X,Z) + g(\overline{\phi}X,\nabla_X Z) = 0. \tag{3.2.13}$$

Substituting (3.2.13) in (3.2.12), gives

$$(\nabla_X h^s)(\overline{\phi}X,Z) = 0. \tag{3.2.14}$$

Similarly,

$$(\nabla_{\overline{\phi}X}h^s)(X,Z) = 0. \tag{3.2.15}$$

Then, substituting (3.2.14) and (3.2.15) in (3.2.10), we get

$$\overline{R}(X,\overline{\phi}X,Z,\overline{\phi}Z)=0. \tag{3.2.16}$$

Substituting (3.2.16) in (3.2.9), gives

$$\overline{c}||X||^2||Z||^2=||(\nabla_Z\overline{\phi})X||^2+\overline{g}(Z,\overline{H}X)^2\geq 0, \tag{3.2.17}$$

which implies that $\overline{c}\geq 0$. When the ambient manifold is cosymplectic, then $\nabla\overline{\phi}=0$ and also $N^{(1)}=0$ and hence $d\eta=0$ from (1.3.14). In this case $\overline{c}=0$. □

Example 3.2.2. Let $(M,g,S(TM),\,S(TM^{\perp}))$ be an ascreen QGCR-lighlike submanifold in Example 4.1.2. Applying (1.2.15) and Koszul's formula (1.1.1) to Example 4.1.2 we obtain

$$\begin{aligned} &h_i^l(X,Y)=0 \quad \forall X,Y\in\Gamma(TM), \quad \text{where} \quad i=1,2,3,\\ &\varepsilon_4 h_4^s(X_1,X_1)=2 \quad \text{and} \quad h_4^s(X,Y)=0, \quad \forall X\neq X_1, Y\neq X_1. \end{aligned} \tag{3.2.18}$$

Using (1.2.22), (3.2.18) and $\varepsilon_4=\overline{g}(W,W)=1+4(y^5)^2$, we also derive

$$h(X_1,X_1)=\frac{2}{1+4(y^5)^2}W. \tag{3.2.19}$$

We remark that M is not totally geodesic. From (3.2.19) and (1.2.32) we note that M is totally umbilical with

$$\mathscr{H}=\frac{2}{(1+4(y^5)^2)^2}W.$$

By straightforward calculations we also have

$$\nabla_{X_1}X_1=4y^5X_1 \quad \text{and} \quad \nabla_{X_i}X_j=0 \quad \forall i,j\neq 1.$$

Thus, D_0 and $\overline{\phi}\mathscr{S}$ are parallel distributions with respect to ∇. Hence, M satisfies Theorem 3.2.1 and $\overline{c}=0$.

Corollary 3.2.3. *Let $(M,g,S(TM),\,S(TM^{\perp}))$ be a totally umbilical or totally geodesic ascreen QGCR-lightlike submanifold of an indefinite cosymplectic space form $\overline{M}(\overline{c})$ of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$. Then, $\overline{c}=0$.*

Definition 3.2.4 ([9])**.** A lightlike submanifold M of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is called irrotational if $\overline{\nabla}_XE\in\Gamma(TM)$, for any $E\in\Gamma(\mathrm{Rad}\,TM))$ and $X\in\Gamma(TM)$. Equivalently, M is irrotational if

$$h^l(X,E)=h^s(X,E)=0, \tag{3.2.20}$$

for all $X \in \Gamma(TM)$ and $E \in \Gamma(\mathrm{Rad}\,TM)$.

Theorem 3.2.5. *Let $(M, g, S(TM),\ S(TM^{\perp}))$ be an irrotational proper ascreen QGCR-lightlike submanifold of an indefinite nearly cosymplectic space form $\overline{M}(c)$ of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$. Then, $\overline{c} \leq 0$ or $\overline{c} \geq 0$. Equality holds when $\overline{M}(\overline{c})$ is an indefinite cosymplectic space form.*

Proof. By setting $\overline{Y} = E$, $\overline{Z} = \overline{\phi}E$, $\overline{X} = X$ and $\overline{W} = E$ in (1.2.34), we get

$$\begin{aligned}\overline{R}(X, \overline{\phi}E, E, E) &= \overline{g}((\nabla_X h^l)(E, \overline{\phi}E), E) - \overline{g}((\nabla_E h^l)(X, \overline{\phi}E), E)\\ &\quad + \overline{g}((\nabla_X h^s)(E, \overline{\phi}E), E) - \overline{g}((\nabla_E h^s)(X, \overline{\phi}E), E),\end{aligned} \tag{3.2.21}$$

for any $X \in \Gamma(TM)$ and $E \in \Gamma(\mathrm{Rad}\,TM)$. Then, using the fact that M is irrotational, (3.2.21) reduces to

$$\overline{R}(X, \overline{\phi}E, E, E) = 0, \ \ \forall\, X \in \Gamma(TM). \tag{3.2.22}$$

On the other hand, setting $\overline{Y} = \overline{W} = E$, $\overline{X} = X$ and $\overline{Z} = \overline{\phi}E$ in (1.3.20), we get

$$\begin{aligned}\overline{R}(X, E, \overline{\phi}E, E) &= \overline{g}((\overline{\nabla}_E\overline{\phi})\overline{\phi}E, (\overline{\nabla}_X\overline{\phi})E) - \overline{g}((\overline{\nabla}_E\overline{\phi})E, (\overline{\nabla}_X\overline{\phi})\overline{\phi}E)\\ &\quad - 2\overline{g}((\overline{\nabla}_E\overline{\phi})X, (\overline{\nabla}_E\overline{\phi})\overline{\phi}E) + \overline{g}(\overline{H}E, \overline{\phi}E)\overline{g}(\overline{H}X, E)\\ &\quad - \overline{g}(\overline{H}E, E)\overline{g}(\overline{H}X, \overline{\phi}E) - 2\overline{g}(\overline{H}E, X)\overline{g}(\overline{H}E, \overline{\phi}E)\\ &\quad - \eta(E)\eta(E)\overline{g}(\overline{H}X, \overline{H}\,\overline{\phi}E) + \eta(W)\eta(\overline{\phi}E)\overline{g}(\overline{H}X, \overline{H}E)\\ &\quad + \eta(E)\eta(X)\overline{g}(\overline{H}E, \overline{H}\,\overline{\phi}E) - \eta(X)\eta(\overline{\phi}E)\overline{g}(\overline{H}E, \overline{H}E)\\ &\quad + \overline{c}\{-\eta(E)\eta(X)\overline{g}(\overline{\phi}E, E) + \eta(E)\eta(E)\overline{g}(X, \overline{\phi}E)\\ &\quad + \overline{g}(\overline{\phi}E, X)\overline{g}(\overline{\phi}^2E, E) + \overline{g}(\overline{\phi}^2E, X)\overline{g}(\overline{\phi}E, E)\\ &\quad - 2\overline{g}(\overline{\phi}E, \overline{\phi}E)\overline{g}(X, \overline{\phi}E)\},\end{aligned}$$

for all $X \in \Gamma(TM)$, which on simplifying leads to

$$\begin{aligned}\overline{R}(X, E, \overline{\phi}E, E) &= -3\overline{g}((\overline{\nabla}_E\overline{\phi})\overline{\phi}E, (\overline{\nabla}_E\overline{\phi})X)\\ &\quad - \eta(E)^2\overline{g}(\overline{H}X, \overline{H}\,\overline{\phi}E) + 4\overline{c}\eta(E)^2\overline{g}(X, \overline{\phi}E).\end{aligned} \tag{3.2.23}$$

Now, using (3.2.22) and (3.2.23), we get

$$4\overline{c}\eta(E)^2\overline{g}(X, \overline{\phi}E) = 3\overline{g}((\overline{\nabla}_E\overline{\phi})\overline{\phi}E, (\overline{\nabla}_E\overline{\phi})X) + \eta(E)^2\overline{g}(\overline{H}X, \overline{H}\,\overline{\phi}E). \tag{3.2.24}$$

