Approximation methods for solutions of some nonlinear problems in Banach spaces

by
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As the candidate's supervisor I have approved this thesis for submission.

Dr O.T Mewomo

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Dedication

To the ever green memory of my late father Simeon Ugwuoke Ogbuisi and late brother Everistus Obiorah Ogbuisi.

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First I recognise the author and finisher of my faith and in souvenir of my life, I say thank you God Almighty for everything.

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Abstract

We study convergence analysis of generalized split feasibility problems in the frame work of Hilbert spaces and introduce a modified iterative algorithm which does not require any prior knowledge of operator norm for solving generalized split feasibility problems and obtained a strong convergence result. Further, we analyse an iterative method for finding a common element of the solution set of the split feasibility problem and the set $F(T)$ of fixed points of a right Bregman strongly nonexpansive mapping T in the setting of p-uniformly convex Banach spaces which are also uniformly smooth. By combining Mann's iterative method and the Halpern's approximation method, we propose an iterative algorithm for finding an element in the intersection of $F(T)$ and the solution set of the split feasibility problem; derive the strong convergence of the proposed algorithm under appropriate conditions and give numerical results to verify the efficiency and implementation of our method. Moreover, we propose an iterative scheme for solving multiple-set split feasiblity problems in p-uniformly convex Banach spaces which are also uniformly smooth using Bregman distance techniques and obtain a strong convergence result for approximating solutions of multiple-set split feasiblity problem in the framework of p-uniformly convex Banach spaces which are also uniformly smooth with a numerical computation. Also, we carried out a study of split variational inclusion problem in real Banach spaces with a view to analyse an iterative method for obtaining a solution of the split variational inclusion problem in Banach spaces. We propose an Halpern type algorithm and with our algorithm, we state and prove a strong convergence theorem for the approximation of solution of split variational inclusion problem in the framework of p-uniformly convex Banach spaces which are also uniformly smooth. We further consider split monotone variational inclusion problem and fixed point problem for multivalued strictly pseudocontractive-type mappings in real Hilbert spaces with a view to finding a point in the intersection of the set of solutions of split monotone variational inclusion problem and the set of solutions of fixed point problem for multivalued strictly pseudocontractive-type mappings. We introduce an iterative algorithm and with this iterative algorithm, we state and prove a strong convergence theorem for approximating a common solution of split monotone variational inclusion problem and fixed point problem for multivalued strictly pseudocontractive-type mappings in the frame work of real Hilbert spaces. We further applied our result to solve split minimization problem and split variational inequality problem. Still on inclusion problem, we give a general iterative algorithm that does not require any knowledge of the operator norm for approximating a solution of a split monotone variational inclusion problem which is also a common element of the set of fixed points of a finite family of strictly pseudocontractive mappings. A strong convergence theorem for approximating a common solution of split monotone variational inclusion problem and a fixed point problem of a finite family of strictly pseudocontractive mappings, which is also a solution of some variational inequality problems was stated and proved in the frame work of Hilbert spaces. An iterative algorithm for approximating a solution of a split equality monotone variational inclusion problem for monotone operators which is also a solution of a split equality fixed point problem for strictly pseudocontractive maps in real Hilbert spaces was given and using our iterative algorithm, we state and prove a strong convergence theorem for approximating a common solution of split equality monotone variational inclusion problem and a split equality fixed point problem for strictly pseudocontractive maps in

the frame work of real Hilbert spaces. A numerical example was also given to demonstrate the efficiency of the result. Again, an iterative algorithm that does not require any knowledge of the operator norm for approximating a solution of split generalised mixed equilibrium problem which is also a fixed point of a κ-strictly pseudocontractive mapping was constructed and a strong convergence theorem for approximating a common solution of a split generalised mixed equilibrium problem and a fixed point problem for κ-strictly pseudocontractive mapping was stated and proved in the frame work of Hilbert spaces. A numerical example was also given. A new simultaneous iterative algorithm for solving split equality for systems of generalised mixed equilibrium problem and split equality fixed point problem in p-uniformly convex Banach spaces which are also uniformly smooth using the Bregmann distance techniques was introduced and with the iterative algorithm, we state and prove a strong convergence theorem for the approximation of a solution of split equality for systems of generalised mixed equilibrium problem and split equality fixed point problem in the frame work of p-uniformly convex Banach spaces which are also uniformly smooth. This result extends results on split equality generalised mixed equilibrium problems from Hilbert spaces to p-uniformly convex Banach spaces which are also uniformly smooth. Furthermore, we introduce an iterative algorithm for approximating a common fixed point of an infinite family of left Bregman strongly nonexpansive mappings which is also a common solution of a finite system of generalised mixed equilibrium problems and a common zero of a finite family of maximal monotone operators in a reflexive real Banach space. A strong convergence theorem is also proved for finding an element in the intersection of the set of solution of a fixed point problem for infinite family of left Bregman strongly nonexpansive mappings, the set of solutions of a system of generalised mixed equilibrium problems and the set of zero points of a finite family of maximal monotone operators in a reflexive real Banach space. Lastly, we study the split hierarchical variational inequality problem, introduce an iterative algorithm to approximate the solution of split hierarchical variational inequality problem for demi-contractive mappings in real Hilbert spaces and obtain a strong convergence result with no compactness assumptions on the space and the map. As well, we propose a method to solve convex minimization problems of the type $\min\{f(x) + g(x)\}$ where f and g are convex, f has a Lipschitz gradient and the proximal mapping of g can be evaluated. Our method mixes together the relaxations and a "viscosity" term, to steer the iterates towards a specific minimizer. We prove that our iterative scheme converges strongly to a minimizer of the sum of two mappings in real Hilbert spaces. Our resulting algorithm and its convergence analysis appear new to this type of convex minimization problem. We give some applications of our results and give some numerical example to illustrate the performance of our algorithm.

Contents

Declaration

This thesis in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used in the text, proper reference has been made.

Ferdinard Udochukwu Ogbuisi

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Contributed papers from the thesis

1. Y. Shehu, F.U. Ogbuisi and O.T. Mewomo, Further investigation into approximation of a common solution of fixed point problems and split feasibility problems, Acta. Math. Sci. Ser. B. Engl. Ed. 36 (3), (2016), 913-930 (MR3479264, ISI Indexed, Published by Elsevier).

2. F.U. Ogbuisi and O.T. Mewomo, Iterative solution of split variational inclusion problem in real Banach space, Afrika Matematika 28 (1-2), (2017), 295-309 (MR3613639, Scopus Indexed, Published by Springer).

3. F.U. Ogbuisi and O.T. Mewomo, Convergence analysis of common solution of certain nonlinear problems, Fixed Point Theory, (2017) (Accepted, to appear, ISI Indexed).

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6. F.U. Ogbuisi and O.T. Mewomo, An operator norm independent solution of monotone variational inclusion problem in Hilbert space. *Proceedings of the Southern Africa Mathematical Sciences Association (SAMSA 2016), Annual Conference, 21 - 24 November 2016, University of Pretoria, South Africa*, (2017), 58-73.

Chapter 1

General Introduction

1.1 Background of the study

The theory of fixed point is one of the most powerful tools of modern mathematics. Theorems concerning the existence and properties of fixed points are known as fixed point theory. Fixed point theory blends analysis, topology and geometry. In particular, it has been applied in such fields as mathematics, engineering, physics, economics, game theory, biology, chemistry and so on.

In 1886, Poincare [177] was the first to work in this field. Then Brouwer [32] in 1912, proved fixed point theorem for a square, a sphere and their n-dimensional counter parts which was further extended by Kakutani [115]. The Banach contraction mapping principle which is one of the fundamental principle in the field of functional analysis was obtained by Stephan Banach in 1922, see Banach [13]. This shows that a contraction mapping on a complete metric space possesses a unique fixed point.

The origin of metric fixed point theory itself, rests in the method of successive approximations for proving existence and uniqueness of solutions of differential equations. This method is associated with the names of such celebrated nineteenth century mathematicians as Cauchy, Liouville, Lipschitz, Peano, and especially, Picard. In fact the precursors of the fixed point theoretic approach are explicit in the work of Picard. However it is the Polish mathematician Stefan Banach who is credited with placing the ideas underlying the method into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations.

The Banach contraction mapping principle is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in analysis. This is because the contractive condition on the mapping is easy to test and it requires only the structure of a complete metric space for its setting.

The key components of the Banach Contraction Mapping Principle as it first appeared in Banach's 1922 thesis [13] are these: (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping. Thus there exists a constant $k < 1$ such that $d(T(x), T(y)) \leq kd(x, y)$ for each $x, y \in X$. From this one draws three conclusions:

(i) T has a unique fixed point, say x_0.
(ii) For each $x \in X$ the Picard sequence $\{T^m(x)\}$ converges to x_0.
(iii) The convergence is uniform.
In fact condition (iii) can be put in much more explicit form in terms of error estimates:
(a) $d(T^n(x), x_0) \leq \dfrac{k^n}{1-k} d(x, T(x))$ for each $x \in X$ and $n \geq 1$;
(b) $d(T^{n+1}(x), x_0) \leq \dfrac{k}{1-k} d(T^{n+1}x, T^n(x))$ for each $x \in X$ and $n \geq 1$.
In particular, there is an explicit rate of convergence:
(iv) $d(T^{n+1}(x), x_0) \leq k d(T^n x, x_0)$.

A new direction of research took place in the field of fixed point theory for approximating fixed points and convergence of iterative sequences after the work of Mann [148] and Ishikawa [109]. The importance of this new direction of research in fixed point theory is that whenever there is no exact solution to an equation or the solution is difficult to obtain, some sort of approximate solution is desired. The fixed point can be obtained either by changing the nature of mappings or by working on the structure of the underlying spaces, such as its topological structures.

In functional analysis, fixed point theory is divided mainly into four branches, namely: set theoretical fixed point theory, topological fixed point theory, fuzzy topological fixed point theory and metric fixed point theory [13, 33, 122, 123]. In this study, we are interested in metric fixed point theory with particular interest on Split Feasibility Problems (SFP) and some of its generalisations.

It is worth mentioning that SFP in finite-dimensional spaces was first introduced by Censor and Elfving [54] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [46]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [52, 55, 58, 62]. SFP and some of its generalisations in infinite-dimensional Hilbert spaces and some Banach spaces can be found in [46, 55, 168, 205, 210, 209, 225, 231, 237, 250]. Moreover, the convex feasibility formalism is at the core of the modeling of many inverse problems and has been used to model significant real-world problems. For some existing results on SFP, see [60, 158, 161, 162, 234].

Censor and Elfving [54] introduced the concept of SFP and used multidistance method to obtain the iterative algorithms for solving this problem. Their algorithms as well as others obtained later involves matrix inverses at each step [8]. Byrne [46, 45] proposed a new iterative method called CQ-method that involves only the orthogonal projections onto C and Q and does not need to compute the matrix inverses, where C and Q are nonempty closed convex subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively. It is one of the main advantages of this method compare to other methods. The CQ algorithm is as follows:

$$x_{n+1} = P_C(x_n - \gamma A^T(I - P_Q)Ax_n), \quad n = 0, 1, \cdots,$$

where $\gamma \in (0, \frac{2}{L})$, A is an $M \times N$ matrix, A^T denotes the transpose of the matrix A, L is the largest eigenvalue of the matrix A^TA, and P_C and P_Q denote the metric projections onto C and Q respectively. Byrne also studied the convergence of the CQ algorithm for arbitrary nonzero matrix A.

Motivated by the work of Byrne [46, 45], Yang [237] proposed a modification of the CQ algorithm, called relaxed CQ algorithm in which he replaced P_C and P_Q by P_{C_n} and P_{Q_n}, respectively, where C_n and Q_n are half-spaces. One common advantage of the CQ algorithm and relaxed CQ algorithm is that the computation of the matrix inverses is not necessary. However, a fixed step-size related to the largest eigenvalue of the matrix $A^T A$ is used. Computing the largest eigenvalue may be hard and conservative estimate of the step-size usually results in slow convergence. So, Qu and Xiu [180] modified the CQ algorithm and relaxed CQ algorithm by adopting Armijo-like searches. The modified algorithm need not compute the matrix inverses and the largest eigenvalue of the matrix $A^T A$, and make a sufficient decrease of the objective function at each iteration.

Zhao *et al.* [251] proposed a modified CQ algorithm by computing step-size adaptively and perform an additional projection step onto some simple closed convex set $X \subset \mathbb{R}^N$ in each iteration. Since all the algorithms have been introduced in finite-dimensional setting, Xu [236] proposed the relaxed CQ algorithm in infinite-dimensional setting, and also proved the weak convergence of the proposed algorithm. In 2011, Li [137] developed some improved relaxed CQ methods with the optimal step-length to solve the split feasibility problem based on the modified relaxed CQ algorithm [180].

Recently, several authors has extended the concept of split feasibility problems to multiple set split feasibility problems, split equilibrium problems, split monotone variational inclusion problems and split variational inequality problems. In this work, we studied multiple set split feasibility problem, split equilibrium problems, split monotone variational inclusion problems, split hierarchical variational inequality problem and minimisation problems and obtained some important convergence results which either compliment or improve some results in the literation as highlighted in chapter 8.

1.2 Research problems and motivation

In this section, we discuss the research problems and motivation for the study.

1.2.1 Research Problems

Recently, many authors have studied and introduced different iterative algorithms for fixed point problems, split monotone variational inclusion problems, convex feasibility problems, split feasibility problems and equilibrium problems see for example [47, 64, 55, 86, 93, 120, 158, 160] and the references therein. These authors have produced weak and strong convergence results in Hilbert spaces and to the best of our knowledge not much has been done on split monotone variational inclusion problems in a more general Banach spaces other than the Hilbert spaces.

In this work, we study the split monotone variational inclusion problem in more general Banach spaces for example q-uniformly smooth Banach spaces which are p-uniformly convex. Also, we extend some of the results on split monotone variational inclusion problem and some of its special cases from Hilbert spaces to higher Banach spaces.

Moreover, most of the results on split monotone variational inclusion problems and split equilibrium problems in Hilbert spaces require a prior knowledge of the operator norm which is not always easy to calculate or estimate. Thus, we introduce some iterative schemes for solving split monotone variational inclusion problem and split generalised mixed equilibrium problems which do not require the knowledge of the operator norm.

We also study variational inclusion and generalised mixed equilibrium problems in reflexive Banach spaces using Bregman distance techniques. Furthermore, we also carry out some studies on fixed point problems, spilt feasibility problems and multiple set split feasibility problems using Bregman distance techniques in reflexive Banach spaces.

Finally, We improve on some recent and important results of Moudafi [160], Kazmi and Rizvi [120], Chang *et al.* [63], Martin-Marquez *et al.* [154], Masad and Reich [158], Moudafi [161], Suantai *et al.* [215], Yao and Cho [240], Yao *et al.* [241], Zhu and Chang [253] and others.

1.2.2 Motivation

Our motivation for this work is discussed under the following headings:

1. Fixed point problems;
2. Split monotone inclusion problems;
3. Generalized mixed Equilibrium problems;
4. Split feasibility problems.

1. Fixed point problems:

   Bruck [36] noted that apart from being an obvious generalization of the contraction mappings, nonexpansive maps are important for the following two reasons:
   (a) Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.
   (b) Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$, where the operators $T(t)$ are in general set-valued and are accretive or dissipative and minimally continuous.
   (c) Many well-known algorithms in signal processing and image reconstruction are iterative in nature and a wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the Krasnoselskii-Mann iteration procedure, for particular choices of the nonexpansive operator, see [45].

   Despite many existing results for nonexpansive type mapping in the literature, there are still much to be done on nonexpansive type mappings using Bregman distance Technique.

2. Split monotone inclusion problems:

   An important and perhaps interesting topic in nonlinear analysis and convex optimization concerns solving inclusions of the form $0 \in A(x)$, where A is a maximal monotone operator on a Banach space E. Its importance in convex optimization is evidenced from the fact that many problems that involve convexity can be formulated as finding zeros of maximal monotone operators. For example, convex minimizations and convex-concave mini-max problems, to mention but a few can be formulated in this way. Furthermore, the variational inclusion problem is important generalization of variational inequality problems and have been extensively studied and generalized in different directions to study a wide class of problems arising in mechanics, optimization, nonlinear programming, economics, finance and applied sciences. In particular, the subdiferential of a proper, convex and lower semi-continuous (lsc) function f on a Banach space E, ∂f is a maximal monotone operator and a point $p \in E$ minimizes f if and only if $0 \in \partial f(p)$.

   Many Mathematicians have studied this split monotone variational inclusion problem extensively in Hilbert spaces by introducing different iterative schemes and proving convergence theorems for solving split monotone variational inclusion problems in Hilbert spaces, (see, [47, 120, 160] and some of the references there in).

   The point of interest here is that as important as the split monotone variational inclusion problem is, much have not been done on it in Banach spaces more general than Hilbert spaces. Also, most of the existing results in Hilbert spaces involve iterative schemes such as shrinking projection algorithm or requires the knowledge of the operator norm which some times may be difficult to compute. Thus, there is the need to introduce simpler and much easier iterative algorithms or iterative algorithms that do not require any prior knowledge of the operator norm for solving split monotone variational inclusion problem in certain Banach spaces.

3. Generalised mixed equilibrium:

   Equilibrium theory represents an important area of mathematical sciences as numerous problems in physics, optimization, operations research, economics, game theory, financial mathematics and mechanics can be formulated as an equilibrium problem. Equilibrium problems include variational inequalities, optimization problems, Nash equilibria problems, saddle point problems, fixed point problems and complementarity problems as special cases, and the generalised mixed equilibrium problems generalize the equilibrium problems.

   In the theory of variational inequalities, variational inclusions, and equilibrium problems, the development of an efficient and implementable iterative algorithms is interesting and important. In past years, some iterative methods have been proposed to solve the equilibrium problem and variational inequality problems in Hilbert spaces and Banach spaces, see, for instance [23, 51, 73, 93, 112, 113, 129] and the references therein.

   In the literature, results on split equilibrium problems are mostly in Hilbert spaces and also depends on a prior knowledge of the operator norm. Hence, It will be important to introduce iterative algorithms for solving split equilibrium problems which do not require knowledge of operator norm. Furthermore, there is also the need to obtain iterative solution of split equilibrium problem or any of its generalisations in Banach spaces more general than Hilbert spaces.

4. Split feasibility problems:

   Recently, The SFP have been extended infinite-dimensional Hilbert spaces, see [46, 55, 225, 235, 239, 250] and have also been applied in solving problems in areas such as image restoration, computer tomograph, and radiation therapy treatment planning, see [52, 55, 58]. For some existing results on split feasibility problem, see [60, 158, 161, 162, 234]. For some existing results on split equality fixed point Problems, split common fixed point problems and split Convex feasibility problems, see [46, 52, 54, 55, 239].

   The study of split feasibility problem and multiple set split feasibility problem in Banach spaces outside Hilbert spaces is still rare. Thus, it is a step in the right direction to study these in the framework of Banach spaces.

To further substantiate the merit of the study of split feasibility, we present following practical application of split feasibility.

Example 1.2.1. (see [245]) We consider an equilibrium-optimization model which can be regarded as an extension of a Nash-Cournot oligopolistic equilibrium model in electricity markets. The latter model has been investigated in some research papers (see e.g. [80, 181]). In this equilibrium model, it is assumed that there are n companies, each company i may posses I_i generating units. Let x denote the vector whose entry x_j stands for the power generating unit j . Similarly as in [245], let the price $p_i(s)$ be a decreasing affine function of s with $s = \sum_{j=1}^{N} x_j$ where N is the number of all generating units, i.e., $p_i(s) = \alpha - \beta_i s$. Then the profit made by company i is given by $f_i(x) = p_i(s)\Big(\sum_{j\in I_i} x_j\Big) - \sum_{j\in I_i} C_j(x_j)$, where $C_j(x_j)$ is the cost for generating x_j by the generating unit j. Suppose that K_i is the strategy set of company i, the condition $\sum_{j\in I_i} x_j \in K_i$ must be satisfied for every i. Then the strategy set of the model is $K := K_1 \times K_2 \times \cdots \times K_n$.
Infact, each company seeks to maximize its profit by choosing the corresponding production level provided that the production of the other companies are parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept.
We recall that a point $x^* \in K = K_1 \times K_2 \times \cdots \times K_n$ is an equilibrium point of the model if $f_i(x^*) \geq f_i(x^*[x_i])\ \ \forall x_i \in K_i,\ \forall i = 1, 2, \cdots, n$, where the vector $x^*[x_i]$ stands for the vector obtained from x^* by replacing x_i^* with x_i . By taking $f(x, y) := \psi(x, y) - \psi(x, x)$ with

$$\psi(x, y) := -\sum_{i=1}^{n} f_i(x[y_i]) \tag{1.1}$$

the problem of finding a Nash equilibrium point of the model can be formulated as

$$x^* \in K : f(x^*, x) \geq 0\ \forall x \in K.$$

We extend this equilibrium model by additionally assuming that to produce electricity the generating units used some materials.
Let $a_{l,j}$ denote the quality of material $l(l = 1, \cdots, m)$ for producing one unit electricity by the generating unit $j(j = 1, \cdots, N)$. Let A be the matrix whose entries are $a_{l,j}$. Then, the entry l of the vector Ax is the quantity of material l for producing x. Using materials for production may cause pollution to environment for which companies have to pay environmental fee. Suppose that $g(Ax)$ is the total environmental fee for producing x. The task now is to find a production x^* such that it is a Nash equilibrium point with minimum environmental fee. This problem can be formulated as a split feasibility problem of the form:
Find

$$x^* \in K : f(x^*, x) \geq 0, \forall x \in K, \quad g(Ax^*) \leq g(Ax) \quad \forall x \in K. \tag{1.2}$$

1.3 Objectives of the study

The main objectives of this study are:

(i) to introduce iterative algorithms and prove strong convergence theorems for solving split feasibility problems and multiple set split feasibility problems in p-uniformly convex Banach spaces which are also uniformly smooth and give some numerical examples with applications;

(ii) to propose an iterative algorithm and with the proposed algorithm state and prove a strong convergence result for approximating a common solution of a fixed point problem, generalised mixed equilibrium problem and variational inclusion problem in the frame work of real reflexive Banach spaces using Bregman distance technique and also give applications to convex feasibility problems, variational inequality problems and the problem of finding zeroes of Bregman inverse strongly monotone operators;

(iii) to extend some existing results on split monotone variational inclusion problems from the frame work of Hilbert spaces to q-uniformly smooth Banach spaces which are also p-uniformly convex;

(iv) to introduce an iterative method that does not require any knowledge of the operator norm for approximating a solution of a split generalised mixed equilibrium problem which is also a fixed point of a κ-strictly pseudocontractive mapping and give some numerical examples with applications;

(v) to state and prove a strong convergence result for approximating common solution of finite family of split equality generalised mixed equilibrium problems and split equality fixed point problems using Bregman distance technique;

(vi) to propose an iterative method for solving convex minimization problems of the form $\min\{f(x) + g(x)\}$ where f and g are convex functions and give some numerical examples with applications to split feasibility problems and LASSO problems;

(vii) to state and prove a strong convergence theorem for approximating a common solution of a monotone variational inclusion problem and a fixed point problem of a finite family of strictly pseudocontractive mappings, which is also a solution of some variational inequality problems.

1.4 Organization of the thesis

The thesis is divided into eight chapters as follows:

In chapter 1, we give a brief historical background of our study, discussed the research problems and motivation for the study, give the objectives of the study and finally describe the organization of the thesis.

In chapter two, we introduce some basic concepts and terms and give some existing results and classical inequalities that are needed in establishing our results in this work. Some notable existing iterative scheme and the concept of Bregmann distance are also discussed.

Our major work begins in Chapter three. This chapter comprises four sections with a strong convergence result given in each of the three sections.
In section 3.1, we give a brief introduction of SFP and Multiple-set split feasibility problems and some existing results on them.
In section 3.2, a non-operator norm dependent iterative solution of generalised split feasibility problems was given with a strong convergence theorem stated and proved.
In Section 3.3, we study and analyse an iterative method for finding a common element of the solution set of the split feasibility problem and the set F(T) of fixed points of a right Bregman strongly nonexpansive mapping T in the setting of p-uniformly convex Banach spaces which are also uniformly smooth. Moreover, we derive the strong convergence of the proposed algorithm under appropriate conditions and give numerical results to verify the efficiency and implementation of our method.
In Section 3.4, we introduce an iterative scheme for solving multiple-set split feasiblity problems in p-uniformly convex Banach spaces which are also uniformly smooth using Bregman distance techniques. We further obtain a strong convergence result for approximating solutions of multiple-set split feasiblity problem in the framework of p-uniformly convex Banach spaces which are also uniformly smooth with a numerical computation.

In chapter four, we introduce an iterative algorithm for approximating a common fixed point of an infinite family of left Bregman strongly nonexpansive mappings which is also a common solution of a finite system of generalised mixed equilibrium problems and a common zero of a finite family of maximal monotone operators in a reflexive real Banach space. A strong convergence theorem is also proved for finding an element in the intersection of the set of solution of a fixed point problem for infinite family of left Bregman strongly nonexpansive mappings, the set of solutions of a system of generalised mixed equilibrium problems and the set of zero points of a finite family of maximal monotone operators in a reflexive real Banach space.

Chapter five was devoted to split equilibrium problems and two strong convergence results, each in a separate section were presented. In section 5.1, we presented the definitions of split equilibrium problem and split generalized mixed equilibrium problems, and some of

the convergence results already obtained on them in the literature.
In section 5.2, we introduce an iterative algorithm that does not require any knowledge of the operator norm for approximating a solution of a split generalised mixed equilibrium problem which is also a fixed point of a κ-strictly pseudocontractive mapping. Furthermore, a strong convergence theorem for approximating a common solution of a split generalised mixed equilibrium problem and a fixed point problem for κ-strictly pseudocontractive mapping was stated and proved in the frame work of Hilbert spaces. A numerical example was also given.
In section 5.3, we introduce a simultaneous iterative algorithm for solving split equality for system of generalised mixed equilibriums problem and split equality fixed point problem in p-uniformly convex Banach spaces which are also uniformly smooth using the Bregmann distance techniques. Furthermore, we state and prove a strong convergence theorem for the approximation of a solution of split equality for systems of generalised mixed equilibrium problem and split equality fixed point problem in the frame work of p-uniformly convex Banach spaces which are also uniformly smooth.

Chapter six was devoted to the study of split inclusion problems and four strong convergence results were discussed with one in each section of the chapter.
In section 6.1, a short description of split monotone variational inclusion problem and a few of the results already obtained for solving split monotone variational inclusion problem. In section 6.2, we study split variational inclusion problem in real Banach spaces with a view to analyse an iterative method for obtaining a solution of the split variational inclusion problem in Banach spaces. We propose a Halpern type algorithm and with our algorithm, we state and prove a strong convergence theorem for the approximation of solution of split variational inclusion problem in the framework of p-uniformly convex Banach spaces which are also uniformly smooth.
In section 6.3, we consider split monotone variational inclusion problem and fixed point problem for multivalued strictly pseudocontractive-type mappings in real Hilbert spaces with a view to finding a point in the intersection of the set of solutions of split monotone variational inclusion problem and the set of solutions of fixed point problem for multivalued strictly pseudocontractive-type mappings. We introduce an iterative algorithm and with this iterative algorithm, we state and prove a strong convergence theorem for approximating a common solution of split monotone variational inclusion problem and fixed point problem for multivalued strictly pseudocontractive-type mappings in the frame work of real Hilbert spaces. We further applied our result to solve split minimization problem and split variational inequality problem.
In section 6.4, we introduce a general iterative algorithm that does not require any knowledge of the operator norm for approximating a solution of a split monotone variational inclusion problem which is also a common element of the set of fixed points of a finite family of strictly pseudocontractive mappings. Furthermore, a strong convergence theorem for approximating a common solution of a split monotone variational inclusion problem and a fixed point problem of a finite family of strictly pseudocontractive mappings, which is also a solution of some variational inequality problems was stated and proved in the frame work of Hilbert spaces.
In section 6.5, an iterative algorithm for approximating a solution of a split equality monotone variational inclusion problem for monotone operators which is also a solution

of a split equality fixed point problem for strictly pseudocontractive maps in real Hilbert spaces. Using our iterative algorithm, we state and prove a strong convergence theorem for approximating a common solution of split equality monotone variational inclusion problem and a split equality fixed point problem for strictly pseudocontractive maps in the frame work of real Hilbert spaces. A numerical example was also given to demonstrate the efficiency of the result.

The following sections make up chapter seven:
Section 7.1: In this section, we study the split hierarchical variational inequality problem and the convergence analysis in Hilbert space. Moreover we introduce an iterative algorithm to approximate the solution of split hierarchical variational inequality problem for demi-contractive mappings in a real Hilbert space and obtain a strong convergence result with no compactness assumptions on the space and the map.
Section 7.2: In this section, we propose a method to solve convex minimization problems of the type $\min\{f(x) + g(x)\}$ where f and g are convex, f has a Lipschitz gradient and the proximal mapping of g can be evaluated. Our method mixes together the relaxations and a "viscosity" term, to steer the iterates towards a specific minimizer. We prove that our iterative scheme converges strongly to a minimizer of the sum of two mappings in real Hilbert spaces. Our resulting algorithm and its convergence analysis appear new to this type of convex minimization problem. We give some applications of our results and give some numerical example to illustrate the performance of our algorithm.

In the last chapter, our contribution to knowledge is discussed and some areas of future research are also pointed out.

Chapter 2

Preliminaries and Some Important Results

2.1 Preliminaries

In this section, some important basic concepts, terms and results that are relevant to the study are introduced and presented. Our main references for this section are [22] and [30].

2.1.1 Reflexive Banach spaces

Let E be a Banach space, the dual space E^* of E is the space of continuous linear functionals on E. E^* is a Banach space with the norm $||.||_{E^*}$ defined by

$$||\xi||_{E^*} := sup\{|\langle \xi, x\rangle| : x \in E, ||x|| \leq 1\}, \quad \xi \in E^*, \tag{2.1}$$

where the paring $\langle \xi, x\rangle$ is defined by the action of $\xi \in E^*$ at $x \in E$, that is $\langle \xi, x\rangle := \xi(x)$. Bilinearity and continuity in both arguments are obvious and since

$$\frac{|\langle \xi, x\rangle|}{||x||_E} = |\langle \xi, \frac{x}{||x||_E}\rangle| \leq ||\xi||_{E^*}, \; x \neq 0,$$

we have that

$$|\langle \xi, x\rangle| \leq ||\xi||_{E^*}||x||_E \text{ for all } \xi \in E^*, x \in E. \tag{2.2}$$

The inequality (2.2) is known as the generalised Cauchy-Schwartz inequality. The mapping ι_E that maps E into its bi-dual E^{**} defined by

$$\langle \iota_E(x), \xi\rangle := \langle \xi, x\rangle, \quad x \in E, \xi \in E^*$$

is called the canonical embedding of E into its bi-dual E^{**}. Clearly ι_E is linear and an isometry but it is not in general surjective. If ι_E is surjective, we say that it is an isometric isomorphism and the Banach space E is reflexive. Moreover, E is reflexive if and only if E^* is reflexive. Examples of reflexive Banach spaces include Hilbert spaces, l_p and L_p spaces $(1 < p < \infty)$ and the Sobolev spaces. Some important spaces in analysis that are not reflexive include L_1 and L_∞.

The following property of reflexive Banach space is fundamental to the convergence analysis in this thesis.

Theorem 2.1.1. *([30] Theorem 3.18) Assume that E is a reflexive Banach space and let $\{x_n\}$ be a bounded sequence in E. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges in the weak topology $\sigma(E, E^*)$.*

Definition 2.1.1. A normed space E is called smooth if for every $x \in E$, $||x|| = 1$, there exists a unique $x^* \in E^*$ such that $||x^*|| = 1$ and $\langle x, x^* \rangle = ||x||$.

Let dim $E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) := \sup\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : ||x|| \leq 1, ||y|| \leq t\}.$$

E is said to be uniformly smooth if and only if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0,$$

and *q-uniformly smooth* if there exists $C_q > 0$ such that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$.

Definition 2.1.2. A normed space E is called uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that for $x, y \in E$ with $||x|| = 1, ||y|| = 1$ and $||x - y|| \geq \epsilon$, then $||\frac{1}{2}(x + y)|| \leq 1 - \delta$.

Let dim $E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left|\left|\frac{x+y}{2}\right|\right| : ||x|| = ||y|| = 1; \epsilon = ||x - y||\right\}.$$

E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and *p-uniformly convex* if there is $C_p > 0$ so that $\delta_E(\epsilon) \geq C_p \epsilon^p$ for any $\epsilon \in (0, 2]$. It is a common knowledge that every uniformly convex Banach space E is reflexive.

Definition 2.1.3. Let E be a real Banach space with the dual E^* and let p be a given real number with $p > 1$. The generalized duality mapping J_p^E from E into 2^{E^*} is defined by

$$J_p^E(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

In particular, $J = J_2^E$ is called the normalized duality mapping and $J_p^E(x) = ||x||^{p-2} J(x)$ for all $x \neq 0$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. The normalised duality mapping J has the following properties:

1. if E is smooth, then J is single-valued;
2. if E is strictly convex, then J is one-to-one;
3. if E is reflexive, then J is surjective;
4. if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
5. if E^* is uniformly convex, then J is uniformly continuous on each bounded subsets of E and J is single-valued and also one-to-one.

It is known that E is p-uniformly convex and uniformly smooth if and only if its dual E^* is q-uniformly smooth and uniformly convex. In a q-uniformly smooth Banach space E, the generalised duality mapping is one-to-one and satisfies $J_p^E = (J_q^{E^*})^{-1}$ where $J_q^{E^*}$ is the generalised duality mapping of E^* (see [6, 74]). The generalised duality mapping J_p^E is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_p^E x_n, y\rangle \to \langle J_p^E x, y\rangle$$

holds true for any $y \in E$. We note here that l_p $(p > 1)$ spaces has such a property, but the L_p $(p > 2)$ does not share this property.

2.1.2 Bounded linear operators on Banach spaces

Let $(X, ||.||)$ and $(Y, ||.||_1)$ be two Banach spaces over the same scalar field $\mathbb{F}$. A mapping $A : D(A) \subset X \to Y$ satisfying

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

for all $x, y \in D(A)$ and $\alpha, \beta \in \mathbb{F}$ is called a linear operator or a linear transformation. If A is linear and there exists $K \geq 0$ such that

$$||Ax||_1 \leq K||x||, \ \forall \ x \in X,$$

then A is called a bounded (continuous) linear operator.

Proposition 2.1.2. *Let X^* and Y^* be the dual spaces of X and Y respectively and let $A : D(A) \subseteq X \to Y$ be densely defined linear operator, then there exists a linear operator $A^* : D(A^*) \subseteq Y^* \to X^*$ such that*

$$\langle \xi, Ax\rangle = \langle A^*\xi, x\rangle, \ \forall \ x \in D(A), \xi \in D(A^*),$$

and any other linear mapping $B : D(B) \subseteq Y^* \to X^*$ satisfying

$$\langle \xi, Ax\rangle = \langle B\xi, x\rangle, \ \forall \ x \in D(A), \xi \in D(B),$$

is a restriction of A^*.

Proof. Let $D(A)$ be dense in X and let

$$S := \{\xi \in Y^* : x \mapsto \langle \xi, Ax\rangle \ \text{ continuous } \ \text{on } D(A)\}.$$

For $\xi \in S$, define $g_\xi : D(A) \to \mathbb{F}$ by

$$g_\xi(x) = \langle \xi, Ax\rangle \ \forall \ x \in D(A).$$

Since $D(A)$ is dense in X, g_ξ has a unique continuous conjugate linear extension to all of X, preserving the norm. Let us denote extension of g_ξ by $\bar{g}_\xi$. Taking $D(A^*) = S$, define $A^* : D(A^*) \to X^*$ by $A^*\xi = \bar{g}_\xi$. It can be seen that A^* is a linear operator and it satisfies

$$\langle A^*\xi, x\rangle = \langle \xi, Ax\rangle, \ \forall \ x \in D(A), \xi \in D(A^*).$$

Now suppose $B : D(B) \subseteq Y^* \to X^*$ is another linear operator such that

$$\langle B\xi, x\rangle = \langle \xi, Ax\rangle, \ \forall \ x \in D(A), \xi \in D(B).$$

Note that if $\xi \in D(B)$, then

$$|\langle \xi, Ax\rangle| = |\langle B\xi, x\rangle| \leq ||B\xi|| ||x||, \ x \in D(A).$$

So that $x \mapsto \langle \xi, Ax\rangle$ is continuous on $D(A)$. Thus, $D(B) \subseteq S = D(A^*)$. Further $\xi \in D(B) \subseteq D(A^*)$ implies

$$\langle B\xi, x\rangle = \langle \xi, Ax\rangle = \langle A^*\xi, x\rangle, \ \forall \ x \in D(A).$$

Hence, $B\xi = A^*\xi$ for all $\xi \in D(B)$, showing that B is a restriction of A^*. □

The operator A^* defined in Proposition 2.1.2 above is called the adjoint of A. Moreover, if $D(A) = X$ and A is a bounded operator, then $A^* : Y^* \to X^*$ is the operator which satisfies

$$\langle A^*\xi, x\rangle = \langle \xi, Ax\rangle, \ \forall \ \xi \in Y^*, x \in X$$

and A^* is a bounded linear operator with $||A^*|| = ||A||$.

2.1.3 Some operators on Hilbert spaces

Let K be a nonempty subset of a real Hilbert space H and 2^K the family of all nonempty subsets of K. A mapping $T : K \to K$ is said to be *nonexpansive* [241] if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall \ \ x, y \in K, \tag{2.3}$$

and $T : K \to K$ is said to be λ-*strictly pseudocontractive* [2, 34], if for $0 \leq \lambda < 1$,

$$||Tx - Ty||^2 \leq ||x - y||^2 + \lambda||(I - T)x - (I - T)y||^2, \quad \forall \ \ x, y \in K. \tag{2.4}$$

It is well known that (2.4) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 - \frac{1 - \lambda}{2}||(I - T)x - (I - T)y||^2. \tag{2.5}$$

A mapping $T : K \to K$ is said to be *demicontractive*, if $\exists\ \beta \in [0, 1)$ such that

$$||Tx - q||^2 \leq ||x - q||^2 + \beta||x - Tx||^2, \quad \forall\ x \in K, q \in F(T). \tag{2.6}$$

Obviously, (2.6) is equivalent to

$$2\langle Tx - x, x - q\rangle \leq (\beta - 1)||x - Tx||^2, \quad \forall\ x \in K, q \in F(T). \tag{2.7}$$

Clearly, every strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.
A mapping $T : K \to K$ is said to be
(i) monotone, if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall\ x, y \in K;$$

(ii) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall\ x, y \in K;$$

(iii) β-inverse strongly monotone (β-ism), if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta\|Tx - Ty\|^2, \quad \forall\ x, y \in K;$$

(iv) firmly nonexpansive, if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \quad \forall\ x, y \in K.$$

A bounded linear operator D on H is called strongly positive if there exists $\bar{\delta} > 0$ such that

$$\langle Dx, x\rangle \geq \bar{\delta}||x||^2, \quad \forall\ x \in H.$$

Let $T : K \to 2^K$ be a multivalued map, then $x \in D(T)$ is a fixed point of T if $x \in Tx$ and the set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of T.