Replacing X with $\overline{\phi}E$ in (3.2.24) and the using (1.3.17) of Lemma 1.3.3 to the resulting equation gives

$$\overline{c}\eta(E)^2\overline{g}(\overline{\phi}E,\overline{\phi}E) = \eta(E)^2\overline{g}(\overline{H}\,\overline{\phi}E,\overline{H}\,\overline{\phi}E). \tag{3.2.25}$$

Since M is ascreen QGCR-lightlike submanifold, there exist $E \in \Gamma(D_2)$ such that $\eta(E) = b \neq 0$, and thus (3.2.25) simplifies to

$$\overline{c} = -\frac{1}{b^2}\overline{g}(\overline{H}E,\overline{H}E) = \frac{1}{b^2}d\eta(E,\overline{H}E). \tag{3.2.26}$$

We observe that $\overline{c} = 0$ if either $d\eta = 0$ (i.e., $\overline{M}(c)$ is cosymplectic space form [4]) or $\overline{H}E$ is a null vector field. The second case implies that $\overline{H}E$ belongs to $\mathrm{Rad}\,TM$ or $l\mathrm{tr}(TM)$. If $\overline{H}E \in \Gamma(\mathrm{Rad}\,TM)$, then there exists a non zero smooth function κ' such that $\overline{H}E = \kappa' E$, for some arbitrary $E \in \Gamma(\mathrm{Rad}\,TM)$. Taking the $\overline{g}$-product of $\overline{H}E = \kappa' E$ with ξ leads to $0 = \kappa'\eta(E)$, from which $\eta(E) = 0$. Since M is ascreen QGCR-lightlike submanifold, then, there is $E \in \Gamma(D_2)$ such that $\eta(E) \neq 0$, hence a contradiction. Similar reasoning can be applied if $\overline{H}E \in \Gamma(l\mathrm{tr}(TM))$. Therefore, $\overline{c} = 0$ only if $\overline{H}E = 0$ (i.e., $d\eta = 0$) which occurs when $\overline{M}(c)$ is cosymplectic space form [4]. It turns out that $\overline{c} \leq 0$ or $\overline{c} \geq 0$ depending on whether $\overline{H}E$ is space-like or time-like vector field respectively. □

Corollary 3.2.6. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an irrotational proper ascreen QGCR-lightlike submanifold of an indefinite cosymplectic space form $\overline{M}(\overline{c})$ of pointwise constant $\overline{\phi}$-sectional curvature $\overline{c}$. Then, $\overline{c} = 0$.*

It is easy to see from (3.2.19) that $h^l(X,E) = h^s(X,E) = 0$ and hence M given in Example 3.2.2 is an irrotational ascreen QGCR-lighlike submanifold of an indefinite cosymplectic space form $\overline{M}(\overline{c})$. As is proved in that example $\overline{c} = 0$.

3.3 Mixed geodesic ascreen QGCR-lightlike submanifolds

Definition 3.3.1. A QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ is called mixed geodesic QGCR-lightlike submanifold if its second fundamental form, h, satisfies $h(X,Y) = 0$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(\widehat{D})$.

We will need the following lemma in the next theorem.

Lemma 3.3.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be any 3-lightlike proper ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. Then,*

$$2\eta(E)\eta(N) = 1,$$

for any $E \in \Gamma(D_2)$ *and* $N \in \Gamma(\mathscr{L})$.

Proof. The proof follows from straightforward calculations using $\overline{g}(\xi,\xi) = 1$ and $\xi = \eta(N)E + \eta(E)N$. $\square$

Theorem 3.3.3. *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a 3-lightlike proper ascreen QGCR-lightlike submanifold of an indefinite nearly* α*-Sasakian manifold* $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$*. Then, M is mixed geodesic if and only if* $h^s_\beta(X,Y) = 0$ *and* "**on**"$A^*_{E_i}X = 0$*, for all* $X \in \Gamma(D)$*,* $Y \in \Gamma(\widehat{D})$*,* $W_\beta \in \Gamma(S(TM^{\perp}))$ *and* $E_i \in \Gamma(\mathrm{Rad}\,TM)$.

Proof. By the defintion of ascreen QGCR-lightlike submanifold, M is mixed geodesic if

$$\overline{g}(h(X,Y), W_\beta) = \overline{g}(h(X,Y), E_i) = 0, \tag{3.3.1}$$

for all $X \in \Gamma(D)$, $Y \in \Gamma(\widehat{D})$, $W_\beta \in \Gamma(S(TM^{\perp}))$ and $E_i \in \Gamma(\mathrm{Rad}\,TM)$. Now, by virtue of (1.2.22) and the first equation of (3.3.1), we have

$$0 = \overline{g}(h(X,Y), W_\beta) = \varepsilon_\beta h^s_\beta(X,Y),$$

from which $h^s_\beta(X,Y) = 0$, since $\varepsilon_\beta \neq 0$. On the other hand, using the second equation of (3.3.1), (1.2.15) and (1.2.19) we derive

$$\overline{g}(h(X,Y), E_i) = \overline{g}(\overline{\nabla}_X Y, E_i) = -\overline{g}(Y, \overline{\nabla}_X E_i) = g(Y, A^*_{E_i}X) = 0. \tag{3.3.2}$$

Since $D = D_0 \perp D_1$ and $\widehat{D} = \{D_2 \perp \overline{\phi}D_2\} \oplus \overline{D}$, we observe that $A^*_{E_i}X \notin \Gamma(\overline{\phi}D_2)$ or $\overline{\phi}\mathscr{L}$. In fact, let suppose that $A^*_{E_i}X \notin \Gamma(\overline{\phi}D_2)$, then there exist a non-vanishing smooth function ℓ such that $A^*_{E_i}X = \ell\overline{\phi}E$, for $E \in \Gamma(D_2)$. Thus,

$$0 = g(Y, A^*_{E_i}X) = \ell g(Y, \overline{\phi}E), \;\; \forall\, Y \in \Gamma(\widehat{D}). \tag{3.3.3}$$

Taking $Y = \overline{\phi}N$ in (3.3.3), where $N \in \Gamma(\mathscr{L})$ and using Lemma 3.3.2, we have

$$0 = g(Y, A^*_{E_i}X) = \ell g(\overline{\phi}N, \overline{\phi}E) = \ell(1 - \eta(E)\eta(N)) = \frac{1}{2}\ell,$$

which is a contradiction, since $\ell \neq 0$. Hence $A^*_{E_i}X \notin \Gamma(\overline{\phi}D_2 \oplus \overline{\phi}\mathscr{L})$. Moreover, $A^*_{E_i}X \notin \Gamma(\overline{\phi}\mathscr{S})$ since if $A^*_{E_i}X \in \Gamma(\overline{\phi}\mathscr{S})$, then there is a non-vanishing smooth function ω such that $A^*_{E_i}X = \omega\overline{\phi}W_\beta$. Taking the $\overline{g}$-product of this equation with respect to $Y = \overline{\phi}W_\beta$ and using the fact that $\eta(W_\beta) = 0$, we get

$$0 = g(Y, A^*_{E_i}X) = \omega\overline{g}(\overline{\phi}W_\beta, \overline{\phi}W_\beta) = \omega\overline{g}(W_\beta, W_\beta) = \omega\varepsilon_\beta,$$

which is a contradiction, since $\varepsilon_\beta \neq 0$ and $\omega \neq 0$. Hence, $A^*_{E_i}X \notin \Gamma(\{\overline{\phi}D_2 \oplus \overline{\phi}\mathscr{L}\} \perp \overline{\phi}\mathscr{S})$, which implies that $A^*_{E_i}X \in \Gamma(D_0)$. Since $A^*_{E_i}X \in \Gamma(D_0)$, then the non-degeneracy of D_0 implies that there exist some $Z \in \Gamma(D_0)$ such that $g(A^*_{E_i}X, Z) \neq 0$. But using (1.2.19) and (1.2.15), together with the fact that M is mixed geodesic we derive

$$g(A^*_{E_i}X, Z) = -g(\nabla_X E_i, Z) = \overline{g}(E_i, \overline{\nabla}_X Z) = \overline{g}(E_i, \nabla_X Z) = 0, \tag{3.3.4}$$

which is a contradiction. Thus $A^*_{E_i}X \notin \Gamma(\{\overline{\phi}D_2 \oplus \overline{\phi}\mathscr{L}\} \perp \overline{\phi}\mathscr{S} \perp D_0)$, i.e., $A^*_{E_i}X = 0$. The converse is obvious. □

Corollary 3.3.4. *Let $(M, g, S(TM), S(TM^\perp))$ be a proper ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. Then, if M is mixed geodesic then $h^l_i(X, E_i) = 0$ and $\varphi_{\beta i}(X) = 0$, for all $X \in \Gamma(D)$ and $E_i \in \Gamma(D_2)$.*

Definition 3.3.5. A QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ is called D-geodesic QGCR-lightlike submanifold if its second fundamental form h satisfies

$$h(X, Y) = 0,$$

for any $X, Y \in \Gamma(D)$.