Definition 2.1.4. Let H be a Hilbert space. A subset K of H is called *proximinal*, if for each $x \in H$ there exists $k \in K$ such that

$$||x - k|| = \inf\{||x - y|| : y \in K\} = d(x, K), \tag{2.8}$$

The family of all proximinal subsets of H will be denoted by $P(H)$. It is known that every closed convex subset of a Hilbert space is proximinal.
Let $H(\cdot, \cdot)$ denote the Hausdorff metric induced by the metric d on H, that is for $A, B \in CB(H)$,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B}(b, A)\right\}, \tag{2.9}$$

where $CB(H)$ is the set of all closed and bounded subset of H. Let H be a Hilbert space and $T : D(T) \subseteq H \to 2^H$ be a multivalued mapping, T is said to be $L-$Lipschzian if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$H(Tx, Ty) \leq L||x - y||. \tag{2.10}$$

In (2.10), if $L \in [0,1)$, T is a contraction while T is nonexpansive if $L = 1$. T is called quasi-nonexpansive if $F(T) = \{x \in D(T) : x \in Tx\} \neq \emptyset$ and for $p \in F(T)$,

$$H(Tx, Tp) \leq ||x - p||. \tag{2.11}$$

T is said to be κ-strictly pseudocontractive-type in the sense of Isiogugu [111], if there exists $\kappa \in [0,1)$ such that, given any pair $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $||u - v|| \leq H(Tx, Ty)$ and

$$H^2(Tx, Ty) \leq ||x - y||^2 + \kappa||x - u - (y - v)||^2. \tag{2.12}$$

$T : D(T) \subseteq H \to CB(H)$ is said to be κ-strictly pseudocontractive in the sense of Chidume *et al.* [68], if there exists $\kappa \in [0,1)$ such that for all $x, y \in D(T)$

$$H^2(Tx, Ty) \leq ||x - y||^2 + \kappa||x - u - (y - v)||^2, \quad \forall\ u \in Tx, v \in Ty. \tag{2.13}$$

It has been observed (see Isiogugu [110]) that every κ-strictly pseudocontractive mapping $T : D(T) \subseteq H \to P(H)$ is κ-strictly pseudocontractive-type.

The study of fixed points for multivalued contractions and nonexpansive mappings (see [221]) was initiated by Nadler [164] and Markin [151] respectively, and by now there exists an extensive literature on multivalued fixed point theory which has applications in convex optimization, differential inclusions, fractals, discontinuous differential equations, optimal control, computing homology of maps, computer-assisted proofs in dynamics, digital imaging and economics (e.g., [100, 114] and references cited therein). There are many classical and well developed areas of applications (see [114] and the references therein) where a multivalued map is used as a generalization of a single valued map.

2.1.4 The metric projection on Hilbert spaces

Let C be a nonempty closed convex subset of a Hilbert space H. The metric projection onto C is the mapping $P_C : H \to C$ which assigns to each $x \in H$ the unique point $P_C x$ in C with the property

$$||x - P_C x|| = \min\{||x - y|| : y \in C\}.$$

The metric projections have the following characterisations (see [14] for details)

Proposition 2.1.3. *Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0$ for all $y \in C$.*

It then follows from Proposition 2.1.3, that
(i) $||P_C x - P_C y||^2 \leq \langle x - y, P_C x - P_C y\rangle$ for all $x, y \in H$, that is, the metric projection is firmly nonexpansive;
(ii) $||x - P_C x||^2 \leq ||x - y||^2 - ||y - P_C x||^2$ for all $x \in H$ and $y \in C$;
(iii) if C is a closed subspace, then P_C coincides with the orthogonal projection from H onto C, that is, for $x \in H$, $x - P_C$ is orthogonal to C (i.e., $\langle x - P_C x, y\rangle = 0$ for $y \in C$).

The following are some examples of closed convex subsets of Hilbert spaces and the projection mapping on them [152].
1. If $C = \{x \in H : ||x - u|| \leq r\}$ is a closed ball centred at $u \in H$ with radius $r > 0$, then

$$P_C x = \begin{cases} u + r\frac{(x-u)}{||x-u||}, & x \notin C \\ x, & x \in C. \end{cases}$$

2. If $C = [a, b]$ is a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, \cdots, a_n)^T$ and $b = (b_1, b_2, \cdots, b_n)^T$, then for $1 \leq i \leq n$, $P_C x$ has the ith coordinate given by

$$(P_C x)_i = \begin{cases} a_i, & x_i < a_i \\ x_i, & x_i \in [a_i, b_i] \\ b_i, & x_i > b_i. \end{cases}$$

3. If $C = \{y \in H : \langle a, y \rangle = \alpha\}$ is a hyperplane, with $a \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_C x = x - \frac{\langle a, x \rangle - \alpha}{||a||^2} a.$$

4. If $C = \{y \in H : \langle a, y \rangle \leq \alpha\}$ is a closed half space, with $a \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_C x = \begin{cases} x - \frac{\langle a, x \rangle - \alpha}{||a||^2} a, & \langle a, x \rangle > \alpha \\ x, & \langle a, x \rangle \leq \alpha. \end{cases}$$

2.1.5 Convex functions

Here, we present a brief and concise study of convex functions that is relevant to this our work.

Definition 2.1.5. A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be

1. proper if its effective domain $D(f) = \{x \in E : f(x) < \infty\}$ is nonempty;
2. convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) \ \forall \lambda \in (0, 1); x, y \in D(f);$$

3. lower semicontinuous at $x_0 \in D(f)$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x),$$

4. upper semicontinuous at $x_0 \in D(f)$ if

$$f(x_0) \geq \limsup_{x \to x_0} f(x).$$

We say that f is lower (upper) semicontinuous on $D(f)$ if it is lower (upper) semicontinuous at every $x_0 \in D(f)$;

Let f be a function on a Banach space E and $x \in int(dom f)$, for any y in E, we define the directional derivative of f at x by

$$f^o(x,y) := \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}. \tag{2.14}$$

If the limit as $t \to 0^+$ in (2.14) exists for each y, then the function f is said to be *Gâteaux differentiable* at x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y \rangle := f^o(x,y)$ for all $y \in E$ (see [57]). The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at each $x \in int(dom f)$. When the limit as $t \to 0$ in (2.14) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is *Fréchet* differentiable at x. A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be subdifferentiable at $x_0 \in D(f)$ if there exists a functional $x^* \in E^*$ called subgradient of f at x_0 such that

$$f(x) \geq f(x_0) + \langle x^*, x - x_0 \rangle, \quad \forall x \in E.$$

The set of all the subgradients of f at x_0,

$$\partial f(x_0) = \{x^* \in E^* : f(x) \geq f(x_0) + \langle x^*, x - x_0 \rangle, \quad \forall x \in E\}$$

is called the subdifferential of f at x_0.

Proposition 2.1.4. *(Cioranescu [74]) Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then*

1. *the function f is subdifferentiable on $intD(f)$, where $intD(f)$ denotes the interior of the domain of f;*
2. *the function f is Gâteaux differentiable at $x \in intD(f)$ if and only if it has a unique subgradient $\partial f(x) = \nabla f(x)$ called the gradient of f.*

The *Fenchel conjugate* function of f is the convex function $f^* : E^* \to \mathbb{R}$ defined by

$$f^*(\xi) = sup\{\langle \xi, x \rangle - f(x) : x \in E\}.$$

It is not difficult to check that whenever f is proper and lower semicontinuous, so is f^*. The function f is said to be *cofinite* if $dom f^* = E^*$.

The function f is said to be *Legendre* if it satisfies the following two conditions:
(L1) $int(dom f) \neq \emptyset$ and the subdifferential ∂f is single-valued on its domain.
(L2) $int(dom f^*) \neq \emptyset$ and ∂f^* is single-valued on its domain.

The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [16]. Their definition is equivalent to conditions (L1) and (L2) because the space E is assumed to be reflexive (see [16], Theorems 5.4 and 5.6, page 634). It is well known that in reflexive spaces $\nabla f = (\nabla f^*)^{-1}$ (see [25], page 83). When this fact is combined with conditions (L1) and (L2), we obtain

$$ran\nabla f = dom\nabla f^* = int(domf)^* \text{ and } ran\nabla f^* = dom\nabla f = int(domf).$$

It also follows that f is Legendre if and only if f^* is Legendre (see [16], Corollary 5.5, page 634) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains.
When the Banach space E is smooth and strictly convex, in particular, a Hilbert space, the function $(\frac{1}{p})\|.\|^p$ with $p \in (1, \infty)$ is Legendre (cf. [16], Lemma 6.2, page 639). For examples and more information regarding Legendre functions, see, for instance, [15, 16].

2.1.6 Bregman distance and some related notions

In 1967, Bregman [28] introduced a nice and effective method for using the so called Bregman distance function D_f (see, Definition 2.1.6) in the process of designing and analysing feasibility and optimization algorithms. This opened a growing area of research in which Bregman distance techniques is applied in various ways in order to design and analyse iterative algorithms for solving equilibria and for computing fixed points of nonlinear mappings (see, e.g.,[5, 6, 7, 15, 19, 28, 38, 39, 41, 42, 44, 88, 125, 186, 188, 189, 190, 191, 192, 193, 214, 66] and the references therein).

Let $f : E \to \mathbb{R}$ be an admissible function, that is, a proper, lower semicontinuous, convex and Gâteaux differentiable function. Under these conditions, we know that f is continuous in $int(domf)$ (see [16]).

Definition 2.1.6. The bifunction $D_f : domf \times int(domf) \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle, \tag{2.15}$$

is called the *Bregman distance* (cf. [28, 57]).

The Bregman distance does not satisfy the well-known properties of a metric, but it does have the following important property, which is called the three point identity: for any $x \in domf$ and $y, z \in int(domf)$

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle. \tag{2.16}$$

The modulus of total convexity of f is the bifunction $v_f : int(domf) \times [0, +\infty) \to [0, +\infty]$ which is defined by

$$v_f(x, t) := inf\{D_f(y, x) : y \in domf, \quad \|y - x\| = t\rangle\}.$$

The function f is said to be *totally convex at a point* $x \in int(domf)$ if $v_f(x, t) > 0$ whenever $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point $x \in int(domf)$. This property is less stringent than uniform convexity (see [38], Section 2.3, page 92). Examples of totally convex functions can be found, for instance, in [27, 38, 41].

We remark that f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [41], Theorem 2.10, page 9).
The Bregman projection (cf. [28]) with respect to f of $x \in int(domf)$ onto a nonempty, closed and convex set $C \subset int(domf)$ is defined as the necessarily unique vector $Proj_C^f(x) \in C$, which satisfies

$$D_f(Proj_C^f(x), x) = inf\{D_f(y, x) : y \in C\}. \tag{2.17}$$

Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. It is known from [41] that $z = Proj_C^f x$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0$ for all $y \in C$. We also have

$$D_f(y, Proj_C^f(x)) + D_f(Proj_C^f(x), x) \leq D_f(y, x), \quad \forall\, x \in E, y \in C. \tag{2.18}$$

Similar to the metric projection in Hilbert spaces, the Bregman projection with respect to totally convex and Gâteaux differentiable functions has a variational characterization (cf. [41], Corollary 4.4, page 23).

Proposition 2.1.5. *(see [188])(Characterization of Bregman Projections). Suppose that $f : E \to \mathbb{R}$ is totally convex and Gâteaux differentiable in $int(domf)$. Let $x \in int(domf)$ and let $C \subset int(domf)$ be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent:*
(i) The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f.
(ii) The vector $\hat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \quad \forall\, y \in C.$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality $D_f(y, z) + D_f(z, x) \leq D_f(y, x) \quad \forall\, y \in C$.

Recall that the function f is said to be *sequentially consistent* [17], if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first is bounded,

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Rightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.19}$$

Let C be a convex subset of $int(domf)$ and let T be a self-mapping of C. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Noting that the Bregman distance is not symmetric, we define the following operators.

Definition 2.1.7. (i) A mapping $T : C \to C$ with a nonempty asymptotic fixed point set is said to be *left Bregman strongly nonexpansive* (see [155]) with respect to a nonempty $\hat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x) \ \ \forall \, x \in C, p \in \hat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

then

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

(ii) An operator $T : C \to int(domf)$ is said to be *left Bregman firmly nonexpansive* (L-BFNE) if,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle$$

for any $x, y \in C$, or equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

See [17, 27, 191] for more information and examples of L-BFNE operators (operators in this class are also called D_f -firm and BFNE). For two recent studies of the existence and approximation of fixed points of left Bregman firmly nonexpansive operators, see [155, 191]. It is also known that if T is left Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then $F(T) = \hat{F}(T)$ and $F(T)$ is closed and convex (see [191]). It also follows that every left Bregman firmly nonexpansive mapping is left Bregman strongly nonexpansive with respect to $F(T) = \hat{F}(T)$.

Definition 2.1.8. (i) A mapping T with a nonempty asymptotic fixed point set is said to be *right Bregman strongly nonexpansive* (see [154]) with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(Tx, p) \leq D_f(x, p), \ \ \forall \, x \in C, \ \ p \in \widehat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n\to\infty} (D_f(x_n, p) - D_f(Tx_n, p)) = 0,$$

then

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0.$$

(ii) An operator $T : C \to \text{int dom} f$ is said to be *right Bregman firmly nonexpansive* (R-BFNE) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(Tx) - \nabla f(Ty), x - y\rangle$$

for any $x, y \in C$, or equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(x, Tx) + D_f(y, Ty) \leq D_f(x, Ty) + D_f(y, Tx).$$

According to Martin-Marquez *et al.* [154], a left (right) Bregman strongly nonexpansive mapping T with respect to a nonempty $\widehat{F}(T)$ is called *strictly left (right) Bregman strongly nonexpansive mapping.* See [154] for more information and examples of R-BFNE operators. From [154], we know that every right Bregman firmly nonexpansive mapping is right Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex, Legendre and Gâteaux differentiable function. Let the function $V_f : E \times E^* \to [0, +\infty)$ associated with f be defined as

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*),\ \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*))\ \forall x \in E, x^* \in E^*,$$

and by the subdifferential inequality, we have

$$V_f(x,x^*)+\langle y^*, \nabla f^*(x^*)-x\rangle \leq V_f(x, x^*+y^*),\ \forall x \in E \text{ and } x^*, y^* \in E^* \text{ (see[126], Lemmas 3.2 and 3.3).}$$

In addition, if f is a proper lower semi-continuous function, then $f^* : E^* \to \mathbb{R} \cup \{+\infty\}$ is a proper *weak** lower semi-continuous and convex function (see [175]). Hence V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i))) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.20}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

The generalised duality mapping J_p^E is actually the subdifferential of the function $f_p(x) = (\frac{1}{p})||x||^p$. Given that $f = f_p$, then the Bregman distance with respect to f_p now becomes

$$\begin{aligned}\Delta_p(x, y) &= \frac{1}{q}||x||^p - \langle J_p^E x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{p}(||y||^p - ||x||^p) + \langle J_p^E x, x - y\rangle \\ &= \frac{1}{q}(||x||^p - ||y||^p) - \langle J_p^E x - J_p^E y, y\rangle.\end{aligned}$$

The Bregman distance also posses the following important properties

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_p^E x - J_p^E y\rangle,\quad \forall\ x, y, z \in E. \tag{2.21}$$

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_p^E x - J_p^E y\rangle,\quad \forall\ x, y \in E. \tag{2.22}$$

For the p-uniformly convex space, the norm metric and Bregman distance has the following relation(see [201]):

$$\tau||x - y||^p \leq \Delta_p(x, y) \leq \langle x - y, J_p^E x - J_p^E y\rangle, \tag{2.23}$$

where $\tau > 0$ is some fixed number. Let C be a nonempty, closed and convex subset of E. The metric projection

$$P_C x := arg\min_{y\in C} ||x-y||,\ x \in E,$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_p^E(x - P_C x), z - P_C x\rangle \leq 0, \quad \forall\ z \in C. \tag{2.24}$$

Similarly, to the metric projection, the Bregman projection is defined as

$$\Pi_C x = arg\min_{y\in C} \Delta_p(x,y), \quad x \in E,$$

the unique minimizer of the Bregman distance (see [200]). The Bregman projection can also be characterized by a variational inequality:

$$\langle J_p^E(x) - J_p^E(\Pi_C x), z - \Pi_C x\rangle \leq 0,\ \forall\ z \in C, \tag{2.25}$$

from which one has

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x,z) - \Delta_p(x, \Pi_C x),\ \forall\ z \in C. \tag{2.26}$$

Let C be a convex subset of int domf_p, where $f_p(x) = (\frac{1}{p})||x||^p, \ \ 2 \leq p < \infty$ and let T be a self-mapping of C. Following [6], we make use of the function $V_p : E^* \times E \to [0, +\infty)$, which is defined by

$$V_p(x,y) := \frac{1}{q}||x||^q - \langle x, y\rangle + \frac{1}{p}||y||^p,\ \forall\ x^* \in E^*, x \in E.$$

V_p is nonnegative and $V_p(x^*, x) = \Delta_p(J_q^{E^*}(x^*), x)$ for all $x \in E^*$ and $y \in E$. Moreover, by the subdifferential inequality,

$$\langle \nabla f(x), y - x\rangle \leq f(y) - f(x),$$

with $f(x) = \frac{1}{q}||x||^q, x \in E^*,$ then $\nabla f(x) = J_q^{E^*}(x)$.
So, we have

$$\langle J_q^{E^*}(x), y\rangle \leq \frac{1}{q}||x+y||^q - \frac{1}{q}||x||^q. \tag{2.27}$$

From (2.27), we obtain that

$$\begin{aligned} V_p(x^*+y^*, x) &= \frac{1}{q}||x^*+y^*||^q - \langle x^*+y^*, x\rangle + \frac{1}{q}||x||^p \\ &\geq \frac{1}{q}||x^*||^q + \langle y^*, J_q^{E^*}(x^*)\rangle - \langle x^*+y^*, x\rangle + \frac{1}{q}||x||^p \\ &= \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p + \langle y^*, J_q^{E^*}(x^*)\rangle - \langle y^*, x\rangle \\ &= \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p + \langle y^*, J_q^{E^*}(x^*) - x\rangle \\ &= V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x\rangle, \end{aligned} \tag{2.28}$$

for all $x \in E$ and $x^*, y^* \in E^*$. In addition, since $f = f_p$ is a proper lower semi-continuous and convex function, we have that $f^* = f_p^*$ is a proper *weak** lower semi-continuous and convex function (see, for example, [175]). Hence V_p is convex in the second variable. Thus for all $z \in E$,

$$\begin{aligned}
\Delta_p\big(J_q^{E^*}\big(\sum_{i=1}^N t_i J_p^E(x_i)\big), z\big) &= V_p\Big(\sum_{i=1}^N t_i J_p^E(x_i), z\Big)\\
&= \frac{1}{q}||\sum_{i=1}^N t_i J_p^E(x_i)||^q - \langle\sum_{i=1}^N t_i J_p^E(x_i), z\rangle + \frac{1}{p}||z||^p\\
&\leq \frac{1}{q}\sum_{i=1}^N t_i ||J_p^E(x_i)||^q - \sum_{i=1}^N t_i\langle J_p^E(x_i), z\rangle + \frac{1}{p}||z||^p\\
&= \frac{1}{q}\sum_{i=1}^N t_i ||(x_i)||^{(p-1)q} - \sum_{i=1}^N t_i\langle J_p^E(x_i), z\rangle + \frac{1}{p}||z||^p\\
&= \frac{1}{q}\sum_{i=1}^N t_i ||(x_i)||^p - \sum_{i=1}^N t_i\langle J_p^E(x_i), z\rangle + \frac{1}{p}||z||^p\\
&= \sum_{i=1}^N t_i \Delta_p(x_i, z), \qquad (2.29)
\end{aligned}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

2.2 Fixed point iteration procedures

In this section, we discuss some notable iteration schemes that have been used in the study of fixed point and optimisation problems. Our main references for this section are [22] and [67].

2.2.1 Picard iteration

Let (X, d) be a metric space and C a closed subset of X. A mapping $T : C \to C$ is said to be a contraction if there exists $\alpha \in (0,1)$ such that $d(Tx, Ty) \leq \alpha d(x, y),\ \forall\ x, y \in C$. Let $T : C \to C$ be a mapping possessing at least one fixed point $p \in F(T)$. For some $x_0 \in X$, the sequence of iteration $\{x_n\}$ given

$$x_{n+1} = T(x_n),\ n \geq 0, \qquad (2.30)$$

is known as the Picard iteration. The Banach contraction mapping principle established that if (X, d) is a complete metric space and $T : X \to X$ is a contraction, then T has a unique fixed point and for arbitrary $x_0 \in X$, the Picard sequence iteration converges to the unique fixed point of T.

The following example shows that the Picard sequence may fail to converge to a fixed point of a nonexpansive mapping with a unique fixed point.

Example 2.2.1. Let $B := \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ and let T denote an anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then T is nonexpansive with the origin as the only fixed point. Moreover, the Picard sequence does not converge to zero. If C is a closed nonempty subset of a Banach space and $T : C \to C$ is nonexpansive (not a strict contraction), it is known that T may not have a fixed point. Even when it has, the Picard iteration may fail to converge to such a fixed point, thus the need to consider some other iteration procedures.

2.2.2 Krasnoselskii Iteration

Krasnoselskii [128] showed that in Example 2.2.1, one can obtain a convergent sequence of successive approximations if instead of T one takes the auxiliary nonexpansive mapping $\frac{1}{2}(I+T)$, where I denotes the identity transformation of the plane, i.e., if the sequence of successive approximations is defined by $x_0 \in K$,

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n), n \geq 0, \tag{2.31}$$

instead of the Picard iterates, $x_{n+1} = Tx_n,\ x_0 \in K, n \geq 0$. It is easy to see that the mappings T and $\frac{1}{2}(I+T)$ have the same set of fixed points, so that the limit of a convergent sequence defined by (2.31) is necessarily a fixed point of T.

More generally, if E is a normed linear space and K is a convex subset of E, a generalization of equation (2.31) which has proved successful in the approximation of fixed points of nonexpansive mappings $T : K \to K$ (when they exist) is the following scheme called *Krasnoselskii iteration*: $x_0 \in K$,

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, n = 0, 1, 2, ..., \tag{2.32}$$

$\lambda \in [0, 1]$ is a constant (see, e.g., Schaefer [202]). The Krasnoselskii iteration is exactly the picard iteration corresponding to the averaged operator $T_\lambda = (1-\lambda)I + \lambda T$, where I is identity operator and clearly, if $\lambda = 1$, the Krasnoselskii iteration process reduces to the Picard iteration process.

2.2.3 Mann Iteration

The Mann iteration is the sequence generated iteratively by $x_0 \in E$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n,\ n \geq 0,$$

where $\{\alpha_n\} \subset [0, 1]$. Observe that if $T_n = (1-\alpha_n)I + \alpha_n T$, then $F(T) = F(T_n),\ \forall \alpha_n \in (0, 1]$. If the sequence $\alpha_n = \lambda$ (constant), then the Mann iteration reduces to the Krasnoselskii iteration. The Mann iteration was originally defined in a matrix formulation (see [22] chapter 4 for more details).

The Mann iteration scheme has only weak convergence in general (see, [95, 218] for example). The Mann iteration method has been successfully employed in approximating fixed points (when they exist) of nonexpansive mappings, see for example [67], but this success has not carried over to the more general class of pseudo-contractions. If K is a compact convex subset of a Hilbert space and $T : K \to K$ is Lipschitz, then, by Schauder fixed point theorem, T has a fixed point in K. All efforts to approximate such a fixed point by means of the Mann sequence when T is also assumed to be pseudo-contractive proved abortive. Hicks and Kubicek [251] gave an example of a discontinuous pseudo-contraction with a unique fixed point for which the Mann iteration does not always converge. Borwein and Borwein [33], (Proposition 8) gave an example of a Lipschitz map (which is not pseudocontractive) with a unique fixed point for which the Mann sequence fails to converge.

It had remained an open question (see e.g., Borwein and Borwein [26], Chidume and Moore [69], Hicks and Kubicek [107]) whether or not the Mann recursion formula converges to a fixed point of T if the operator T is pseudo-contractive and Lipschitz until it was eventually resolved in the negative by Chidume and Mutangadura [148] in the following example.

Example 2.2.2. Let X be the real Hilbert space $\mathbb{R}^2$ under the usual Euclidean inner product. If $x = (a, b) \in X$, we define $x^\perp \in X$ to be $(b, -a)$. Trivially, we have $\langle x, x^\perp \rangle = 0$ and $||x^\perp|| = ||x||$, $\langle x^\perp, y^\perp \rangle = \langle x, y\rangle$, $||x^\perp - y^\perp|| = ||x - y||$ and $\langle x^\perp, y\rangle + \langle x, y^\perp\rangle = 0$ for all $x, y \in X$. We take our closed and bounded convex set K to be the closed unit ball in X and put $K_1 = \{x \in X : ||x|| \leq \frac{1}{2}\}$, $K_2 = \{x \in X : \frac{1}{2} \leq ||x|| \leq 1\}$. Define the map $T : K \to K$ as follows:

$$Tx = \begin{cases} x + x^\perp, & \text{if } x \in K_1 \\ \dfrac{x}{||x||} - x + x^\perp, & \text{if } x \in K_2. \end{cases}$$

Then, T is a Lipschitz pseudo-contractive map of a compact convex set into itself with a unique fixed point for which the Mann sequence does not converge.

2.2.4 Ishikawa Iteration

The Ishikawa iteration scheme is defined as follows: for $x_0 \in E$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n],$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. Ishikawa iteration was first used to establish the convergence to a fixed point for Lipschtzian and pseudocontrative selfmap of a convex compact subset of a Hilbert space. Ishikawa [109] introduced the iteration scheme and proved the following theorem.

Theorem 2.2.3. *If K is a compact convex subset of a Hilbert space H, $T : K \to K$ is a Lipschitzian pseudo-contractive map and x_0 is any point of K, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer $n \geq 0$ by the Ishikawa iteration scheme where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions*

1. $0 \leq \alpha_n \leq \beta_n < 1$;

2. $\lim_{n\to\infty} \beta_n = 0$;

3. $\sum_{n\geq 0} \alpha_n \beta_n = \infty$.

Although the Mann and Ishikawa iterations are related in the sense that if $\beta_n = 0$, Ishikawa iteration reduces to the Mann iteration, there is no general dependence between convergence results for Mann iteration and Ishikawa iteration.

Several authors have recently started considering the modified Mann iteration, respectively modified Ishikawa iteration by replacing the operator T by its nth iterate T^n. The modified Ishikawa iteration is defined by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad \forall n \geq 0. \end{cases} \tag{2.33}$$

The modified Mann iteration is obtained from (2.33) by setting $\beta_n = 0$.

Recently, the Ishikawa and Mann iterations with errors for nonlinear mappings were introduced as follows; Let $x_0 \in C$ and

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \quad \forall n \geq 0, \end{cases} \tag{2.34}$$

where C is a nonempty subset of a Banach space E, $T : C \to E$ an operator, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{u_n\}, \{v_n\}$ are sequences in C such that

$$\sum ||u_n|| < \infty, \ \sum ||v_n|| < \infty. \tag{2.35}$$

Iterative sequence (2.34) is known as the Ishikawa iteration with error and if we set $\beta_n = 0$ in (2.34), we obtain the Mann iteration with errors. It was noted in [22] that in spite of the fact that the fixed point iteration procedures are designed for numerical purposes and hence the consideration of errors is of both theoretical and practical importance, however it seems that the iteration process with errors introduced by (2.34) is not satisfactory from practical point of view. Condition (2.35) indeed imply that the errors tend to zero, which is not suitable for the randomness of the occurrence of errors in practical computations. To correct this loop hole in the iterative scheme (2.34) another Ishikawa interation with error was given as follows: Let C be a nonempty convex subset of a Banach space E and $T : C \to E$ be a mapping. For a given $x_0 \in C$, the sequence $\{x_n\}$ defined iteratively by

$$\begin{cases} y_n = \alpha_n x_n + \beta_n T x_n + \gamma_n v_n, \\ x_{n+1} = \alpha'_n x_n + \beta'_n T y_n + \gamma'_n u_n, \quad \forall n \geq 0, \end{cases} \tag{2.36}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$, and $\{\gamma'_n\}$ are sequences in the interval $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

2.2.5 Kirk iteration

Let E be a Banach space and T a mapping of E into E. The Kirk's iteration is the iteration defined as follows: for $x_0 \in E$,

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \cdots + \alpha_k T^k x_n,$$

where k is a fixed integer, $k \geq 1$, $\alpha_i \geq 0$ for $0 \leq i \leq k$, $\alpha_1 > 0$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1$. This iteration becomes the Picard iteration if $\alpha_0 = 0$, $k = 1$ and the Krasnoselskii iteration if $k = 1$.

The iteration processes of Kirk, Mann, Krasnoselskii and Ishikawa are mainly used to generate successive approximations for fixed points in normed linear spaces for which the Picard iteration does not converge.

2.2.6 Figueiredo Iteration

Let H be a Hilbert space and C a closed, bounded and convex subset of H containing 0. The Figueiredo sequence $\{x_n\}$ is defined by $x_0 \in C$ and

$$x_n = T_n^{n^2} x_{n-1},\ n \geq 1,$$

where $T_n x = \frac{n}{n+1} T x,\ n \geq 1$. It is known that the Figueiredo iteration converges strongly to a fixed point of nonexpansive operators.

2.2.7 Halpern Iteration

Let C be a closed convex subset of a real Banach space E and $T : C \to C$ a nonexpansive mapping. For fixed $t \in (0,1)$ and arbitrary $u \in C$, let $z_t \in C$ denote the unique fixed point of T_t defined by $T_t x := tu + (1-t)Tx,\quad x \in C$. Assume $F(T) := \{x \in C : Tx = x\} \neq \emptyset$. Browder [35] established that in a Hilbert space, $\lim_{t \to 0} z_t$ exists and is a fixed point of T. Reich [184] extended this result to uniformly smooth Banach spaces and Kirk [124] obtained the same result in arbitrary Banach spaces under the additional assumption that T has pre-compact range. Let $\alpha_n \in [0,1]$ and arbitrary $u \in C$, the Halpern iteration method is given by $\{x_n\}$ in C defined by $x_0 \in C$,

$$x_{n+1} := \alpha_n u + (1-\alpha_n) T x_n, n \geq 0. \tag{2.37}$$

Reich in [184] posed the following question.
Question: Let E be a Banach space. Is there a sequence $\{\alpha_n\}$ such that whenever a weakly compact convex subset C of E has the fixed point property for nonexpansive mappings, then the sequence $\{x_n\}$ defined by (2.37) converges to a fixed point of T for arbitrary fixed $u \in C$ and all nonexpansive $T : C \to C$?
Halpern [102] was the first to study the convergence of the algorithm (2.37) in the framework of Hilbert spaces. He proved the following Theorem.

Theorem 2.2.4. *(Halpern, [102]) Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be arbitrary. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}, \theta \in (0,1)$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$, $x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, n \geq 1$. Then, $\{x_n\}$ converges strongly to the element of $F(T) := x \in C : Tx = x$ nearest to u.*

Lions [138] improved Theorem 2.2.4 in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies the following conditions: (i) $\lim \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iii) $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} = 0$.

Reich [184] gave an affirmative answer to the above question in the case when E is uniformly smooth and $\alpha_n = n^{-a}$ with $0 < a < 1$. It was observed that both Halpern's and Lions' conditions on the real sequence $\{\alpha_n\}$ excluded the natural choice, $\alpha_n := (n+1)^{-1}$. This was overcome by Wittmann [229] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the following conditions:

$$\lim \alpha_n = 0; \quad (ii) \sum_{n=1}^{\infty} \alpha_n = \infty; \quad (iii) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{2.38}$$

Reich [185] extended this result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality maps (e.g., $l_p(1 < p < \infty)$), where the sequence $\{\alpha_n\}$ is required to satisfy conditions (i) and (ii) of (2.38) and to be decreasing (and hence also satisfies (iii) of (2.38)). Shioji and Takahashi [212] extended Wittmann's result to Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex bounded subset of C has the fixed point property for nonexpansive mappings (e.g., L_p spaces $(1 < p < \infty)$). They proved the following theorem.

Theorem 2.2.5. *Let E be a Banach space whose norm is uniformly Gâateaux differentiable and let C be a closed convex subset of E. Let T be a nonexpansive mapping from C into C such that the set of fixed points of T is nonempty. Let $\{\alpha_n\}$ be a sequence which satisfies the following conditions:* $\lim \alpha_n = 0$; $\quad (ii) \sum_{n=1}^{\infty} \alpha_n = \infty$; $\quad (iii) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. *Let $u \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 \in C$, $x_{n+1} := \alpha_n u + (1-\alpha_n)Tx_n, n \geq 0$. Assume that $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \downarrow 0$, where for $0 < t < 1$, z_t is the unique element of C which satisfies $z_t = tu + (1-t)Tz_t$. Then, $\{x_n\}$ converges strongly to z.*

Halpern showed that the conditions (i) $\lim \alpha_n = 0$; *and* $(ii) \sum_{n=0}^{\infty} \alpha_n = \infty$ are necessary for the convergence of the sequence $\{x_n\}$ defined by (2.37). It is not known if in general they are sufficient. Some authors have established that if in the recursion formula (2.37), Tx_n is replaced with $T_n x_n := \sum_{k=0}^{n-1} T^k x_n$ then conditions (i) and (ii) are sufficient.

2.3 Some important concepts and results

In this section, we present further definitions and state some important results that will be needed in the proof of our convergence theorems.

Let T be a mapping on a Banach space E, then a point $x \in E$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$.

Definition 2.2 *(Demiclosedness property) Let $T : E \to E$ be a nonlinear mapping. Then T is said to be demiclosed at $y \in E$, if $x_n \rightharpoonup x \in E$ and $Tx_n \to y$, then $y = Tx$.*

Lemma 2.3.1. *(Demiclosedness principle) [172] Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be κ-strictly pseudocontractive mapping. Then $I - T$ is demi closed at 0, i.e., if $x_n \rightharpoonup x \in K$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.3.2. *[150] Assume that D is a strongly positive linear operator on a Hilbert space H with a coefficient $\delta > 0$ and $0 < \rho < ||D||^{-1}$. Then $||I - \rho D|| \leq 1 - \rho\delta$.*

Lemma 2.3.3. *[2] Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to H,\ (i = 1, 2, \cdots , N)$ be a finite family of k_i-strictly pseudocontractive mappings and suppose $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 2.3.4. *[2] Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_i : C \to H,\ (i = 1, 2, \cdots , N)$ be a finite family of k_i-strictly pseudocontractive mappings and suppose $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^N F(T_i)$.*

Lemma 2.3.5. *[2] Let C be a nonempty, closed and convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with a coefficient $\rho \in (0, 1)$ and D is a strongly positive linear bounded operator with a coefficient $\bar{\delta} > 0$. Then, for $0 < \delta < \frac{\bar{\delta}}{\rho}$,*

$$\langle x - y, (D - \delta f)x - (D - \delta f)y\rangle \geq (\bar{\delta} - \delta\rho)||x - y||^2, \quad \forall\ x, y \in H.$$

That is, $D - \delta f$ is strongly monotone with coefficient $\bar{\delta} - \delta\rho$.

Lemma 2.3.6. *[110] Let K be a nonempty subset of a real Hilbert space H and let $T : K \to P(K)$ be a κ-strictly pseudocontractive-type mapping such that $F_s(T)$ is nonempty. Then $F_s(T)$ is closed and convex.*

Lemma 2.3.7. *[81, 82] Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping, then for all $x, y \in H$,*

$$\langle (x - Tx) - (y - Ty), Ty - Tx\rangle \leq \frac{1}{2}||(Tx - x) - (Ty - y)||^2, \tag{2.39}$$

and consequently if $y \in F(T)$ then

$$\langle x - Tx, Ty - Tx\rangle \leq \frac{1}{2}||Tx - x||^2. \tag{2.40}$$

Lemma 2.3.8. *Let H be a real Hilbert space. Then the following results hold*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle, \forall\ x, y \in H$$

and

$$||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2, \quad \forall\ x, y \in H.$$

Lemma 2.3.9. *Let H be a Hilbert space, then $\forall x, y \in H$ and $\alpha \in (0,1)$, we have*

$$||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2.$$

Let E be a real Banach space and let E^* be the dual space of E. Denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x^*, x\rangle$. Let $B : E \to 2^{E^*}$ be a set-valued mapping, then the domain of B is defined as $domB := \{x \in E : Bx \neq \emptyset\}$ and the graph of B is given as $G(B) := \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. A set valued mapping B is said to be monotone if $\langle x^* - y^*, x - y\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in G(B)$ and B is said to be maximal monotone if its graph is not contained in the graph of any other monotone operator on E. It is known that if B is maximal monotone, then the set $B^{-1}(0) = \{\bar{x} \in E : 0 \in B(\bar{x})\}$ is closed and convex.

Lemma 2.3.10. *[134] Let $M : H \to 2^H$ be a maximal monotone mapping and $f : H \to H$ be a Lipschitz continuous mapping. Then the mapping $G = M + f : H \to 2^H$ is a maximal monotone mapping.*

A mapping $T : H \to H$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T := (1-\beta)I + \beta S$$

where $\beta \in (0,1)$, $S : H \to H$ is a nonexpansive mapping and I is the identity mapping on H. Every averaged mapping is nonexpansive and every firmly nonexpansive mapping is averaged. Thus since the resolvent of maximal monotone operators are firmly nonexpansive, they are averaged and therefore nonexpansive. For details, see [18, 45, 142, 160].