Since M is ascreen QGCR-lightlike submanifold, we have $\overline{g}(X, \xi) = 0$ for all $X \in \Gamma(D)$. Applying $\overline{\nabla}_Y$ to $\overline{g}(X, \xi) = 0$ we get

$$\eta(\overline{\nabla}_Y X) = -\overline{g}(X, \overline{\nabla}_Y \xi). \tag{3.3.5}$$

Interchanging X and Y in (3.3.5), and then adding the resulting equation to (3.3.5), gives

$$\eta(\overline{\nabla}_X Y) + \eta(\overline{\nabla}_Y X) = \alpha\{\overline{g}(Y, \overline{\phi}X) + \overline{g}(X, \overline{\phi}Y)\} + \overline{g}(Y, \overline{H}X) + \overline{g}(X, \overline{H}Y) = 0. \tag{3.3.6}$$

Theorem 3.3.6. *Let $(M, g, S(TM), S(TM^\perp))$ be a proper ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. Then, M is D-geodesic if and only if $\overline{\phi}h^l(X, \overline{\phi}E)$ and $\overline{\phi}h^s(X, \overline{\phi}W)$ respectively have no components along $l\mathrm{tr}(TM)$ and $S(TM^\perp)$, while both $\nabla_X\overline{\phi}E$ and $\nabla_X\overline{\phi}W \notin \Gamma(D_0)$ for all $X \in \Gamma(D)$, $E \in \Gamma(\mathrm{Rad}\,TM)$ and $W \in \Gamma(\mathscr{S})$.*

Proof. By the defintion of an ascreen QGCR-lightlike submanifold, M is D geodesic if and only if $\overline{g}(h(X, Y), E) = \overline{g}(h(X, Y), W) = 0$, for all $X, Y \in \Gamma(D)$, $W_\beta \in \Gamma(S(TM^\perp))$ and $E \in \Gamma(\mathrm{Rad}\,TM)$.

Using (1.2.15) and (1.3.2), we derive

$$\begin{aligned}\overline{g}(h(X,Y),E) &= \overline{g}(\overline{\nabla}_X Y,E)\\ &= \overline{g}(\overline{\phi}\,\overline{\nabla}_X Y,\overline{\phi}E) - \overline{g}(Y,\overline{\nabla}_X \xi)\overline{g}(E,\xi),\end{aligned}$$

from which when we apply (1.3.10) we get

$$\overline{g}(h(X,Y),E) = \overline{g}(\overline{\phi}\,\overline{\nabla}_X Y,\overline{\phi}E) + \alpha\overline{g}(Y,\overline{\phi}X)\overline{g}(E,\xi) + \overline{g}(Y,\overline{H}X)\overline{g}(E,\xi). \tag{3.3.7}$$

Interchanging X and Y in (3.3.7) and considering the fact that h is symmetric we get

$$\overline{g}(h(X,Y),E) = \overline{g}(\overline{\phi}\,\overline{\nabla}_Y X,\overline{\phi}E) + \alpha\overline{g}(X,\overline{\phi}Y)\overline{g}(E,\xi) + \overline{g}(X,\overline{H}Y)\overline{g}(E,\xi). \tag{3.3.8}$$

Summing (3.3.7) and (3.3.8) and then applying (3.3.6), we have

$$2\overline{g}(h(X,Y),E) = \overline{g}(\overline{\phi}\,\overline{\nabla}_X Y,\overline{\phi}E) + \overline{g}(\overline{\phi}\,\overline{\nabla}_Y X,\overline{\phi}E). \tag{3.3.9}$$

Now, applying condition in (1.3.4) to (3.3.9), leads to

$$2\overline{g}(h(X,Y),E) = \overline{g}(\overline{\nabla}_X \overline{\phi}Y,\overline{\phi}E) + \overline{g}(\overline{\nabla}_Y \overline{\phi}X,\overline{\phi}E). \tag{3.3.10}$$

From (3.3.10) and (1.2.15) we derive

$$\begin{aligned}2\overline{g}(h(X,Y),E) &= \overline{g}(\overline{\nabla}_X \overline{\phi}Y,\overline{\phi}E) + \overline{g}(\overline{\nabla}_Y \overline{\phi}X,\overline{\phi}E)\\ &= -\overline{g}(\overline{\phi}Y,\nabla_X \overline{\phi}E) - \overline{g}(\overline{\phi}X,\nabla_Y \overline{\phi}E)\\ &\quad - \overline{g}(\overline{\phi}Y,h(X,\overline{\phi}E)) - \overline{g}(\overline{\phi}X,h(Y,\overline{\phi}E)).\end{aligned} \tag{3.3.11}$$

If we let $X,Y \in \Gamma(D_1)$ in (3.3.11), we obtain

$$2\overline{g}(h(X,Y),E) = \overline{g}(Y,\overline{\phi}h(X,\overline{\phi}E)) + \overline{g}(X,\overline{\phi}h(Y,\overline{\phi}E)). \tag{3.3.12}$$

On the other hand, when $X,Y \in \Gamma(D_0)$, we get

$$2\overline{g}(h(X,Y),E) = -\overline{g}(\overline{\phi}Y,\nabla_X \overline{\phi}E) - \overline{g}(\overline{\phi}X,\nabla_Y \overline{\phi}E). \tag{3.3.13}$$

It is easy to see from (3.3.12) and (3.3.13) that if $\overline{\phi}h(X,\overline{\phi}E) \notin \Gamma(l\mathrm{tr}(TM))$ and $\nabla_X \overline{\phi}E \notin \Gamma(D_0)$, then $\overline{g}(h(X,Y),E) = 0$. The other assertions follow in the same way. The converse is obvious. □

Corollary 3.3.7. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a proper ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. If M is D-geodesic then $\nabla^*_X \overline{\phi} E$, $\nabla^*_X \overline{\phi} W \notin \Gamma(D_0)$, for all $X \in \Gamma(D)$, $E \in \Gamma(D_2)$ and $W \in \Gamma(\mathscr{S})$.*

Corollary 3.3.8. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a proper ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. If M is D-geodesic then D defines a totally geodesic folliation in M.*

3.4 Minimal ascreen QGCR-lightlike submanifolds

Consider a quasi-orthonormal frame a long $T\overline{M}$ given by

$$\{E_1, \cdots, E_r, X_1, \cdots, X_m, W_1, \cdots, W_n, N_1, \cdots, N_r\}, \tag{3.4.1}$$

such that $\{E_1, \cdots, E_q, X_1, \cdots, X_m\} \in TM$. Let suppose that $\{E_1, \cdots, E_{2p}\}$, $\{E_{2p+1}, \cdots, E_r\}$ and $\{X_1, \cdots, X_{2l}\}$ are the bases of D_1, D_2 and D_0 respectively. Further, let $\{W_{r+1}, \cdots, W_k\}$ and $\{N_{2p+1}, \cdots, N_r\}$, respectively be the bases of $\mathscr{S}$ and $\mathscr{L}$.

Definition 3.4.1 ([16]). A lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be *ascreen* if the structure vector field, ξ, belongs to $\mathrm{Rad}\, TM \oplus l\mathrm{tr}(TM)$.