Lemma 2.3.11. *Let $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$||x - y||^q \leq ||x||^q - q\langle J_q(x), y\rangle + C_q||y||^q. \tag{2.41}$$

Lemma 2.3.12. *(Xu [230]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \ n \geq 0,$$

where
(i) $\{\alpha_n\} \subset [0,1], \ \sum \alpha_n = \infty$;
(ii) $\limsup \sigma_n \leq 0$;
(iii) $\gamma_n \geq 0, \ \sum \gamma_n < \infty$.
Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.3.13. *(Mainge [147]) Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.3.14. *(Reich and Sabach [189]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.*

Lemma 2.3.15. *(Reich and Sabach [188]) If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.3.16. *(Butnariu and Iusem [38]) The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 2.3.17. *(Reich and Sabach [189]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.*

Lemma 2.3.18. *(Suantai et al.[215]) Let E be a reflexive real Banach space. Let C be a nonempty, closed and convex function of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Suppose T is a left Bregman strongly nonexpansive mappings of C into E such that $F(T) = \hat{F}(T) \neq 0$. If $\{x_n\}_{n=0}^{\infty}$ is bounded sequence such that $x_n - Tx_n \to 0$ and $z := \overleftarrow{P}roj_{\Omega}^{f}u$, then*

$$\limsup_{n\to\infty}\langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

Lemma 2.3.19. *([189]) Let $A : E \to 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0^*) \neq \emptyset$. Then*

$$D_f(p, Res_{\lambda A}^{f}(x) + D_f(Res_{\lambda A}^{f}(x), x) \leq D_f(p, x)$$

for all $\lambda > 0, p \in A^{-1}(0^*)$ and $x \in E$.

Lemma 2.3.20. *([146]) Let $\{a_n\}$ be a sequence of non-negative numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n r_n,$$

where $\{r_n\}$ is a bounded sequence of real numbers and $\{\gamma_n\} \subset [0, 1]$ satisfies $\sum \gamma_n = \infty$. Then it follows that

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} r_n.$$

The following lemma was given by Wang and Xu [224]. We include its proof for the sake of completeness.

Lemma 2.3.21. *([224], Lemma 2.15) Assume $\gamma' \geq \bar{\gamma} \geq 0$. Then*

$$||\text{prox}_{\bar{\gamma}g}(I - \bar{\gamma}\nabla f)x - x|| \leq 2||\text{prox}_{\gamma' g}(I - \gamma'\nabla f)x - x||.$$

Chapter 3

Split Feasibility and Fixed Point Problems

3.1 Introduction

The Convex Feasibility Problem (CFP), as an important optimization problem is to find a common element of the intersection of finitely many convex sets (see [70]).
Let E_1 and E_2 be Banach spaces and let C and Q be nonempty closed and convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_2$ be a bounded linear operator, then the Split Feasibility Problem (SFP), see [54], is to find an element $x \in E_1$ satisfying

$$x \in C \text{ such that } Ax \in Q. \tag{3.1}$$

Let $\{C_i\}_{i=1}^{t}$ and $\{Q_j\}_{j=1}^{r}$ be nonempty, closed and convex subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively, where $t \geq 1$ and $r \geq 1$ are integers. Let A be an $M \times N$ real matrix and A^* the transpose of A. The Multiple-Sets Split Feasibility Problem (MSFP) introduced by Censor *et al.*[55] is to find a point $x^* \in \mathbb{R}^N$ such that

$$x^* \in C := \cap_{i=1}^{t} C_i, \quad Ax^* \in Q := \cap_{j=1}^{r} Q_j. \tag{3.2}$$

The MSFP can be applied as a unified model for various practical problems which include signal processing and image reconstruction [21], intensity-modulated radiation therapy [52, 53, 55, 135, 136] and other fields of applied sciences. If we let $t = r = 1$ in MSFP (3.2), we obtain the following problem: Find $x^* \in \mathbb{R}^N$ such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q$$

which is the SFP, for example, see [46, 45, 54, 85, 144, 158, 171, 216, 235, 236, 238, 239, 249].
Let S be the solution set of MSFP (3.2), then the MSFP corresponds to the following minimisation problem:

$$\min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2}\|x - P_C(x)\|^2 + \frac{1}{2}\|Ax - P_Q(Ax)\|^2 \right\},$$

where P_C and P_Q are the orthogonal projections onto C and Q respectively.
Censor *et al.* [55] defined the proximity function $p(x)$ to measure the distance of a point to all sets:

$$p(x) := \frac{1}{2}\sum_{i=1}^{t} l_i\|x - P_{C_i}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j\|Ax - P_{Q_j}(Ax)\|^2, \tag{3.3}$$

where $l_i (i = 1, 2, ..., t)$ and $\lambda_j (j = 1, 2, ..., r)$ are positive constants such that $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r} \lambda_j = 1$. Censor *et al.* [55] showed that

$$\nabla p(x) := \sum_{i=1}^{t} l_i(x - P_{C_i}(x)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_j})Ax. \tag{3.4}$$

Let $\Omega \subseteq \mathbb{R}^N$ be an auxiliary simple nonempty, closed and convex set such that $\Omega \cap S \neq \emptyset$, Censor *et al.* [55] studied the constrained MSFP as follows:

$$\text{find } \; x^* \in \Omega \; \text{ such that } \; x^* \; \text{ solves } \; (3.2). \tag{3.5}$$

They proposed the following projection algorithm

$$x_{n+1} = P_\Omega(x_n - s\nabla p(x_n)), \tag{3.6}$$

where s is a positive scalar and obtained the global convergence of this algorithm under the condition that $0 < a \leq s \leq b < \frac{2}{L}$, for some real numbers a, b and with L being the Lipschitz constant of ∇p. Algorithm (3.6) involves a fixed stepsize that is restricted by Lipschitz constant L of ∇p. But the Lipschitz constant L of ∇p depends on the largest eigenvalue (spectral radius) of the matrix A^*A, which is very hard to compute. To overcome the set back of computing the spectral radius of A^*A, Zhang *et al.* [248] inspired by the self adaptive method given by He *et al.* [104] for solving variational inequality problems, proposed a self adaptive projection method for solving the MSFP, which does not require the computation or estimation of the spectral radius of A^*A.
Zhao *et al.* [147], introduced a modified relaxed CQ algorithm to solve the MSFP that can be implemented easily, since it computes projections onto half-spaces and does not involve the computation or estimation of the spectral radius of A^*A.
Recently, authors have started studying SFP in infinite dimensional Hilbert spaces instead of $\mathbb{R}^N$ and $\mathbb{R}^M$ for example see [135, 236, 158].
Let the closed and convex subsets C_i $(i = 1, 2, ..., t)$ and Q_j $(j = 1, 2, ..., r)$ of an infinite dimensional Hilbert space be the level sets of convex functions given as follows:

$$C_i = \{x \in H_1 : c_i(x) \leq 0\} \;\text{ and }\; Q_j = \{y \in H_2 : q_j(y) \leq 0\}, \tag{3.7}$$

where $c_i : H_1 \to \mathbb{R}$, $\; i = 1, 2, ..., t$, and $q_j : H_2 \to \mathbb{R}$, $\; j = 1, 2, ..., r$, are convex functions. Let c_i $(i = 1, 2, ..., t)$ and q_j $(j = 1, 2, ..., r)$ be subdifferentiable on H_1 and H_2 respectively, and ∂c_i $(i = 1, 2, ..., t)$ and ∂q_j $(j = 1, 2, ..., r)$ be bounded operators (i.e. bounded on bounded sets).
Set

$$C_i^n = \{x \in H_1 : c_i(x_n) \leq \langle \xi_i^n, x_n - x\rangle\}, \tag{3.8}$$

where $\xi_i^n \in \partial c_i(x_n)$ for $i = 1, 2, ..., t$ and

$$Q_j^n = \{y \in H_2 : q_j(Ax_n) \leq \langle \zeta_j^n, Ax_n - y \rangle\}, \tag{3.9}$$

where $\zeta_j^n \in \partial q_j(Ax_n)$ for $j = 1, 2, ..., r$.
Obviously, C_i^n $(i = 1, 2, ..., t)$ and Q_j^n $(j = 1, 2, ..., r)$ are half spaces and it is easy to verify that $C_i^n \supset C_i$ $(i = 1, 2, ..., t)$ and $Q_j^n \supset Q_j$ $(j = 1, 2, ..., r)$ holds for every $n \geq 0$, noting the subdifferentiable inequality. Let p_n be defined as follows:

$$p_n(x) := \frac{1}{2}\sum_{i=1}^{t} l_i \|x - P_{C_i^n}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_i \|Ax - P_{Q_j^n}(Ax)\|^2 \tag{3.10}$$

and thus we have that

$$\nabla p_n(x) := \sum_{i=1}^{t} l_i \Big(x - P_{C_i^n}(x)\Big) + \sum_{j=1}^{r} \lambda_i A^* \Big(I - P_{Q_j^n}\Big) A(x), \tag{3.11}$$

where A^* is the adjoint of A. Lόpez *et al.* [135] introduced a relaxed CQ algorithm with a new adaptive way of determining the stepsize sequence (τ_n) for solving the SFP in infinite dimensional Hilbert spaces, where the closed and convex subsets C and Q are level sets of convex functions, and obtained a weak convergence result.
Precisely, Lόpez *et al.* [135] proved the following result:

Theorem 3.1.1. *Let H_1 and H_2 be real Hilbert spaces. Let $C = \{x \in H_1 : c(x) \leq 0\}$ and $Q = \{x \in H_2 : q(x) \leq 0\}$. Let $C_n = \{x \in H_1 : c_i(x_n) \leq \langle \xi_n, x_n - x \rangle\}, \xi_n \in \partial c(x_n)$; $Q_n = \{x \in H_2 : q_i(Ax_n) \leq \langle \zeta_n, Ax_n - y \rangle\}, \zeta_n \in \partial q(Ax_n)$; $f_n(x) = \frac{1}{2}\|(I - P_{Q_n})Ax\|^2, \nabla f_n(x) = A^*(I - P_{Q_n})Ax$ and*

$$\tau_n = \frac{\rho_n f_n(x_n)}{\Big\|\nabla f_n(x_n)\Big\|^2}, \ 0 < \rho_n < 4. \tag{3.12}$$

The sequence $\{x_n\}$ is constructed as follows:
Choose an arbitrary $x_0 \in H_1$. Assume x_n has been constructed. If $\nabla f_n(x_n) = 0$, then stop; otherwise continue and construct x_{n+1} via the formula

$$x_{n+1} = P_{C_n}(x_n - \tau_n \nabla f_n(x_n)).$$

Then under some certain conditions, $\{x_n\}$ converges weakly to a solution of SFP.

Since the computation τ_n does not depend on the operator norm of A, it is implementable. Motivated by the work of Zhao *et al.* [251] and Lόpez *et al* [135], He *et al.* [105] introduced a new relaxed CQ algorithm for the MSFP in infinite dimensional Hilbert spaces.

Let $u \in H_1$, and start an initial guess $x_0 \in H_1$ arbitrarily. Assume that x_n has been constructed. If $\nabla p_n(x_n) = 0$, then stop (x_n is an approximate solution of MSFP (3.2)). Otherwise continue and calculate x_{n+1} by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(x_n - \tau_n \nabla p_n(x_n)), \tag{3.13}$$

where the sequence $\alpha_n \in (0,1)$, ∇p_n is given in (3.11), $\tau_n = \dfrac{\rho_n p_n(x_n)}{\|\nabla p_n(x_n)\|^2}$, $0 < \rho_n < 4$. He *et al.* [105] with some mild conditions on the sequences $\{\alpha_n\}$ and $\{\rho_n\}$ proved that the above algorithm converges in norm to $P_S u$. For some other recent results on MSFP see [219, 158] and some of the references therein.
Schöpfer *et al.* [201] introduced in p-uniformly smooth Banach spaces the following algorithm: for $x_0 \in E_1$ and $n \geq 0$, set

$$x_{n+1} = \Pi_C J_q^{E_1^*}[J_p^{E_1}(x_n) - tA^* J_p^{E_2}(Ax_n - P_Q(Ax_n))], \tag{3.14}$$

where Π_C denotes the Bregman projection and J_p^E the duality mapping. Using algorithm (3.14), Schöpfer *et al.* [201] obtained a weak convergence result in a p-uniformly convex and uniformly smooth Banach space, with the condition that the duality mapping of E is sequentially weak-to-weak continuous. To obtain strong convergence, Wang [222] based on an idea in Nakajo-Takahashi [165], introduced the following algorithm: for any initial guess x_0, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = T_n x_n, \\ D_n = \{u \in E : \Delta_p(y_n, u) \leq \Delta_p(x_n, u)\}, \\ E_n = \{u \in E : \langle x_n - u, J_p^E x_0 - J_p^E x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{D_n \cap E_n}(x_0), \end{cases} \tag{3.15}$$

where T_n is defined for each $n \in \mathbb{N}$ by

$$T_n x = \begin{cases} \Pi_{C_{i(n)}}(x), & 1 \leq i(n) \leq r, \\ J_q^{E_1^*}[J_p^{E_1} x - t_n A^* J_p^{E_2}(I - P_{Q_{i(n)}})Ax], & r+1 \leq i(n) \leq r+s, \end{cases} \tag{3.16}$$

$i : \mathbb{N} \to I$ is the cyclic control mapping $i(n) = n mod(r+s) + 1$, and t_n satisfies

$$0 < t \leq t_n \leq \left(\frac{q}{C_q ||A||^p}\right)^{\frac{1}{q-1}}.$$

Censor and Segal [60] in 2008 proposed the Split Common Fixed Point Problem (SCFPP) for directed operators in finite dimensional Hilbert spaces. They developed an algorithm for the two set SCFPP which generated a sequence $\{x_n\}$ according to the iterative procedure: Let $x_0 \in \mathbb{R}^n$ be arbitrary

$$x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k), \quad k \in \mathbb{N}$$

where $\gamma \in \left(0, \dfrac{2}{\lambda}\right)$ with λ being the spectral radius of the operator A^*A and proved the convergence of the algorithm in finite dimensional spaces. Cui et al. [83] proposed a damped projection method for SCFPP and studied its convergence result. Inspired by the work of Censor and Segal, Moudafi [161] introduced the following algorithm for SCFPP for demicontractive operators T and U in Hilbert spaces:

$$\begin{cases} x_0 \in H_1, \\ u_k = x_k + \gamma A^*(T - I)Ax_k, \\ x_{n+1} = (1 - t_k)u_k + t_k U y_k, \quad k \in \mathbb{N}, \end{cases} \tag{3.17}$$

where $\gamma \in \left(0, \dfrac{1-\mu}{\lambda}\right)$, with λ being the spectral radius of the operator A^*A, $t_k \in (0,1)$. Using the iterative scheme (3.17), Moudafi proved *weak convergence result* for the SCFPP. We observe that strong convergence result of SCFPP using iterative scheme (3.17) can be obtained by putting compactness assumption on the space or on the operator or by modifying the iterative scheme (3.17).

Furthermore, Cui and Wang [84] give a new algorithm that does not require any prior information on the operator norm to find a solution of SCFPP and obtained a weak convergence result. Also in 2011, Moudafi [162] considered the relaxed algorithm for computing the approximate solutions of SCFPP for quasi-nonexpansive operators and studied the weak convergence of the proposed algorithm. Kraikaew and Saejung [127] considered SCFPP for quasi-nonexpansive operators and obtained a strong convergence result with their proposed algorithm.

Let $B : H_1 \to 2^{H_1}$ be a multivalued mapping, $T : H_2 \to H_2$ a mapping, and $A : H_1 \to H_2$ a bounded linear operator. Takahashi *et al.*[217] studied the following problem: find $x \in H_1$ such that

$$0 \in B(x) \quad \text{and} \quad Ax \in F(T). \tag{3.18}$$

We shall denote by Ω the solution set of (3.18).
Takahashi *et al.*[217] stated and proved the following two weak convergence results.

Theorem 3.1.2. *[217] Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda^B = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and $A : H_1 \to H_2$ a bounded linear operator. Suppose $B^{-1}(0) \cap A^{-1}F(T) \neq \emptyset$. For any $x_1 = x \in H_1$, define*

$$x_{n+1} = J_{\lambda_n}^B (I - \gamma_n A^*(I-T)A)x_n, \quad \forall n \in \mathbb{N},$$

where the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:
(i) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \infty$,
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \dfrac{1}{||A||^2}$.
Then $\{x_n\}$ converges weakly to a point $z_0 \in B^{-1}(0) \cap A^{-1}F(T)$, which is a strong limit of the projection of $\{x_n\}$ onto $B^{-1} \cap A^{-1}F(T)$, that is $z_0 = \lim_{n\to\infty} P_{B^{-1}(0)\cap A^{-1}F(T)} x_n$.

Theorem 3.1.3. *[217] Let H_1 and H_2 be Hilbert spaces. Let $B : H \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda^B = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and $A : H_1 \to H_2$ a bounded linear operator. Suppose $B^{-1}(0) \cap A^{-1}F(T) \neq \emptyset$, for any $x_1 = x \in H_1$, define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J_{\lambda_n}^B (I - \lambda_n A^*(I-T)A)x_n, \quad \forall\ n \in \mathbb{N},$$

where the sequences $\{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions:
(C1) $\sum_{n=1}^\infty \beta_n(1-\beta_n) < \infty$,
(C2) $0 < a \leq \lambda_n \leq \dfrac{1}{||A||^2}$ and $\sum_{n=1}^\infty |\lambda_n - \lambda_{n+1}| < \infty$.
Then $x_n \rightharpoonup z_0 \in B^{-1}(0) \cap A^{-1}F(T)$, where $z_0 = \lim_{n\to\infty} P_{B^{-1}(0)\cap A^{-1}F(T)} x_n$.

In this chapter, we give iterative solution of generalised split feasibility problems which do not depend on any knowledge of the operator norm. Also iterative solutions of split feasibility and multiple set split feasibility problems were given with numerical computations in the frame work of p-uniformly convex Banach spaces which are uniformly smooth.

3.2 A non operator norm dependent iterative solution of generalised split feasibility problems

In this section, we consider problem (3.18) for the case where T is a demicontractive mapping and obtained a strong convergence result. Since there exists some linear operator A such that $||A||$ is very hard if not impossible to calculate or to estimate, we made a proper choice of the step length γ_n, introduced an iterative scheme that do not require any prior knowledge of the operator norm and obtained a strong convergence result.

Theorem 3.2.1. *Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be κ-demicontractive mapping and suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let the step size γ_n be chosen such that for some $\epsilon > 0$, $\gamma_n \in \Big(\epsilon, \dfrac{(1-\kappa)||TAy_n - Ay_n||^2}{||A^*(T-I)Ay_n||^2} - \epsilon\Big)$, if $TAy_n \neq Ay_n$; otherwise $\gamma_n = \gamma$(γ being any nonnegative real number). Suppose $\{\alpha_n\}$ is a sequence in $(0,1)$, and the following conditions are satisfied:*
(1) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(2) $T - I$ are demiclosed at the origin.
Then the sequence $\{x_n\}$ generated for any $x_0, u \in H_1$ by

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n u, \\ x_{n+1} = J_\lambda^B(y_n + \gamma_n A^*(T-I)Ay_n), \quad n \geq 0, \end{cases} \tag{3.19}$$

converges strongly to $\bar{x} = P_\Omega u$.

Proof. Let $p \in \Omega$, from (3.19), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||J_\lambda^B(y_n + \gamma_n A^*(T-I)Ay_n) - p||^2 \\ &\leq ||y_n + \gamma_n A^*(T-I)Ay_n - p||^2 \\ &= ||y_n - p||^2 + \gamma_n^2||A^*(T-I)Ay_n||^2 + 2\gamma_n\langle y_n - p, A^*(T-I)Ay_n\rangle \\ &= ||y_n - p||^2 + \gamma_n^2||A^*(T-I)Ay_n||^2 + 2\gamma_n\langle A(y_n - p), (T-I)Ay_n\rangle \\ &= ||y_n - p||^2 + \gamma_n^2||A^*(T-I)Ay_n||^2 \\ &\quad +2\gamma_n[\langle Ay_n - Ap, TAy_n - Ap\rangle + \langle Ay_n - Ap, Ap - Ay_n\rangle] \\ &= ||y_n - p||^2 + \gamma_n^2||A^*(T-I)Ay_n||^2 \\ &\quad +2\gamma_n[\langle Ay_n - Ap, TAy_n - Ap\rangle - ||Ay_n - Ap||^2] \end{aligned}$$

$$
\begin{aligned}
&\leq ||y_n - p||^2 + \gamma_n^2||A^*(T-I)Ay_n||^2 \\
&\quad +\gamma_n[(\kappa-1)||TAy_n - Ay_n||^2 - 2||Ay_n - Ap||^2] \\
&\leq ||y_n - p||^2 + \gamma_n^2||A^*(T-I)Ay_n||^2 + \gamma_n(\kappa-1)||TAy_n - Ay_n||^2 \\
&\leq ||y_n - p||^2 + \gamma_n[\gamma_n||A^*(T-I)Ay_n||^2 \\
&\quad +(\kappa-1)||TAy_n - Ay_n||^2]. \qquad (3.20)
\end{aligned}
$$

From the choice of γ_n, (3.20) and (3.19), we get

$$
\begin{aligned}
||x_{n+1} - p|| &\leq ||y_n - p|| \\
&\leq (1-\alpha_n)||x_n - p|| + \alpha_n||p - u|| \\
&\leq \max\{||x_n - p||, ||p - u||\} \\
&\vdots \\
&\leq \max\{||x_0 - p||, ||p - u||\}. \qquad (3.21)
\end{aligned}
$$

Hence, $\{||x_n - p||\}$ is bounded and therefore $\{x_n\}$ and $\{y_n\}$ are also bounded.

We now divide into two cases to establish strong convergence.
Case 1. Suppose $\{||x_n - p||\}$ is monotonically decreasing, then obviously

$$
||x_{n+1} - p|| - ||x_n - p|| \to 0, n \to \infty. \qquad (3.22)
$$

If $TAy_n = Ay_n$, then

$$
\lim_{n\to\infty} ||A^*(T-I)Ay_n|| = 0.
$$

Suppose $TAy_n \neq Ay_n$, then $\gamma_n \in \Big(\epsilon, \dfrac{(1-\kappa)||TAy_n - Ay_n||^2}{||A^*(T-I)Ay_n||^2} - \epsilon\Big)$ and from (3.20), we have

$$
\begin{aligned}
||x_{n+1} - p||^2 &\leq ||y_n - p||^2 + \gamma_n[\gamma_n||A^*(T-I)Ay_n||^2 + (\kappa-1)||TAy_n - Ay_n||^2] \\
&= ||x_n - p||^2 + \alpha_n^2||x_n - u||^2 - 2\alpha_n\langle x_n - p, x_n - u\rangle \\
&\quad +\gamma_n[\gamma_n||A^*(T-I)Ay_n||^2 + (\kappa-1)||TAy_n - Ay_n||^2] \\
&\leq ||x_n - p||^2 + \alpha_n^2||x_n - u||^2 - 2\alpha_n\langle x_n - p, x_n - u\rangle \\
&\quad -\gamma_n\epsilon||A^*(T-I)Ay_n||^2. \qquad (3.23)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\gamma_n\epsilon||A^*(T-I)Ay_n||^2 &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n^2||x_n - u||^2 \\
&\quad -2\alpha_n\langle x_n - p, x_n - u\rangle \to 0, n \to \infty. \qquad (3.24)
\end{aligned}
$$

Thus

$$
\lim_{n\to\infty} ||A^*(T-I)Ay_n|| = 0. \qquad (3.25)
$$

Again from (3.23), we have

$$
\begin{aligned}
||x_{n+1} - p||^2 &\leq ||x_n - p||^2 + \alpha_n^2||x_n - u||^2 - 2\alpha_n\langle x_n - p, x_n - u\rangle \\
&\quad +\gamma_n[\gamma_n||A^*(T-I)Ay_n||^2 + (\kappa-1)||TAy_n - Ay_n||^2], \qquad (3.26)
\end{aligned}
$$

which implies

$$\begin{aligned}\gamma_n(1-\kappa)||TAy_n - Ay_n||^2 &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 \\ &+\alpha_n^2||x_n - u||^2 - 2\alpha_n\langle x_n - p, x_n - u\rangle \\ &+\gamma_n^2||A^*(T-I)Ay_n||^2 \to 0, n \to \infty. \qquad (3.27)\end{aligned}$$

Hence,

$$\lim_{n\to\infty} ||TAy_n - Ay_n|| = 0. \qquad (3.28)$$

From (3.19)

$$||x_n - y_n|| = \alpha_n||x_n - u|| \to 0, n \to \infty. \qquad (3.29)$$

Now, from (3.20)

$$\begin{aligned}||x_{n+1} - p||^2 &= ||J_\lambda^B(y_n + \gamma_n A^*(T-I)Ay_n) - p||^2 \\ &\leq \langle x_{n+1} - p, y_n + \gamma_n A^*(T-I)Ay_n - p\rangle \\ &= \frac{1}{2}[||x_{n+1} - p||^2 + ||y_n + \gamma_n A^*(T-I)Ay_n - p||^2 \\ &-||x_{n+1} - p - (y_n + \gamma_n A^*(T-I)Ay_n) - p)||^2] \\ &\leq \frac{1}{2}[||x_{n+1} - p||^2 + ||y_n - p||^2 + \gamma_n[\gamma_n||A^*(T-I)Ay_n||^2 \\ &+(\kappa - 1)||TAy_n - Ay_n||^2] - ||x_{n+1} - y_n - \gamma_n A^*(T-I)Ay_n - p)||^2] \\ &\leq \frac{1}{2}[||x_{n+1} - p||^2 + ||y_n - p||^2 - (||x_{n+1} - y_n||^2 + \gamma_n^2||A^*(T-I)Ay_n||^2 \\ &-2\gamma_n\langle x_{n+1} - y_n, A^*(T-I)Ay_n\rangle)] \\ &\leq \frac{1}{2}[||x_{n+1} - p||^2 + ||y_n - p||^2 - ||x_{n+1} - y_n||^2 \\ &+2\gamma_n||A(x_{n+1} - y_n)||||(T-I)Ay_n||]. \qquad (3.30)\end{aligned}$$

That is,

$$\begin{aligned}||x_{n+1} - p||^2 &\leq ||y_n - p||^2 - ||x_{n+1} - y_n||^2 \\ &+2\gamma_n||A(x_{n+1} - y_n)||||(T-I)Ay_n||, \qquad (3.31)\end{aligned}$$

which implies that

$$\begin{aligned}||x_{n+1} - y_n||^2 &\leq ||y_n - p||^2 - ||x_{n+1} - p||^2 + 2\gamma_n||A(x_{n+1} - y_n)||||(T-I)Ay_n|| \\ &= ||(1-\alpha_n)x_n + \alpha_n u - p||^2 - ||x_{n+1} - p||^2 + 2\gamma_n||A(x_{n+1} - y_n)||||(T-I)Ay_n|| \\ &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n^2||p - u||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, p - u\rangle \\ &+2\gamma_n||A(x_{n+1} - y_n)||||(T-I)Ay_n|| \to 0 \quad as \quad n \to \infty. \qquad (3.32)\end{aligned}$$

Therefore,

$$||x_{n+1} - y_n|| \to 0 \quad as \quad n \to \infty. \qquad (3.33)$$

It follows from (3.29) and (3.33) that

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - y_n|| + ||y_n - x_n|| \to 0, n \to \infty.$$

Let $u_n = y_n + \gamma_n A^*(T - I)Ay_n$.

$$||y_n - u_n|| = \gamma_n ||A^*(T - I)Ay_n|| \to 0, n \to \infty.$$

Thus

$$||x_{n+1} - u_n|| \leq ||x_{n+1} - y_n|| + ||y_n - u_n|| \to 0, n \to \infty.$$

Since $\{x_n\}$ is bounded and H_1 is a Hilbert space, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$, for some $q \in H_1$. Furthermore, $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$, implies that $\{x_{n_i+1}\} \rightharpoonup q$.
We now show that $0 \in B(q)$.
Let $(v, z) \in G(B)$, that is $z \in B(v)$. From $x_{n_i+1} = J_\lambda^B u_{n_i}$ we obtain that

$$u_{n_i} \in (I + \lambda B)x_{n_i+1}.$$

That is

$$\frac{1}{\lambda}(u_{n_i} - x_{n_i+1}) \in B(x_{n_i+1}).$$

Using the maximal monotonicity of B, we have

$$\langle v - x_{n_i+1}, z - \frac{1}{\lambda}(u_{n_i} - x_{n_i+1})\rangle \geq 0.$$

Therefore,

$$\langle v - x_{n_i+1}, z\rangle \quad \geq \quad \langle v - x_{n_i+1}, \frac{1}{\lambda}(u_{n_i} - x_{n_i+1})\rangle. \tag{3.34}$$

Since $x_{n_i+1} \rightharpoonup q$, we have

$$\lim_{i\to\infty} \langle v - x_{n_i+1}, z\rangle = \langle v - p, z\rangle.$$

Thus from (3.34)

$$\langle v - q, z\rangle \geq 0.$$

Since B is maximally monotone, we have $0 \in B(q)$.
Moreover, since $\|y_n - x_{n+1}\| \to 0$, we have that Ay_{n_i} converges weakly to Aq and by (3.28) and the fact that $I - T$ is demiclosed at 0, we get that

$$Aq \in F(T).$$

Hence, $q \in \Omega$.
Next, we prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to $\bar{x}$.
Choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - \bar{x}, u - \bar{x}\rangle = \lim_{i\to\infty} \langle x_{n_i} - \bar{x}, u - \bar{x}\rangle.$$

Since $x_{n_i} \rightharpoonup q$ then it follows from Proposition2.1.3 that

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - \bar{x}, u - \bar{x}\rangle &= \lim_{i\to\infty}\langle x_{n_i} - \bar{x}, u - \bar{x}\rangle \\ &= \langle q - \bar{x}, u - \bar{x}\rangle \le 0. \end{aligned} \tag{3.35}$$

From (3.20)

$$\begin{aligned} ||x_{n+1} - \bar{x}|| &\le ||y_n - \bar{x}||^2 + \gamma_n[\gamma_n||A^*(T-I)Ay_n||^2 \\ &\quad +(\kappa - 1)||TAy_n - Ay_n||^2] \\ &\le ||y_n - \bar{x}||^2 \\ &= ||(1-\alpha_n)(x_n - \bar{x}) - \alpha_n(\bar{x} - u)||^2 \\ &= (1-\alpha_n)^2||x_n - q||^2 + \alpha_n^2||\bar{x} - u||^2 \\ &\quad -2\alpha_n(1-\alpha_n)\langle x_n - \bar{x}, \bar{x} - u\rangle \\ &\le (1-\alpha_n)||x_n - q||^2 + \alpha_n[\alpha_n||\bar{x} - u||^2 \\ &\quad -2(1-\alpha_n)\langle x_n - \bar{x}, \bar{x} - u\rangle]. \end{aligned} \tag{3.36}$$

Clearly,

$$\alpha_n||\bar{x} - u||^2 - 2(1-\alpha_n)\langle x_n - \bar{x}, \bar{x} - u\rangle \to 0.$$

Applying Lemma 2.3.12 to (3.36), we obtain that $\{x_n\}$ converges strongly to q.

Case 2. Assume that $\{||x_n - p||\}$ is not monotonically decreasing sequence.
Set $\Gamma_n = ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$(for some n_0 large enough)

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \quad for \quad n \ge n_0.$$

From (3.24), we have

$$\begin{aligned}\gamma_{\tau(n)}\epsilon||A^*(T-I)Ay_{\tau(n)}||^2 &\le \alpha_{\tau(n)}^2||x_{\tau(n)} - u|| \\ &\quad -2\alpha_{\tau(n)}\langle x_{\tau(n)} - p, x_{\tau(n)} - u\rangle \to 0, \tau(n) \to \infty. \end{aligned} \tag{3.37}$$

Thus

$$||A^*(T-I)Ay_{\tau(n)}|| \to 0, \text{ as } n \to \infty.$$

Again from (3.23), we have

$$\begin{aligned} ||x_{\tau(n)+1} - p||^2 &\le ||x_{\tau(n)} - p||^2 + \alpha_{\tau(n)}^2||x_{\tau(n)} - u||^2 \\ &\quad -2\alpha_{\tau(n)}\langle x_{\tau(n)} - p, x_{\tau(n)} - u\rangle \\ &\quad +\gamma_{\tau(n)}[\gamma_{\tau(n)}||A^*(T-I)Ay_{\tau(n)}||^2 \\ &\quad +(\kappa - 1)||TAy_{\tau(n)} - Ay_{\tau(n)}||^2], \end{aligned} \tag{3.38}$$

which implies

$$\begin{aligned}\gamma_{\tau(n)}(1-\kappa)||TAy_{\tau(n)}-Ay_{\tau(n)}||^2 &\leq \alpha^2_{\tau(n)}||x_{\tau(n)}-u||^2-2\alpha_{\tau(n)}\langle x_{\tau(n)}-p, x_{\tau(n)}-u\rangle \\ &\quad +\gamma^2_{\tau(n)}||A^*(T-I)Ay_{\tau(n)}||^2 \to 0, n\to\infty. \qquad (3.39)\end{aligned}$$

Hence,

$$||TAy_{\tau(n)}-Ay_{\tau(n)}|| \to 0.$$

By same argument as in case 1, we get that there is a subsequence of $\{x_{\tau(n)}\}$ also denoted as $\{x_{\tau(n)}\}$ which converges weakly to $q \in \Omega$ as $\tau(n) \to \infty$ and

$$\limsup_{n\to\infty}\langle x_{\tau(n)}-\bar{x}, u-\bar{x}\rangle \leq 0.$$

Now, for all $n \geq n_0$,

$$\begin{aligned}0 &\leq ||x_{\tau(n)+1}-\bar{x}||^2-||x_{\tau(n)}-\bar{x}||^2 \\ &\leq \alpha_{\tau(n)}[\alpha_{\tau(n)}||\bar{x}-u||^2-2(1-\alpha_{\tau(n)})\langle x_{\tau(n)}-\bar{x}, \bar{x}-u\rangle-||x_{\tau(n)}-\bar{x}||^2],\end{aligned}$$

i.e

$$||x_{\tau(n)}-\bar{x}||^2 \leq \alpha_{\tau(n)}||\bar{x}-u||^2-2(1-\alpha_{\tau(n)})\langle x_{\tau(n)}-\bar{x}, \bar{x}-u\rangle \to 0.$$

Hence,

$$\lim_{n\to\infty}||x_{\tau(n)}-\bar{x}|| = 0.$$

Therefore,

$$\lim_{n\to\infty}||x_{\tau(n)}-\bar{x}||^2 = \lim_{n\to\infty}||x_{\tau(n)+1}-\bar{x}||^2 = 0.$$

Furthermore, for $n \geq n_0$, it is easily observed that $||x_{\tau(n)}-\bar{x}||^2 \leq ||x_{\tau(n)+1}-\bar{x}||^2$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $||x_j-\bar{x}||^2 > ||x_{j+1}-\bar{x}||^2$ for $\tau(n)+1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq ||x_n-\bar{x}||^2 \leq \max\{||x_{\tau(n)}-\bar{x}||^2, ||x_{\tau(n)+1}-\bar{x}||^2\} = ||x_{\tau(n)+1}-\bar{x}||^2.$$

So, $\lim_{n\to\infty}||x_n-\bar{x}||^2 = 0$, that is $\{x_n\}$ and $\{y_n\}$ converge strongly to $\bar{x}$.

□

Corollary 3.2.2. *Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J^B_\lambda = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be quasi-nonexpansive mapping and suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let the step size γ_n be chosen such that for some $\epsilon > 0$, $\gamma_n \in \Big(\epsilon, \dfrac{(1-\kappa)||TAy_n-Ay_n||^2}{||A^*(T-I)Ay_n||^2}-\epsilon\Big)$, if $TAy_n \neq Ay_n$; otherwise $\gamma_n = \gamma$(γ being any nonnegative real number). Suppose $\{\alpha_n\}$ is a sequence in $(0,1)$, and the following conditions are satisfied:*
(1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$,
(2) $I-T$ is demiclosed at 0.
Then the sequence $\{x_n\}$ generated for any $x_0, u \in H_1$ by

$$\begin{cases} y_n = (1-\alpha_n)x_n+\alpha_n u, \\ x_{n+1} = J^B_\lambda(y_n+\gamma_n A^*(T-I)Ay_n), \quad n \geq 0, \end{cases} \qquad (3.40)$$

converges strongly to $\bar{x} = P_\Omega u$.

Corollary 3.2.3. *Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be nonexpansive mapping and suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let the step size γ_n be chosen such that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \dfrac{(1-\kappa)||TAy_n - Ay_n||^2}{||A^*(T-I)Ay_n||^2} - \epsilon\right)$, if $TAy_n \neq Ay_n$; otherwise $\gamma_n = \gamma$(γ being any nonnegative real number). Suppose $\{\alpha_n\}$ is a sequence in $(0,1)$, and the following condition are satisfied:*
$\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
Then the sequence $\{x_n\}$ generated for any $x_0, u \in H_1$ by

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n u, \\ x_{n+1} = J_\lambda^B(y_n + \gamma_n A^*(T-I)Ay_n), \quad n \geq 0, \end{cases} \tag{3.41}$$

converges strongly to $\bar{x} = P_\Omega u$.

3.3 Convergence analysis of iterative method for multiple-set split feasibility problems in certain Banach spaces

Our aim here is to construct an iterative scheme for approximating a solution of MSFP (3.2) and prove strong convergence of the sequence generated by our scheme in p-uniformly convex real Banach spaces which are also uniformly smooth. Our result complements some related results in the literature.

Theorem 3.3.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C_i, $i = 1,2,...,t$ and Q_j, $j = 1,2,...,r$ be finite families of nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that MSFP* (3.2) *(see also (1) in [219]) has a nonempty solution set Ω_{MS}. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_{i,n}\}$ be sequences in $(0,1)$. For a fixed $u \in H_1$, let sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_1 \in E_1$,*

$$\begin{cases} x_n = J_q^{E_1^*}[J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}((I - P_{\cap_{j=1}^r Q_j})Au_n)], \\ u_{n+1} = \Pi_C J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^t \gamma_{i,n} J_p^{E_1}(\Pi_{C_i} x_n)\Big)\Big], \end{cases} \tag{3.42}$$

with the conditions
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < a \leq \beta_n \leq b < 1$;
(iv) $\beta_n + \sum_{i=1}^t \gamma_{i,n} = 1$;
(v) $0 < t \leq t_n \leq k < \left(\dfrac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.
Then, $\{u_n\}$ and $\{x_n\}$ both converge strongly to $\bar{x}$, where $\bar{x} = \Pi_{\Omega_{MS}} u$.

Proof. Let $z \in \Omega_{MS}$, then it follows from (2.24) that

$$\begin{aligned}
\langle J_p^{E_2}((I - P_{\cap_{j=1}^r Q_j})Au_n), Au_n - Az\Big\rangle &= \|Au_n - P_{\cap_{j=1}^r Q_j}(Au_n)\|^p \\
&\quad + \langle J_p^{E_2}((I - P_{\cap_{j=1}^r Q_j})Au_n), P_{\cap_{j=1}^r Q_j}(Au_n) - Az\rangle \\
&\geq \|(I - P_{\cap_{j=1}^r Q_j})Au_n\|^p. \qquad (3.43)
\end{aligned}$$

Also, from (3.43) and Lemma 2.3.8, we obtain

$$\begin{aligned}
\Delta_p(x_n, z) &\leq \Delta_p(J_q^{E_1^*}[J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}((I - P_{\cap_{j=1}^r Q_j})Au_n)], z) \\
&= \frac{1}{q}||J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}(I - P_{\cap_{j=1}^r Q_j})Au_n||^q - \langle J_p^{E_1}(u_n), z\rangle \\
&\quad + t_n\langle J_p^{E_2}((I - P_{\cap_{j=1}^r Q_j})Au_n), Az\rangle + \frac{1}{p}||z||^p \\
&\leq \frac{1}{q}||J_p^{E_1}(u_n)||^q - t_n\langle Au_n, J_p^{E_2}(I - P_{\cap_{j=1}^r Q_j})Au_n)\rangle \\
&\quad + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_2}((I - P_{\cap_{j=1}^r Q_j})Au_n)||^q \\
&\quad - \langle J_p^{E_1}(u_n), z\rangle + t_n\langle Az, J_p^{E_2}((I - P_{\cap_{j=1}^r Q_j})Au_n)\rangle + \frac{1}{p}||z||^P \\
&= \frac{1}{q}||u_n||^p - \langle J_p^{E_1}(u_n), z\rangle + \frac{1}{p}||z||^P + t_n\langle J_p^{E_2}(I - P_{\cap_{j=1}^r Q_j})Au_n, Az - Au_n\rangle \\
&\quad + \frac{C_q(t_n||A||)^q}{q}||(I - P_{\cap_{j=1}^r Q_j})Au_n||^p \\
&= \Delta_p(u_n, z) + t_n\langle J_p^{E_2} w_n, Az - Au_n\rangle + \frac{C_q(t_n||A||)^q}{q}||(I - P_{\cap_{j=1}^r Q_j})Au_n||^p \\
&\leq \Delta_p(u_n, z) - \Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big)||(I - P_{\cap_{j=1}^r Q_j})Au_n||^p. \qquad (3.44)
\end{aligned}$$

Now, using condition (iv) in (3.44), we have

$$\Delta_p(x_n, z) \leq \Delta_p(u_n, z).$$

Furthermore, using (2.26) in (3.42), we have that

$$\begin{aligned}
\Delta_p(u_{n+1}, z) &\leq \Delta_p\Big(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^t \gamma_{i,n} J_p^{E_1}(\Pi_{C_i} x_n)\Big)\Big], z\Big) \\
&= \Delta_p\Big(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\beta_n J_p^{E_1}(x_n) + (1-\alpha_n)\sum_{i=1}^t \gamma_{i,n} J_p^{E_1}(\Pi_{C_i} x_n)\Big], z\Big) \\
&\leq \alpha_n \Delta_p(u, z) + (1-\alpha_n)\beta_n \Delta_p(x_n, z) + (1-\alpha_n)\sum_{i=1}^t \gamma_{i,n}\Delta_p(\Pi_{C_i} x_n, z)
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n \Delta_p(u,z) + (1-\alpha_n)\beta_n \Delta_p(x_n,z) + (1-\alpha_n)\sum_{i=1}^{t} \gamma_{i,n}\Delta_p(x_n,z) \\
&= \alpha_n \Delta_p(u,z) + (1-\alpha_n)\Delta_p(x_n,z) \\
&\leq \alpha_n \Delta_p(u,z) + (1-\alpha_n)\Delta_p(u_n,z) \\
&\leq \max\{\Delta_p(u,z), \Delta_p(u_n,z)\} \\
&\vdots \\
&\leq \max\{\Delta_p(u,z), \Delta_p(u_0,z)\}.
\end{aligned} \tag{3.45}
$$

Therefore, $\{\Delta_p(u_n,z)\}$ is bounded and consequently, we have that $\{\Delta_p(x_n,z)\}$ is bounded. Thus, the sequences $\{u_n\}$ and $\{x_n\}$ are bounded.
Now,

$$
\begin{aligned}
\Delta_p(x_{n+1},z) &\leq \Delta_p(u_{n+1},z) \\
&\leq \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^{t}\gamma_{i,n} J_p^{E_1}(\Pi_{C_i}x_n)\Big)\Big], z) \\
&= V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)\sum_{i=1}^{t}\gamma_{i,n} J_p^{E_1}(\Pi_{C_i}x_n)\Big), z) \\
&\leq V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^{t}\gamma_{i} J_p^{E_1}(\Pi_{C_i}x_n)\Big) \\
&\quad -\alpha_n(J_p^{E_1}u - J_p^{E_1}z), z) - \langle -\alpha_n(J_p^{E_1}u - J_p^{E_1}z), J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) \\
&\quad +(1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^{t}\gamma_{i,n} J_p^{E_1}(\Pi_{C_i}x_n)\Big)\Big] - z\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= V_p(\alpha_n J_p^{E_1}(z) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^{t}\gamma_{i,n} J_p^{E_1}(\Pi_{C_i}x_n)\Big) \\
&\quad +\alpha_n\langle J_p^{E_1}u - J_p^{E_1}z, u_{n+1} - z\rangle \\
&= \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(z) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^{t}\gamma_{i,n} J_p^{E_1}(\Pi_{C_i}x_n)\Big)\Big], z) \\
&\quad +\alpha_n\langle J_p^{E_1}u - J_p^{E_1}z, u_{n+1} - z\rangle \\
&\leq \alpha_n\Delta_p(z,z) + (1-\alpha_n)\beta_n\Delta_p(x_n,z) + (1-\alpha_n)\sum_{i=1}^{t}\gamma_{i,n}\Delta_p(\Pi_{C_i}x_n,z) \\
&\quad +\alpha_n\langle J_p^{E_1}u - J_p^{E_1}z, u_{n+1} - z\rangle \\
&\leq (1-\alpha_n)\Delta_p(x_n,z) + \alpha_n\langle J_p^{E_1}u - J_p^{E_1}z, u_{n+1} - z\rangle.
\end{aligned} \tag{3.46}
$$

We now consider two cases to prove the strong convergence.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Delta_p(x_n,z)\}_{n=n_0}^{\infty}$ is monotonically non-increasing. Then, obviously $\{\Delta_p(x_n,z)\}$ converges and

$$
\Delta_p(x_{n+1},z) - \Delta_p(x_n,z) \to 0, \quad n \to \infty. \tag{3.47}
$$

Let $s_n = J_q^{E_1^*}(\beta_n J_p^{E_1} x_n + \sum_{i=1}^t \gamma_{i,n}\Delta_p(\Pi_{C_i} x_n), n \geq 1.$
Then

$$\begin{aligned}
\Delta_p(s_n, z) &= \Delta_p(J_q^{E_1^*}(\beta_n J_p^{E_1} x_n + \sum_{i=1}^t \gamma_{i,n}\Delta_p(\Pi_{C_i} x_n, z)) \\
&\leq \beta_n \Delta_p(x_n, z) + \sum_{i=1}^t \gamma_{i,n}\Delta_p(\Pi_{C_i} x_n, z) \\
&\leq \beta_n \Delta_p(x_n, z) + \sum_{i=1}^t \gamma_{i,n}\Delta_p(x_n, z) \\
&= \Delta_p(x_n, z). \qquad (3.48)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
0 &\leq \Delta_p(x_n, z) - \Delta_p(s_n, z) \\
&= \Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) + \Delta_p(x_{n+1}, z) - \Delta_p(s_n, z) \\
&\leq \Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) + \Delta_p(u_{n+1}, z) - \Delta_p(s_n, z) \\
&\leq \Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) + \alpha_n \Delta_p(u, z) + (1-\alpha_n)\Delta_p(s_n, z) - \Delta_p(s_n, z) \\
&= \Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) + \alpha_n(\Delta_p(u, z) - \Delta_p(s_n, z)) \to 0, \quad n \to \infty. \qquad (3.49)
\end{aligned}$$

Again, we obtain

$$\begin{aligned}
\Delta_p(s_n, z) &\leq \beta_n \Delta_p(x_n, z) + \sum_{i=1}^t \gamma_{i,n}\Delta_p(\Pi_{C_i} x_n, z) \\
&= \Delta_p(x_n, z) - \sum_{i=1}^t \gamma_{i,n}\Delta_p(x_n, z) + \sum_{i=1}^t \gamma_{i,n}\Delta_p(\Pi_{C_i} x_n, z) \\
&= \Delta_p(x_n, z) + \sum_{i=1}^t \gamma_{i,n}(\Delta_p(\Pi_{C_i} x_n, z) - \Delta_p(x_n, z)). \qquad (3.50)
\end{aligned}$$

Therefore,

$$\sum_{i=1}^t \gamma_{i,n}(\Delta_p(x_n, z) - \Delta_p(\Pi_{c_i} x_n, z)) \leq \Delta_p(x_n, z) - \Delta_p(s_n, z) \to 0, n \to \infty.$$

By condition (iii), we have that

$$\Delta_p(x_n, z) - \Delta_p(\Pi_{C_i} x_n, z) \to 0, n \to \infty.$$

Since Π_{C_i} for each i is right Bregman strongly nonexpansive, we have that

$$\lim_{n\to\infty} \Delta_p(\Pi_{C_i} x_n, x_n) = 0$$

which implies by (2.23) that

$$\lim_{n\to\infty} ||\Pi_{C_i} x_n - x_n|| = 0. \qquad (3.51)$$

Since the sequence $\{x_n\}$ is bounded and E_1 is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to $x^* \in E_1$. From (3.51), it follows that $x^* \in F(\Pi_{C_i})$ for $i = 1, 2, ..., t$ since $F(\Pi_{C_i}) = \hat{F}(\Pi_{C_i})$. Thus, $x^* \in C_i$ for all $i = 1, 2, ..., t$, that is $x^* \in \cap_{i=1}^t C_i$.
Next, we show that $Ax^* \in \cap_{j=1}^r Q_j$.
From (3.44), we obtain for some $M_1 > 0$ that

$$\begin{aligned}\Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big)||(I - P_{\cap_{j=1}^r Q_j})Au_n||^p &\leq \Delta_p(u_n, z) - \Delta_p(x_n, z) \\ &\leq \alpha_{n-1}M_1 + \Delta_p(x_{n-1}, z) \\ &\quad -\Delta_p(x_n, z) \to 0, \quad n \to \infty, \quad (3.52)\end{aligned}$$

and since

$$\begin{aligned}0 &< t\Big(1 - \frac{C_q k^{q-1}||A||^q}{q}\Big) \\ &\leq \Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big),\end{aligned}$$

we have that $||w_n||^p \to 0, \quad n \to \infty$, which implies

$$||Au_n - P_{\cap_{j=1}^r Q_j}Au_n|| \to 0, \quad n \to \infty.$$

Hence, we obtain from the definition of y_n that

$$\begin{aligned}0 &\leq ||J_p^{E_1}x_n - J_p^{E_1}u_n|| \\ &\leq t_n||A^*||||J_p^{E_2}(Au_n - P_Q(Au_n)|| \\ &\leq \Big(\frac{q}{C_q||A||^q}\Big)^{\frac{1}{q-1}}||A^*||||Au_n - P_Q(Au_n)|| \to 0, n \to \infty.\end{aligned}$$

Since $J_q^{E_1^*}$ is norm- to- norm uniformly continuous on bounded subsets of E_1^*, we have that

$$\lim_{n\to\infty}||x_n - u_n|| \to 0, \quad n \to \infty. \quad (3.53)$$

Now, using (2.24), we get

$$\begin{aligned}||(I - P_{\cap_{j=1}^r Q_j})Ax^*||^p &= \langle J_p^{E_2}(Ax^* - P_{\cap_{j=1}^r Q_j}Ax^*), Ax^* - P_{\cap_{j=1}^r Q_j}Ax^*\rangle \\ &= \langle J_p^{E_2}(Ax^* - P_{\cap_{j=1}^r Q_j}Ax^*), Ax^* - Au_{n_k}\rangle \\ &\quad +\langle J_p^{E_2}(Ax^* - P_{\cap_{j=1}^r Q_j}Ax^*), Au_{n_k} - P_{\cap_{j=1}^r Q_j}Au_{n_k}\rangle \\ &\quad +\langle J_p^{E_2}(Ax^* - P_{\cap_{j=1}^r Q_j}Ax^*), P_{\cap_{j=1}^r Q_j}Au_{n_k} - P_{\cap_{j=1}^r Q_j}Ax^*\rangle \\ &\leq \langle J_p^{E_2}(Ax^* - P_{\cap_{j=1}^r Q_j}Ax^*), Ax^* - Au_{n_k}\rangle \\ &\quad +\langle J_p^{E_2}(Ax^* - P_{\cap_{j=1}^r Q_j}Ax^*), Au_{n_k} - P_{\cap_{j=1}^r Q_j}Au_{n_k}\rangle. \quad (3.54)\end{aligned}$$

By the continuity of A, $\quad Ax_{n_k} \rightharpoonup Ax^*, k \to \infty$ and $||x_n - u_n|| \to 0, n \to \infty$, we have that $Au_{n_k} \rightharpoonup Ax^*, k \to \infty$. Hence, letting $k \to \infty$ in (3.54), we have that

$$||Ax^* - P_{\cap_{j=1}^r Q_j}Ax^*|| = 0.$$

Therefore, $Ax^* = P_{\cap_{j=1}^r Q_j} Ax^*$, that is, $Ax^* \in Q_j, \quad j = 1, 2, .., r$. Hence, we have that $x^* \in \Omega_{MS}$.

Next, we show that the sequence $\{x_n\}$ converges strongly to $\Pi_{\Omega_{MS}} u$. Now

$$\begin{aligned} \Delta_p(u_{n+1}, x_n) &= \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}(x_n) + \sum_{i=1}^t \gamma_{i,n} J_p^{E_1}(\Pi_{C_i} x_n)\Big)\Big], x_n) \\ &\leq \alpha_n \Delta_p(u, x_n) + (1-\alpha_n)\beta_n \Delta_p(x_n, x_n) \\ &\quad +(1-\alpha_n)\sum_{i=1}^t \gamma_{i,n} \Delta_p(\Pi_{C_i} x_n, x_n) \to 0, \quad n \to \infty. \end{aligned}$$

Hence,

$$||u_{n+1} - x_n|| \to 0, n \to \infty.$$

Let $\bar{x} = \Pi_{\Omega_{MS}} u$. Then, we have from (3.46) that

$$\Delta_p(x_{n+1}, \bar{x}) \leq (1-\alpha_n)\Delta_p(x_n, \bar{x}) + \alpha_n \langle J_p^{E_1} u - J_p^{E_1} \bar{x}, u_{n+1} - \bar{x} \rangle.$$

We now choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1} \bar{x}, x_n - \bar{x} \rangle = \lim_{j\to\infty} \langle J_p^{E_1} u - J_p^{E_1} \bar{x}, x_{n_j} - \bar{x} \rangle.$$

Then, we have from (2.25) that

$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1} \bar{x}, x_n - \bar{x} \rangle = \lim_j \langle J_p^{E_1} u - J_p^{E_1} \bar{x}, x_{n_j} - \bar{x} \rangle = \langle J_p^{E_1} u - J_p^{E_1} \bar{x}, x^* - \bar{x} \rangle \leq 0.$$

Since $||x_n - u_{n+1}|| \to 0, \quad n \to \infty$, we have

$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1} \bar{x}, u_{n+1} - \bar{x} \rangle = \limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1} \bar{x}, x_n - \bar{x} \rangle \leq 0.$$

Therefore by Lemma 2.3.12, we conclude that $\Delta_p(x_n, \bar{x}) \to 0, \quad n \to \infty$. Thus, $x_n \to \bar{x}, \quad n \to \infty$. By $||x_n - u_n|| \to 0, \quad n \to \infty$, we have that $u_n \to \bar{x}, \quad n \to \infty$.

Case 2. Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\Delta_p(x_{n_i,z}) < \Delta_p(x_{n_i+1}, z)$ for all $i \in \mathbb{N}$. Then by Lemma 2.3.13, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$,

$$\Delta_p(x_{m_k}, z) \leq \Delta_p(x_{m_k+1}, z),$$

and

$$\Delta_p(x_k, z) \leq \Delta_p(x_{m_k+1}, z).$$

Using the same line of arguments in (3.48), (3.49) and (3.50) (noting that $\Delta_p(x_{m_k}, z) \leq \Delta_p(x_{m_k+1}, z)$), we can show that

$$\lim_{k\to\infty} ||\Pi_{C_i} x_{m_k} - x_{m_k}|| = 0.$$

By same arguments as in case 1, we obtain that

$$\limsup_{k\to\infty}\langle J_p^{E_1}u - J_p^{E_1}\bar{x}, u_{m_k+1} - \bar{x}\rangle \leq 0.$$

Again from (3.46), we have

$$\Delta_p(x_{m_k+1}, \bar{x}) \;\leq\; (1-\alpha_{m_k})\Delta_p(x_{m_k}, \bar{x}) + \alpha_{m_k}\langle J_p^{E_1}u - J_p^{E_1}\bar{x}, u_{m_k+1} - \bar{x}\rangle,$$

which implies

$$\alpha_{m_k}\Delta_p(x_{m_k}, \bar{x}) \;\leq\; \Delta_p(x_{m_k}, \bar{x}) - \Delta_p(x_{m_k+1}, \bar{x}) + \alpha_{m_k}\langle J_p^{E_1}u - J_p^{E_1}\bar{x}, u_{m_k+1} - \bar{x}\rangle.$$

That is,

$$\Delta_p(x_{m_k}, \bar{x}) \leq \langle J_p^{E_1}u - J_p^{E_1}\bar{x}, u_{m_k+1} - \bar{x}\rangle.$$

Therefore

$$\lim_{k\to\infty}\Delta_p(x_{m_k}, \bar{x}) = 0$$

and since

$$\Delta_p(x_k, \bar{x}) \leq \Delta_p(x_{m_k+1}, \bar{x}),$$

for all $k \in \mathbb{N}$, we conclude that

$$x_k \to \bar{x}, k \to \infty.$$

□

3.3.1 Numerical example

We present some numerical results. More precisely, we give a numerical example in $(\mathbb{R}^3, ||.||_2)$ of the problem considered in Theorem 3.3.1 in this section. Now take

$$C_i := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x\rangle \geq bi\},$$

where $a = (8, -3, -7)$ and $b = -5$, then

$$\Pi_{C_i}(x) = \frac{bi - \langle a, x\rangle}{||a||_2^2}a + x.$$

Let

$$Q_j := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x\rangle = bj\},$$

where $a = (-5, 6, -3)$ and $b = 7$ then

$$P_{Q_j}(x) = \max\Big\{0, \frac{bj - \langle a, x\rangle}{||a||_2^2}\Big\}a + x.$$

$\alpha_n = \frac{1}{n+1}, \beta_n = 1 - \frac{1-(\frac{1}{2n})^t}{2n-1}), \gamma_{i,n} = \frac{1}{(2n)^i}$ and $A = \begin{pmatrix} 6 & -5 & -8 \\ -3 & 2 & -4 \\ -6 & -2 & 5 \end{pmatrix}$, then our iterative scheme (3.42) becomes

$$\begin{cases} x_n = [u_n - t_n A^T(Au_n - P_{\cap_{j=1}^r Q_j}(Au_n))], \\ u_{n+1} = \Big(\frac{u}{n+1} + (1 - \frac{1}{n+1})((1 - \frac{1-(\frac{1}{2n})^t}{2n-1}))x_n + \sum_{i=1}^t \frac{1}{(2n)^i}\Pi_{C_i}x_n\Big), \quad n \geq 1. \end{cases} \tag{3.55}$$

We make different choices of u_1, u, t_n and use $\frac{||x_{n+1}-x_n||}{||x_2-x_1||} < 10^{-2}$ for stopping criterion.

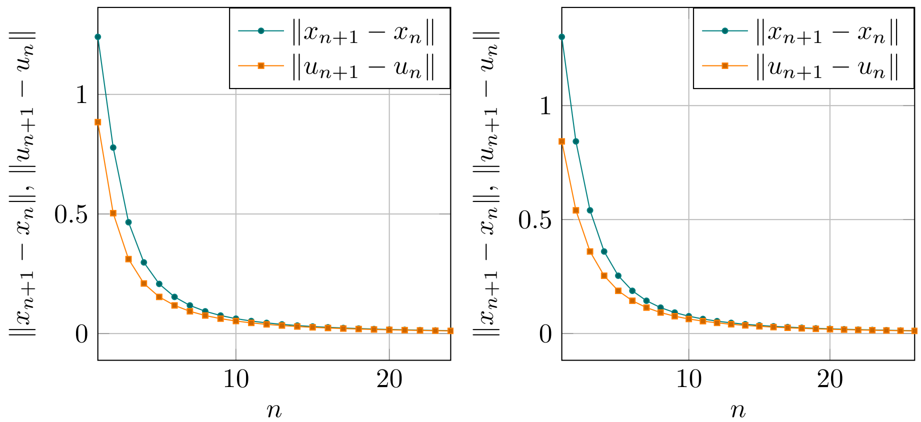


Figure 3.1: Example 1: example 1a left, example 1b right

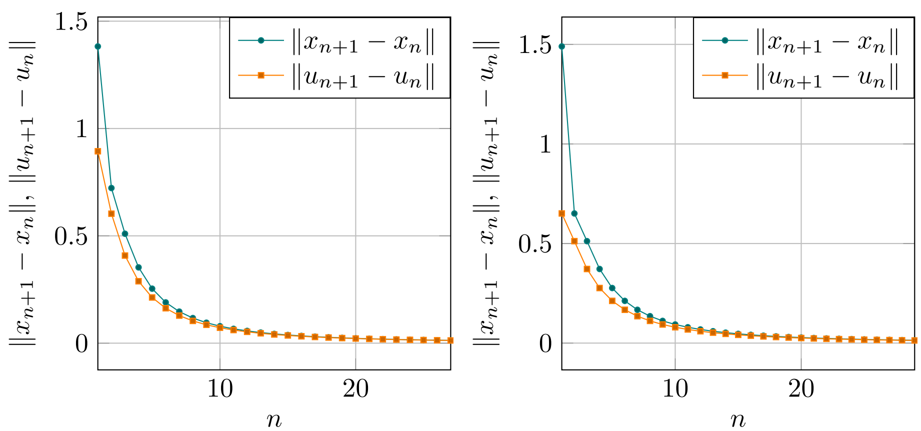


Figure 3.2: Example 2: example 2a left, example 2b right

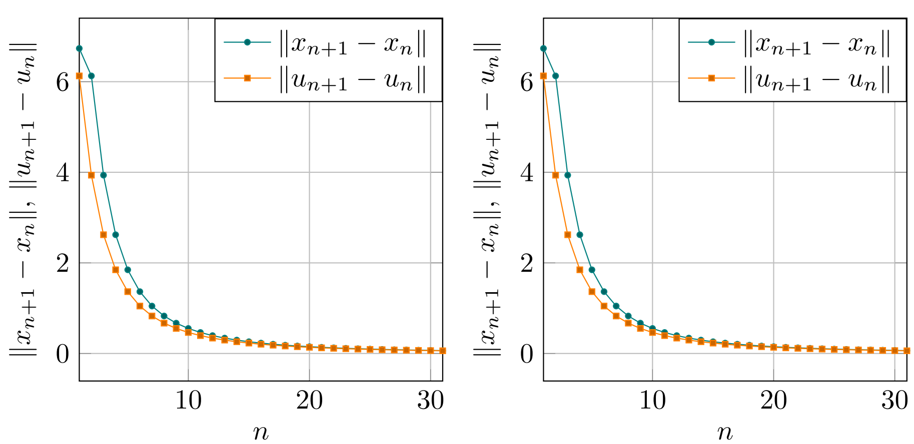


Figure 3.3: Example 3: example 3a left, example 3b right

Example 1:

Take $u = (1, 1, -1)$ and $u_1 = (3, -1, 2)$.
The numerical result of this problem using our algorithm (3.55) with;
case 1a (fig 3.1 left): $t_n = 0.0078$,
and
case 1b(fig 3.1 right): $t_n = 0.0000001$.

Example 2:

Take $u = (2, 0, 1)$ and $u_1 = (4, -2, 3)$.
case 2a (fig 3.2 left): $t_n = 0.0078$,
and
case 2b(fig 3.2 right): Take $t_n = 0.0000001$.

Example 3:

Take $u = (-3, 5, -6)$ and $u_1 = (15, 13, 17)$.
case 3a (fig 3.3 left): $t_n = 0.0078$,
and
case 3b(fig 3.3 right): $t_n = 0.0000001$.

The Mathlab version used is R2014a and the execution times are as follows:

1. Example 1a (case 1a) 0.0810 seconds.
2. Example 1b (case 1b) 0.0551 seconds.
3. Example 2a (case 2a) 0.0728 seconds.
4. Example 2b (case 2b) 0.0568 seconds.
5. Example 3a (case 3a) 0.0635 seconds.
6. Example 3b (case 3b) 0.0700 seconds.

3.4 Further investigation into approximation of a common solution of fixed point problems and split feasibility problems

By combining Mann's iterative method and the Halpern's approximation method, we propose an iterative method and obtain strong convergence result for finding a common element of the set of solutions of the SFP and the set of fixed points of a right Bregman strongly nonexpansive mapping in the setting of p-uniformly convex Banach spaces which are also uniformly smooth.

Theorem 3.4.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP (3.1) has a nonempty solution set Ω_S. Let T be a right Bregman strongly nonexpansive mapping of C into C such that $F(T) = \widehat{F}(T) \neq \emptyset$ and $F(T) \cap \Omega_S \neq \emptyset$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1) such that $\alpha_n + \beta_n + \gamma_n = 1, \alpha_n \leq b < 1, (1-\alpha_n)a < \gamma_n, a > 0$. For a fixed $u \in C$, let sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$,*

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}[J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}(I - P_Q)Au_n], \\ u_{n+1} = \Pi_C J_q^{E_1^*}[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)], \end{cases} \tag{3.56}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < t \leq t_n \leq k \leq \Big(\dfrac{q}{C_q\|A\|^q}\Big)^{\frac{1}{q-1}}$.
Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap\Omega_S} u$.

Proof. Let $w_n := Au_n - P_Q(Au_n)$. Then for any $z \in F(T)\cap\Omega_S$ if follows from (2.24) that

$$\begin{aligned} \langle J_p^{E_2} w_n, Au_n - Az\rangle &= \|Au_n - P_Q(Au_n)\|^p + \langle J_p^{E_2} w_n, P_Q(Au_n) - Az\rangle \\ &\geq \|Au_n - P_Q(Au_n)\|^p \\ &= \|w_n\|^p. \end{aligned} \tag{3.57}$$

From (3.57) and Lemma 2.3.8, we obtain that

$$\begin{aligned} \Delta_p(x_n, z) &\leq \Delta_p(J_q^{E_1^*}[J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}(I - P_Q)Au_n], z) \\ &= \frac{1}{q}||J_p^{E_1}(u_n) - t_n A^* J_p^{E_2} w_n||^q - \langle J_p^{E_1}(u_n), z\rangle + t_n\langle J_p^{E_2}(w_n), Az\rangle + \frac{1}{p}||z||^p \\ &\leq \frac{1}{q}||J_p^{E_1}(u_n)|| - t_n\langle Au_n, J_p^{E_2}(w_n)\rangle + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_2}(w_n)||^p - \langle J_p^{E_1}(u_n), z\rangle \\ &\quad + t_n\langle Az, J_p^{E_2}(w_n)\rangle + \frac{1}{p}||z||^P \\ &= \frac{1}{q}||u_n||^p - \langle J_p^{E_1}(u_n), z\rangle + \frac{1}{p}||z||^P + t_n\langle J_p^{E_2} w_n, Az - Au_n\rangle + \frac{C_q(t_n||A||)^q}{q}||(w_n)||^p \\ &= \Delta(u_n, z) + t_n\langle J_p^{E_2} w_n, Az - Au_n\rangle + \frac{C_q(t_n||A||)^q}{q}||(w_n)||^p \\ &\leq \Delta(u_n, z) - \Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big)||(w_n)||^p. \end{aligned} \tag{3.58}$$

Then from condition (iv), we have that

$$\Delta_p(x_n, z) \leq \Delta_p(u_n, z).$$

Now

$$\begin{aligned}
\Delta_p(u_{n+1},z) &\le \Delta_p(J_q^{E_1^*}[\alpha_n J_p^{E_1}(u)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n)],z)\\
&\le \alpha_n\Delta_p(u,z)+\beta_n\Delta_p(x_n,z)+\gamma_n\Delta_p(Tx_n,z)\\
&\le \alpha_n\Delta_p(u,z)+\beta_n\Delta_p(x_n,z)+\gamma_n\Delta_p(x_n,z)\\
&= \alpha_n\Delta_p(u,z)+(\beta_n+\gamma_n)\Delta_p(x_n,z)\\
&\le \alpha_n\Delta_p(u,z)+(\beta_n+\gamma_n)\Delta_p(u_n,z)\\
&\le \max\{\Delta_p(u,z),\Delta_p(u_n,z)\}\\
&\vdots\\
&\le \max\{\Delta_p(u,z),\Delta_p(u_1,z)\}. \qquad (3.59)
\end{aligned}$$

Therefore $\{\Delta_p(u_n,z)\}$ is bounded and consequently, we have that $\{\Delta_p(x_n,z)\}$ is bounded. Thus the sequences $\{u_n\}$ and $\{x_n\}$ are bounded.
Set $v_n := J_q^{E_1^*}[\alpha_n J_p^{E_1}(z)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n)]$, then

$$\begin{aligned}
\Delta_p(x_{n+1},z) &\le \Delta_p(u_{n+1},z)\\
&\le \Delta_p(J_q^{E_1^*}[\alpha_n J_p^{E_1}(u)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n)],z)\\
&= V_f(\alpha_n J_p^{E_1}(u)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n),z)\\
&\le V_f(\alpha_n J_p^{E_1}(u)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n)-\alpha_n(J_p^{E_1}u-J_p^{E_1}z),z)\\
&\quad -\langle -\alpha_n(J_p^{E_1}u-J_p^{E_1}z),J_q^{E_1^*}[\alpha_n J_p^{E_1}(u)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n)]-z\rangle\\
&= V_f(\alpha_n J_p^{E_1}(z)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n),z)+\alpha_n\langle J_p^{E_1}u-J_p^{E_1}z,v_n-z\rangle\\
&= \Delta_p(J_q^{E_1^*}[\alpha_n J_p^{E_1}(z)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n)],z)+\alpha_n\langle J_p^{E_1}u-J_p^{E_1}z,v_n-z\rangle\\
&\le \alpha_n\Delta_p(z,z)+\beta_n\Delta_p(x_n,z)+\gamma_n\Delta_p(Tx_n,z)+\alpha_n\langle J_p^{E_1}u-J_p^{E_1}z,v_n-z\rangle\\
&\le (1-\alpha_n)\Delta_p(x_n,z)+\alpha_n\langle J_p^{E_1}u-J_p^{E_1}z,v_n-z\rangle. \qquad (3.60)
\end{aligned}$$

We now consider two cases to prove the strong convergence.

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\Delta_p(x_n,z)\}$ is monotonically nonincreasing. Then, $\{\Delta_p(x_n,z)\}$ converges and

$$\Delta_p(x_{n+1},z)-\Delta_p(x_n,z)\to 0, n\to\infty. \qquad (3.61)$$

Let $s_n := J_q^{E_1^*}(\frac{\beta_n}{1-\alpha_n}J_p^{E_1}x_n+\frac{\gamma_n}{1-\alpha_n}Tx_n)$. Then

$$\begin{aligned}
\Delta_p(s_n,z) &= \Delta_p(J_q^{E_1^*}(\frac{\beta_n}{1-\alpha_n}J_p^{E_1}x_n+\frac{\gamma_n}{1-\alpha_n}Tx_n),z)\\
&\le \frac{\beta_n}{1-\alpha_n}\Delta_p(x_n,z)+\frac{\gamma_n}{1-\alpha_n}\Delta_p(Tx_n,z)\\
&\le \frac{\beta_n+\gamma_n}{1-\alpha_n}\Delta_p(x_n,z)\\
&= \Delta_p(x_n,z). \qquad (3.62)
\end{aligned}$$

Thus

$$\begin{aligned}
0 &\leq \Delta_p(x_n, z) - \Delta_p(s_n, z) \\
&= \Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) + \Delta_p(x_{n+1}, z) - \Delta_p(s_n, z) \\
&\leq \Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) + \Delta_p(u_{n+1}, z) - \Delta_p(s_n, z) \\
&\leq \Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) + \alpha_n\Delta_p(u, z) - (1-\alpha_n)\Delta_p(s_n, z) - \Delta_p(s_n, z) \\
&= \Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) + \alpha_n(\Delta_p(u, z) - \Delta_p(s_n, z)) \to 0, \quad n \to \infty. \qquad (3.63)
\end{aligned}$$

Again

$$\begin{aligned}
\Delta_p(s_n, z) &\leq \frac{\beta_n}{1-\alpha_n}\Delta_p(x_n, z) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(Tx_n, z) \\
&= \Big(1 - \frac{\gamma_n}{1-\alpha_n}\Big)\Delta_p(x_n, z) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(Tx_n, z) \\
&= \Delta_p(x_n, z) + \frac{\gamma_n}{1-\alpha_n}\Big(\Delta_p(Tx_n, z) - \Delta_p(x_n, z)\Big). \qquad (3.64)
\end{aligned}$$

Hence, since $\alpha_n + \gamma_n \leq 1$ and $\alpha_n \leq b < 1$, we obtain

$$\begin{aligned}
a(\Delta_p(x_n, z) - \Delta_p(Tx_n, z)) &< \frac{\gamma_n}{1-\alpha_n}\Big(\Delta_p(x_n, z) - \Delta_p(Tx_n, z)\Big) \\
&\leq \Delta_p(x_n, z) - \Delta_p(s_n, z) \to 0 \to \infty. \qquad (3.65)
\end{aligned}$$

Therefore, since T is right Bregman strongly nonexpansive, we have that

$$\lim_{n\to\infty} \Delta_p(Tx_n, x_n) = 0$$

which implies that

$$\lim_{n\to\infty} ||Tx_n - x_n|| = 0. \qquad (3.66)$$

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $x^* \in C$. From (3.66), it follows that $x^* \in F(T)$ since $F(T) = \hat{F}(T)$.

Next, we show that $Ax^* \in Q$, that is $x^* \in \Omega_S$. Set $y_n = J_q^{E_1^*}[J_p^{E_1}(u_n) - t_nA^*J_p^{E_2}(I - P_Q)Au_n]$, then from (2.26), (3.58) and (3.60) we have

$$\begin{aligned}
\Delta_p(y_n, x_n) &= \Delta_p(y_n, \Pi_C y_n) \\
&\leq \Delta_p(y_n, z) - \Delta_p(x_n, z) \\
&\leq \Delta_p(u_n, z) - \Delta_p(x_n, z) \\
&\leq \alpha_{n-1}M + \Delta_p(x_{n-1}, z) - \Delta_p(x_n, z) \to 0, n \to \infty, \qquad (3.67)
\end{aligned}$$

where $M > 0$ is such that $\Delta_p(x_{n-1}, z) + \langle J_p^{E_1}u - J_p^{E_1}z, v_{n-1} - z\rangle \leq M$. Thus, we have that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \qquad (3.68)$$

Also from (3.58), we obtain

$$\begin{aligned}\Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big)||w_n||^P &\leq \Delta_p(u_n, z) - \Delta_p(x_n, z) \\ &\leq \alpha_{n-1}M + \Delta_p(x_{n-1}, z) \\ &\quad -\Delta_p(x_n, z) \to 0, \quad n \to \infty, \end{aligned} \tag{3.69}$$

and since

$$0 < t\Big(1 - \frac{C_q k^{q-1}(||A||)^q}{q}\Big) \leq \Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big),$$

we have that $||w_n||^p \to 0, n \to \infty$, which implies

$$||Au_n - P_Q(Au_n)|| \to 0, n \to \infty.$$

Hence, we obtain from the definition of y_n that

$$\begin{aligned} 0 &\leq ||J_p^{E_1}y_n - J_p^{E_1}u_n|| \\ &\leq t_n||A^*||||J_p^{E_2}(Au_n - P_Q(Au_n)|| \\ &\leq \Big(\frac{q}{C_q||A||^q}\Big)^{\frac{1}{q-1}}||A^*||||Au_n - P_Q(Au_n)|| \to 0, n \to \infty. \end{aligned}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have that

$$\lim_{n\to\infty}||y_n - u_n|| = \lim_{n\to\infty}||J_q^{E_1^*}J_p^{E_1}y_n - J_q^{E_1^*}J_p^{E_1}u_n|| \to 0, n \to \infty. \tag{3.70}$$

It follows from (3.68) and (3.70) that

$$||x_n - u_n|| \leq ||x_n - y_n|| + ||y_n - u_n|| \to 0, \quad n \to \infty.$$

Now from (2.24), we have

$$\begin{aligned} ||(I - P_Q)Ax^*||^p &= \langle J_p^{E_2}(Ax^* - P_QAx^*), Ax^* - P_QAx^*\rangle \\ &= \langle J_p^{E_2}(Ax^* - P_QAx^*), Ax^* - Au_{n_j}\rangle + \langle J_p^{E_2}(Ax^* - P_QAx^*), Au_{n_j} - P_QAu_{n_j}\rangle \\ &\quad +\langle J_p^{E_2}(Ax^* - P_QAx^*), P_QAu_{n_j} - P_QAx^*\rangle \\ &\leq \langle J_p^{E_2}(Ax^* - P_QAx^*), Ax^* - Au_{n_j}\rangle + \langle J_p^{E_2}(Ax^* - P_QAx^*), Au_{n_j} - P_QAu_{n_j}\rangle. \end{aligned}$$

By the continuity of A and $||x_n - u_n|| \to 0, n \to \infty$, we have that $Au_{n_j} \rightharpoonup Ax^*, j \to \infty$. Hence, letting $j \to \infty$, we have that

$$||Ax^* - P_QAx^*|| = 0.$$

Therefore, $Ax^* = P_QAx^*$, that is $Ax^* \in Q$. Hence, we have that $x^* \in F(T) \cap \Omega_S$.