The following result for ascreen QGCR-lightlike submanifolds is well-known (see Lemma 3.6 and Theorem 3.7 of [22]).

Theorem 3.4.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $\overline{M}$, then $\xi \in \Gamma(D_2 \oplus \mathscr{L})$. Further, if M is a 3-lightlike QGCR submanifold, then M is ascreen lightlike submanifold if and only if $\overline{\phi}\mathscr{L} = \overline{\phi} D_2$.*

From (3.4.1) and (2.1.1), we can write the generalized structure vector field of an ascreen QGCR-lightlike submanifold as

$$\xi = \sum_{i=2p+1}^{r} a_i E_i + \sum_{i=2p+1}^{r} b_i N_i, \tag{3.4.2}$$

where $a_i = \overline{g}(N_i, \xi)$ and $b_i = \overline{g}(E_i, \xi)$.

Now, using (3.4.1) and Theorem 3.4.2 above, we deduce the following for an r-lightlike ascreen QGCR-submanifold (M, g).

Proposition 3.4.3. *Let M be a proper r-lightlike ascreen QGCR submanifold, where $r \geq 3$, of an indefinite almost contact manifold $\overline{M}$. Then, there exist at least one pair $\{E_u, N_u\} \subset \mathscr{L} \oplus D_2$ and a corresponding non-vanishing real valued smooth function σ_u, where $u \in \{2p+1, \cdots, r\}$, such that $\overline{\phi} N_u = \sigma_u \overline{\phi} E_u$ and $\dim(\overline{\phi}\mathscr{L} \oplus \overline{\phi} D_2) \geq 1$. Equality occurs when $r = 3$.*

Proof. The proof follows from Theorem 3.4.2 above. □

As an example, we construct a 4-lightlike ascreen QGCR submanifold. Let us consider the case $\alpha = 0$ and $\overline{H} = 0$. That is, $\overline{M} = (\mathbb{R}_q^{2m+1}, \overline{\phi}_0, \xi, \eta, \overline{g})$ is an indefinite cosymplectic manifold with the usual cosympletic structure given in (1.3.5).

Example 3.4.4. Let $\overline{M} = (\mathbb{R}_6^{15}, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,-,-,+,+,+,+,-,-,-,+,+,+,+,+)$ with respect to the canonical basis

$$(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7,\ \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial z).$$

Let (M, g) be a submanifold of $\overline{M}$ given by

$$y^1 = -x^4,\ y^2 = x^5,\ y^3 = \sqrt{z^2 - (x^3)^2},\ y^4 = x^1,\ y^6 = x^6,\ x^3, y^3 > 0.$$

By direct calculations, one can easily check that the vector fields

$$\begin{aligned}
&E_1 = \partial x_4 + \partial y_1, \quad E_2 = \partial x_1 - \partial y_4, \quad E_3 = \partial x_5 + \partial y_2, \\
&E_4 = x^3 \partial x_3 + y^3 \partial y_3 + z \partial z, \quad X_1 = \partial x_6 + \partial y_6, \quad X_2 = \partial x_2 - \partial y_5, \\
&X_3 = y^3 \partial x_3 - x^3 \partial y_3, \quad X_4 = -\partial x_2 - \partial y_5, \quad X_5 = \partial y_7, \quad X_6 = \partial x_7,
\end{aligned}$$

form a local frame of TM. With reference to the above frame, we see that $\mathrm{Rad}\,TM$ is spanned by $\{E_1, E_2, E_3, E_4\}$, and therefore M is a 4-lightlike submanifold. Further more, $\overline{\phi}_0 E_1 = E_2$, therefore we set $D_1 = \mathrm{span}\{E_1, E_2\}$. Notice that $\overline{\phi}_0 E_3 = X_2$ and $\overline{\phi}_0 E_4 = X_3$ thus, $D_2 = \mathrm{span}\{E_3, E_4\}$. Also, $\overline{\phi}_0 X_5 = X_6$, so we set $D_0 = \mathrm{span}\{X_5, X_6\}$. Further, by following direct calculations, we have

$$\begin{aligned}
&N_1 = \frac{1}{2}(\partial x_4 - \partial y_1), \quad N_2 = \frac{1}{2}(-\partial x_1 - \partial y_4), \quad N_3 = \frac{1}{2}(\partial x_5 - \partial y_2) \\
&N_4 = \frac{1}{2z^2}(-x^3 \partial x_3 - y^3 \partial y_3 + z \partial z), \quad W = \partial x_6 - \partial y_6.
\end{aligned}$$

Note that $l\mathrm{tr}(TM) = \mathrm{span}\{N_1, N_2, N_3, N_4\}$ and $\mathscr{S} = \mathrm{span}\{W\}$. It is easy to see that $\overline{\phi}_0 N_2 = -N_1$ and $\overline{\phi}_0 N_3 = X_4$. Notice $\overline{\phi}_0 N_4 = -\frac{1}{2z^2} X_3 = -\frac{1}{2z^2} \overline{\phi}_0 E_4$ and, hence, $\sigma_4 = -\frac{1}{2z^2}$ (see

Proposition 3.4.3). Therefore, $\mathscr{L} = \mathrm{span}\{N_3, N_4\}$. Also, $\overline{\phi}_0 W = -X_1$ and hence $\mathscr{S} = \mathrm{span}\{W\}$. Observe that $\overline{\phi}_0\mathscr{L} \oplus \overline{\phi}_0 D_2 = \mathrm{span}\{X_2, X_3, X_4\}$ and therefore $\dim(\overline{\phi}_0\mathscr{L} \oplus \overline{\phi}_0 D_2) = 3$. Applying $\overline{\phi}_0$ to (3.4.2) and substituting the corresponding $\overline{\phi}_0 E_i$s and $\overline{\phi}_0 N_i$s for $i = 3,4$ we obtain $a_3 + b_3 = 0$, $a_3 - b_3 = 0$ and $2z^2 a_4 = b_4$. Finally, we get $\xi = \frac{1}{2z} E_4 + z N_4$. Since $\overline{\phi}_0 \xi = 0$ and $\overline{g}(\xi, \xi) = 1$, we see that (M, g) is a 4-lightlike ascreen QGCR submanifold of $\overline{M}$ satisfying the hypothesis of Proposition 3.4.3.

Next, we adapt the definition of minimal lightlike submanifolds given by [10].

Definition 3.4.5. A lightlike submanifold $(M, g, S(TM))$ of a semi-Riemannian $(\overline{M}, \overline{g})$ is called minimal if;

1. $h^s = 0$ on $\mathrm{Rad}\, TM$ and,
2. $\mathrm{trace}(h) = 0$, where trace is writen with respect to g restricted to $S(TM)$.

It is well-known that the Definition 3.4.5 is independent of the choice of the screen distribution $S(TM)$ [10].

Now, we construct a minimal ascreen QGCR-lightlike submanifold, which is note totally geodesic, of a nearly α-Sasakian manifold with $\alpha = 0$ and $\overline{H} = 0$ (i.e., the ambient space is a cosymplectic manifold).