Next, we show that $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap\Omega_S}u$. Now

$$\begin{aligned} \Delta_p(v_n, x_n) &= \Delta_p(J_q^{E_1^*}[\alpha_nJ_p^{E_1}(u) + \beta_nJ_p^{E_1}(x_n) + \gamma_nJ_p^{E_1}(Tx_n)], x_n) \\ &\leq \alpha_n\Delta_p(u, x_n) + \beta_n\Delta_p(x_n, x_n) + \gamma_n\Delta_p(Tx_n, x_n) \to 0, n \to \infty. \end{aligned}$$

Hence,
$$||v_n - x_n|| \to 0, \quad n \to \infty.$$
Let $\bar{x} = \Pi_{F(T)\cap\Omega_S} u$. Then from (3.60), we have
$$\Delta_p(x_{n+1}, \bar{x}) \le (1-\alpha_n)\Delta_p(x_n, \bar{x}) + \alpha_n \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, v_n - \bar{x}\rangle.$$
We now choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that
$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_n - \bar{x}\rangle = \lim_{j\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{n_j} - \bar{x}\rangle.$$
Then from (2.26), we have
$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_n - \bar{x}\rangle = \lim_{j\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{n_j} - \bar{x}\rangle = \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x^* - \bar{x}\rangle \le 0.$$
Hence, since $||x_n - v_n|| \to 0, n \to \infty$, we have
$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, v_n - \bar{x}\rangle \le \limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_n - \bar{x}\rangle \le 0.$$
Therefore, by Lemma 2.3.12, we conclude that $\Delta_p(x_n, \bar{x}) \to 0, \quad n \to \infty$, that is $||x_n - \bar{x}|| \to 0, \quad n \to \infty$. Therefore, $x_n \to \bar{x}$.

Case 2. Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\Delta_p(x_{n_i,z}) < \Delta_p(x_{n_i+1}, z)$ for all $i \in \mathbb{N}$. Then, by Lemma 2.3.13 there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$.
$$\Delta_p(x_{m_k}, z) \le \Delta_p(x_{m_k+1}, z),$$
and
$$\Delta_p(x_k, z) \le \Delta_p(x_{m_k+1}, z).$$
Also, we have
$$\begin{aligned} \Delta_p(x_{m_k}, z) - \Delta_p(Tx_{m_k}, z) &= \Delta_p(x_{m_k}, z) - \Delta_p(x_{m_k+1}, z) + \Delta_p(x_{m_k+1}, z) - \Delta_p(Tx_{m_k}, z) \\ &\le \Delta_p(x_{m_k}, z) - \Delta_p(x_{m_k+1}, z) \\ &\quad + \alpha_{m_k}(\Delta_p(u, z) - \Delta_p(x_{m_k}, z)) \to 0, k \to \infty. \end{aligned}$$
Thus
$$\lim_{k\to\infty} ||Tx_{m_k} - x_{m_k}|| = 0.$$
By same arguments as in case 1, we obtain that
$$\limsup_{k\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, v_{m_k} - \bar{x}\rangle \le 0.$$
Again from (3.60), we have
$$\Delta_p(x_{m_k+1}, \bar{x}) \le (1-\alpha_{m_k})\Delta_p(x_{m_k}, \bar{x}) + \alpha_{m_k} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, v_{m_k} - \bar{x}\rangle,$$

which implies that

$$\alpha_{m_k}\Delta_p(x_{m_k},\bar{x}) \leq \Delta_p(x_{m_k},\bar{x}) - \Delta_p(x_{m_k+1},\bar{x}) + \alpha_{m_k}\langle J_p^{E_1}u - J_p^{E_1}\bar{x}, v_{m_k} - \bar{x}\rangle.$$

That is,

$$\Delta_p(x_{m_k},\bar{x}) \leq \langle J_p^{E_1}u - J_p^{E_1}\bar{x}, v_{m_k} - \bar{x}\rangle.$$

Therefore

$$\lim_{k\to\infty}\Delta_p(x_{m_k},\bar{x}) = 0$$

and since

$$\Delta_p(x_k,\bar{x}) \leq \Delta_p(x_{m_k+1},\bar{x})$$

for all $k \in \mathbb{N}$, we conclude that

$$x_k \to \bar{x}, \quad k \to \infty.$$

$\square$

Corollary 3.4.2. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP (3.1) has a nonempty solution set Ω_S. Suppose $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\alpha_n \leq b < 1$. For a fixed $u \in C$, let sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$,*

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}[J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}(I - P_Q)Au_n], \\ u_{n+1} = \Pi_C J_q^{E_1^*}[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(x_n)], \end{cases} \tag{3.71}$$

with the conditions
(i) $\lim\limits_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(iii) $0 < t \leq t_n \leq k \leq \left(\dfrac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.
Then, $\{x_n\}$ converges strongly to $\Pi_{\Omega_S}u$.

Next, using the idea in [153], we consider the mapping $T : C \to C$ defined by $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$, where $T_i (i = 1, 2, \ldots, N)$ are right Bregman strongly nonexpansive mappings on E. Using Theorem 3.4.1, we have the following corollary.

Corollary 3.4.3. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP (3.1) has a nonempty solution set Ω_S. Let $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$ where $T_i : C \to C$ for $i = 1, 2, ..., N$ is a finite family of right Bregman strongly nonexpansive mappings such that $F(T_i) = \widehat{F}(T_i)$ and $(\cap_{i=1}^N F(T_i)) \cap \Omega_S \neq \emptyset$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1) such that $\alpha_n + \beta_n + \gamma_n = 1, \alpha_n \leq b < 1, (1-\alpha_n)a < \gamma_n, a > 0$. For a fixed $u \in C$, let sequences $\{x_n\}$ and $\{u_n\}$ be iteratively generated by $u_0 \in E_1$,*

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}[J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}(I - P_Q)Au_n], \\ u_{n+1} = \Pi_C J_q^{E_1^*}[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)], \end{cases} \tag{3.72}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < t \le t_n \le k \le \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.
Then, $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\cap\Omega_S} u$.

Corollary 3.4.4. *Let* H_1 *and* H_2 *be two real Hilbert spaces. Let* C *and* Q *be nonempty, closed and convex subsets of* H_1 *and* H_2 *respectively,* $A : H_1 \to H_2$ *be a bounded linear operator and* $A^* : H_2 \to H_1$ *be the adjoint of* A. *Suppose that SFP* (3.1) *has a nonempty solution set* Ω_S. *Let* T *be a strongly quasi-nonexpansive operator of* C *into* C *such that* $F(T) \neq \emptyset$, $I - T$ *is demi-closed at zero and* $F(T)\cap\Omega_S \neq \emptyset$. *Suppose* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in (0,1) such that* $\alpha_n + \beta_n + \gamma_n = 1, \alpha_n \le b < 1, (1-\alpha_n)a < \gamma_n, a > 0$. *For a fixed* $u \in C$, *let sequences* $\{x_n\}$ *and* $\{u_n\}$ *be iteratively generated by* $u_0 \in H_1$,

$$\begin{cases} x_n = P_C[u_n - t_n A^*(I - P_Q)Au_n], \\ u_{n+1} = P_C[\alpha_n u + \beta_n x_n + \gamma_n(Tx_n)], \end{cases} \tag{3.73}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < t \le t_n \le k < \frac{2}{||A||^2}$.
Then, $\{u_n\}$ *and* $\{x_n\}$ *both converge strongly to* $\bar{x}$, *where* $\bar{x} = P_{F(T)\cap\Omega_S} u$.

Remark Prototype for the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{t_n\}$ in Theorem 3.4.1 are:

$$\alpha_n = \frac{1}{n+1}, \quad \beta_n = 1 - \frac{2an+1}{n+1}, \quad \gamma_n = \frac{2an}{n+1},$$

$$a \in (0, \frac{1}{2}), (\text{a} = \frac{1}{4} \ \text{ in our next numerical example}) \ \forall \text{n} \ge 0.$$

and

$$t_n = (\frac{n}{n+1})\left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}}, \quad \forall n \ge 0.$$

3.4.1 Numerical examples

In this subsection, we present some numerical results. All codes were written in Matlab 2012b and run on Hp i-5 dual core laptop.

Example
Here, we take $E_1 = L_2([0,1]) = E_2$ with inner product given as

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Now, let

$$C := \{x \in L_2([0,1]) : \langle x, a\rangle = b\},$$

where $a = 2t^2$, $b = 0$. Then

$$P_C(x) = \max\{0, \frac{b - \langle a, x\rangle}{||a||_2^2}\}a + x.$$

Also, let

$$Q := \{x \in L_2([0,1]) : \langle x, c\rangle \geq d\},$$

where $c = \frac{t}{3}$, $d = -1$. Then

$$\Pi_Q(x) = P_Q(x) = \frac{d - \langle c, x\rangle}{||c||_2^2}c + x.$$

Let us assume that

$$A : L_2([0,1]) \to L_2([0,1]), \quad Ax(t) = \frac{x(t)}{2}.$$

Then A is a bounded linear operator with $||A|| = \frac{1}{2}$ and $A^* = A$. Suppose that we take the operator T in Theorem 3.4.1 as $T = P_C$, the metric projection onto C. Then the problem considered in Theorem 3.4.1 reduces to:

$$\text{Find } x \in F(T) \cap C (= C) \text{ such that } Ax \in Q. \tag{3.74}$$

We observe that if Ω denotes the set of solutions of (3.74),then $\Omega_S \neq \emptyset$, since $x^* = 0 \in \Omega_S$. Furthermore, our iterative scheme (3.56) becomes $u_1 \in L_2([0,1])$,

$$\begin{cases} x_n = P_C[u_n - t_n A^*(I - P_Q)Au_n], \\ u_{n+1} = P_C[\frac{u}{n+1} + \frac{n}{2(n+1)}x_n + \frac{n}{2(n+1)}(P_C x_n)], n \geq 1. \end{cases} \tag{3.75}$$

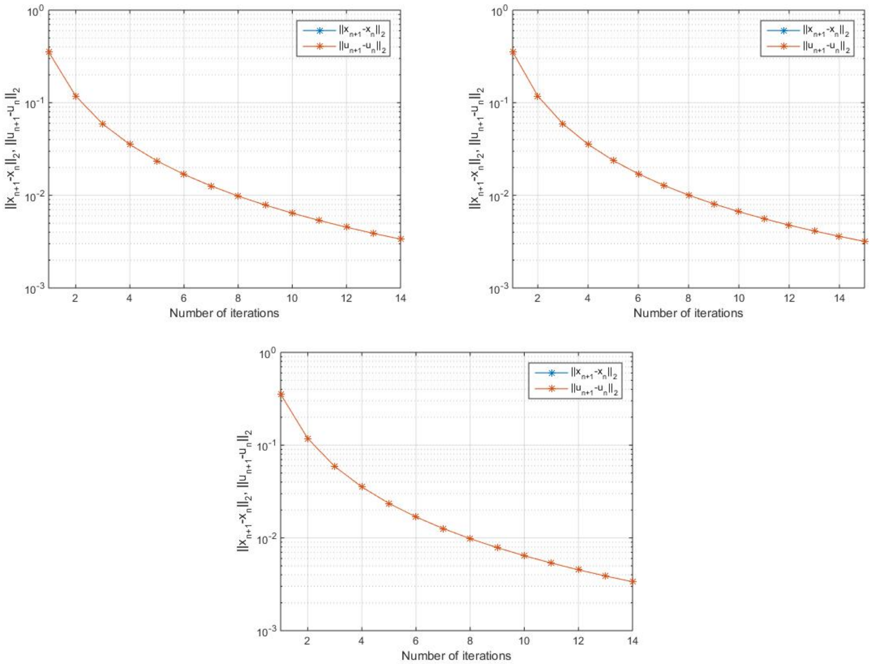
We make different choices of u_1 and t_n with a choice of $u = \frac{5}{2}t^2 - 2t$ and take $\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-4}$ as our stopping criterion.

Case1: $t_n = 0.0000001$, $u_1 = t$. We have the numerical analysis tabulated in table 1 and graph shown below.

Case2: $t_n = 0.0002$, $u_1 = t$. We have the numerical analysis tabulated in table 2 and graph shown below.

Case3: $t_n = 0.0000002$, $u_1 = t$. We have the numerical analysis tabulated in table 3 and graph shown below.

Remark We remark from this numerical example that different choices of u_1 and t_n within the specified spaces and range have no effect on the number of iterations required for convergence and with very insignificant effect on the cpu run time as can be seen from



the Tables and the corresponding Figures. The representation of $||x_{n+1} - x_n||$ against the number of iterations (blue line) is the same as the representation of $||u_{n+1} - u_n||$ against the number of iterations (red line) in the figures. This is because the values of $||x_{n+1}-x_n||$ and $||u_{n+1}-u_n||$ are either the same or very close to each other and so the curves (blue and red lines) overlapped each other. Also, we see from the tables and graphs that the closer the values of t_n to zero, the less the number of iterations required for the convergence.

Table 3.1: Example 4, Case I

Time taken	No. of iterations	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$
1.882076	2	0.353553	0.353553
	3	0.117851	0.117851
	4	0.0589256	0.0589256
	5	0.0353553	0.0353553
	6	0.0235702	0.0235702
	7	0.0168359	0.0168359
	8	0.0126269	0.0126269
	9	0.00982093	0.00982093
	10	0.00785674	0.00785674
	11	0.00642824	0.00642824
	12	0.00535687	0.00535687
	13	0.00453274	0.00453274
	14	0.0038852	0.0038852

Table 3.2: Example 4, Case II

Time taken	No. of iterations	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$
2.173481	2	0.35379	0.353805
	3	0.118088	0.118093
	4	0.0591626	0.0591651
	5	0.0355924	0.0355939
	6	0.0238074	0.0238084
	7	0.0170731	0.0170738
	8	0.0128642	0.0128648
	9	0.0100584	0.0100588
	10	0.0080943	0.00809464
	11	0.00666593	0.00666622
	12	0.0055947	0.00559494
	13	0.00477072	0.00477092
	14	0.00412335	0.00412352
	15	0.00360549	0.00360565

Table 3.3: Example 4, Case III

Time taken	No. of iterations	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$
2.321461	2	0.353554	0.353554
	3	0.117851	0.117851
	4	0.0589258	0.0589258
	5	0.0353556	0.0353556
	6	0.0235705	0.0235705
	7	0.0168361	0.0168361
	8	0.0126271	0.0126271
	9	0.00982116	0.00982116
	10	0.00785698	0.00785698
	11	0.00642848	0.00642848
	12	0.00535711	0.00535711
	13	0.00453297	0.00453297
	14	0.00388544	0.00388544

Chapter 4

Convergence Analysis of Common Solution of Certain Nonlinear Problems

4.1 Introduction

The problem of finding zero points for maximal monotone operators plays an important role in optimizations because it can be reduced to a convex minimization problem and a variational inequality problem. We observe here that if we denote by $N_C(v)$, the normal cone of some nonempty, closed and convex set C of a Hilbert space H at a point $v \in C$, i.e.,

$$N_C(v) := \{d \in H : \langle d, y - v\rangle \leq 0, \forall\, y \in C\},$$

and define the set valued mapping B by

$$B(v) := \begin{cases} f(v) + N_C(v), v \in C, \\ \emptyset, \qquad\qquad otherwise, \end{cases}$$

where f is some given operator, then under certain continuity assumption on f, Rockafellar (see Theorem 3 of [197]) showed that B is maximal monotone mapping and $B^{-1}(0)$ solves the variational inequality problem which is to find $x^* \in C$ such that
$\langle f(x^*), x - x^*\rangle \geq 0, \forall\, x \in C$.

Let $f : E \to E^*$ be inverse strongly monotone mapping and let $B : E \to 2^{E^*}$ be maximal monotone mapping. The Monotone Variational Inclusion Problem (MVIP) is to find $x \in E$ such that

$$0 \in f(x) + B(x), \tag{4.1}$$

where 0 is the zero vector in E. If $f \equiv 0$, then we have the variational inclusion problem. The set of solutions to the MVIP (4.1) is denoted by $I(f, M)$. For further studies on MVIP see for example [174] and some of the references therein.

Let A be a maximal monotone operator, the resolvent of A denoted by $Res_A^f : E \to 2^E$ is defined as follows [17]:

$$Res_A^f(x) = ((\nabla f + A)^{-1} \circ \nabla f)(x).$$

It is known that $F(Res_A^f) = A^{-1}(0^*)$ and Res_A^f is single valued (see [17]). If f is Legendre function which is bounded, uniformly Fréchet differetiable on bounded subsets of E, then $\hat{F}(Res_A^f) = F(Res_A^f)$ (see [17]). The Yosida approximation $A_\lambda : E \to E^*, \quad \lambda > 0$ is also defined by

$$A_\lambda(x) = \frac{1}{\lambda}(\nabla f(x) - \nabla f(Res_{\lambda A}^f(x))),$$

for all $x \in E$. From Proposition 2.7 [189], it is known that $(Res_{\lambda A}^f(x), A_\lambda(x)) \in G(A)$ and $0^* \in Ax$ if and only if $0^* \in A_\lambda x$ for all $x \in E$ and $\lambda > 0$.

Zegeye and Shahzad [246] proved a strong convergence theorem for a common fixed point of a finite family of right Bregman strongly nonexpansive mappings in the framework of real reflexive Banach spaces. Furthermore, they applied their method to approximate a common zero of a finite family of maximal monotone mappings and a solution of a finite family of convex feasibility problems in reflexive real Banach spaces. In particular, they proved the following theorem:

Theorem 4.1.1. *(Zegeye and Shahzad, [246]) Let $f : E \to \mathbb{R}$ be a cofinite function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom f) and let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be a finite family of right Bregman strongly nonexpansive mappings such that $F(T_i) = \widehat{F}(T_i)$, for each $i \in \{1, 2, \ldots, N\}$. Assume that $\mathfrak{F} := \cap_{i=1}^N F(T_i)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots$$

where $T = T_N \circ T_{N-1} \circ \cdots \circ T_2 \circ T_1$, $\{\alpha_n\} \subset (0, 1)$ satisfy (i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and (ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point p in $\mathfrak{F}$.

Rockafeller [195], motivated by the work of Martinet [156], introduced in a Hilbert space the following proximal point iterative algorithm:

$$\begin{cases} x_1 = x \in E, \\ x_{n+1} = J_{\lambda_n}^B x_n, \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J_{\lambda_n}^B$ is the resolvent of B defined by $J_\lambda^B = (I + \lambda B)^{-1}$ for all $\lambda > 0$, and B is a maximal monotone operator on E. He proved that the sequence $\{x_n\}$ generated by (4.2) converges weakly to an element in $B^{-1}(0)$ provided $\liminf\limits_{n\to\infty} \lambda_n > 0$.

A weak convergence result was also obtained by Kamimura and Takahashi [116] in a real Hilbert space with the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)J_{\lambda_n}^B x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ under some suitable conditions on $\{\lambda_n\}$.
Inspired by the result of Kamimura and Takahashi [116], Kohsaka and Takahashi [126] in reflexive Banach space introduced the following iterative algorithm:

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(J^B_{\lambda_n} x_n)), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$, $f : E \to \mathbb{R}$ is a Bregman function and $J^B_\lambda = (\nabla f + \lambda B)^{-1} \circ \nabla f$ for all $\lambda > 0$. They obtained a weak convergence result with the proposed algorithm. For some other existing results for finding zero points of maximal monotone operators see for example [29, 43, 71, 96, 173, 189, 213] and some of the references therein.

Let E be a reflexive Banach space and C a nonempty, closed and convex subset of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function and $\Phi : C \to E^*$ be a nonlinear mapping. The *Generalised Mixed Equilibrium Problem* (GMEP) is to find $u \in C$ such that

$$g(u,y) + \langle \Phi u, y-u\rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall\, y \in C. \tag{4.3}$$

Denote the set of solutions of problem (4.3) by $GMEP(g,\Phi,\varphi)$. That is

$$GMEP(g,\Phi,\varphi) = \{u \in C : g(u,y) + \langle \Phi u, y-u\rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall\, y \in C\}.$$

If $\Phi = 0$, then the generalised mixed equilibrium problem (4.3) reduces to the following *mixed equilibrium problem*, find $u \in C$ such that

$$g(u,y) + \varphi(y) - \varphi(u) \geq 0, \quad \forall\, y \in C.$$

If $\varphi = 0$, then the generalised mixed equilibrium problem (4.3) becomes the *generalised equilibrium problem*, find $u \in C$ such that

$$g(u,y) + \langle \Phi u, y-u\rangle \geq 0, \quad \forall\, y \in C.$$

Again, if $\Phi = \varphi = 0$, then the generalised mixed equilibrium problem (4.3) becomes the *equilibrium problem*, find $u \in C$ such that

$$g(u,y) \geq 0, \quad \forall\, y \in C. \tag{4.4}$$

We denote the solution set of (4.4) as $EP(g)$. Equilibrium problems and their generalizations have been widely applied to solve problems in various fields such as: linear or nonlinear programming, variational inequalities, complementary problems, optimization problems, fixed point problems and have also been widely applied to physics, structural analysis, management sciences, economics, etc (see, for example [23, 77, 179, 178]).

In solving equilibrium problem (4.4), the bifunction g is assumed to satisfy the following conditions:
(A1) $g(x,x) = 0$ for all $x \in C$;
(A2) g is monotone, i.e., $g(x,y) + g(y,x) \geq 0$ for all $x, y \in C$;
(A3) for each $x, y \in C$, $\lim_{t \to 0} g(tz + (1-t)x, y) \leq g(x,y)$;
(A4) for each $x \in C$; $y \mapsto g(x,y)$ is convex and lower semicontinuous.

It is known (see [247]) that if $g(x,y)$ satisfies $(A1)-(A4)$, then the function $F(x,y) := g(x,y) + \langle \Phi x, y-x\rangle + \varphi(y) - \varphi(x)$ satisfies $(A1)-(A4)$ and $GMEP(g,B,\varphi,)$ is closed and convex. The resolvent of a bifunction $g : C\times C \to \mathbb{R}$ (see [190]) is the operator $Res_g^f : E \to C$ defined by

$$Res_g^f(x) = \{z \in C : g(z,y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \ \ \forall \ y \in C\}. \tag{4.5}$$

For any $x \in E$, there exists $z \in C$ such that $z = Res_g^f(x)$ (see [190]).

Lemma 4.1.2. *([247]) Let C be nonempty closed convex subset of a Hilbert space H. Let $B : C \to H$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, and $f : C \times C \to \mathbb{R}$ be a bifunction that satisfies $(A1)-(A4)$ (Conditions $A1 - A4$ is stated in chapter 4 of this thesis). For $r > 0$ and $x \in H$; then there exists $u \in C$ such that*

$$f(u,y) + \langle Bu, y-u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0, \quad \forall \ y \in C. \tag{4.6}$$

Define a mapping $T_r^f : C \to C$ as follows:

$$\begin{aligned} T_r^f(x) \ &= \ \{u \in C : f(u,y) + \langle Bu, y-u\rangle + \varphi(y) - \varphi(u) \\ &\quad + \frac{1}{r}\langle y-u, u-x\rangle \geq 0, \quad \forall \ y \in C\}. \end{aligned} \tag{4.7}$$

Then, the following conclusions hold:
1. T_r^f is single-valued,
2. T_r^f is firmly nonexpansive, i.e., for any $x,y \in H$; $||T_r^f(x) - T_r^f(y)||^2 \leq \langle T_r^f(x) - T_r^f(y), x-y\rangle$,
3. $F(T_r^f) = GMEP(f;B;\varphi)$,
4. $GMEP(F;B;\varphi)$ is closed and convex.

Lemma 4.1.3. *(Reich and Sabach [190]) Let $f : E \to (-\infty, +\infty)$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of E. If the bifunction $g : C\times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), then,*
1. Res_g^f is single-valued;
2. Res_g^f is a Bregman firmly nonexpansive mapping;
3. $F(Res_g^f) = EP(g)$;
4. $EP(g)$ is a closed and convex subset of C;
5. for all $x \in E$ and $q \in F(Res_g^f)$,

$$D_f(q, Res_g^f(x)) + D_f(Res_g^f(x), x) \leq D_f(q,x).$$

Many authors have proposed some efficient and implementable algorithms and obtain some convergence theorems for solving equilibrium problems and some of their generalizations, (see for example, [3, 72, 77, 129, 130, 141, 163, 187, 206, 207, 223, 227, 242, 244] and the references therein). Cholamjiak and Suantai [72] proposed a hybrid iterative scheme for finding a common element in the solution set of system of equilibrium problems and the common fixed points set of an infinitely countable family of quasi-nonexpansive mappings and prove the following strong convergence theorem.

Theorem 4.1.4. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed and convex subset of E. Let $\{f_j\}_{j=1}^M$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions $(A1)-(A4)$, and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself. Assume that $F := (\cap_{i=1}^\infty F(T_i)) \cap (\cap_{j=1}^M EP(f_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_0}x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_{n,i} = J_q^{E^*}(\alpha_n J_p^E x_n + (1-\alpha_n)J_p^E T_i x_n), \\ u_{n,i} = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \ldots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} = \{z \in C_n : \sup_{i\geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 0. \end{cases}$$

Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, ..., M$ are sequences which satisfy the following conditions:
(B1) $\limsup\limits_{n\to\infty} \alpha_n < 1$,
(B2) $\liminf\limits_{n\to\infty} r_{j,n} > 0$.
Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Inspired and motivated by the researches going on in this direction, in this chapter, we propose an iterative algorithm for approximating a fixed point of an infinite family of left Bregman strongly nonexpansive mappings, which is also a common solution to a finite system of equilibrium problems and finite system of variational inclusion problems in a reflexive real Banach space and give some applications.

4.2 Main results

Theorem 4.2.1. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^\infty$ be an infinite family of left Bregman strongly nonexpasive mappings from C into itself and $F(T_j) = \hat{F}(T_j), \ \ \forall j \geq 1$. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be bifunctions satisfying conditions $(A1)-(A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ be a proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ and $A_i : E \to 2^{E^*} (i = 1, 2, ..., N)$ be maximal monotone operators, such that $\Omega_{AS} := \cap_{j=1}^\infty F(T_j) \cap (\cap_{k=1}^N EP(G_k) \cap (\cap_{i=1}^N A_i^{-1}(0)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} y_n = Res_{\lambda_n^N A_N}^f \circ Res_{\lambda_n^{N-1} A_{N-1}}^f \circ \cdots \circ Res_{\lambda_n^2 A_2}^f \circ Res_{\lambda_n^1 A_1}^f x_n, \\ u_n = Res_{G_N}^f \circ Res_{G_{N-1}}^f \circ \cdots \circ Res_{G_2}^f \circ Res_{G_1}^f y_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^\infty \gamma_{nj} \nabla f(T_j u_n)), \quad \forall n \geq 1, \end{cases} \tag{4.8}$$

converges strongly to a point $p = \overleftarrow{P}roj_{\Omega_{AS}}^f u \in \Omega_{AS}$, where $G(x, y) := g(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{nj}\}$ and $\{\lambda_n\}$ satisfy the following conditions:
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

A prototype example of the control sequences are; $\lambda_n^k = \frac{n^k}{2(n+1)^k}$, $\alpha_n = \frac{1}{n+6}$, $\beta_n = \frac{n^2+7n+9}{(n+6)(n+3)}$ and $\gamma_{nj} = \frac{1}{2^j(n+3)}$.

Proof. It is known (see [247]) that the function $G(x,y) := g(x,y) + \langle Bx, y-x\rangle + \varphi(y) - \varphi(x)$ satisfies $(A1)-(A4)$ and $GMEP(g,\varphi,B)$ is closed and convex.
For any $x^* \in \Omega_{AS}$, then from (4.8), we have that

$$\begin{aligned} D_f(x^*, y_n) &= D_f(x^*, Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n) \\ &\leq D_f(x^*, Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n) \\ &\vdots \\ &\leq D_f(x^*, x_n). \end{aligned} \tag{4.9}$$

Also from (4.8), we have

$$\begin{aligned} D_f(x^*, u_n) &= D_f(x^*, Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} y_n) \\ &\leq D_f(x^*, Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} y_n) \\ &\vdots \\ &\leq D_f(x^*, y_n). \end{aligned} \tag{4.10}$$

Again from (2.20),(4.8),(4.9) and (4.10), we have

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n))) \\ &\leq \alpha_n D_f(x^*, u) + \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, T_j(u_n)) \\ &\leq \alpha_n D_f(x^*, u) + \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, u_n) \\ &= \alpha_n D_f(x^*, u) + (\beta_n + \sum_{j=1}^{\infty} \gamma_{nj}) D_f(x^*, u_n) \\ &\leq \alpha_n D_f(x^*, u) + (\beta_n + \gamma_{nj}) D_f(x^*, y_n) \\ &\leq \alpha_n D_f(x^*, u) + (\beta_n + \sum_{j=1}^{\infty} \gamma_{nj}) D_f(x^*, x_n) \\ &= \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, x_n) \\ &\leq \max\{D_f(x^*, u), D_f(x^*, x_n)\} \\ &\vdots \\ &\leq \max\{D_f(x^*, u), D_f(x^*, x_1)\}. \end{aligned} \tag{4.11}$$

Therefore, $\{D_f(x^*, x_n)\}$ is bounded and so also are $\{D_f(x^*, u_n)\}$ and $\{D_f(x^*, y_n)\}$. Consequently, we have that the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are bounded.
Moreover,

$$\begin{aligned}
D_f(x^*, u_{n+1}) &\leq D_f(x^*, x_{n+1}) \\
&= V_f(x^*, \alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)) \\
&\leq V_f(x^*, \alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n) - \alpha_n(\nabla f(u) - \nabla f(x^*))) \\
&\quad -\langle \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)) - x^*, -\alpha_n(\nabla f(u) - \nabla f(x^*))\rangle \\
&= V_f(x^*, \alpha_n \nabla f(x^*) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)) \\
&\quad +\alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle \\
&= D_f(x^*, \nabla f^*(\alpha_n \nabla f(x^*) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)) \\
&\quad +\alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle \\
&\leq \alpha_n D_f(x^*, x^*) + \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, T_j u_n) \\
&\quad +\alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle \\
&= \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, T_j u_n) + \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle \\
&\leq \beta_n D_f(x^*, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(x^*, u_n) + \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle \\
&= (1-\alpha_n) D_f(x^*, u_n) + \alpha_n \langle x_{n+1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle. \qquad (4.12)
\end{aligned}$$

We consider two cases to obtain strong convergence.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(x^*, u_n)\}_{n=1}^{\infty}$ is monotonically nonincreasing. Then $\{D_f(x^*, u_n)\}_{n=1}^{\infty}$ converges and $D_f(x^*, u_{n+1}) - D_f(x^*, u_n) \to 0, n \to \infty$.
Let $s_n := \nabla f^*\big(\frac{\beta_n}{1-\alpha_n} \nabla f(u_n) + \frac{\sum_{j=1}^{\infty} \gamma_{nj}}{1-\alpha_n} \nabla f(T_j u_n)\big)$.

Then,

$$\begin{aligned}
D_f(x^*, s_n) &= D_f(x^*, \nabla f^*\big(\frac{\beta_n}{1-\alpha_n}\nabla f(u_n) + \frac{\sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n}\nabla f(T_j u_n)\big) \\
&\leq \frac{\beta_n}{1-\alpha_n} D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n} D_f(x^*, T_j u_n) \\
&\leq \frac{\beta_n}{1-\alpha_n} D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n} D_f(x^*, u_n) \\
&\leq \frac{\beta_n + \sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n} D_f(x^*, u_n). \qquad (4.13)
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\leq D_f(x^*, u_n) - D_f(x^*, s_n) \\
&= D_f(x^*, u_n) - D_f(x^*, u_{n+1}) + D_f(x^*, u_{n+1}) - D_f(x^*, s_n) \\
&\leq D_f(x^*, u_n) - D_f(x^*, u_{n+1}) + D_f(x^*, x_{n+1}) - D_f(x^*, s_n) \\
&\leq D_f(x^*, u_n) - D_f(x^*, u_{n+1}) + \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, s_n) - D_f(x^*, s_n) \\
&= D_f(x^*, u_n) - D_f(x^*, u_{n+1}) + \alpha_n (D_f(x^*, u) - D_f(x^*, s_n)) \to 0, n \to \infty. \qquad (4.14)
\end{aligned}$$

Furthermore

$$\begin{aligned}
D_f(x^*, s_n) &\leq \frac{\beta_n}{1-\alpha_n} D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n} D_f(x^*, T_j u_n) \\
&= \big(1 - \frac{\sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n}\big) D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n} D_f(x^*, T_j u_n) \\
&= D_f(x^*, u_n) + \frac{\sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n} (D_f(x^*, T_j u_n) - D_f(x^*, u_n)). \qquad (4.15)
\end{aligned}$$

Therefore from (4.15), we have

$$\frac{\sum_{j=1}^{\infty}\gamma_{nj}}{1-\alpha_n}(D_f(x^*, u_n) - D_f(x^*, T_j u_n)) \leq D_f(x^*, u_n) - D_f(x^*, s_n) \to 0, n \to \infty. \qquad (4.16)$$

Since T_j is left Bregman strongly nonexpansive, we obtain that

$$\lim_{n\to\infty} D_f(u_n, T_j u_n) = 0,$$

which implies that

$$\lim_{n\to\infty} ||u_n - T_j u_n|| = 0. \qquad (4.17)$$

Since $\{u_n\}$ is bounded and E is a reflexive Banach space, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ that converges weakly to $p \in C$. It then follows from (4.17) that $p \in \cap_{j=1}^{\infty} F(T_j)$, since $F(T_j) = \hat{F}(T_j)$.