Example 3.4.6. Let $\overline{M} = (\mathbb{R}^{13}_4, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,-,+,+,+,+,-,-,+,+,+,+,+)$ with respect to the canonical basis

$$(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z).$$

Let (M, g) be a submanifold of $\overline{M}$ given by

$$\begin{aligned} &x^1 = \omega^1,\ x^2 = \omega^2,\ x^3 = \omega^3,\ x^4 = \omega^4,\ x^5 = \cos\omega^5 \cosh\omega^6, \\ &x^6 = \sin\omega^5 \cosh\omega^6,\ y^1 = -\omega^4,\ y^2 = \sqrt{2}\omega^8 - \omega^2,\ \ y^3 = \omega^7, \\ &y^4 = \omega^1,\ y^5 = \cos\omega^5 \sinh\omega^6,\ y^6 = \sin\omega^5 \sinh\omega^6, \quad z = \omega^8. \end{aligned}$$

By direct calculations, we can easily check that the vector fields

$$\begin{aligned} &E_1 = \partial x_4 + \partial y_1, \quad E_2 = \partial x_1 - \partial y_4, \\ &E_3 = \partial x_2 + \partial y_2 + \sqrt{2}\partial z, \quad X_1 = -x^6 \partial x_5 + x^5 \partial x_6 - y^6 \partial y_5 + y^5 \partial y_6 \\ &X_2 = -\partial x_2 + \partial y_2, \quad X_3 = y^5 \partial x_5 + y^6 \partial x_6 + x^5 \partial y_5 + x^6 \partial y_6, \\ &X_4 = \partial y_3,\ X_5 = \partial x_3, \end{aligned}$$

form a local frame of TM. From the above frame, we can see that $\mathrm{Rad}\,TM$ is spanned by $\{E_1, E_2, E_3\}$, and therefore, M is a 3-lightlike submanifold. Further, $\overline{\phi}_0 E_1 = E_2$, therefore we set $D_1 = \mathrm{span}\{E_1, E_2\}$. Also $\overline{\phi}_0 E_3 = -X_2$ and thus $D_2 = \mathrm{span}\{E_3\}$. It is easy to see that $\overline{\phi}_0 X_4 = X_5$, so we set $D_0 = \mathrm{span}\{X_4, X_5\}$. On the other hand, following direct calculations, we have

$$\begin{aligned}
N_1 &= \frac{1}{2}(\partial x_4 - \partial y_1),\ N_2 = \frac{1}{2}(-\partial x_1 - \partial y_4),\\
N_3 &= \frac{1}{4}(-\partial x_2 - \partial y_2 + \sqrt{2}\partial z),\ W_1 = -y^6\partial x_5 + y^5\partial x_6 + x^6\partial y_5 - x^5\partial y_6,\\
W_2 &= x^5\partial x_5 + x^6\partial x_6 - y^5\partial y_5 - y^6\partial y_6,
\end{aligned}$$

from which $l\mathrm{tr}(TM) = \mathrm{span}\{N_1, N_2, N_3\}$ and $S(TM^{\perp}) = \mathrm{span}\{W_1, W_2\}$. Clearly, $\overline{\phi}_0 N_2 = -N_1$. Further, $\overline{\phi}_0 N_3 = \frac{1}{4}X_2$ and thus $\mathscr{L} = \mathrm{span}\{N_3\}$. Notice that $\overline{\phi}_0 N_3 = -\frac{1}{4}\overline{\phi}_0 E_3$, which implies that $\sigma_3 = -\frac{1}{4}$ and therefore, $\overline{\phi}_0\mathscr{L} = \overline{\phi}_0 D_2$. Also, $\overline{\phi}_0 W_1 = -X_1$ and $\overline{\phi}_0 W_2 = -X_3$. Therefore $\mathscr{S} = \mathrm{span}\{W_1, W_2\}$. Now, we calculate ξ as follows: Using (3.4.2) we have $\xi = a_3E_3 + b_3N_3$. Applying $\overline{\phi}_0$ to this equation we obtain $a_3\overline{\phi}_0E_3 + b_3\overline{\phi}_0N_3 = 0$. Now, substituting for $\overline{\phi}_0E_3$ and $\overline{\phi}_0N_3$ in this equation we get $4a_3 = b_3$, from which we get $\xi = \frac{1}{2\sqrt{2}}E_3 + \sqrt{2}N_3$. Since $\overline{\phi}_0\xi = 0$ and $\overline{g}(\xi, \xi) = 1$, we see that (M, g) is a proper ascreen QGCR-lightlike submanifold of $\overline{M}$. Finally, we verify the minimality of (M, g). By simple calculations one can verify easily that the following vectors;

$$\begin{aligned}
&\widehat{X}_1 = \frac{1}{\sqrt{\rho}}X_1,\quad \widehat{X}_2 = \frac{1}{\sqrt{2}}X_2,\ \ \widehat{X}_3 = \frac{1}{\sqrt{\rho}}X_3,\quad \widehat{X}_4 = X_4,\ \ \widehat{X}_5 = X_5,\\
&\widehat{W}_1 = \frac{1}{\sqrt{\rho}}W_1,\quad \widehat{W}_2 = \frac{1}{\sqrt{\rho}}W_2,\quad \text{where}\ \ \rho := \cosh 2\omega^6,
\end{aligned}$$

are unit vector fields. Moreover, $\varepsilon_2 = g(\widehat{X}_2, \widehat{X}_2) = -1$, $\varepsilon_i = g(\widehat{X}_i, \widehat{X}_i) = 1$, for $i = 1,3,4,5$ and $\varepsilon_\beta = \overline{g}(\widehat{W}_\beta, \widehat{W}_\beta) = 1$, for $\beta = 1,2$. Now, applying (1.2.15) and Koszul's formula (see [7]) one gets $h(E_1,E_1) = h(E_2,E_2) = h(E_3,E_3) = h(\widehat{X}_2,\widehat{X}_2) = 0$, $h(\widehat{X}_4,\widehat{X}_4) = h(\widehat{X}_5,\widehat{X}_5) = 0$, $h^l(\widehat{X}_1,\widehat{X}_1) = h^l(\widehat{X}_3,\widehat{X}_3) = 0$, $h^s(\widehat{X}_1,\widehat{X}_1) = -\frac{1}{\rho\sqrt{\rho}}\widehat{W}_2$ and $h^s(\widehat{X}_3,\widehat{X}_3) = \frac{1}{\rho\sqrt{\rho}}\widehat{W}_2$. Hence, M is not a totally geodesic ascreen QGCR-lightlike submanifold. Also, we have $\mathrm{trace}(h)|_{S(TM)} = \varepsilon_1 h^s(\widehat{X}_1,\widehat{X}_1) + \varepsilon_2 h^s(\widehat{X}_3,\widehat{X}_3) = 0$. Therefore, M is a minimal proper ascreen QGCR-lightlike submanifold of $\overline{M}$.

Definition 3.4.7 ([9])**.** A lightlike submanifold M of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called irrotational if $\overline{\nabla}_X E \in \Gamma(TM)$, for any $E \in \Gamma(\mathrm{Rad}\,TM))$ and $X \in \Gamma(TM)$. Equivalently,

M is irrotational if

$$h^l(X,E) = h^s(X,E) = 0, \; \forall\, X \in \Gamma(TM), \; E \in \Gamma(\mathrm{Rad}\, TM). \tag{3.4.3}$$

Theorem 3.4.8. *Let (M,g) be an irrotational ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{g})$. If ∇ is a metric connection, then M is minimal if* $\mathrm{trace}(A_{W_\beta})|_{S(TM)} = 0$.

Proof. First, we notice that when M is irrotational then (3.4.3) implies that $h^s = 0$ on $\mathrm{Rad}\, TM$. Thus, condition (i) of Definition 3.4.5 is satisfied. Now using Definition 2.2.1 we can see that the screen distribution $S(TM)$ is generally spanned by

$$\{X_1, \cdots, X_{2l}, \overline{\phi}E_{2p+1}, \cdots, \overline{\phi}E_r, \overline{\phi}N_{2p+1}, \cdots, \overline{\phi}N_r, \overline{\phi}W_{r+1}, \cdots, \overline{\phi}W_k\}. \tag{3.4.4}$$

Since M is ascreen QGCR-lightlike submanifold, the dimension of the frame in (3.4.4) is lower than that of a comparable GCR-lightlike subamanifold due to existence of some $u \in \{2p+1, \cdots, r\}$ and non-vanishing smooth function(s) σ_u such that $\overline{\phi}N_u = \sigma_u \overline{\phi}E_u$ (see Proposition 3.4.3 above). Furthermore, the vectors $\overline{\phi}E_u$ and $\overline{\phi}N_u$ are non-null, since $g(\overline{\phi}E_u, \overline{\phi}E_u) = -a_u b_u \neq 0$ and $g(\overline{\phi}N_u, \overline{\phi}N_u) = -a_u b_u \neq 0$. Thus, by setting $Z_u = \overline{\phi}N_u = \sigma_u \overline{\phi}E_u$ we have

$$\begin{aligned} \mathrm{trace}(h)|_{S(TM)} &= \sum_{t=1}^{2l} \varepsilon_t h(X_t, X_t) + \sum_{j=2p+1}^{\kappa} h(\overline{\phi}E_j, \overline{\phi}E_j) \\ &\sum_{j=2p+1}^{\kappa} h(\overline{\phi}N_j, \overline{\phi}N_j) + \sum_{u=\kappa+1}^{r} \varepsilon_u h(Z_u, Z_u) + \sum_{d=r+1}^{k} \varepsilon_d h(\overline{\phi}W_d, \overline{\phi}W_d). \end{aligned} \tag{3.4.5}$$

Replacing Z with E_i in (1.2.23) we derive

$$(\nabla_X g)(Y, E_i) = \sum_{i=1}^{r} h_i^l(X,Y)\lambda_i(E_i) = \overline{g}(h^l(X,Y), E_i), \tag{3.4.6}$$

for any $X, Y \in \Gamma(S(TM))$. Then using (3.4.6) and the assumption ∇ is a metric connection we get

$$\begin{aligned}\operatorname{trace}(h)|_{S(TM)} &= \sum_{t=1}^{2l} \frac{\varepsilon_t}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta \overline{g}(h^s(X_t, X_t), W_\beta) W_\beta \\ &+ \sum_{j=2p+1}^{\kappa} \frac{1}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta \overline{g}(h^s(\overline{\phi} E_j, \overline{\phi} E_j), W_\beta) W_\beta \\ &+ \sum_{j=2p+1}^{\kappa} \frac{1}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta \overline{g}(h^s(\overline{\phi} N_j, \overline{\phi} N_j), W_\beta) W_\beta \\ &+ \sum_{d=r+1}^{k} \frac{\varepsilon_d}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta \overline{g}(h^s(\overline{\phi} W_d, \overline{\phi} W_d), W_\beta) W_\beta \\ &+ \sum_{u=\kappa+1}^{r} \frac{\varepsilon_u}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta \overline{g}(h^s(Z_u, Z_u), W_\beta) W_\beta. \end{aligned} \tag{3.4.7}$$

Then using (1.2.22) we derive

$$\overline{g}(h^s(X,Y), W_\beta) = \varepsilon_\beta h^s_\beta(X,Y) = g(A_{W_\beta} X, Y), \tag{3.4.8}$$

for any $X, Y \in \Gamma(S(TM))$. Finally, replacing (3.4.8) in (3.4.7) we get

$$\begin{aligned}\operatorname{trace}(h)|_{S(TM)} &= \sum_{t=1}^{2l} \frac{\varepsilon_t}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta g(A_{W_\beta} X_t, X_t) W_\beta \\ &+ \sum_{j=2p+1}^{\kappa} \frac{1}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta g(A_{W_\beta} \overline{\phi} E_j, \overline{\phi} E_j) W_\beta \\ &+ \sum_{j=2p+1}^{\kappa} \frac{1}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta g(A_{W_\beta} \overline{\phi} N_j, \overline{\phi} N_j) W_\beta \\ &+ \sum_{d=r+1}^{k} \frac{\varepsilon_d}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta g(A_{W_\beta} \overline{\phi} W_d, \overline{\phi} W_d) W_\beta \\ &+ \sum_{u=\kappa+1}^{r} \frac{\varepsilon_u}{n} \sum_{\beta=r+1}^{n} \varepsilon_\beta \overline{g}(A_{W_\beta} Z_u, Z_u) W_\beta, \end{aligned} \tag{3.4.9}$$

from which our assertion follows. Hence the proof. □

Example 3.4.9. Let (M, g) be a submanifold of $\mathbb{R}^{2m+1}_4$ given in Example 4.1.2. We have shown that $h^l(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$. Hence, from (1.2.23) we can see that the induced connection ∇ is a metric connection. Further, we have also seen that $h(X,Y) = 0$

for all $X,Y \in \Gamma(\mathrm{Rad}\,TM)$ and thus, $h^s(X,Y)=0$ on $\mathrm{Rad}\,TM$ and also $h^s(X,E)=0$ for all $X \in \Gamma(TM)$. Therefore, M is an irrotational minimal ascreen QGCR-lightlike submanifold of $\mathbb{R}_4^{2m+1}$ with $\mathrm{trace}(A_{W_\beta})|_{S(TM)}=0$ and thus satisfying the above theorem.

Corollary 3.4.10. *Let (M,g) be a totally umbilical irrotational ascreen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{g})$. If ∇ is a metric connection, then M is minimal if the mean curvature vectors $\mathscr{H}^s=0$.*

Chapter 4

Co-screen QGCR-lightlike submanifolds of nearly α-Sasakian manifod

4.1 Introduction

In this chapter, we study a special class of QGCR-lightlike submanifolds of indefinite nearly α-Sasakian manifolds, called *co-screen QGCR-lightlike submanifold*. We discuss the integrability of distributions and also establish the necessary conditions for such distributions to be nearly parallel and nearly auto-parallel.

From the proof of Theorem 2.3.5 we can see that when $\alpha \neq 0$ and one assumes that ξ is normal, then $a_k = \eta(N_k) = 0$. This makes ξ to be in the co-screen distribution. Thus, we will say that a QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold, with $\alpha \neq 0$, is co-screen QGCR-lightlike submanifold if $\xi \in \Gamma(S(TM^{\perp}))$.

From Definition 2.2.1 of QGCR-lightlike submanifold we notice that if M is a co-screen QGCR-lightlike submanifold then the direct sum in (2.2.3) reduces to the orthogonal sum $\overline{\phi}\,\overline{D} = \mathscr{S} \perp \mathscr{L}$. Note that this condition is also satisfied by GCR-lightlike submanifols though $\xi \in \Gamma(S(TM))$. In the case of co-screen QGCR-lightlike submanifolds we have $\xi \in \Gamma(S(TM^{\perp}))$ and therefore, we have the following definition.

Definition 4.1.1. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. We say that M is co-screen QGCR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) there exist two distributions D_1 and D_2 of $\mathrm{Rad}\,TM$ such that

$$\mathrm{Rad}\,TM = D_1 \oplus D_2, \quad \overline{\phi} D_1 = D_1, \quad \overline{\phi} D_2 \subset S(TM), \tag{4.1.1}$$

(ii) there exist vector bundles D_0 and $\overline{D}$ over $S(TM)$ such that

$$S(TM) = \{\overline{\phi}D_2 \oplus \overline{D}\} \perp D_0, \tag{4.1.2}$$

$$\text{with} \quad \overline{\phi}D_0 \subseteq D_0, \quad \overline{\phi}\,\overline{D} = \mathscr{S} \perp \mathscr{L}, \tag{4.1.3}$$

where D_0 is a non-degenerate and invariant distribution on M, $\mathscr{L}$ and $\mathscr{S}$ are respectively vector subbundles of $l\mathrm{tr}(TM)$ and $S(TM^{\perp})$.

If $D_1 \neq \{0\}$, $D_0 \neq \{0\}$, $D_2 \neq \{0\}$ and $\mathscr{S} \neq \{0\}$, then M is called a *proper co-screen QGCR lightlike submanifold*. The tangent bundle of M is decomposed as follows;

$$TM = D \oplus \overline{D}, \text{ with } D = \mathrm{Rad}\,TM \perp D_0 \perp \overline{\phi}D_2. \tag{4.1.4}$$

The transversal bundle can also be decomposed as

$$\mathrm{tr}(TM) = \overline{\phi}\,\overline{D} \perp \mathscr{G} \perp \langle \xi \rangle, \tag{4.1.5}$$

where $\overline{\phi}\mathscr{G} = \mathscr{G}$.

It is well known from [9] that GCR-lightlike submanifolds, tangent to the structure vector field ξ, include real hypersurfaces. However, its easy to see that co-screen QGCR-lightlike submanifolds exclude real lightlike hypersurfaces, since in a real lightlike hypersurface $\mathrm{tr}(TM) = l\mathrm{tr}(TM)$, implying that $S(TM^{\perp}) = \{0\}$.