We next show that $p \in \cap_{k=1}^{\infty} EP(G_k) = \cap_{k=1}^{\infty} GMEP(F_k, \varphi_k, B_k)$. Denote $\Theta_k = Res_{G_k}^f \circ Res_{G_{k-1}}^f \circ, ..., \circ Res_{G_1}^f$ for $k = 1, 2, ...N$ and $\Theta_0 = I$. We note that $u_n = \Theta_N y_n$. Now, by using the fact that $Res^f_{G_k}, k = 1, 2, \cdots, N$; is properly left Bregman nonexpansive mapping, we have

$$\begin{aligned} D_f(x^*, u_n) &= D_f(x^*, \Theta_N y_n) \\ &= D_f(x^*, Res^f_{G_N}\Theta_{N-1}y_n) \\ &\leq D_f(x^*, \Theta_{N-1}y_n) \leq \cdots \leq D_f(x^*, y_n) \\ &\leq D_f(x^*, x_n). \end{aligned} \tag{4.18}$$

Since $x^* \in EP(G_N) = F(Res^f_{G_n})$, then from Lemma 4.1.3, (4.10) and (4.18), we have

$$\begin{aligned} D_f(u_n, \Theta_{N-1}y_n) &= D_f(Res^f_{G_N}\Theta_{N-1}y_n), \Theta_{N-1}y_n) \\ &\leq D_f(x^*, \Theta_{N-1}y_n) - D_f(x^*, u_n) \\ &\leq D_f(x^*, y_n) - D_f(x^*, u_n) \\ &\leq D_f(x^*, x_n) - D_f(x^*, u_n) \\ &\leq (1-\alpha_{n-1})D_f(x^*, u_{n-1}) + \alpha_{n-1}\langle x_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle - D_f(x^*, u_n) \\ &\leq \alpha_{n-1}M + D_f(x^*, u_{n-1}) - D_f(x^*, u_n) \to 0, n \to \infty, \end{aligned} \tag{4.19}$$

where $M > 0$ is such that $D_f(x^*, u_{n-1}) + \langle x_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle \leq M$.
Therefore,

$$\lim_{n\to\infty} D_f(\Theta_N y_n, \Theta_{N-1}y_n) = \lim_{n\to\infty} D_f(u_n, \Theta_{N-1}y_n) = 0.$$

From Lemma 2.3.12, we have

$$\lim_{n\to\infty} ||\Theta^N y_n - \Theta^{N-1}y_n|| = \lim_{n\to\infty} ||u_n - \Theta^{N-1}y_n|| = 0. \tag{4.20}$$

Thus, we have from (4.20) that

$$\lim_{n\to\infty} ||\nabla f(\Theta^N y_n) - \nabla f(\Theta^{N-1}y_n)|| = 0. \tag{4.21}$$

Again, since $x^* \in EP(G_{N-1}) = F(Res^f_{G_{N-1}})$, it follows from (4.18) and Lemma 4.1.3 that

$$\begin{aligned} D_f(\Theta^{N-1}y_n, \Theta^{N-2}y_n) &= D_f(Res^f_{G_{N-1}}\Theta^{N-2}y_n, \Theta^{N-2}y_n) \\ &\leq D_f(x^*, \Theta^{N-2}y_n) - D_f(x^*, \Theta^{N-1}y_n) \\ &\leq D_f(x^*, y_n) - D_f(x^*, u_n) \\ &\leq D_f(x^*, x_n) - D_f(x^*, u_n) \\ &\leq \alpha_{n-1}M + D_f(x^*, u_{n-1}) - D_f(x^*, u_n) \to 0, n \to \infty. \end{aligned} \tag{4.22}$$

That is

$$\lim_{n\to\infty} D_f(\Theta^{N-1}y_n, \Theta^{N-2}y_n) = 0.$$

Hence from Lemma 2.3.12, we have

$$\lim_{n\to\infty} ||\Theta^{N-1}y_n - \Theta^{N-2}y_n|| = 0, \tag{4.23}$$

and consequently, we have

$$\lim_{n\to\infty} ||\nabla f(\Theta^{N-1}y_n) - \nabla f(\Theta^{N-2}y_n)|| = 0. \tag{4.24}$$

In a similar way, we can verify that

$$\lim_{n\to\infty} ||\Theta^{N-2}y_n - \Theta^{N-3}y_n|| = \cdots = \lim_{n\to\infty} ||\Theta^1 y_n - y_n|| = 0. \tag{4.25}$$

It is now easily seen from (4.20),(4.23) and (4.25), that

$$\lim_{n\to\infty} ||\Theta^k y_n - \Theta^{k-1}y_n|| = 0, k = 1, 2, \cdots, N. \tag{4.26}$$

and

$$\lim_{n\to\infty} ||u_n - y_n|| = 0.$$

Now since $u_{n_j} \rightharpoonup p$ and $\lim_{n\to\infty} ||u_n - y_n|| = 0$, we have that $y_{n_j} \rightharpoonup p$. Also from(4.20),(4.23), (4.25) and $y_{n_j} \rightharpoonup p$, we have that $\Theta^k y_{n_j} \rightharpoonup p, j \to \infty$, for each $k = 1, 2, \cdots, N$. Again using (4.26), we get that

$$\lim_{n\to\infty} ||\nabla f(\Theta^k y_n) - \nabla f(\Theta^{k-1}y_n)|| = 0, k = 1, 2, \cdots, N. \tag{4.27}$$

Therefore by (4.5), we have that for each $k = 1, 2, \cdots, N$,

$$G_k(\Theta^k y_{n_j}, y) + \langle y - \Theta^k y_{n_j}, \nabla f(\Theta^k y_{n_j}) - \nabla f(\Theta^{k-1}y_{n_j})\rangle \geq 0, \quad \forall y \in C.$$

Again using (A2), we obtain

$$\langle y - \Theta^k y_{n_j}, \nabla f(\Theta^k y_{n_j}) - \nabla f(\Theta^{k-1}y_{n_j})\rangle \geq G_k(y, \Theta^k y_{n_j}). \tag{4.28}$$

Thus, a combination of (A4),(4.27),(4.28) and $\Theta^k y_{n_j} \rightharpoonup p, j \to \infty$, gives us that for each $k = 1, 2, \cdots, N$,

$$G_k(y, p) \leq 0, \quad \forall\ y \in C.$$

Then for fixed $y \in C$, let $z_{t,y} := ty + (1-t)p$ for all $t \in (0, 1]$. This implies that $z_{t,y} \in C$ and further yields that $G_k(z_{t,y}, p) \leq 0$. It then follows from (A1) and (A4) that

$$\begin{aligned} 0 &= G_k(z_{t,y}, z_{t,y}) \\ &\leq tG_k(z_{t,y}, y) + (1-t)G_k(z_{t,y}, p) \\ &\leq tG_k(z_{t,y}, y) \end{aligned}$$

and hence, from condition (A3), we obtain $G_k(p, y) \geq 0, \quad \forall\ y \in C$, which implies that

$$p \in \cap_{k=1}^N EP(G_k).$$

Next, we show that $p \in \cap_{i=1}^N A_i^{-1}(0) = \cap_{i=1}^N F(Res^f_{\lambda_n^i A_i})$.
Set $\Phi^i = Res^f_{\lambda_n^i A_i} \circ Res^f_{\lambda_n^{i-1} A_{i-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1}$, for each $i = 1, 2, \cdots, N$, and $\Phi^0 = I$. We note that $y_n = \Phi^i x_n$.
Since $x^* \in A_N^{-1}(0)$, by Lemma 2.3.19, we have

$$\begin{aligned} D_f(y_n, \Phi^{N-1}(x_n)) &\leq D_f(x^*, \Phi^{N-1}(x_n)) - D_f(x^*, y_n) \\ &\leq D_f(x^*, x_n) - D_f(x^*, y_n) \\ &= (1-\alpha_{n-1})D_f(x^*, u_{n-1}) + \alpha_{n-1}\langle z_{n-1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle - D_f(x^*, y_n) \\ &\leq \alpha_{n-1}M_1 + D_f(x^*, u_{n-1} - D_f(x^*, u_n) \to 0, n \to \infty, \end{aligned} \tag{4.29}$$

where M_1 is such that $D_f(x^*, u_{n-1}) + \alpha_{n-1}\langle z_{n-1} - x^*, \nabla f(u) - \nabla f(x^*)\rangle \leq M_1$. Since f is sequentially consistent, then we have from (4.29) that

$$\lim_{n\to\infty} ||y_n - \Phi^{N-1}x_n|| = 0, \tag{4.30}$$

and hence

$$\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(\Phi^{N-1}x_n)|| = 0. \tag{4.31}$$

Again, since $x^* \in A_{N-1}^{-1}(0)$, by Lemma 2.3.19, we have

$$\begin{aligned} D_f(\Phi^{N-1}(x_n), \Phi^{N-2}(x_n)) &\leq D_f(x^*, \Phi^{N-2}(x_n)) - D_f(x^*, \Phi^{N-1}(x_n)) \\ &\leq D_f(x^*, x_n) - D_f(x^*, y_n) \\ &\leq \alpha_{n-1}M_1 \\ &\quad + D_f(x^*, u_{n-1}) - D_f(x^*, u_n) \to 0, n \to \infty. \end{aligned} \tag{4.32}$$

Thus since f is sequentially consistent, we have

$$\lim_{n\to\infty} ||\Phi^{N-1}(x_n) - \Phi^{N-2}(x_n)|| = 0, \tag{4.33}$$

and hence

$$\lim_{n\to\infty} ||\nabla f(\Phi^{N-1}(x_n)) - \nabla f(\Phi^{N-2}(x_n))|| = 0.$$

Following the same procedure, we have that

$$\lim_{n\to\infty} ||\Phi^{N-2}(x_n) - \Phi^{N-3}(x_n)|| = \cdots = \lim_{n\to\infty} ||\Phi^1(x_n) - x_n|| = 0. \tag{4.34}$$

Therefore, from (4.30),(4.33) and (4.34), we conclude that

$$\lim_{n\to\infty} ||\Phi^i(x_n) - \Phi^{i-1}(x_n)|| = 0, \quad i = 1, 2, \cdots, N$$

and

$$\lim_{n\to\infty} ||\nabla f(\Phi^i(x_n)) - \nabla f(\Phi^{i-1}(x_n))|| = 0. \tag{4.35}$$

Thus, we have that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Since $y_{n_j} \rightharpoonup p$ and $\lim_{n\to\infty} ||y_n - x_n|| = 0$, we have that $x_{n_j} \rightharpoonup p$. For each $i = 1, 2, \cdots, N$, we note that $\Phi^i(x_n) = Res^f_{\lambda_n^i A_i}\Phi^{i-1}(x_n)$ and therefore

$$||A_{\lambda_n^i}\Phi^i x_n|| = \frac{1}{\lambda_n^i}||\nabla f(\Phi^{i-1}(x_n)) - \nabla f(\Phi^i(x_n))||.$$

Hence from (4.35) and the condition $\lim_{n\to\infty} \lambda_n^i > 0$, we have

$$\lim_{n\to\infty} ||A_{\lambda_n^i}\Phi^i x_n|| = 0. \tag{4.36}$$

Now, since $(\Phi^i x_n, A_{\lambda_n^i}\Phi^{i-1}(x_n)) \in G(A_i)$ for each $i = 1, 2, \cdots, N$. If $(w, w^*) \in G(A_i)$ for each $i = 1, 2, \cdots, N$, then it follows from the monotonicity of $A_i, i = 1, 2, \cdots, N$, that

$$\langle w^* - A_{\lambda_n^i}\Phi^{i-1}(x_n), w - A_{\lambda_n^i}\Phi^i(x_n)\rangle \geq 0.$$

Since $x_{n_j} \rightharpoonup p$, then $\Phi^i(x_{n_j}) \rightharpoonup p$ for each $i = 1, 2, \cdots, N$. Thus from (4.36), we have

$$\langle w^*, w - p\rangle \geq 0,$$

and since A_i is maximally monotone for each $i = 1, 2, \cdots, N$, we conclude that $p \in \cap_{i=1}^N A_i^{-1}(0)$.
Thus we have

$$p \in \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^N EP(G_k)) \cap (\cap_{i=1}^N A_{i=1}^{-1}(0)),$$

that is

$$p \in \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^N GMEP(g_k, \varphi_k, B_k)) \cap (\cap_{i=1}^N A_{i=1}^{-1}(0)).$$

We now show that $\{x_n\}$ converges strongly to $z := \overleftarrow{P}roj_{\Omega}^f u$.

$$\begin{aligned} D_f(u_n, x_{n+1}) &= D_f(u_n, \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(Tu_n))) \\ &\leq \alpha_n D_f(u_n, u) + \beta_n D_f(u_n, u_n) + \sum_{j=1}^{\infty} \gamma_{nj} D_f(u_n, Tu_n) \to 0, n \to \infty. \end{aligned}$$

Therefore, by Lemma 2.3.12, it follows that

$$||u_n - x_{n+1}|| \to 0, n \to \infty.$$

Now,

$$||x_n - x_{n+1}|| \leq ||x_n - y_n|| + ||y_n - u_n|| + ||u_n - x_{n+1}|| \to 0, n \to \infty.$$

Let $z := \overleftarrow{P}roj_{\Omega}^f u$, we now show that

$$\limsup_{n\to\infty} \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle = \lim_{j\to\infty} \langle x_{n_j} - z, \nabla f(u) - \nabla f(z)\rangle.$$

Then, from $||x_n - x_{n+1}|| \to 0, n \to \infty$ and Lemma 2.3.18, we have

$$\limsup_{n\to\infty} \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle = \limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

Now, from (4.12),

$$\begin{aligned} D_f(z, x_{n+1}) &\leq (1 - \alpha_n) D_f(z, u_n) + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\leq (1 - \alpha_n) D_f(z, y_n) + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\leq (1 - \alpha_n) D_f(z, x_n) + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle. \end{aligned}$$

Hence by Lemma 2.3.4, we obtain $D_f(z, x_n) \to 0, n \to \infty$ and so

$$||x_n - z|| \to 0.$$

That is $\{x_n\}$ converges strongly to $z := \overleftarrow{P}roj_{\Omega}^{f} u$.

Case 2. Suppose there exists a subsequence $\{n_\iota\}$ of $\{x_n\}$ such that

$$D_f(x^*, x_{n_\iota}) \leq D_f(x^*, x_{n_\iota+1}) \;\; \forall \; \iota \in \mathbb{N}.$$

Then by Lemma 2.3.5, there exists a nondecreasing sequence $\{m_\tau\} \subset \mathbb{N}$ such that $m_\tau \to \infty, \tau \to \infty$,

$$D_f(x^*, x_{m_\tau}) \leq D_f(x^*, x_{m_\tau+1})$$

and

$$D_f(x^*, x_\tau) \leq D_f(x^*, x_{m_\tau+1}) \;\; \forall \; \tau \in \mathbb{N}.$$

Again, let $s_{n_\tau} := \nabla f^*(\frac{\beta_{n_\tau}}{1-\alpha_{n_\tau}} \nabla f(u_{n_\tau}) + \frac{\sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} \nabla f(T_j u_{n_\tau}))$.
Then

$$\begin{aligned} D_f(x^*, s_{n_\tau}) \;\; &+ \;\; D_f(x^*, \nabla f^*(\frac{\beta_{n_\tau}}{1-\alpha_{n_\tau}} \nabla f(u_{n_\tau}) + \frac{\sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} \nabla f(T_j u_{n_\tau}))) \\ &\leq \;\; (\frac{\beta_{n_\tau}}{1-\alpha_{n_\tau}}) D_f(x^*, u_{n_\tau}) + \frac{\sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}}) D_f(x^*, T_j u_{n_\tau}) \\ &\leq \;\; (\frac{\beta_{n_\tau} + \sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}}) D_f(x^*, u_{n_\tau}) \\ &\leq \;\; D_f(x^*, u_{n_\tau}). \end{aligned}$$

Therefore,

$$\begin{aligned} 0 \;\; &\leq \;\; D_f(x^*, u_{n_\tau}) - D_f(x^*, s_{n_\tau}) \\ &= \;\; D_f(x^*, u_{n_\tau}) - D_f(x^*, u_{n_\tau+1}) + D_f(x^*, u_{n_\tau+1}) - D_f(x^*, s_{n_\tau}) \\ &\leq \;\; D_f(x^*, u_{n_\tau}) - D_f(x^*, u_{n_\tau+1}) + D_f(x^*, x_{n_\tau+1}) - D_f(x^*, s_{n_\tau}) \\ &\leq \;\; D_f(x^*, u_{n_\tau}) - D_f(x^*, u_{n_\tau+1}) + \alpha_{n_\tau} D_f(x^*, u) + (1-\alpha_{n_\tau}) D_f(x^*, s_{n_\tau}) - D_f(x^*, s_{n_\tau}) \\ &= \;\; D_f(x^*, u_{n_\tau}) - D_f(x^*, u_{n_\tau+1}) + \alpha_{n_\tau}(D_f(x^*, u) - D_f(x^*, s_{n_\tau}) \to 0, n \to \infty. \end{aligned}$$

Furthermore

$$\begin{aligned} D_f(x^*, s_{n_\tau}) \;\; &\leq \;\; \frac{\beta_{n_\tau}}{1-\alpha_{n_\tau}} D_f(x^*, u_{n_\tau}) + \frac{\sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} D_f(x^*, T_j u_{n_\tau}) \\ &= \;\; \big(1 - \frac{\sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}}) D_f(x^*, u_{n_\tau}) + \frac{\sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} D_f(x^*, T_j u_{n_\tau}) \\ &= \;\; D_f(x^*, u_{n_\tau}) + \frac{\sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} (D_f(x^*, T_j u_{n_\tau}) - D_f(x^*, u_{n_\tau})). \qquad (4.37) \end{aligned}$$

Thus from (4.37), we have

$$\frac{\sum_{j=1}^{\infty} \gamma_{n_\tau j}}{1-\alpha_{n_\tau}} (D_f(x^*, u_{n_\tau}) - D_f(x^*, T_j u_{n_\tau})) \leq D_f(x^*, u_{n_\tau}) - D_f(x^*, s_{n_\tau}) \to 0, n \to \infty. (4.38)$$

Since T_j for each j is left Bregman strongly nonexpansive, we obtain that

$$\lim_{\tau\to\infty} D_f(u_{n_\tau}, T_j u_{n\tau}) = 0,$$

which implies that

$$\lim_{\tau\to\infty} ||u_{n_\tau} - T_j u_{n_\tau}|| = 0. \tag{4.39}$$

By the same arguments as in Case 1, we obtain that

$$\limsup_{\tau\to\infty} \langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0. \tag{4.40}$$

and

$$D_f(z, x_{n_\tau+1}) \leq (1-\alpha_{n_\tau})D_f(z, x_{n_\tau}) + \alpha_{n_\tau}\langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle,$$

which since $D_f(z, x_{n_\tau}) \leq D_f(z, x_{n_\tau+1})$ implies

$$\begin{aligned} \alpha_{n_\tau} D_f(z, x_{n_\tau}) &\leq D_f(z, x_{n_\tau}) - D_f(z, x_{n_\tau+1}) + \alpha_{n_\tau}\langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\leq \alpha_{n_\tau}\langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle. \end{aligned}$$

Since $\alpha_{n_\tau} > 0$, we have

$$D_f(z, x_{n_\tau}) \leq \langle x_{n_\tau+1} - z, \nabla f(u) - \nabla f(z)\rangle.$$

Hence it follows from (4.40) that

$$\lim_{\tau\to\infty} D_f(z, x_{n_\tau}) = 0.$$

Since $D_f(z, x_\tau) \leq D_f(z, x_{m_\tau+1})$ for all $\tau \in \mathbb{N}$, we conclude that $x_\tau \to z, \tau \to \infty$. This implies that $x_n \to z, n \to \infty$, which completes the proof. □

Corollary 4.2.2. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^{\infty}$ be an infinite family of left Bregman nonexpasive mappings from C into itself and $F(T_j) = \hat{F}(T_j)$, $\forall j \geq 1$. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ and $A_i : E \to 2^{E^*}(i = 1, 2, ..., N)$ be maximal monotone operators such that $\Omega_A := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{i=1}^{N} A_i^{-1}(0)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j y_n)), \quad \forall n \geq 1, \end{cases} \tag{4.41}$$

converges strongly to a point $p = \overleftarrow{P}roj^f_{\Omega_A} u \in \Omega_A$, where the sequences $\alpha_n, \beta_n, \gamma_{nj}$ and λ_n satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

Corollary 4.2.3. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^{\infty}$ be an infinite family of left Bregman nonexpasive mappings from C into itself and $F(T_j) = \hat{F}(T_j), \quad \forall j \geq 1$. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be bifunction satisfying conditions $(A1) - (A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ be proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ with $\Omega_S := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} x_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)), \quad \forall n \geq 1, \end{cases} \tag{4.42}$$

converges strongly to a point $p = \overleftarrow{Proj}^f_{\Omega_S} u \in \Omega_S$, where $G(x, y) := g(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ and the sequences α_n, β_n, and γ_{nj} satisfies the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$.

4.3 Applications

4.3.1 Convex feasibility problem

Let $\{K_j\}_{j=1}^{\infty}$ be nonempty closed and convex subsets of E such that $\cap_{j=1}^{\infty} K_j \neq \emptyset$. The Convex Feasibility Problem (CFP) is to find $x \in \cap_{j=1}^{\infty} K_j$. Obviously $F(\overleftarrow{Proj}^f_{K_j}) = K_j$ for all $j \geq 1$. If the Legendre function is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the Bregman projection $\overleftarrow{Proj}^f_{K_j}$ is BFNE, hence BSNE and $F(\overleftarrow{Proj}^f_{K_j}) = \hat{F}(\overleftarrow{Proj}^f_{K_j})$ (see, [191] Lemma 1.2.3). Thus, if we take $T_j = \overleftarrow{Proj}^f_{K_j}$ in Theorem 4.2.1, we get a strong convergence theorem for approximating the solution of CFPs, a common solution to a finite system of generalised mixed equilibrium problems and a common element of the set of zeros of a finite family of maximal monotone operators.

Theorem 4.3.1. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $T_j = \overleftarrow{Proj}^f_{K_j}$, where $\{K_j\}_{j=1}^{\infty}$, are nonempty closed and convex subsets of C. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be bifunctions satisfying conditions $(A1) - (A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ be proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ and $A_i : E \to 2^{E^*} (i = 1, 2, ..., N)$ be maximal monotone operators, such that $\Omega_{AS} := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \cap (\cap_{i=1}^{N} A_i^{-1}(0)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u,*

$x_1 \in E$ by

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} y_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)), \quad \forall n \geq 1, \end{cases} \tag{4.43}$$

converges strongly to a point $p = \overleftarrow{P}roj^f_{\Omega_{AS}} u \in \Omega_{AS}$, where $G(x,y) := g(x,y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ and the sequences $\alpha_n, \beta_n, \gamma_{nj}$ and λ_n satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$*;*
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$*;*
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

4.3.2 Zeroes of Bregman inversely strongly monotone operators

Let the Legendre function f be such that

$$ran(\nabla f - A) \subseteq ran(\nabla f). \tag{4.44}$$

The operator $A : E \to 2^{E^*}$ is called *Bregman inversly strongly monotone* (BISM) if

$$(domA) \cap (int(domf) \neq \emptyset$$

and for any $x, y \in int(domf)$, and each $\xi \in Ax, \eta \in Ay$, we have

$$\langle \xi - \eta, \nabla f^*(\nabla f(x) - \xi) - \nabla f^*(\nabla f(y) - \eta)\rangle \geq 0.$$

This class of operators was introduced by Butnariu and Kassey (see [183]). For any operator $A : E \to 2^{E^*}$, the anti resolvent $A^f : E \to 2^E$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A).$$

Observe that $domA^f \subseteq (domA) \cap (int(domf)$ and $ranA^f \subseteq int(domf)$. The operator A [183] is BISM if and only if the anti-resolvent A^f is a single valued BFNE operator. Some examples of BISM operator can be seen in [183]. From the definition of anti-resolvent and ([183], Lemma 3.5), we obtain the following proposition.

Proposition 4.3.2. *Let $f : E \to (-\infty, +\infty]$ be a Legendre function and let $A : E \to 2^{E^*}$ be a BISM operator such that $A^{-1}(0)^* \neq \emptyset$. Then the following statements holds;*
(i) $A^{-1}(0)^* = F(A^f)$,
(ii) for any $u \in A^{-1}(0)^*$ and $x \in domA^f$, we have

$$D_f(u, A^f) + D_f(A^f x, x) \leq D_f(u, x).$$

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the resolvent of A^f of A is a single-valued BSNE operator which satisfies $F(A^f) = \hat{F}(A^f)$ ([191] Lemma 1.3.2).

In Theorem 4.2.1, if we let $T_i = A_i^f$ and let f be the Legendre function such that (4.44) is satisfied then we obtain the following result for approximating a common zeroes of infinite family Bregman Inversely Strongly Monotone Operators, a common solution to a finite system of generalised mixed equilibrium problems and a common element of the set of zeros of a finite family of maximal monotone operators.

Theorem 4.3.3. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^{\infty} = \{A_j^f\}_{j=1}^{\infty}$. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be bifunctions satisfying conditions $(A1) - (A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ be proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ and $A_i : E \to 2^{E^*} (i = 1, 2, ..., N)$ be maximal monotone operators with $\Omega_{AS} := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \cap (\cap_{i=1}^{N} A_i^{-1}(0)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} y_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)), \quad \forall n \geq 1, \end{cases} \tag{4.45}$$

converges strongly to a point $p = \overleftarrow{Proj}^f_{\Omega_{AS}} u \in \Omega_{AS}$, where $G(x, y) := g(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ and the sequences $\alpha_n, \beta_n, \gamma_{nj}$ and λ_n satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

4.3.3 Variational inequalities

Let $A : E \to E^*$ be a BISM operator and let C be a nonempty, closed and convex subset of $domA$. The variational inequality problem corresponding to A is to find $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall\ y \in C. \tag{4.46}$$

The set of solutions of (4.46) is denoted by VI(A,C).

Proposition 4.3.4. *([190]Proposition 8). Let $f : E \to (-\infty, +\infty]$ be a Legendre and totally convex function which satisfies the range condition* (4.44). *Let $A : E \to E^*$ be a BISM operator. If C is a nonempty, closed and convex subset of $domA \cap int(domf)$, then $VI(A, C) = F(\overleftarrow{Proj}^f_C \circ A^f)$.*

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, the anti-resolvent A^f is single-valued ([183], Lemma 3.5(d))and BSNE operator (see Section 2 and [183] Lemma 3.5(c)) which satisfies $F(A^f) = \hat{F}(A^f)$. Since the

Bregman projection $\overleftarrow{Proj}_C^f$ is a BFNE operator, it is a BSNE which satisfies $F(\overleftarrow{Proj}_C^f) = \hat{F}(\overleftarrow{Proj}_C^f)$. It now follows (see [186] Lemma 2) that $\overleftarrow{Proj}_C^f \circ A^f$ is also a BSNE operator which satisfies $F(\overleftarrow{Proj}_C^f \circ A^f) = \hat{F}(\overleftarrow{Proj}_C^f \circ A^f)$. From Proposition 4.4 we know that $F(\overleftarrow{Proj}_C^f \circ A^f) = V(A, C)$. Therefore in Theorem 4.2.1, if we let $T_i = \overleftarrow{Proj}_C^f \circ A^f$, we get an algorithm for finding a common solution to the variational inequality problem corresponding to infinitely many BISM operators and system of equilibrium problem.

Theorem 4.3.5. *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. Let $A_j : E \to E^*$, $j \geq 1$, be an infinite family of BISM operators such that $C \subset domA_j$ and $\{T_j\}_{n=1}^{\infty} = \{\overleftarrow{Proj}_C^f \circ A_j^f\}_{j=1}^{\infty}$. Let $g_k : C \times C \to \mathbb{R}, (k = 1, 2, ..., N)$ be bifunction satisfying conditions $(A1) - (A4)$. Let $B_k : E \to E^*, (k = 1, 2, ..., N)$ be continuous and monotone mappings, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}, (k = 1, 2, ..., N)$ be proper lower semicontinuous and convex functions. Let $f : E \to \mathbb{R}$ be strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$ and $A_i : E \to 2^{E^*}(i = 1, 2, ..., N)$ be maximal monotone operators with $\Omega_{AS} := \cap_{j=1}^{\infty} F(T_j) \cap (\cap_{k=1}^{N} EP(G_k) \cap (\cap_{i=1}^{N} A_i^{-1}(0)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated for arbitrary u, $x_1 \in E$ by*

$$\begin{cases} y_n = Res^f_{\lambda_n^N A_N} \circ Res^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^2 A_2} \circ Res^f_{\lambda_n^1 A_1} x_n, \\ u_n = Res^f_{G_N} \circ Res^f_{G_{N-1}} \circ ... \circ Res^f_{G_2} \circ Res^f_{G_1} y_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(u_n) + \sum_{j=1}^{\infty} \gamma_{nj} \nabla f(T_j u_n)), \quad \forall n \geq 1, \end{cases} \tag{4.47}$$

converges strongly to a point $p = \overleftarrow{Proj}^f_{\Omega_{AS}} u \in \Omega_{AS}$, where $G(x, y) := g(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ and the sequences $\alpha_n, \beta_n, \gamma_{nj}$ and λ_n satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{nj} = 1$;
(iv) $0 < a < \beta_n, \sum_{j=1}^{\infty} \gamma_{nj} < b < 1$;
(v) $\liminf_{n\to\infty} \lambda_n^k > 0$ *for each* $k = 1, 2, ..., N$.

Chapter 5

Split Equilibrium and Fixed Point Problems

5.1 Introduction

Let H_1 and H_2 be real Hilbert spaces and C and Q nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $f_1 : C \times C \to \mathbb{R}$, $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions, $\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$, $\varphi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be functions and $B_1 : C \to H_1$, $B_2 : Q \to H_2$ be nonlinear mappings. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split generalised mixed equilibrium problem* is to find $x^* \in C$ such that

$$f_1(x^*, x) + \langle B_1 x^*, x - x^* \rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall\, x \in C, \tag{5.1}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) + \langle B_2 y^*, y - y^* \rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall\, y \in Q. \tag{5.2}$$

We shall denote the solution set of (5.1)-(5.2) by
$\Omega_{GMEP} = \{x^* \in GMEP(f_1, B_1, \varphi_1) : Ax^* \in GMEP(f_2, B_2, \varphi_2)\}$.
If $B_1 = 0$ and $B_2 = 0$, then (5.1)-(5.2) reduces to the following split mixed equilibrium problem, find $x^* \in C$ such that

$$f_1(x^*, x) + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall\, x \in C, \tag{5.3}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall\, y \in Q, \tag{5.4}$$

with solution set $\Omega_\varphi = \{x^* \in MEP(f_1, \varphi_1) : Ax^* \in MEP(f_2, \varphi_2)\}$.
Again in (5.1)-(5.2) if $\varphi_1 = \varphi_2 = 0$, we obtain the following split generalised equilibrium problem, find $x^* \in C$ such that

$$f_1(x^*, x) + \langle B_1 x^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C, \tag{5.5}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) + \langle B_2 y^*, y - y^* \rangle \geq 0, \quad \forall\, y \in Q, \tag{5.6}$$

with solution set $\Omega_{GEP} = \{x^* \in GEP(f_1, B_1) : Ax^* \in GEP(f_2, B_2)\}$.
Moreover, if $B_1 = B_2$ and $\varphi_1 = \varphi_2 = 0$, we have the split equilibrium problem, find $x^* \in C$ such that

$$f_1(x^*, x) \geq 0, \quad \forall\, x \in C, \tag{5.7}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) \geq 0, \quad \forall\, y \in Q, \tag{5.8}$$

with solution set $\Omega_0 = \{x^* \in EP(f_1) : Ax^* \in EP(f_2)\}$.

Kazmi and Rizvi [119] studied the pair of equilibrium problems (5.7) and (5.8) called split equilibrium problem.

Bnouhachem [24] stated and proved the following strong convergence result.

Theorem 5.1.1. *Let H_1 and H_2 be two real Hilbert spaces, and let $C \subset H_1$ and $Q \subset H_2$ be nonempty closed and convex subset of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying $A1 - A4$ and f_2 is upper semicontinuous in the first argument. Let $S, T : C \to C$ be a nonexpansive mapping such that $\Omega_0 \cap F(T) \neq \emptyset$. Let $f : C \to C$ be a k-Lipschitzian mapping and η-strongly monotone, and let $U : C \to C$ be τ-Lipschitzian mapping. For a given arbitrary $x_0 \in C$, let the iterative sequence $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} u_n = T_{r_n}^{f_1}(x_n + \gamma A^*(T_{r_n}^{f_2} - I)Ax_n), \\ y_n = \beta_n S x_n + (1 - \beta_n) u_n, \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu f)(T(y_n))] \quad \forall\, n \geq 0, \end{cases} \tag{5.9}$$

where $\{r_n\} \subset (0, 2\zeta)$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A. Suppose the parameters satisfy $0 < \mu < \left(\dfrac{2\eta}{k^2}\right)$, $0 \leq \rho\eta < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty} (\frac{\beta_n}{\alpha_n}) = 0$,
(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2\zeta$ *and* $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.
Then $\{x_n\}$ converges strongly to $z \in \Omega_0 \cap F(T)$.

In this chapter, we present an operator norm independent iterative solution of split generalised mixed equilibrium problems with numerical example and a simultaneous iterative scheme for approximating a common solution of split equality for finite family of equilibrium problems and split equality fixed point problems for left Bregmann strongly nonexpansive mappings.

5.2 On split generalised mixed equilibrium problems and fixed point problems with no prior knowledge of operator norm

It is our intention here to introduce an iterative scheme which does not require any knowledge of the operator norm and obtain a strong convergence theorem for approximating solution of split generalised mixed equilibrium problem which also solves a fixed point problem for κ-strictly pseudocontractive mapping.

Precisely, we consider the following problem: find $x^* \in F(S)$ such that

$$f_1(x^*, x) + \langle B_1 x^*, x - x^* \rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall\, x \in C, \tag{5.10}$$

and $y^* = Ax^* \in Q$ solves

$$f_2(y^*, y) + \langle B_2 y^*, y - y^* \rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall\, y \in Q, \tag{5.11}$$

where S is a strictly pseudocontractive mapping on C, C and Q being nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively.

Theorem 5.2.1. *Let H_1 and H_2 be two real Hilbert spaces, let C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(A1)-(A4)$ and f_2 is upper semicontinuous in first argument. Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be continuous and monotone mappings, $\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex function. Let $S : C \to C$ be a κ-strictly pseudocontraction, such that $\Omega_{GMEP} \cap F(S) \neq \emptyset$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$; $\gamma_n \in \left(\epsilon, \frac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\right)$ for $T_{r_n}^{f_2} Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequence $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} w_n = (1 - \alpha_n)x_n + \alpha_n u, \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n S y_n, \quad \forall\, n \geq 0, \end{cases} \tag{5.12}$$

converges strongly to the point $x^* = P_{\Omega_{GMEP} \cap F(S)} u$, where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$.

Proof. Let $p \in \Omega_{GMEP} \cap F(S)$, then from (5.12), we have

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\beta_n)y_n + \beta_n S y_n - p||^2 \\
&= ||(1-\beta_n)(y_n - p) + \beta_n(Sy_n - p)||^2 \\
&= (1-\beta_n)^2||y_n - p||^2 + \beta_n^2||Sy_n - p||^2 + 2\beta_n(1-\beta_n)\langle y_n - p, Sy_n - p\rangle \\
&\leq (1-\beta_n)^2||y_n - p||^2 + \beta_n^2[||y_n - p||^2 + \kappa||y_n - Sy_n||^2] \\
&\quad +2\beta_n(1-\beta_n)[||y_n - p||^2 - \frac{1-\kappa}{2}||y_n - Sy_n||^2] \\
&= (1 - 2\beta_n + \beta_n^2)||y_n - p||^2 + \beta_n^2[||y_n - p||^2 + \kappa||y_n - Sy_n||^2] \\
&\quad +2\beta_n||y_n - p||^2 - 2\beta_n^2||y_n - p||^2 - \beta_n(1-\beta_n)(1-\kappa)||y_n - Sy_n||^2 \\
&= ||y_n - p||^2 + [\beta_n^2\kappa - \beta_n(1-\beta_n)(1-\kappa)]||y_n - Sy_n||^2 \\
&= ||y_n - p||^2 + \beta_n[\kappa + \beta_n - 1]||y_n - Sy_n||^2 \\
&\leq ||y_n - p||^2. \qquad (5.13)
\end{aligned}$$

Again, from (5.12), we have

$$\begin{aligned}
||y_n - p||^2 &= ||T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) - p||^2 \\
&\leq ||w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n - p||^2 \\
&= ||w_n - p||^2 + \gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 \\
&\quad +2\gamma_n\langle w_n - p, A^*(T_{r_n}^{f_2} - I)Aw_n\rangle, \qquad (5.14)
\end{aligned}$$

but from Lemma 2.3.8, we have

$$\begin{aligned}
2\gamma_n\langle w_n - p, &A^*(T_{r_n}^{f_2} - I)Aw_n\rangle \\
&= 2\gamma_n\langle A(w_n - p) + (T_{r_n}^{f_2} - I)Aw_n - (T_{r_n}^{f_2} - I)Aw_n, (T_{r_n}^{f_2} - I)Aw_n\rangle \\
&= 2\gamma_n[\langle T_{r_n}^{f_2}Aw_n - Ap, (T_{r_n}^{f_2} - I)Aw_n\rangle - ||(T_{r_n}^{f_2} - I)Aw_n||^2 \\
&\leq 2\gamma_n\Big[\frac{1}{2}||(T_{r_n}^{f_2} - I)Aw_n||^2 - ||(T_{r_n}^{f_2} - I)Aw_n||^2\Big] \\
&= -\gamma_n||(T_{r_n}^{f_2} - I)Aw_n||^2. \qquad (5.15)
\end{aligned}$$

Therefore, from (5.14),(5.15) and the condition $\gamma_n \in \Big(\epsilon, \frac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\Big)$, we obtain

$$\begin{aligned}
||y_n - p||^2 &\leq ||w_n - p||^2 + \gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - \gamma_n||(T_{r_n}^{f_2} - I)Aw_n||^2 \\
&= ||w_n - p||^2 + \gamma_n[\gamma_n||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - ||(T_{r_n}^{f_2} - I)Aw_n||^2 \\
&\leq ||w_n - p||^2. \qquad (5.16)
\end{aligned}$$

Thus, from (5.13) and (5.16)

$$\begin{aligned}
||x_{n+1}-p|| &\leq ||w_n - p|| \\
&= ||(1-\alpha_n)x_n + \alpha_n u - p|| \\
&= ||(1-\alpha_n)(x_n - p) + \alpha_n(u-p)|| \\
&\leq (1-\alpha_n)||x_n - p|| + \alpha_n||u-p|| \\
&\leq \max\{||x_n - p||, ||u-p||\} \\
&\vdots \\
&\leq \max\{||x_0 - p||, ||u-p||\}. \qquad (5.17)
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded and so also are $\{y_n\}, \{w_n\}$ and $\{Sy_n\}$ bounded.
Since S is a κ-strictly pseudo-contraction then,

$$\begin{aligned}
&||Sx-p||^2 \leq ||x-p||^2 + \kappa||x-Sx||^2 \\
&\Rightarrow \langle Sx-p, Sx-p\rangle \leq \langle x-p, x-p\rangle + \kappa||x-Sx||^2 \\
&\Rightarrow \langle Sx-p, Sx-x\rangle + \langle Sx-p, x-p\rangle \leq \langle x-p, x-p\rangle + \kappa||x-Sx||^2 \\
&\Rightarrow \langle Sx-p, Sx-x\rangle \leq \langle x-Sx, x-p\rangle + \kappa||x-Sx||^2 \\
&\Rightarrow \langle Sx-x, Sx-x\rangle + \langle x-p, Sx-x\rangle \leq \langle x-Sx, x-p\rangle + \kappa||x-Sx||^2 \\
&\Rightarrow ||Sx-x||^2 \leq \langle x-p, x-Sx\rangle - \langle x-p, Sx-x\rangle + +\kappa||x-Sx||^2 \\
&\Rightarrow (1-\kappa)||Sx-x||^2 \leq 2\langle x-p, x-Sx\rangle. \qquad (5.18)
\end{aligned}$$

It follows from (5.12) and (5.18) that

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||(1-\beta_n)y_n + \beta_n Sy_n - p||^2 \\
&= ||y_n - p + \beta_n(Sy_n - y_n)||^2 \\
&= ||y_n - p||^2 + \beta_n^2||Sy_n - p||^2 - 2\beta_n\langle y_n - p, Sy_n - y_n\rangle \\
&\leq ||y_n - p||^2 + \beta_n(\beta_n - (1-\kappa))||Sy_n - p||^2 \\
&\leq ||w_n - p||^2 + \beta_n(\beta_n - (1-\kappa))||Sy_n - p||^2 \\
&= ||(1-\alpha_n)x_n + \alpha_n u - p||^2 + \beta_n(\beta_n - (1-\kappa))||Sy_n - p||^2 \\
&= (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||u-p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u-p\rangle \\
&\quad +\beta_n(\beta_n - (1-\kappa))||Sy_n - p||^2. \qquad (5.19)
\end{aligned}$$

We now consider two cases to establish the strong convergence of $\{x_n\}$ to $x^* = P_{\Omega_{GMEP}\cap F(S)}u$.
Case 1. Assume that $\{||x_n - p||\}$ is monotonically decreasing sequence. Then $\{x_n\}$ is convergent and clearly

$$\lim_{n\to\infty} ||x_n - p|| = \lim_{n\to\infty} ||x_{n+1} - p||. \qquad (5.20)$$

Thus, from (5.19), we have

$$\begin{aligned}
\beta_n((1-\kappa) - \beta_n)||Sy_n - y_n||^2 &\leq (1-\alpha_n)^2||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n^2||u-p||^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle x_n - p, u-p\rangle \to 0, n\to\infty. \qquad (5.21)
\end{aligned}$$

Therefore,

$$||Sy_n - y_n|| \to 0, n\to\infty. \qquad (5.22)$$

From (5.12),

$$||w_n - x_n|| = \alpha_n||u - x_n|| \to 0, n \to \infty. \tag{5.23}$$

Again from (5.12), we obtain

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\beta_n)y_n + \beta_n Sy_n - p||^2 \\
&= (1-\beta_n)^2||y_n - p||^2 + \beta_n^2||Sy_n - p||^2 + 2\beta_n(1-\beta_n)\langle y_n - p, Sy_n - p\rangle \\
&\leq ||y_n - p||^2 + \beta_n[-1 + \kappa + \beta_n]||y_n - Sy_n||^2 \\
&\leq ||y_n - p||^2 \\
&= ||T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) - p||^2 \\
&\leq ||w_n - p||^2 + \gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - \gamma_n||(T_{r_n}^{f_2} - I)Aw_n||^2 \\
&\leq (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||u - p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle \\
&\quad + \gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - \gamma_n||(T_{r_n}^{f_2} - I)Aw_n||^2 \\
&\leq (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||u - p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle \\
&\quad + \gamma_n[\gamma_n||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - ||(T_{r_n}^{f_2} - I)Aw_n||^2].
\end{aligned} \tag{5.24}$$

It then follows from (5.24) and the condition

$$\gamma_n \in \Big(\epsilon, \frac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\Big), \tag{5.25}$$

that

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||u - p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle \\
&\quad - \epsilon||A^*(T_{r_n}^{f_2} - I)Aw_n||^2,
\end{aligned} \tag{5.26}$$

which implies

$$\begin{aligned}
\epsilon||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 &\leq (1-\alpha_n)^2||x_n - p||^2 - ||x_n - p||^2 \\
&\quad + \alpha_n^2||u - p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle.
\end{aligned} \tag{5.27}$$

Hence,

$$\lim_{n\to\infty} ||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 = 0. \tag{5.28}$$

Furthermore, from (5.24) and (5.28)

$$\begin{aligned}
\gamma_n||(T_{r_n}^{f_2} - I)Aw_n||^2 &\leq (1-\alpha_n)^2||x_n - p||^2 - ||x_{n+1} - p||^2 \\
&\quad + \alpha_n^2||u - p||^2 + \gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle \to 0, n \to \infty.
\end{aligned} \tag{5.29}$$

Therefore

$$\lim_{n\to\infty} ||(T_{r_n}^{f_2} - I)Aw_n|| = 0. \tag{5.30}$$

On the other hand, if $T_{r_n}^{f_2}Aw_n = Aw_n$, then obviously

$$\lim_{n\to\infty} ||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 = 0, \tag{5.31}$$

and

$$\lim_{n\to\infty} ||(T_{r_n}^{f_2} - I)Aw_n||^2 = 0. \tag{5.32}$$

Also,

$$\begin{aligned}
||y_n - p||^2 &= ||T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) - p||^2 \\
&\leq \langle y_n - p, w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n - p\rangle \\
&= \frac{1}{2}[||y_n - p||^2 + ||w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n - p||^2 \\
&\quad -||y_n - p - (w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n) - p)||^2] \\
&\leq \frac{1}{2}[||y_n - p||^2 + ||w_n - p||^2 + \gamma_n(\gamma_n||A^*(T_{r_n}^{f_2} - I)Aw_n||^2 - ||(T_{r_n}^{f_2} - I)Aw_n||^2) \\
&\quad -||y_n - p - (w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n - p)||^2] \\
&\leq \frac{1}{2}[||y_n - p||^2 + ||w_n - p||^2 - (||y_n - w_n||^2 + \gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n|| \\
&\quad -2\gamma_n\langle y_n - w_n, A^*(T_{r_n}^{f_2} - I)Aw_n\rangle)] \\
&\leq \frac{1}{2}[||y_n - p||^2 + ||w_n - p||^2 - ||y_n - w_n||^2 + \gamma_n^2||A^*(T_{r_n}^{f_2} - I)Aw_n|| \\
&\quad +2\gamma_n||y_n - w_n||||A^*(T_{r_n}^{f_2} - I)Aw_n||].
\end{aligned} \tag{5.33}$$

That is

$$||y_n - p||^2 \leq ||w_n - p||^2 - ||y_n - w_n||^2 + 2\gamma_n||y_n - w_n||||A^*(T_{r_n}^{f_2} - I)Aw_n||. \tag{5.34}$$

It then follows from (5.24) and (5.34) that

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq ||w_n - p||^2 - ||y_n - w_n||^2 \\
&\quad +2\gamma_n||y_n - w_n||||A^*(T_{r_n}^{f_2} - I)Aw_n||,
\end{aligned} \tag{5.35}$$

which implies that

$$\begin{aligned}
||y_n - w_n||^2 &\leq ||w_n - p||^2 - ||x_{n+1} - p||^2 + 2\gamma_n||y_n - w_n||||A^*(T_{r_n}^{f_2} - I)Aw_n|| \\
&= ||(1-\alpha_n)x_n + \alpha_n u - p||^2 - ||x_{n+1} - p||^2 + 2\gamma_n||y_n - w_n||||A^*(T_{r_n}^{f_2} - I)Aw_n|| \\
&\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n^2||u - p||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle \\
&\quad +2\gamma_n||y_n - w_n||||A^*(T_{r_n}^{f_2} - I)Aw_n|| \to 0, n \to \infty.
\end{aligned} \tag{5.36}$$

From (5.23) and (5.36), we obtain that

$$||x_n - y_n|| \leq ||x_n - w_n|| + ||w_n - y_n|| \to 0, n \to \infty. \tag{5.37}$$

Again,

$$||x_{n+1} - y_n|| = \beta_n||Sy_n - y_n|| \to 0, \quad n \to \infty. \tag{5.38}$$

From (5.38), we have

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - y_n|| + ||y_n - x_n|| \to 0, \quad n \to \infty. \tag{5.39}$$

Let $u_n = w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n$.
Then,

$$||u_n - w_n|| = \gamma_n ||A^*(T_{r_n}^{f_2} - I)Aw_n|| \to 0, n \to \infty. \tag{5.40}$$

Combining (5.36) and (5.40), we get

$$||y_n - u_n|| \leq ||y_n - w_n|| + ||w_n - u_n|| \to 0, n \to \infty. \tag{5.41}$$

Since $\{y_n\}$ is bounded, the there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ that converges weakly to $q \in C$ and it follows from (5.22) and Lemma 2.3.1 that $q \in F(T)$. Moreover, $\{x_{n_k}\}$ and $\{w_{n_k}\}$ converges weakly to q.
Next, we show that $q \in GMEP(f_1, B_1, \varphi_1)$. Since $y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n)$, we have

$$\begin{aligned} & f_1(y_n, y) + \langle B_1 y_n, y - y_n \rangle + \varphi_1(y) - \varphi_1(y_n) \\ & + \frac{1}{r_n}\langle y - y_n, y_n - w_n \rangle - \frac{1}{r_n}\langle y - y_n, \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n \rangle \geq 0, \forall\, y \in C. \end{aligned} \tag{5.42}$$

Thus, from the monotonicity of $F_1(x,y) := f_1(x,y) + \langle B_1 x, y - x\rangle + \varphi_1(y) - \varphi_1(x)$, we have

$$\begin{aligned} \frac{1}{r_n}\langle y - y_n, y_n - w_n \rangle - \frac{1}{r_n}\langle y - y_n, \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n \rangle \;\; & \geq \;\; f_1(y, y_n) + \langle B_1 y, y_n - y \rangle \\ & \quad + \varphi_1(y_n) - \varphi_1(y), \end{aligned} \tag{5.43}$$

which implies that

$$\begin{aligned} \frac{1}{r_{n_k}}\langle y - y_{n_k}, y_{n_k} - w_{n_k} \rangle - \frac{1}{r_{n_k}}\langle y - y_{n_k}, \gamma_n A^*(T_{r_{n_k}}^{f_2} - I)Aw_{n_k} \rangle \;\; & \geq \;\; f_1(y, y_{n_k}) + \langle B_1 y, y_{n_k} - y \rangle \\ & \quad + \varphi_1(y_{n_k}) - \varphi_1(y). \end{aligned} \tag{5.44}$$

Since $y_{n_k} \rightharpoonup q$, then it follows from (5.23),(5.30),(5.35), (5.37) and $A4$ that,

$$f_1(y, q) + \langle B_1 y, q - y \rangle + \varphi_1(q) - \varphi_1(y) \leq 0, \forall\, y \in C. \tag{5.45}$$

Now, for fixed $y \in C$, let $y_t := ty + (1-t)q$ for all $t \in (0,1)$. This implies that $y_t \in C$. Thus from $A1$ and $A4$

$$\begin{aligned} 0 &= f_1(y_t, y_t) + \langle B_1 y_t, y_t - y_t \rangle + \varphi_1(y_t) - \varphi_1(y_t) \\ &\leq t[f_1(y_t, y) + \langle B_1 y_t, y - y_t \rangle + \varphi_1(y) - \varphi_1(y_t)] \\ &\quad + (1-t)[f_1(y_t, q) + \langle B_1 y_t, q - y_t \rangle + \varphi_1(q) - \varphi_1(y_t)] \\ &\leq t[f_1(y_t, y) + \langle B_1 y_t, y - y_t \rangle + \varphi_1(y) - \varphi_1(y_t)]. \end{aligned} \tag{5.46}$$

Therefore,

$$f_1(y_t, y) + \langle B_1 y_t, y - y_t \rangle + \varphi_1(y) - \varphi_1(y_t) \geq 0. \tag{5.47}$$

Furthermore, from $A3$, we have

$$f_1(q,y) + \langle B_1 q, y - q\rangle + \varphi_1(y) - \varphi_1(q) \geq 0, \tag{5.48}$$

which implies that $q \in GMEP(f_1, B_1, \varphi_1)$. Now we show that $Aq \in GMEP(f_2, B_2, \varphi_2)$. Since $w_{n_k} \rightharpoonup q$ and since A is a bounded linear operator, $Aw_{n_k} \to Aq$.
Set $z_{n_k} = Aw_{n_k} - T^{f_2}_{r_{n_k}} Aw_{n_k}$. Then we have that $Aw_{n_k} - z_{n_k} = T^{f_2}_{r_{n_k}} Aw_{n_k}$, and from (5.30), we have

$$\lim_{n\to\infty} z_{n_k} = 0. \tag{5.49}$$

Therefore, from the definition of $T^{f_2}_{r_{n_k}}$, we observe that

$$\begin{aligned} &f_2(Aw_{n_k} - z_{n_k}, y) + \langle B_2 Aw_{n_k} - z_{n_k}, y - Aw_{n_k} + z_{n_k}\rangle + \varphi_2(y) - \varphi_2(Aw_{n_k} - z_{n_k}) \\ &+\frac{1}{r_{n_k}}\langle y - (Aw_{n_k} - z_{n_k}), (Aw_{n_k} - z_{n_k}) - Aw_{n_k}\rangle \geq 0, \forall y \in C. \end{aligned} \tag{5.50}$$

Since f_2 is upper semicontinuous in first argument, then F_2 defined as

$$F_2(x,y) := f_2(x,y) + \langle B_2 x, y - x\rangle + \varphi_2(y) - \varphi_2(x), \tag{5.51}$$

is also upper semicontinuous in first argument. Thus, taking lim sup to the inequality (5.50) as $k \to \infty$ and using assumption A3, we have

$$f_2(Aq,y) + \langle B_2 Aq, y - Aq\rangle + \varphi_2(y) - \varphi_2(Aq) \geq 0 \quad \forall y \in C, \tag{5.52}$$

which implies $Aq \in GMEP(f_2, B_2, \varphi_2)$. Hence $q \in \Omega_{GMEP} \cap F(S)$.

We now show that $\{x_n\}$ converges strongly to $x^* = P_{\Omega_{GMEP}\cap F(S)} u$.

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||(1-\beta_n)y_n + \beta_n S y_n - x^*||^2 \\ &\leq ||y_n - x^*||^2 \\ &\leq ||w_n - x^*||^2 \\ &= ||(1-\alpha_n)x_n + \alpha_n u - x^*||^2 \\ &= ||(1-\alpha_n)(x_n - x^*) + \alpha_n(u - x^*)||^2 \\ &= (1-\alpha_n)^2||x_n - x^*||^2 + \alpha_n^2||u - x^*||^2 + 2\alpha_n(1-\alpha_n)\langle x_n - x^*, u - x^*\rangle \\ &\leq (1-\alpha_n)||x_n - x^*||^2 + \alpha_n[\alpha_n||u - x^*||^2 \\ &\quad +2(1-\alpha_n)\langle x_n - x^*, u - x^*\rangle]. \end{aligned} \tag{5.53}$$

Choose subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - x^*, u - x^*\rangle = \lim_{k\to\infty}\langle x_{n_k} - x^*, u - x^*\rangle,$$

Since $x_{n_k} \rightharpoonup q$ then it follows from Proposition2.1.3

$$\begin{aligned} \limsup_{n\to\infty}\langle x_n - x^*, u - x^*\rangle &= \lim_{k\to\infty}\langle x_{n_k} - x^*, u - x^*\rangle \\ &= \langle q - x^*, u - x^*\rangle \leq 0. \end{aligned} \tag{5.54}$$

Therefore, by Lemma 2.3.12 and (5.53), we obtain $x_n \to x^*$, $n \to \infty$.

Case 2. Assume that $\{||x_n - p||\}$ is not monotonically decreasing sequence. Set $\Gamma_n = ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}. \tag{5.55}$$

Clearly τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for $n \geq n_0$. It follows from (5.19) that

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \\ &\leq ||x_{\tau(n)+1} - p||^2 - (1-\alpha_{\tau(n)})^2||x_{\tau(n)} - p||^2 \\ &\leq \alpha_n^2||u-p||^2 + \alpha_n(1-\alpha_n)\langle x_n - p, u - p\rangle \\ &\quad +\beta_n(\beta_n - (1-\kappa))||y_n - Sy_n||^2. \end{aligned} \tag{5.56}$$

That is,

$$\begin{aligned} K_{\tau(n)} &\leq \alpha_{\tau(n)}^2||u-p||^2 \\ &\quad +2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - p, u - p\rangle \to 0, \quad as \quad n \to \infty, \end{aligned} \tag{5.57}$$

where $K_{\tau(n)} = \beta_{\tau(n)}((1-\kappa) - \beta_{\tau(n)})||Sy_{\tau(n)} - y_{\tau(n)}||^2$.
By the same argument as (5.22) to (5.41) in case 1, we conclude that $\{x_{\tau(n)}\}$, $\{y_{\tau(n)}\}$ and $\{w_{\tau(n)}\}$ converge weakly to $q \in F(S) \cap \Omega_{GMEP}$. Now for all $n \geq n_0$, and

$$\limsup_{n\to\infty}\langle x_{\tau(n)} - x^*, u - x^*\rangle \leq 0. \tag{5.58}$$

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1} - x^*||^2 - ||x_{\tau(n)} - x^*||^2 \\ &\leq (1-\alpha_{\tau(n)})||x_{\tau(n)} - x^*||^2 + \alpha_{\tau(n)}^2||u - x^*||^2 + 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)}\langle x_{\tau(n)} - x^*, u - x^*\rangle - ||x_{\tau(n)} - x^*||^2 \\ &= \alpha_{\tau(n)}[\alpha_{\tau(n)}||u - x^*||^2 + 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - x^*, u - x^*\rangle - ||x_{\tau(n)} - x^*||^2]. \end{aligned} \tag{5.59}$$

Therefore,

$$\begin{aligned} ||x_{\tau(n)} - x^*||^2 &\leq \alpha_{\tau(n)}||u - x^*||^2 \\ &\quad +2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - x^*, u - x^*\rangle \to 0, \quad n \to \infty. \end{aligned} \tag{5.60}$$

Thus,

$$\lim_{n\to\infty} ||x_{\tau(n)} - x^*||^2 = 0, \tag{5.61}$$

and

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1}. \tag{5.62}$$

Furthermore, for $n \geq n_0$, it is observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}. \tag{5.63}$$

So $\lim_{n\to\infty}\Gamma_n = 0$, that is $\{x_n\}$,$\{y_n\}$ and $\{w_n\}$ converge strongly to $x^* \in F(S) \cap \Omega_{GMEP}$, $\forall\, n \geq 0$. □

Corollary 5.2.2. *Let H_1 and H_2 be two real Hilbert spaces, let C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(A1)-(A4)$ and f_2 is upper semicontinuous in first argument. Let $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ be continuous and monotone mappings, $\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex function. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega_{GMEP} \cap F(S) \neq \emptyset$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$; $\gamma_n \in \Big(\epsilon, \frac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\Big)$ for $T_{r_n}^{f_2} Aw_n \neq Aw_n$ and $\gamma_n = \gamma$ otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} w_n = (1 - \alpha_n)x_n + \alpha_n u, \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n S y_n, \quad \forall n \geq 0, \end{cases} \tag{5.64}$$

converges strongly to a point $x^* \in P_{\Omega_{GMEP} \cap F(S)} u$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences in $(0, 1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

Corollary 5.2.3. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(A1) - (A4)$ and f_2 is upper semicontinuous in first argument. Let $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ be continuous and monotone mappings. Let $S : C \to C$ be a κ strictly pseudo contraction such that $\Omega_{GEP} \cap F(S) \neq \emptyset$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$; $\gamma_n \in \Big(\epsilon, \frac{||(T_{r_n}^{f_2} - I)Aw_n||^2}{||A^*(T_{r_n}^{f_2} - I)Aw_n||^2} - \epsilon\Big)$ for $T_{r_n}^{f_2} Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} w_n = (1 - \alpha_n)x_n + \alpha_n u, \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n S y_n, \quad \forall n \geq 0, \end{cases} \tag{5.65}$$

converges strongly to a point $x^* \in P_{\Omega_{GEP} \cap F(S)} u$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences in $(0, 1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$.

Corollary 5.2.4. *Let H_1 and H_2 be two real Hilbert spaces, let C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying* *conditions $(A1)-(A4)$ and f_2 is upper semicontinuous in first argument. Let $\varphi_1 : C \to \mathbb{R}\cup\{+\infty\}$ and $\varphi_2 : Q \to \mathbb{R}\cup\{+\infty\}$ be proper lower semicontinuous and convex function. Let $S : C \to C$ be a κ strictly pseudo contraction, such that $\Omega_\varphi \cap F(S) \neq \emptyset$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$; $\gamma_n \in \Big(\epsilon, \dfrac{||(T_{r_n}^{f_2}-I)Aw_n||^2}{||A^*(T_{r_n}^{f_2}-I)Aw_n||^2} - \epsilon\Big)$ for $T_{r_n}^{f_2}Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \quad \forall\ n \geq 0, \end{cases} \tag{5.66}$$

converges strongly to a point $x^* \in P_{\Omega_\varphi \cap F(S)}$ where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1-\kappa$.

Corollary 5.2.5. *Let H_1 and H_2 be two real Hilbert spaces, let C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(A1)-(A4)$ and f_2 is upper semicontinuous in first argument. Let $S : C \to C$ be a κ strictly pseudo contraction, such that $\Omega_0 \cap F(S) \neq \emptyset$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$; $\gamma_n \in \Big(\epsilon, \dfrac{||(T_{r_n}^{f_2}-I)Aw_n||^2}{||A^*(T_{r_n}^{f_2}-I)Aw_n||^2} - \epsilon\Big)$ for $T_{r_n}^{f_2}Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = T_{r_n}^{f_1}(w_n + \gamma_n A^*(T_{r_n}^{f_2} - I)Aw_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \quad \forall\ n \geq 0, \end{cases} \tag{5.67}$$

converges strongly to a point $x^* \in P_{\Omega_0 \cap F(S)}u$, where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1-\kappa$.

5.2.1 Numerical example and application

We present here in this section an example, a numerical result and an application to split convex minimization problem.

Example
Let $H_1 = H_2 = L^2([0,1])$ with inner product given as $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$. Now take

$f_1(x,y) := ||y||_{L^2} - ||x||_{L^2}$; $B_1 x := 2x$; $\varphi_1(x) = ||x||_{L^2}$ and $Sx = x$. Suppose $A : L^2([0,1]) \to L^2([0,1])$ is defined by

$$Ax(s) = \int_0^1 V(s,t)x(t)dt, \forall\, x \in L^2([0,1]),$$

where $V : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. Then A is a bounded linear operator and the adjoint A^* of A is defined by

$$A^*x(s) = \int_0^1 V(t,s)x(t)dt, \forall\, x \in L^2([0,1]).$$

Here we take $V(s,t) = e^{st}$. Finally take $f_2(x,y) := ||y||^2_{L^2} - ||x||^2_{L^2}$; $B_2 x := 3x$; $\varphi_2(x) = ||x||^2_{L^2}$. We consider the problem; find $x^* \in H_1$ such that

$$Sx^* = x^*, \tag{5.68}$$

$$f_1(x^*, x) + \langle B_1 x^*, x - x^* \rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall\, x \in H_1, \tag{5.69}$$

and $y^* = Ax^* \in H_2$ solves

$$f_2(y^*, y) + \langle B_2 y^*, y - y^* \rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall\, y \in H_2. \tag{5.70}$$

The set of solutions of problem (5.68)-(5.70) is nonempty (since $x(t) = 0$, a.e. is in the set of solutions). Take $\alpha_n = \frac{1}{n+3}$, $\beta_n = \frac{1}{2}(1 - \frac{1}{n+2})$ and let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \dfrac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n} Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number) in iterative scheme (5.12) to obtain

$$\left\{\begin{array}{l} w_n = (1 - \frac{1}{n+3})x_n + \frac{1}{n+3}u, \\ y_n = T^{f_1}_{r_n}(w_n + \gamma_n A^*(T^{f_2}_{r_n} - I)Aw_n), \\ x_{n+1} = (1 - \frac{1}{2}(1 - \frac{1}{n+2}))y_n + \frac{1}{2}(1 - \frac{1}{n+2})y_n, \quad \forall n \geq 0. \end{array}\right. \tag{5.71}$$

Example with numerical computation

Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = \mathbb{R}$. Let $f_1(x,y) = -5x^2 + xy + 4y^2$, $\phi_1(x) = x^2$ and $B_1(x) = 4x$, then $T^{f_1}_r(x) = \dfrac{x}{15r+1}$. Also Let $f_2(x,y) = -3x^2 + xy + 2y^2$, $\phi_2(x) = 2x^2$ and $B_2(x) = 2x$, then $T^{f_2}_r(x) = \dfrac{x}{11r+1}$. Furthermore, let $Ax = 8x$, $A^*x = 8x$ and $S(x) = -2x$. We make difference choices of x_0, u and use $\frac{||x_{n+1} - x_n||}{||x_1 - x_0||} < 10^{-4}$ for stopping criterion. Take $\alpha_n = \frac{1}{n+2}$, $\beta_n = \frac{1}{6}(1 - \frac{1}{n+2})$, $r_n = \dfrac{n}{n+1}$ and let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \dfrac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n} Aw_n \neq Aw_n$ and γ_n any positive real number otherwise, in iterative scheme (5.12) to obtain

$$\left\{\begin{array}{l} w_n = (1 - \frac{1}{n+2})x_n + \frac{1}{n+2}u, \\ y_n = \dfrac{w_n + \gamma_n 8\left(\frac{-11(\frac{n}{n+1})w_n}{11(\frac{n}{n+1})+1}\right)}{15(\frac{n}{n+1}) + 1}, \\ x_{n+1} = (1 - \frac{1}{6}(1 - \frac{1}{n+2}))y_n - \frac{1}{3}(1 - \frac{1}{n+2})y_n, \quad \forall\, n \geq 0. \end{array}\right. \tag{5.72}$$

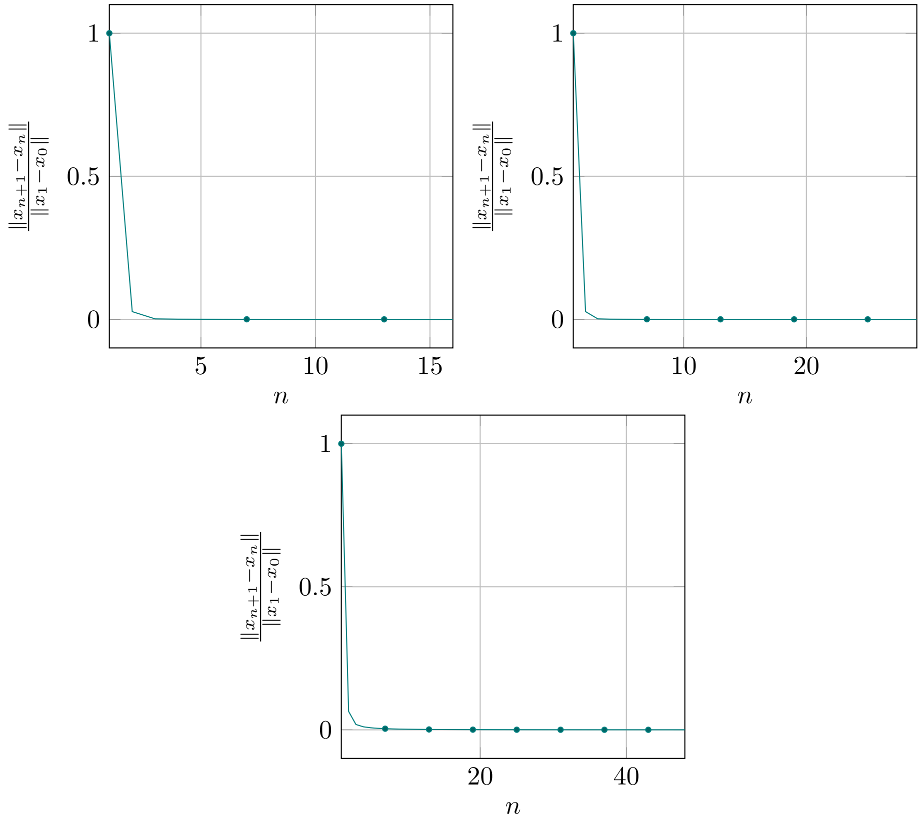


Figure 5.1: Errors: Case 1, $\varepsilon = 10^{-4}$ (top left; 0.010sec); Case 2, $\varepsilon = 10^{-4}$ (top right; 0.011sec); Case 3, $\varepsilon = 10^{-4}$ (bottom; 0.013sec).

Case 1. $x_0 = 2$, $u = 1$ and $\gamma_n \in \left(\epsilon, \dfrac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n}Aw_n \neq Aw_n$ and $\gamma_n = 0.0000021$ otherwise.

Case 2. $x_0 = 6$, $u = 3$ and $\gamma_n \in \left(\epsilon, \dfrac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n}Aw_n \neq Aw_n$ and $\gamma_n = 0.0000222$ otherwise.

Case 3. $x_0 = 1$, $u = 8$ and $\gamma_n \in \left(\epsilon, \dfrac{||(T^{f_2}_{r_n} - I)Aw_n||^2}{||A^*(T^{f_2}_{r_n} - I)Aw_n||^2} - \epsilon\right)$ for $T^{f_2}_{r_n}Aw_n \neq Aw_n$ and $\gamma_n = 0.0003$ otherwise.

The Mathlab version used is R2014a and the execution times are as follows:

1. (case 1, $\varepsilon = 10^{-4}$) and execution time is 0.010 sec.
2. (case 2, $\varepsilon = 10^{-4}$) and execution time is 0.011 sec.
3. (case 3, $\varepsilon = 10^{-4}$) and execution time is 0.013 sec.

Applications to split convex minimisation problem

Here, we apply our result to study the following split convex minimisation problem:find

$$x^* \in F(S) \text{ such that } x^* = \arg\min_{x\in C}(h_1(x) + \phi_1(x) + \varphi_1(x)), \tag{5.73}$$

and such that

$$Ax^* = \arg\min_{y\in Q}(h_2(x) + \phi_2(x) + \varphi_2(x)), \tag{5.74}$$

where C and Q are nonempty closed and convex subset of H_1 and H_2. Also $h_1, \varphi_1 : C \to \mathbb{R}$ and $h_2, \varphi_2 : Q \to \mathbb{R}$ are four convex and lower semi-continuous functionals. Furthermore, $\phi_1 : C \to \mathbb{R}$ and $\phi_2 : Q \to \mathbb{R}$ are convex continuously differentiable functions and $A : H_1 \to H_2$ a bounded linear operator. Let $f_i(x, y) = h_i(y) - h_i(x)$ and $B_i = \nabla\phi_i, \;\; i = 1, 2$ and $\nabla\phi$ denotes the gradient of ϕ.

Then the split convex minimisation problem (5.73)-(5.74) can be formulated as the following split generalised mixed equilibrium problem: find $x^* \in F(S)$ such that

$$h_1(x) - h_1(x^*) + \langle\nabla\phi_1 x^*, x - x^*\rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \;\; \forall\, x \in C, \tag{5.75}$$

and $y^* = Ax^* \in Q$ solves

$$h_2(y) - h_2(y^*) + \langle\nabla\phi_2 y^*, y - y^*\rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \;\; \forall\, y \in Q. \tag{5.76}$$

Thus, Theorem 5.2.1 provides a strong convergence theorem for solving split convex minimisation problem (5.73)-(5.74).

5.3 On split equality for finite family of generalised mixed equilibrium problem and fixed point problem in real Banach spaces

Let E_1, E_2 and E_3 be three real Banach spaces and C, Q be nonempty closed and convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $g_1^i : C \times C \to \mathbb{R} \;\; (i = 1, 2, ..., N)$ and $g_2^j : Q \times Q \to \mathbb{R} \;\; (j = 1, 2, ..., M)$ be two finite families of bifunctions satisfying conditions $(A1) - (A4)$. Let $\Phi_1^i : C \to E_1^* \;\; (i = 1, 2, ..., N)$ and $\Phi_2^j : Q \to E_2^* \;\; (j = 1, 2, ..., M)$ be two finite families of continuous and monotone mappings, $\varphi_1^i : C \to \mathbb{R} \cup +\{\infty\} \;\; (i = 1, 2, ..., N)$ and $\varphi_2^j : Q \to \mathbb{R} \cup +\{\infty\} \;\; (j = 1, 2, ..., M)$ be two finite families of proper lower semicontinuous and convex functions. Let $T : C \to C$ and $S : Q \to Q$ be nonlinear mappings. Then, we consider the following problem: find $\bar{x} \in F(T)$ and $\bar{y} \in F(S)$ such that

$$g_1^i(\bar{x}, x) + \langle\Phi_1^i \bar{x}, x - \bar{x}\rangle + \varphi_1^i(x) - \varphi_1^i(\bar{x}) \geq 0, \;\; \forall\, x \in C, i = 1, 2, \cdots, N; \tag{5.77}$$

$$g_2^j(\bar{y}, y) + \langle\Phi_2^j \bar{y}, y - \bar{y}\rangle + \varphi_2^j(y) - \varphi_2^j(\bar{y}) \geq 0, \;\; \forall\, y \in Q, j = 1, 2, \cdots, M; \tag{5.78}$$

and $A\bar{x} = B\bar{y}$. We shall denote the solution set of (5.77)-(5.78) by $\Omega_{Eq} = \{(\bar{x}, \bar{y}) : \bar{x} \in F(T) \cap (\cap_{i=1}^{N} GMEP(g_1^i, \Phi_1^i, \varphi_1^i)), \bar{y} \in F(S) \cap (\cap_{j=1}^{M} GMEP(g_2^j, \Phi_2^j, \varphi_2^j)), A\bar{x} = B\bar{y}\}$.
This problem (5.77)-(5.78) that we are considering has as special cases the split equality equilibrium problem, the split equality variational inequality problem, the split equality convex minimisation problem and the split generalised mixed equilibrium problem.

Theorem 5.3.1. *Let E_1, E_2 and E_3 be three real Banach spaces which are $p-$uniformly convex and uniformly smooth and C, Q be nonempty closed and convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators, $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$ the adjoints of A and B respectively. Let $g_1^i : C \times C \to \mathbb{R}$ $(i = 1, 2, ..., N)$ and $g_2^j : Q \times Q \to \mathbb{R}$ $(j = 1, 2, ..., M)$ be two finite families of bifunctions satisfying conditions $(A1) - (A4)$. Let $\Phi_1^i : C \to E_1^*$ $(i = 1, 2, ..., N)$ and $\Phi_2^j : Q \to E_2^*$ $(j = 1, 2, ..., M)$ be two finite families of continuous and monotone mappings, $\varphi_1^i : C \to \mathbb{R} \cup \{+\infty\}$ $(i = 1, 2, ..., N)$ and $\varphi_2^j : Q \to \mathbb{R} \cup \{+\infty\}$ $(j = 1, 2, ..., M)$ be two finite families of proper lower semicontinuous and convex functions. Let $T : C \to C$ and $S : Q \to Q$ be left Bregman strongly nonexpansive mappings such that $\Omega_{Eq} \neq \emptyset$ and let $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. For a fixed $u \in E_1$ and a fixed $v \in E_2$, let the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by $x_0 \in E_1$ and $y_0 \in E_2$:*

$$\begin{cases} u_n = Res^f_{G_1^N} \circ Res^f_{G_1^{N-1}} \circ ... \circ Res^f_{G_1^2} \circ Res^f_{G_1^1} J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)], \\ v_n = Res^f_{G_2^M} \circ Res^f_{G_2^{M-1}} \circ ... \circ Res^f_{G_2^2} \circ Res^f_{G_2^1} J_q^{E_2^*}[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)], \\ x_{n+1} = J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], \\ y_{n+1} = J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_1}(S(v_n))\Big], \end{cases} \tag{5.79}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$*;*
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(iii) $0 < a \le \beta_n, \gamma_n \le d < 1$;
(iv) $0 < t \le t_n \le k \le \left(\frac{q}{2C_q\|A\|^q}\right)^{\frac{1}{q-1}}, \left(\frac{q}{2D_q\|B\|^q}\right)^{\frac{1}{q-1}}$*;*
$G_\iota(x, y) := g_\iota(x, y) + \langle \Phi_\iota x, y - x\rangle + \varphi_\iota(y) - \varphi_\iota(x), \quad (\iota = 1, 2)$.
Then, the sequence $\{(x_n, y_n)\}$ strongly converges to $(\bar{x}, \bar{y}) = (\Pi_{\Gamma_1} u, \Pi_{\Gamma_2} v)$, where $(\Gamma_1, \Gamma_2) = \{(\bar{x}, \bar{y}) : \bar{x} \in F(T) \cap (\cap_{i=1}^{N} GMEP(g_1^i, \Phi_1^i, \varphi_1^i)), \bar{y} \in F(S) \cap (\cap_{j=1}^{M} GMEP(g_2^j, \Phi_2^j, \varphi_2^j))$ and $A\bar{x} = B\bar{y}\}$ and $\Pi_{\Gamma_1}, \Pi_{\Gamma_2}$ are the Bregman projections onto Γ_1 and Γ_2 respectively.

Proof. It is known (see [247]), that the function $G(x, y) := g(x, y) + \langle \Phi x, y - x\rangle + \varphi(y) - \varphi(x)$ satisfies $(A1) - (A4)$ and $GMEP(g, \Phi, \varphi)$ is closed and convex.