Next, we construct an example of a co-screen QGCR-lightlike submanifold of a special nearly α-Sasakian manifold $\overline{M}$ in which $\overline{H} = 0$ and $\alpha = 1$. More precisely, we take $\overline{M}$ to be a Sasakian manifold.

Example 4.1.2. Let $\overline{M} = (\mathbb{R}^{13}_4, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,-,+,+,+,+,-,-,+,+,+,+,+)$ with respect to the canonical basis

$$(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z).$$

Let (M, g) be a submanifold of $\overline{M}$ given by

$$x^1 = y^4, \; y^1 = -x^4, \; x^2 = y^3 \text{ and } y^5 = (x^5)^{\frac{1}{2}}.$$

By direct calculations, we can easily check that the vector fields

$$\begin{aligned}
&E_1 = \partial x_4 + \partial y_1 + y^4 \partial z, \quad E_2 = \partial x_1 - \partial y_4 + y^1 \partial z,\\
&E_3 = \partial x_2 + \partial y_3 + y^2 \partial z, \quad X_1 = 2y^5 \partial x_5 + \partial y_5 + 2(y^5)^2 \partial z,\\
&X_2 = \partial x_3 - \partial y_2 + y^3 \partial z, \quad X_3 = 2(\partial x_3 + \partial y_2 + y^3 \partial z),\\
&X_4 = 2\partial y_6 \quad \text{and} \quad X_5 = 2(\partial x_6 + y^6 \partial z),
\end{aligned}$$

form a local frame of TM. From this, we can see that $\mathrm{Rad}\,TM$ is spanned by $\{E_1, E_2, E_3\}$, and therefore, M is 3-lightlike. Further, $\overline{\phi}_0 E_1 = E_2$, therefore we set $D_1 = \mathrm{span}\{E_1, E_2\}$. Also $\overline{\phi}_0 E_3 = X_2$ and thus $D_2 = \mathrm{span}\{E_3\}$. It is easy to see that $\overline{\phi}_0 X_4 = X_5$, so we set $D_0 = \mathrm{span}\{X_4, X_5\}$. On the other hand, following direct calculations, we have

$$\begin{aligned}
&N_1 = 2(\partial x_4 - \partial y_1 + y^4 \partial z), \quad N_2 = 2(-\partial x_1 - \partial y_4 + y^1 \partial z),\\
&N_3 = 2(-\partial x_2 + \partial y_3 + y^2 \partial z), \quad W_1 = \partial x_5 - 2y^5 \partial y_5 + y^5 \partial z,\\
&\text{and } W_2 = 2\partial z,
\end{aligned}$$

from which $l\mathrm{tr}(TM) = \mathrm{span}\{N_1, N_2, N_3\}$ and $S(TM^{\perp}) = \mathrm{span}\{W_1, W_2\}$. Clearly, $\overline{\phi}_0 N_2 = -N_1$. Further, $\overline{\phi}_0 N_3 = X_3$ and thus $\mathscr{L} = \mathrm{Span}\{N_3\}$. Also, $\overline{\phi}_0 W_1 = -X_1$ and therefore $\mathscr{S} = \mathrm{span}\{W_1\}$. Clearly, M is a co-screen QGCR-lightlike submanifold of $\overline{M}$.

4.2 Integrability of distributions

Let $(M, g, S(TM), S(TM^{\perp}))$ be a co-screen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold, $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$, and let S and R be the projections of TM on to D and $\overline{D}$ respectively, while F and Q are projections of $\mathrm{tr}(TM)$ on to $\overline{\phi}\,\overline{D}$ and $\mathscr{G}$ respectively. Then,

$$X = SX + RX \quad \text{and} \quad V = FV + QV + \eta(V)\xi, \tag{4.2.1}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}(TM))$.

Applying $\overline{\phi}$ to the two equations of (4.2.1), we respectively derive

$$\overline{\phi} X = \phi_1 X + \varphi_1 X \quad \text{and} \quad \overline{\phi} V = \phi_2 V + \varphi_2 V, \tag{4.2.2}$$

where $\{\phi_1 X, \phi_2 V\}$ and $\{\varphi_1 X, \varphi_2 V\}$ respectively belongs to TM and $\mathrm{tr}(TM)$.

Using the nearly α-Sasakian condition (1.3.4) and equations (4.2.2) and (1.2.15)-(1.2.17), we derive

$$\begin{aligned}
&-A_{\varphi_1 X}Y - A_{\varphi_1 Y}X + \nabla_X \phi_1 Y + \nabla_Y \phi_1 X \\
&+ \nabla^t_X \varphi_1 Y + \nabla^t_Y \varphi_1 X + h(X, \phi_1 Y) + h(Y, \phi_1 X) \\
&- \phi_1 \nabla_X Y - \phi_1 \nabla_X Y - \varphi_1 \nabla_X Y - \varphi_1 \nabla_X Y \\
&- 2\phi_2 h(X,Y) - 2\varphi_2 h(X,Y) - 2\alpha \overline{g}(X,Y)\xi,
\end{aligned} \tag{4.2.3}$$

for all $X, Y \in \Gamma(TM)$. Then, comparing the tangential and transversal components in (4.2.3), we get;

Tangential components;

$$\begin{aligned}
&\nabla_X \phi_1 Y + \nabla_Y \phi_1 X - A_{\varphi_1 X}Y - A_{\varphi_1 Y}X \\
&- \phi_1 \nabla_X Y - \phi_1 \nabla_X Y - 2\phi_2 h(X,Y) = 0.
\end{aligned} \tag{4.2.4}$$

Transversal components;

$$\begin{aligned}
&\nabla^t_X \varphi_1 Y + \nabla^t_Y \varphi_1 X + h(X, \phi_1 Y) + h(Y, \phi_1 X) \\
&-\varphi_1 \nabla_X Y - \varphi_1 \nabla_X Y - 2\varphi_2 h(X,Y) - 2\alpha \overline{g}(X,Y)\xi = 0.
\end{aligned} \tag{4.2.5}$$

Theorem 4.2.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a co-screen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. Then,*

1. *D is integrable if and only if*

$$h(X, \phi_1 Y) + h(Y, \phi_1 X) = 2\varphi_1 \nabla_Y X + 2\varphi_2 h(X,Y) + 2\alpha \overline{g}(X,Y)\xi,$$

for all $X, Y \in \Gamma(D)$.

2. *$\overline{D}$ is integrable if and only if*

$$A_{\varphi_1 X}Y + A_{\varphi_1 Y}X = -2\phi_1 \nabla_Y X - 2\phi_2 h(X,Y),$$

for all $X, Y \in \Gamma(\overline{D})$.

Proof. Using (4.2.4) and (4.2.5), we derive

$$\begin{aligned}
h(X, \phi_1 Y) + h(Y, \phi_1 X) = {}& \varphi_1 [X,Y] + 2\varphi_1 \nabla_Y X + 2\varphi_2 h(X,Y) \\
&+ 2\alpha \overline{g}(X,Y)\xi,
\end{aligned} \tag{4.2.6}$$

for all $X,Y \in \Gamma(D)$ and

$$A_{\varphi_1 X}Y + A_{\varphi_1 Y}X = -\phi_1[X,Y] - 2\phi_1\nabla_Y X - 2\phi_2 h(X,Y), \tag{4.2.7}$$

for all $X,Y \in \Gamma(\overline{D})$. Then, the assertions follows from (4.2.6) and (4.2.7), which completes the proof.

□

4.3 Nearly parallel and nearly auto-parallel distributions

Definition 4.3.1. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a submanifold of a semi-Riemannian manifold $(\overline{M},\overline{g})$ and let ∇ be the connection induced on its tangent bundle. Then a distribution D on M will be called nearly parallel if*

$$\nabla_X Y + \nabla_Y X \in \Gamma(D), \ \forall X \in \Gamma(TM) \ \text{and} \ Y \in \Gamma(D).$$

Lemma 4.3.2. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a co-screen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$. Then*

$$\eta(\overline{\nabla}_X Y) + \eta(\overline{\nabla}_Y X) = 0,$$

for all $X,Y \in \Gamma(TM)$.

Proof. By direct calculations using 1.3.10 and the anti-symmetry of $\overline{\phi}$ and $\overline{H}$ we have

$$\begin{aligned}\eta(\overline{\nabla}_X Y) + \eta(\overline{\nabla}_Y X) &= -\overline{g}(X,\overline{\nabla}_Y \xi) - \overline{g}(Y,\overline{\nabla}_X \xi) \\ &= \alpha\{\overline{g}(\overline{\phi}X,Y) + \overline{g}(X,\overline{\phi}Y)\} + \overline{g}(\overline{H}X,Y) + \overline{g}(X,\overline{H}Y),\end{aligned}$$

from which our assertion follows. □

Now, using Definition 4.3.1 and Lemma 4.3.2, we have the following;

Theorem 4.3.3. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a co-screen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$. If D is nearly parallel, then $h(X,\overline{\phi}Y) + h(Y,\phi_1 X) + \nabla^t_Y \varphi_1 X$ has no component in $(\mathscr{L} \perp \mathscr{S})$ for all $Y \in \Gamma(D)$ and $X \in \Gamma(TM)$.*

Proof. Using (1.3.4), equations (4.2.2) and Lemma 4.3.2, we derive

$$\begin{aligned}
&-\phi_1 A_{\varphi_1 X}Y-\varphi_1 A_{\varphi_1 X}Y+\phi_1\nabla_X\overline{\phi}Y+\varphi_1\nabla_X\overline{\phi}Y\\
&+\phi_1\nabla_Y\phi_1 X+\varphi_1\nabla_Y\phi_1 X+\nabla_Y X+\nabla_X Y+\phi_2\nabla^t_Y\varphi_1 X\\
&+\varphi_2\nabla^t_Y\varphi_1 X+2h(X,Y)+\phi_2 h(X,\overline{\phi}Y)+\varphi_2 h(X,\overline{\phi}Y)\\
&+\phi_2 h(\phi_1 X,Y)+\varphi_2 h(\phi_1 X,Y)=0, \qquad (4.3.1)
\end{aligned}$$

for all $Y\in\Gamma(D)$ and $X\in\Gamma(TM)$. Then, taking the tangential components of (4.3.1), we get

$$\begin{aligned}
&-\phi_1 A_{\varphi_1 X}Y+\phi_1\nabla_X\overline{\phi}Y+\phi_1\nabla_Y\phi_1 X+\phi_2\nabla^t_Y\varphi_1 X\\
&+\nabla_Y X+\nabla_X Y+\phi_2 h(X,\overline{\phi}Y)+\phi_2 h(\phi_1 X,Y)=0, \qquad (4.3.2)
\end{aligned}$$

for all $Y\in\Gamma(D)$ and $X\in\Gamma(TM)$. The result follows from (4.3.2), using the fact that D is nearly parallel. □

Corollary 4.3.4. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a co-screen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$. If $\overline{D}$ is nearly parallel, then $-A_{\varphi_1 X}Y-A_{\overline{\phi}Y}X+\nabla_Y\phi_1 X$ has no component in D for all $Y\in\Gamma(\overline{D})$ and $X\in\Gamma(TM)$.*

Usung the idea of [25], we define nearly auto-parallel distributons on submanifolds of semi-Riemannian manifolds.

Definition 4.3.5. *Let (M,g) be a submanifold of a semi-Riemannian manifold $(\overline{M},\overline{g})$ and let ∇ be the connection induced in its tangent bundle. Then a distribution D on M will be called nearly auto-parallel if*

$$\nabla_X Y+\nabla_Y X\in\Gamma(D),\ \ \forall X,Y\in\Gamma(D).$$

Theorem 4.3.6. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a co-screen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$. If D is nearly auto-parallel, then $h(X,\overline{\phi}Y)+h(Y,\phi_1 X)$ has no component in $(\mathscr{L}\perp\mathscr{S})$ for all $X,Y\in\Gamma(D)$.*

Proof. Using (1.3.4), equations (4.2.2) and Lemma 4.3.2, we derive

$$\begin{aligned}
&\phi_1\nabla_X\overline{\phi}Y+\varphi_1\nabla_X\overline{\phi}Y+\phi_1\nabla_Y\overline{\phi}X+\varphi_1\nabla_Y\overline{\phi}X\\
&+\phi_2 h(X,\overline{\phi}Y)+\varphi_2 h(X,\overline{\phi}Y)+\phi_2 h(Y,\overline{\phi}X)+\varphi_2 h(Y,\overline{\phi}X)\\
&+\nabla_X Y+\nabla_Y X+2h(X,Y)=0,\ \ \forall X,Y\in\Gamma(D). \qquad (4.3.3)
\end{aligned}$$

Considering the tangential components of (4.3.3) we get

$$\begin{aligned}&\phi_1\nabla_X\overline{\phi}Y+\phi_1\nabla_Y\overline{\phi}X+\phi_2 h(X,\overline{\phi}Y)+\phi_2 h(Y,\overline{\phi}X)\\&\quad+\nabla_XY+\nabla_YX=0,\ \ \forall X,Y\in\Gamma(D).\end{aligned}\tag{4.3.4}$$

Since D is nearly auto-parallel, (4.3.4) leads to

$$\phi_2 h(X,\overline{\phi}Y)+\phi_2 h(Y,\overline{\phi}X)=0,$$

from which our assertion follows. Hence, the proof is complete. □

Corollary 4.3.7. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a co-screen QGCR-lightlike submanifold of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$. If $\overline{D}$ is nearly auto-parallel, then $A_{\varphi_1 X}Y+A_{\overline{\phi}Y}X$ has no component in D for all $X,Y\in\Gamma(\overline{D})$.*

Chapter 5

Conclusions and future work

This dissertation has provided a new view and approach to contact CR-lightlike submanifolds by introducing a special class of CR-lightlike submanifold of indefinite nearly α-Sasakian manifolds, called, quasi generalized CR (QGCR)-lightlike submanifold which is not necessarily tangent to the structure vector field. Generalizing the structure vector field offers additional computational work but at the same time opening way to new and interesting classes of submanifolds. For instance, we have showed that QGCR-lightlike submanifolds include ascreen QGCR, co-screen QGCR and the classical GCR-lightlike submanifolds. Some of the results of our findings have already appeared in our two papers [22] and [23].

Chapter 2 is entirely dedicated to QGCR-lightlike submanifolds of indefinite nearly α-Sasakian manifolds. Section 2.2 introduced the main idea of QGCR-lightlike submanifolds, clearly giving conditions under which a QGCR-lightlike submanifold is a GCR-lightlike submanifold. In Section 2.3, we proposed numerous characterization theorems concerning totally umbilical and totally geodesic QGCR-lightlike submanifolds. In Section 2.4, we prove the necessary and sufficient conditions for the integrability of distributions on any QGCR-lightlike submanifold.

Chapter 3 studied ascreen QGCR-lightlike submanifolds. In Section 3.1, we discussed ascreen QGCR-lightlike submanifolds of an indefinite nearly cosymplectic space form. In Section 3.2, we focussed on totally umbilical and totally geodesic ascreen QGCR-lightlike submanifolds of an indefinite nearly cosymplectic space form. In Section 3.3, we focussed on the mixed geodesity and auto-parallelism of distributions on an ascreen QGCR-lightlike submanifold. In Section 3.4, we focussed on minimal ascreen QGCR-lightlike submanifolds.

Chapter 4 studied co-screen QGCR-lightlike submanifolds. In Section 4.2 we discuss the integrability of distributions on such submanifolds. In Section 4.3, we introduced the concept of nearly parallel and nearly auto-parallel distributions.

While this dissertation has opened a way to a new class of CR-lightlike submanifolds, we stress that their applications have not been investigated yet. Our future work will focus on finding specific applications of this class of submanifold in other closely related fields such as mathematical physics.

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