For any $(x, y) \in \Omega_{Eq}$, it follows from (5.79) that

$$\begin{aligned}
\Delta_p(x_{n+1}, x) + \Delta_p(y_{n+1}, y) &= \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], x) \\
&\quad + \Delta_p(J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_1}(S(v_n))\Big], y) \\
&\leq \alpha_n \Delta_p(u, x) + \beta_n \Delta_p(u_n, x) + \gamma_n \Delta_p(T(u_n), x) \\
&\quad + \alpha_n \Delta_p(v, y) + \beta_n \Delta_p(v_n, y) + \gamma_n \Delta_p(S(v_n), y) \\
&\leq \alpha_n \Delta_p(u, x) + \beta_n \Delta_p(u_n, x) + \gamma_n \Delta_p(u_n, x) \\
&\quad + \alpha_n \Delta_p(v, y) + \beta_n \Delta_p(v_n, y) + \gamma_n \Delta_p(v_n, y) \\
&= \alpha_n(\Delta_p(u, x) + \Delta_p(v, y)) \\
&\quad + (1 - \alpha_n)(\Delta_p(u_n, x) + \Delta_p(v_n, y)). \qquad (5.80)
\end{aligned}$$

Noting that $Ax = By$, we obtain from (5.79)

$$\begin{aligned}
& \Delta_p(u_n, x) + \Delta_p(v_n, x) \\
= \; & \Delta_p\Big(Res^f_{G_1^N} \circ Res^f_{G_1^{N-1}} \circ ... \circ Res^f_{G_1^2} \circ Res^f_{G_1^1} J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)], x\Big) \\
& + \Delta_p\Big(Res^f_{G_2^M} \circ Res^f_{G_2^{M-1}} \circ ... \circ Res^f_{G_2^2} \circ Res^f_{G_2^1} J_q^{E_2^*}[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)], y\Big) \\
\leq \; & \Delta_p\Big(J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)], x\Big) \\
& + \Delta_p\Big(J_q^{E_2^*}[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)], y\Big) \\
= \; & \frac{1}{q}||J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)||^q - \langle J_p^{E_1}(x_n), x\rangle + t_n\langle J_p^{E_3}(Ax_n - By_n), Ax\rangle + \frac{1}{p}||x||^p \\
& + \frac{1}{q}||J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)||^q - \langle J_p^{E_2}(y_n), x\rangle - t_n\langle J_p^{E_3}(Ax_n - By_n), By\rangle + \frac{1}{p}||y||^p \\
\leq \; & \frac{1}{q}||J_p^{E_1}(x_n)||^q - t_n\langle J_p^{E_3}(Ax_n - By_n), Ax_n\rangle + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
& - \langle J_p^{E_1}(x_n), x\rangle + t_n\langle J_p^{E_3}(Ax_n - By_n), Ax\rangle + \frac{1}{p}||x||^p \\
& + \frac{1}{q}||J_p^{E_2}(y_n)||^q + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n\rangle + \frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
& - \langle J_p^{E_2}(y_n), y\rangle - t_n\langle J_p^{E_3}(Ax_n - By_n), By\rangle + \frac{1}{p}||y||^p \\
= \; & \frac{1}{q}||x_n||^p - \langle J_p^{E_1}(x_n), x\rangle + \frac{1}{p}||x||^p + t_n\langle J_p^{E_3}(Ax_n - By_n), Ax - Ax_n\rangle \\
& + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
& + \frac{1}{q}||y_n||^p - \langle J_p^{E_2}(y_n), y\rangle + \frac{1}{p}||y||^p + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n - By\rangle \\
& + \frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q
\end{aligned}$$

$$
\begin{aligned}
&= \Delta_p(x_n,x) + t_n\langle J_p^{E_3}(Ax_n - By_n), Ax - Ax_n\rangle + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&\quad +\Delta_p(y_n,y) + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n - By\rangle + \frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&= \Delta_p(x_n,x) + \Delta_p(y_n,y) + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n - Ax_n\rangle \\
&\quad +\frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q + \frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q. \qquad (5.81)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Delta_p(u_n,x) + \Delta_p(v_n,x) \\
\leq\ & \Delta_p(x_n,x) + \Delta_p(y_n,y) + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n - Ax_n\rangle \\
&+\frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q + \frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
=\ & \Delta_p(x_n,x) + \Delta_p(y_n,y) \\
&-[t_n - (\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q})]||(Ax_n - By_n)||^p, \qquad (5.82)
\end{aligned}
$$

which implies

$$
\Delta_p(u_n,x) + \Delta_p(v_n,x) \leq \Delta_p(x_n,x) + \Delta_p(y_n,y). \qquad (5.83)
$$

Substituting (5.83) into (5.80), we have

$$
\begin{aligned}
\Delta_p(x_{n+1},x) + \Delta_p(y_{n+1},y) &\leq \alpha_n(\Delta_p(u,x) + \Delta_p(v,y)) + (1-\alpha_n)(\Delta_p(x_n,x) + \Delta_p(y_n,y)) \\
&\leq \max\{(\Delta_p(u,x) + \Delta_p(v,y)), (\Delta_p(x_n,x) + \Delta_p(y_n,y))\} \\
&\ \ \vdots \\
&\leq \max\{(\Delta_p(u,x) + \Delta_p(v,y)), (\Delta_p(x_0,x) + \Delta_p(y_0,y))\}. (5.84)
\end{aligned}
$$

Therefore, $(\{\Delta_p(x_n,x)\},\{\Delta_p(x_n,x)\})$ are bounded and consequently we have that $\{x_n\}$, $\{y_n\}$,$\{u_n\}$, $\{v_n\}$, $\{T(u_n)\}$ and $\{S(v_n)\}$ are all bounded.
Moreover,

$$
\begin{aligned}
\Delta_p(x_{n+1},x) &= \Delta_p\Big(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], x\Big) \\
&= V_p\Big(\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n)), x\Big) \\
&\leq V_p\Big(\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n)) - \alpha_n\Big(J_p^{E_1}(u) - J_p^{E_1}(x)\Big), x\Big) \\
&\quad -\langle -\alpha_n(J_p^{E_1}(u) - J_p^{E_1}(x)), J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(y_n) + \gamma_n J_p^{E_1}(T(u_n))\Big] - x\rangle \\
&= V_p\Big(\alpha_n J_p^{E_1}(x) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n)), x\Big) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle \\
&= \Delta_p\Big(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(x) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], x\Big) \\
&\quad +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle \\
&\leq \alpha_n\Delta_p(x,x) + \beta_n\Delta_p(u_n,z) + \gamma_n\Delta_p(T(u_n),x) \\
&\quad +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle \\
&\leq (1-\alpha_n)\Delta_p(u_n,x) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle \\
&\leq (1-\alpha_n)\Delta_p(x_n,x) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle. \qquad (5.85)
\end{aligned}
$$

Similarly, we have

$$\Delta_p(y_{n+1}, y) \;\; \leq \;\; (1-\alpha_n)\Delta_p(y_n, y) + \alpha_n \langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle. \tag{5.86}$$

We divide into two cases to obtain the strong convergence.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Delta_p(x_n, x) + \Delta_p(y_n, y)\}$ is monotonically non-increasing. Then obviously $\{\Delta_p(x_n, x) + \Delta_p(y_n, y)\}$ converges and

$$(\Delta_p(x_{n+1}, x) + \Delta_p(y_{n+1}, y)) - (\Delta_p(x_n, x) + \Delta_p(y_n, y)) \to 0, n \to \infty. \tag{5.87}$$

Let $w_n := J_q^{E_1^*}\Big(\dfrac{\beta_n}{1-\alpha_n} J_p^{E_1}(u_n) + \dfrac{\gamma_n}{1-\alpha_n} T(u_n)\Big)$ and $z_n := J_q^{E_2^*}\Big(\dfrac{\beta_n}{1-\alpha_n} J_p^{E_2}(v_n) + \dfrac{\gamma_n}{1-\alpha_n} S(v_n)\Big)$. Then,

$$\begin{aligned}
\Delta_p(w_n, x) + \Delta_p(z_n, y) \;\; &= \;\; \Delta_p(J_q^{E_1^*}\Big(\frac{\beta_n}{1-\alpha_n} J_p^{E_1}(u_n) + \frac{\gamma_n}{1-\alpha_n}(T(u_n))\Big), x) \\
&\quad\; + \Delta_p(J_q^{E_1^*}\Big(\frac{\beta_n}{1-\alpha_n} J_p^{E_1}(v_n) + \frac{\gamma_n}{1-\alpha_n}(S(v_n))\Big), y) \\
&\leq \;\; \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n, x) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(T(u_n), x) \\
&\quad\; + \frac{\beta_n}{1-\alpha_n}\Delta_p(v_n, y) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(S(v_n), y) \\
&\leq \;\; \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n, x) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(u_n, x) \\
&\quad\; + \frac{\beta_n}{1-\alpha_n}\Delta_p(v_n, y) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(v_n, y) \\
&= \;\; \Delta_p(u_n, x) + \Delta_p(v_n, y).
\end{aligned} \tag{5.88}$$

Therefore,

$$\begin{aligned}
0 \;\; &\leq \;\; (\Delta_p(u_n, x) + \Delta_p(v_n, y)) - (\Delta_p(w_n, x) + \Delta_p(z_n, y)) \\
&= \;\; \Delta_p(u_n, x) - \Delta_p(x_{n+1}, x) + \Delta_p(x_{n+1}, x) - \Delta_p(w_n, x) \\
&\quad\; + \Delta_p(v_n, y) - \Delta_p(y_{n+1}, y) + \Delta_p(y_{n+1}, y) - \Delta_p(z_n, y) \\
&\leq \;\; \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + \Delta_p(x_{n+1}, x) - \Delta_p(w_n, x) \\
&\quad\; + \Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) + \Delta_p(y_{n+1}, y) - \Delta_p(z_n, y) \\
&\leq \;\; \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + \alpha_n\Delta_p(u, x) + (1-\alpha_n)\Delta_p(w_n, x) - \Delta_p(w_n, x) \\
&\quad\; + \Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) + \alpha_n\Delta_p(v, y) + (1-\alpha_n)\Delta_p(z_n, y) - \Delta_p(z_n, y) \\
&= \;\; (\Delta_p(x_n, x) + \Delta_p(y_n, y)) - (\Delta_p(x_{n+1}, x) + \Delta_p(y_{n+1}, y)) \\
&\quad\; + \alpha_n((\Delta_p(u, x) + \Delta_p(v, y)) - (\Delta_p(w_n, x) + \Delta_p(z_n, y))) \to 0, \quad n \to \infty.
\end{aligned} \tag{5.89}$$

Furthermore,

$$\begin{aligned}
\Delta_p(w_n, x) + \Delta_p(z_n, y) &\leq \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n, x) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(T(u_n), x) \\
&\quad + \frac{\beta_n}{1-\alpha_n}\Delta_p(v_n, y) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(S(v_n), y) \\
&= \Delta_p(u_n, x) - (1 - \frac{\beta_n}{1-\alpha_n})\Delta_p(u_n, x) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(T(u_n), x) \\
&\quad + \Delta_p(v_n, y) - (1 - \frac{\beta_n}{1-\alpha_n})\Delta_p(v_n, y) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(S(v_n), y) \\
&= \Delta_p(u_n, x) + \Delta_p(v_n, y) + \frac{\gamma_n}{1-\alpha_n}\Big(\Delta_p(T(u_n), x) - \Delta_p(u_n, x)\Big) \\
&\quad + \frac{\gamma_n}{1-\alpha_n}\Big(\Delta_p(S(v_n), y) + \Delta_p(v_n, y)\Big). \qquad (5.90)
\end{aligned}$$

Thus, from (5.90)

$$\begin{aligned}
&\frac{\gamma_n}{1-\alpha_n}\Big[(\Delta_p(u_n, x) - \Delta_p(T(u_n), x)) + (\Delta_p(v_n, y) - \Delta_p(S(v_n), y))\Big] \\
\leq\ &\Big((\Delta_p(u_n, x) + \Delta_p(v_n, y)) - \Delta_p(w_n, x) + \Delta_p(z_n, y)\Big) \to 0, n \to \infty, \qquad (5.91)
\end{aligned}$$

which by condition (iii) implies

$$\Delta_p(u_n, x) - \Delta_p(T(u_n), x) \to 0, n \to \infty,$$

and

$$\Delta_p(v_n, y) - \Delta_p(S(v_n), y) \to 0, n \to \infty.$$

Since T and S are left Bregman strongly nonexpansive, we have

$$\lim_{n\to\infty} \Delta_p(Tu_n, u_n) = 0,$$

and

$$\lim_{n\to\infty} \Delta_p(Sv_n, v_n) = 0,$$

which implies

$$\lim_{n\to\infty} ||Tu_n - u_n|| = 0 \qquad (5.92)$$

and

$$\lim_{n\to\infty} ||Sv_n - v_n|| = 0 \qquad (5.93)$$

respectively. Since $\{u_n\}$ is bounded and E_1 is reflexive, there exists subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to $x^* \in E_2$. From (5.92), it follows that $x^* \in F(T)$ since $F(T) = \hat{F}(T)$. Also since $\{u_n\}$ is bounded and E_2 is reflexive, there exists subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that converges weakly to $y^* \in E_2$. From (5.93), it follows that $y^* \in F(S)$ since $F(S) = \hat{F}(S)$.

Next, we show that $Ax^* = By^*$.
Now from (5.82), we obtain

$$\begin{aligned}
& \Big[t_n - \Big(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\Big)\Big]||(Ax_n - By_n)||^p \\
\leq\ & \Delta_p(x_n, x) + \Delta_p(y_n, y) - (\Delta_p(u_n, x) + \Delta_p(v_n, y)) \\
=\ & \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + \Delta_p(x_{n+1}, x) - \Delta_p(u_n, x) \\
& +\Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) + \Delta_p(y_{n+1}, y) - \Delta_p(v_n, y) \\
\leq\ & \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + (1-\alpha_n)\Delta_p(u_n, x) \\
& +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle - \Delta_p(u_n, x) \\
& +\Delta_p(y_n, x) - \Delta_p(y_{n+1}, y) + (1-\alpha_n)\Delta_p(v_n, y) \\
& +\alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle - \Delta_p(v_n, y) \\
=\ & \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) \\
& +\alpha_n(-\Delta_p(u_n, x) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle) \\
& +\Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) \\
& +\alpha_n(-\Delta_p(v_n, y) + \alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle) \to 0, \quad n \to \infty, \qquad (5.94)
\end{aligned}$$

and since

$$0 < t\Big(1 - \Big(\frac{C_q k^{q-1}(||A||)^q}{q} + \frac{D_q k^{q-1}(||B||)^q}{q}\Big)\Big) \ \leq\ \Big(t_n - \Big(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\Big)\Big),$$

we have that $||(Ax_n - By_n)||^p \to 0, n \to \infty$.
Let $\mu_n = J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)]$ and $\nu_n = J_q^{E_2^*}[J_p^{E_2}(y_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)]$. Denote $\Theta_i = Res^f_{G_1^i} \circ Res^f_{G_1^{i-1}} \circ, ..., \circ Res^f_{G_1^1}$ for $i = 1, 2, ...N$ and $\Theta_0 = I$. We note that $u_n = \Theta_N \mu_n$. Also denote $\Psi_j = Res^f_{G_2^j} \circ Res^f_{G_2^{j-1}} \circ, ..., \circ Res^f_{G_2^1}$ for $j = 1, 2, ...M$ and $\Psi_0 = I$. We note that $v_n = \Psi_N \nu_n$. Since $(x, y) \in \cap_{i=1}^N EP(G_1^i) \times \cap_{j=1}^M EP(G_2^j)$, then from (5.3.1) and Lemma 4.1.3(5),

$$\begin{aligned}
& \Delta_p(\Theta_{N-1}\mu_n, u_n) + \Delta_p(\Psi_{M-1}\nu_n, v_n) \\
=\ & \Delta_p(\Theta_{N-1}\mu_n, Res^f_{G_1^N}\Theta_{N-1}\mu_n) + \Delta_p(\Psi_{M-1}\nu_n, Res^f_{G_2^M}\Psi_{M-1}\nu_n) \\
\leq\ & \Delta_p(\Theta_{N-1}\mu_n, x) - \Delta_p(u_n, x) + \Delta_p(\Psi_{M-1}\nu_n, y) - \Delta_p(v_n, y) \\
\leq\ & \Delta_p(\mu_n, x) - \Delta_p(u_n, x) + \Delta_p(\nu_n, y) - \Delta_p(v_n, y) \\
\leq\ & \Delta_p(x_n, x) - \Delta_p(u_n, x) + \Delta_p(y_n, y) - \Delta_p(v_n, y) \\
=\ & \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + \Delta_p(x_{n+1}, x) - \Delta_p(u_n, x) \\
& +\Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) + \Delta_p(y_{n+1}, y) - \Delta_p(v_n, y) \\
\leq\ & \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + (1-\alpha_n)\Delta_p(u_n, x) \\
& +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle - \Delta_p(u_n, x) \\
& +\Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) + (1-\alpha_n)\Delta_p(v_n, y) \\
& +\alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle - \Delta_p(v_n, y) \\
=\ & \Delta_p(x_n, x) + \Delta_p(y_n, y) - (\Delta_p(x_{n+1}, x) + \Delta_p(y_{n+1}, y)) \\
& +\alpha_n(-\Delta_p(u_n, x) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle) \\
& +\alpha_n(-\Delta_p(v_n, y) + \alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle) \to 0, \quad n \to \infty, \qquad (5.95)
\end{aligned}$$

which implies

$$||\Theta_{N-1}\mu_n - u_n|| \to 0, n \to \infty, \tag{5.96}$$

and

$$||\Psi_{M-1}\nu_n - v_n|| \to 0, n \to \infty. \tag{5.97}$$

Consequently, we have

$$||J_p^{E_1}(\Theta_{N-1}\mu_n) - J_p^{E_1}(\Theta_N\mu_n)|| \to 0, n \to \infty, \tag{5.98}$$

and

$$||J_p^{E_2}(\Psi_{M-1}\nu_n) - J_p^{E_2}(\Psi_M\nu_n)|| \to 0, n \to \infty. \tag{5.99}$$

Again

$$\begin{aligned}
& \Delta_p(\Theta_{N-2}\mu_n, \Theta_{N-1}\mu_n) + \Delta_p(\Psi_{M-2}\nu_n, \Psi_{M-1}\nu_n) \\
\leq\ & \Delta_p(\Theta_{N-2}\mu_n, x) - \Delta_p(\Theta_{N-1}\mu_n, x) + \Delta_p(\Psi_{M-2}\nu_n, y) - \Delta_p(\Psi_{M-1}\nu_n, y) \\
\leq\ & \Delta_p(\mu_n, x) - \Delta_p(u_n, x) + \Delta_p(\nu_n, y) - \Delta_p(v_n, y) \\
\leq\ & \Delta_p(x_n, x) - \Delta_p(u_n, x) + \Delta_p(y_n, y) - \Delta_p(v_n, y) \\
=\ & \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + \Delta_p(x_{n+1}, x) - \Delta_p(u_n, x) \\
& +\Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) + \Delta_p(y_{n+1}, y) - \Delta_p(v_n, y) \\
\leq\ & \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + (1-\alpha_n)\Delta_p(u_n, x) \\
& +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle - \Delta_p(u_n, x) \\
& +\Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) + (1-\alpha_n)\Delta_p(v_n, y) \\
& +\alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle - \Delta_p(v_n, y) \\
=\ & \Delta_p(x_n, x) + \Delta_p(y_n, y) - (\Delta_p(x_{n+1}, x) + \Delta_p(y_{n+1}, y)) \\
& +\alpha_n(-\Delta_p(u_n, x) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle) \\
& +\alpha_n(-\Delta_p(v_n, y) + \alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle) \to 0, \quad n \to \infty,
\end{aligned} \tag{5.100}$$

which implies

$$||\Theta_{N-2}\mu_n - \Theta_{N-1}\mu_n|| \to 0, n \to \infty, \tag{5.101}$$

and

$$||\Psi_{M-2}\nu_n - \Psi_{M-1}\nu_n|| \to 0, n \to \infty. \tag{5.102}$$

Consequently, we have

$$||J_p^{E_1}(\Theta_{N-2}\mu_n) - J_p^{E_1}(\Theta_{N-1}\mu_n)|| \to 0, n \to \infty, \tag{5.103}$$

and

$$||J_p^{E_2}(\Psi_{M-2}\nu_n) - J_p^{E_2}(\Psi_{M-1}\nu_n)|| \to 0, n \to \infty. \tag{5.104}$$

In a similar way, we can verify that

$$\lim_{n\to\infty}||\Theta_{N-2}\mu_n - \Theta_{N-3}\mu_n|| = \cdots = \lim_{n\to\infty}||\Theta_1\mu_n - \mu_n|| = 0, \tag{5.105}$$

and

$$\lim_{n\to\infty}||\Psi_{M-2}\nu_n - \Psi_{M-3}\nu_n|| = \cdots = \lim_{n\to\infty}||\Psi_1\nu_n - \nu_n|| = 0. \tag{5.106}$$

Hence, it follows that

$$\lim_{n\to\infty}||\Theta_i\mu_n - \Theta_{i-1}\mu_n|| = 0, i = 1, 2, \cdots, N, \tag{5.107}$$

and

$$\lim_{n\to\infty}||u_n - \mu_n|| = 0.$$

Moreover

$$\lim_{n\to\infty}||\Psi_j\nu_n - \Psi_{j-1}\nu_n|| = 0, j = 1, 2, \cdots, M, \tag{5.108}$$

and

$$\lim_{n\to\infty}||v_n - \nu_n|| = 0.$$

Again, we obtain from the definition of μ_n that

$$\begin{aligned} 0 &\leq ||J_p^{E_1}\mu_n - J_p^{E_1}x_n|| \\ &\leq t_n||A^*||||J_p^{E_2}(Ax_n - By_n)|| \\ &\leq \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}}||A^*||||(Ax_n - By_n)|| \to 0, n \to \infty. \end{aligned}$$

Since $J_q^{E_1^*}$ is norm to norm uniformly continuous on bounded subsets of E_1^*, we have that

$$\lim_{n\to\infty}||\mu_n - x_n|| \;=\; \lim_{n\to\infty}||J_q^{E_1^*}J_p^{E_1}v_n - J_q^{E_1^*}J_p^{E_1}u_n|| \to 0, n \to \infty. \tag{5.109}$$

Thus, from (5.96) and (5.109), we have

$$||x_n - u_n|| \leq ||x_n - \mu_n|| + ||\mu_v - u_n|| \to 0, n \to \infty.$$

Similarly, we have $\lim_{n\to\infty}||\nu_n - y_n|| = 0$ and $||y_n - v_n|| \to 0, n \to \infty$.
Thus, $Ax^* - By^* \in w_w(Ax_n - By_n)$ and since the norm is weakly lower semicontinuous, we obtain

$$||Ax^* - By^*|| \leq \liminf_{n\to\infty}||Ax_n - By_n|| = 0.$$

We next show that $(x^*, y^*) \in \cap_{i=1}^N EP(G_1^i) \times \cap_{j=1}^M EP(G_2^j)$.

Now since $u_{n_k} \rightharpoonup x^*$ and $\lim_{n\to\infty}||u_n - \mu_n|| = 0$, we have that $\mu_{n_k} \rightharpoonup x^*$. Also from (5.96),(5.101), (5.105) and $\mu_{n_k} \rightharpoonup x^*$, we have that $\Theta_i\mu_{n_k} \rightharpoonup x^*, k \to \infty$, for each $i = 1, 2, \cdots, N$. Again using (5.107), we get that

$$\lim_{n\to\infty}||J_p^{E_1}(\Theta_i\mu_n) - J_p^{E_1}(\Theta_{i-1}\mu_n)|| = 0, i = 1, 2, \cdots, N. \tag{5.110}$$

Therefore, by (4.5), we have that for each $i = 1, 2, \cdots, N$,

$$G_1^i(\Theta_i\mu_{n_k}, z) + \langle z - \Theta_i\mu_{n_k}, J_p^{E_1}(\Theta_i\mu_{n_k}) - J_p^{E_1}(\Theta_{i-1}\mu_{n_k})\rangle \geq 0, \quad \forall\ z \in C.$$

Again using (A2), we obtain

$$\langle z - \Theta_i\mu_{n_k}, J_p^{E_1}(\Theta_i\mu_{n_k}) - J_p^{E_1}(\Theta_{i-1}y_{n_k})\rangle \geq G_1^i(z, \Theta_i\mu_{n_k}). \tag{5.111}$$

Thus, a combination of (A4), (5.110), (5.111) and $\Theta_i\mu_{n_k} \rightharpoonup x^*, k \to \infty$, gives us that for each $i = 1, 2, \cdots, N$,

$$G_1^i(z, x^*) \leq 0, \quad \forall\ z \in C.$$

Then for fixed $z \in C$, let $a_{t,z} := tz + (1-t)x^*$ for all $t \in (0, 1]$. This implies that $a_{t,z} \in C$ and further yields that $G_1^i(z_{t,y}, x^*) \leq 0$. It then follows from (A1) and (A4) that

$$\begin{array}{rcl} 0 & = & G_1^i(a_{t,z}, a_{t,z}) \\ & \leq & tG_1^i(a_{t,z}, y) + (1-t)G_1^i(a_{t,z}, x^*) \\ & \leq & tG_1^i(a_{t,z}, z), \end{array}$$

and hence, from condition (A3), we obtain $G_1^i(x^*, z) \geq 0, \quad \forall\ z \in C$, which implies that

$$x^* \in \cap_{i=1}^N EP(G_1^i).$$

Similarly, we have

$$y^* \in \cap_{j=1}^M EP(G_2^j).$$

Next, we show that $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*).
Now, we observe that

$$\begin{array}{rcl} \Delta_p(x_{n+1}, u_n) + \Delta_p(y_{n+1}, v_n) & = & \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], u_n) \\ & & +\Delta_p(J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(u) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_2}(S(v_n))\Big], v_n) \\ & \leq & \alpha_n\Delta_p(u, u_n) + \beta_n\Delta_p(u_n, u_n) + \gamma_n\Delta_p(T(u_n), u_n) \\ & & +\alpha_n\Delta_p(u, v_n) + \beta_n\Delta_p(v_n, v_n) + \gamma_n\Delta_p(S(v_n), v_n) \to 0, n \to \infty. \end{array}$$

Hence,

$$||x_{n+1} - u_n|| \to 0, \quad n \to \infty, \quad \text{and} \quad ||y_{n+1} - v_n|| \to 0, \quad n \to \infty.$$

Thus

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - u_n|| + ||u_n - x_n|| \to 0, n \to \infty,$$

and

$$||y_{n+1} - y_n|| \leq ||y_{n+1} - v_n|| + ||v_n - y_n|| \to 0, n \to \infty.$$

From (5.85) and (5.86), we obtain

$$\begin{array}{rcl} \Delta_p(x_{n+1}, \bar{x}) + \Delta_p(y_{n+1}, \bar{y}) & \leq & (1-\alpha_n)(\Delta_p(x_n, \bar{x}) + \Delta_p(y_n, \bar{y})) \\ & & +\alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle \\ & & +\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle). \end{array} \tag{5.112}$$

Choose subsequences $\{x_{n_j}\}$ of $\{x_n\}$ and $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle = \lim_{j\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{n_j+1} - \bar{x}\rangle,$$

and

$$\limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle = \lim_{j\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{n_j+1} - \bar{y}\rangle.$$

Since $x_{n_j} \rightharpoonup x^*$ and $y_{n_j} \rightharpoonup y^*$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle &= \lim_{j\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{n_j+1} - \bar{x}\rangle \\ &= \langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x^* - \bar{x}\rangle \leq 0, \qquad (5.113)\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle &= \lim_{j\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{n_j+1} - \bar{y}\rangle \\ &= \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y^* - \bar{y}\rangle \leq 0. \qquad (5.114)\end{aligned}$$

Therefore, by Lemma 2.3.12 and (5.112), we conclude that $\Delta_p(x_n, \bar{x})+\Delta_p(y_n, \bar{y}) \to 0,\ n \to \infty$, that is $||x_n - \bar{x}|| \to 0,\ \ n \to \infty$ and $||y_n - \bar{y}|| \to 0,\ n \to \infty$. Therefore, $x_n \to \bar{x}$ and $y_n \to \bar{y}$.

Case 2. Suppose there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\Delta_p(x_{n_k,x})+\Delta_p(y_{n_k,y}) < \Delta_p(x_{n_k+1}, x) + \Delta_p(y_{n_k+1}, y)$ for all $k \in \mathbb{N}$. Then, by Lemma 2.3.13 there exists a nondecreasing sequence $\{m_\tau\} \subseteq \mathbb{N}$ such that $m_\tau \to \infty$.

$$\Delta_p(x_{m_\tau}, x) + \Delta_p(y_{m_\tau}, y) \leq \Delta_p(x_{m_\tau+1}, x) + \Delta_p(y_{m_\tau+1}, y),$$

and

$$\Delta_p(x_k, x) \leq \Delta_p(x_{m_k+1}, x).$$

Using the same line of arguments as in (5.88),(5.89),(5.90),(5.91) and noting that $\Delta_p(x_{m_\tau}, x)+\Delta_p(y_{m_\tau}, y) \leq \Delta_p(x_{m_\tau+1}, x) + \Delta_p(y_{m_\tau+1}, y)$, we can show that

$$\lim_{\tau\to\infty}||Tu_{m_\tau} - u_{m_\tau}|| = 0 \ \text{ and } \ \lim_{\tau\to\infty}||Sv_{m_\tau} - v_{m_\tau}|| = 0.$$

Moreover, as in (5.113) and (5.114), we have

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_\tau+1} - \bar{x}\rangle \leq 0, \qquad (5.115)$$

and

$$\limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_\tau+1} - \bar{y}\rangle \leq 0. \qquad (5.116)$$

Again, from (5.85) and (5.86), we have

$$\begin{aligned}\Delta_p(x_{m_\tau+1}, \bar{x}) + \Delta_p(y_{m_\tau+1}, \bar{y}) \ \leq\ & (1 - \alpha_{m_\tau})(\Delta_p(x_{m_k}, \bar{x}) + \Delta_p(y_{m_\tau}, \bar{y})) \\ & +\alpha_{m_\tau}(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_\tau+1} - \bar{x}\rangle \\ & +\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_\tau+1} - \bar{y}\rangle),\end{aligned}$$

which implies

$$\begin{aligned}\alpha_{m_\tau}(\Delta_p(x_{m_\tau},\bar{x})+\Delta_p(y_{m_\tau},\bar{y})) \quad \le \quad & (\Delta_p(x_{m_\tau},\bar{x})+\Delta_p(y_{m_\tau},\bar{y}))-(\Delta_p(x_{m_\tau+1},\bar{x})+\Delta_p(y_{m_\tau+1},\bar{y}) \\ & +\alpha_{m_\tau}(\langle J_p^{E_1}(u)-J_p^{E_1}(\bar{x}),x_{m_\tau+1}-\bar{x}\rangle \\ & +\langle J_p^{E_2}(v)-J_p^{E_2}(\bar{y}),y_{m_\tau+1}-\bar{y}\rangle).\end{aligned}$$

That is,

$\Delta_p(x_{m_\tau},\bar{x})+\Delta_p(y_{m_\tau},\bar{y}) \le \langle J_p^{E_1}(u)-J_p^{E_1}(\bar{x}),x_{m_\tau+1}-\bar{x}\rangle+\langle J_p^{E_2}(v)-J_p^{E_2}(\bar{y}),y_{m_\tau+1}-\bar{y}\rangle).$

Therefore

$$\lim_{\tau\to\infty}(\Delta_p(x_{m_\tau},\bar{x})+\Delta_p(y_{m_\tau},\bar{y}))=0,$$

and since

$$\Delta_p(x_\tau,\bar{x})+\Delta_p(y_\tau,\bar{y}) \le \Delta_p(x_{m_\tau+1},\bar{x})+\Delta_p(y_{m_\tau+1},\bar{y}), \quad \text{for all} \quad \tau\in\mathbb{N},$$

we conclude that

$$x_\tau \to \bar{x} \quad \text{and} \quad y_\tau \to \bar{y}, \quad \tau\to\infty.$$

□

Corollary 5.3.2. *Let E_1, E_2 and E_3 be three real Banach spaces which are $p-$uniformly convex and uniformly smooth and C, Q be nonempty closed and convex subsets of E_1 and E_2 respectively. Let $A: E_1 \to E_3$ and $B: E_2 \to E_3$ be bounded linear operators, $A^*: E_3^* \to E_1^*$ and $B^*: E_3^* \to E_2^*$ the adjoints of A and B respectively. Let $g_1^i: C\times C\to\mathbb{R}$ $(i=1,2,...,N)$ and $g_2^j: Q\times Q\to\mathbb{R}$ $(j=1,2,...,M)$ be two finite families of bifunctions satisfying conditions $(A1)-(A4)$. Let $\varphi_1^i: C\to\mathbb{R}\cup\{+\infty\}$ $(i=1,2,...,N)$ and $\varphi_2^j: Q\to\mathbb{R}\cup\{+\infty\}$ $(j=1,2,...,M)$ be two finite families of proper lower semicontinuous and convex functions. Let $T: C\to C$ and $S: Q\to Q$ be left Bregman strongly nonexpansive mappings such that $\Omega_{E\varphi}=\{(\bar{x},\bar{y}): \bar{x}\in F(T)\cap(\cap_{i=1}^N GMEP(g_1^i,\varphi_1^i)), \bar{y}\in F(S)\cap(\cap_{j=1}^M GMEP(g_2^j,\varphi_2^j)): A\bar{x}=B\bar{y}\}\neq\emptyset$ and let $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$. For a fixed $u\in E_1$ and a fixed $v\in E_2$, let the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by $x_0\in E_1$ and $y_0\in E_2$:*

$$\begin{cases} u_n = Res^f_{G_1^N}\circ Res^f_{G_1^{N-1}}\circ ...\circ Res^f_{G_1^2}\circ Res^f_{G_1^1} J_q^{E_1^*}[J_p^{E_1}(x_n)-t_nA^*J_p^{E_3}(Ax_n-By_n)], \\ v_n = Res^f_{G_2^M}\circ Res^f_{G_2^{M-1}}\circ ...\circ Res^f_{G_2^2}\circ Res^f_{G_2^1} J_q^{E_2^*}[J_p^{E_2}(y_n)+t_nB^*J_p^{E_3}(Ax_n-By_n)], \\ x_{n+1} = J_q^{E_1^*}\Big[\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}(u_n)+\gamma_nJ_p^{E_1}(T(u_n))\Big], \\ y_{n+1} = J_q^{E_2^*}\Big[\alpha_nJ_p^{E_2}(v)+\beta_nJ_p^{E_2}(v_n)+\gamma_nJ_p^{E_1}(S(v_n))\Big], \end{cases} \tag{5.117}$$

with the conditions
(i) $\lim_{n\to\infty}\alpha_n=0$;
(ii) $\sum_{n=1}^\infty \alpha_n=\infty$;
(iii) $0<a\le\beta_n,\gamma_n\le d<1$;
(iv) $0<t\le t_n\le k\le\Big(\frac{q}{2C_q\|A\|^q}\Big)^{\frac{1}{q-1}},\Big(\frac{q}{2D_q\|B\|^q}\Big)^{\frac{1}{q-1}}$;
$G_\iota(x,y):=g_\iota(x,y)+\varphi_\iota(y)-\varphi_\iota(x), \quad (\iota=1,2)$.
Then, the sequence $\{(x_n,y_n)\}$ strongly converges to $(\bar{x},\bar{y})=(\Pi_{\Gamma_1}u,\Pi_{\Gamma_2}v)$, where $(\Gamma_1,\Gamma_2)=\{(\bar{x},\bar{y}): \bar{x}\in F(T)\cap(\cap_{i=1}^N GMEP(g_1^i,\varphi_1^i)), \bar{y}\in F(S)\cap(\cap_{j=1}^M GMEP(g_2^j,\varphi_2^j))$ and $A\bar{x}=B\bar{y}\}$ and $\Pi_{\Gamma_1},\Pi_{\Gamma_2}$ are the Bregman projections onto Γ_1 and Γ_2 respectively.

Corollary 5.3.3. *Let E_1, E_2 and E_3 be three real Banach spaces which are $p-$uniformly convex and uniformly smooth and C, Q be nonempty closed and convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators, $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$ the adjoints of A and B respectively. Let $g_1^i : C \times C \to \mathbb{R}$ $(i = 1, 2, ..., N)$ and $g_2^j : Q \times Q \to \mathbb{R}$ $(j = 1, 2, ..., M)$ be two finite families of bifunctions satisfying conditions $(A1) - (A4)$. Let $\Phi_1^i : C \to E_1^*$ $(i = 1, 2, ..., N)$ and $\Phi_2^j : Q \to E_2^*$ $(j = 1, 2, ..., M)$ be two finite families of continuous and monotone mappings. Let $T : C \to C$ and $S : Q \to Q$ be left Bregman strongly nonexpansive mappings such that $\Omega_\Phi = \{(\bar{x}, \bar{y}) : \bar{x} \in F(T) \cap (\cap_{i=1}^N GMEP(g_1^i, \Phi_1^i)), \bar{y} \in F(S) \cap (\cap_{j=1}^M GMEP(g_2^j, \Phi_2^j)) : A\bar{x} = B\bar{y}\} \neq \emptyset$ and let $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. For a fixed $u \in E_1$ and a fixed $v \in E_2$, let the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by $x_0 \in E_1$ and $y_0 \in E_2$:*

$$\begin{cases} u_n = Res^f_{G_1^N} \circ Res^f_{G_1^{N-1}} \circ ... \circ Res^f_{G_1^2} \circ Res^f_{G_1^1} J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)], \\ v_n = Res^f_{G_2^M} \circ Res^f_{G_2^{M-1}} \circ ... \circ Res^f_{G_2^2} \circ Res^f_{G_2^1} J_q^{E_2^*}[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)], \\ x_{n+1} = J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], \\ y_{n+1} = J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_1}(S(v_n))\Big], \end{cases} \tag{5.118}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$;
(iii) $0 < a \leq \beta_n, \gamma_n \leq d < 1$;
(iv) $0 < t \leq t_n \leq k \leq \Big(\dfrac{q}{2C_q\|A\|^q}\Big)^{\frac{1}{q-1}}, \Big(\dfrac{q}{2D_q\|B\|^q}\Big)^{\frac{1}{q-1}}$;
$G_\iota(x, y) := g_\iota(x, y) + \langle \Phi_\iota x, y - x\rangle, \quad (\iota = 1, 2)$.
Then, the sequence $\{(x_n, y_n)\}$ strongly converges to $(\bar{x}, \bar{y}) = (\Pi_{\Gamma_1} u, \Pi_{\Gamma_2} v)$, where $(\Gamma_1, \Gamma_2) = \{(\bar{x}, \bar{y}) : \bar{x} \in F(T) \cap (\cap_{i=1}^N GMEP(g_1^i, \Phi_1^i)), \bar{y} \in F(S) \cap (\cap_{j=1}^M GMEP(g_2^j, \Phi_2^j))$ and $A\bar{x} = B\bar{y}\}$ and $\Pi_{\Gamma_1}, \Pi_{\Gamma_2}$ are the Bregman projections onto Γ_1 and Γ_2 respectively.

5.3.1 Numerical Example

In this subsection, we present an numerical example of Theorem 5.3.1 as follows:
Let $E_1 = E_2 = E_3 = \mathbb{R}$ and $C = Q = [-1, 1]$. Take $g_1^i(x, y) := -9ix^2 + xy + (9i - 1)y^2$, $\Phi_1^i(x) = (9i - 3)x$, $\varphi_1^i(x) := (9i - 6)x$, $i = 1, 2, 3, \cdots, M$, we have $Res^f_{G_1^i}(x) = \dfrac{x}{5(9i - 3)}$. Also, we take $g_2^j(x, y) := -7ix^2 + xy + (7i - 1)y^2$, $\Phi_2^j(x) = (7i - 3)x$, $\varphi_2^j(x) := (7i - 6)x$, $j = 1, 2, 3, \cdots, N$, and obtain $Res^f_{G_2^j}(x) = \dfrac{x}{5(7i - 3)}$. Furthermore, let $Ax := 2x$, $Bx := 3x$ and let $T(x) = S(x) = \Pi_c(x) = \Pi_Q(x) = P_C(x)$, with

$$P_C(x) = P_Q(x) = \begin{cases} -1, & x < -1, \\ x, & x \in [-1, 1], \\ 1, & x > 1. \end{cases}$$

Let $\alpha_n = \dfrac{2}{n + 2}$, $\beta_n = \dfrac{n + 1}{2(n + 2)}$ and $\gamma_n = \dfrac{n + 1}{2(n + 2)}$. Then, letting $M = N = 5$ the iteration scheme (5.79) becomes

$$\begin{cases} u_n = \Pi_{i=1}^{5} \dfrac{1}{5(9i-3)}[x_n - 2t_n(2x_n - 3y_n)], \\ v_n = \Pi_{j=1}^{5} \dfrac{1}{5(7j-3)}[y_n - 3t_n(2x_n - 3y_n)], \\ x_{n+1} = \dfrac{2}{n+1}u + \dfrac{n+1}{2(n+2)}(u_n) + \dfrac{n+1}{2(n+2)}(P_C(u_n)), \\ y_{n+1} = \dfrac{1}{n+1}v + \dfrac{n+1}{2(n+2)}(v_n) + \dfrac{n+1}{2(n+2)}(P_Q(v_n)). \end{cases} \tag{5.119}$$

5.3.2 Example 1:

Take $u = 1$, $v = \frac{1}{2}$, $x_0 = 0.1$ and $y_0 = 0.22$.
The numerical result of this problem using our algorithm (5.119) with;
case 1a: $t_n = 0.0000032$
and
case 1b: $t_n = 0.00000051$.

5.3.3 Example 2:

Take $u = 2$, $v = 0.1$, $x_0 = 0.3$ and $y_0 = 0.02$.
case 2a: $t_n = 0.00018$
and
case 2b: $t_n = 0.00000071$.

5.3.4 Example 3:

Take $u = 1$, $v = 1$, $x_0 = 0.1$ and $y_0 = 0.1$.
case 3a: Taken $t_n = 0.00008$
and
case 3b: $t_n = 0.00000011$.

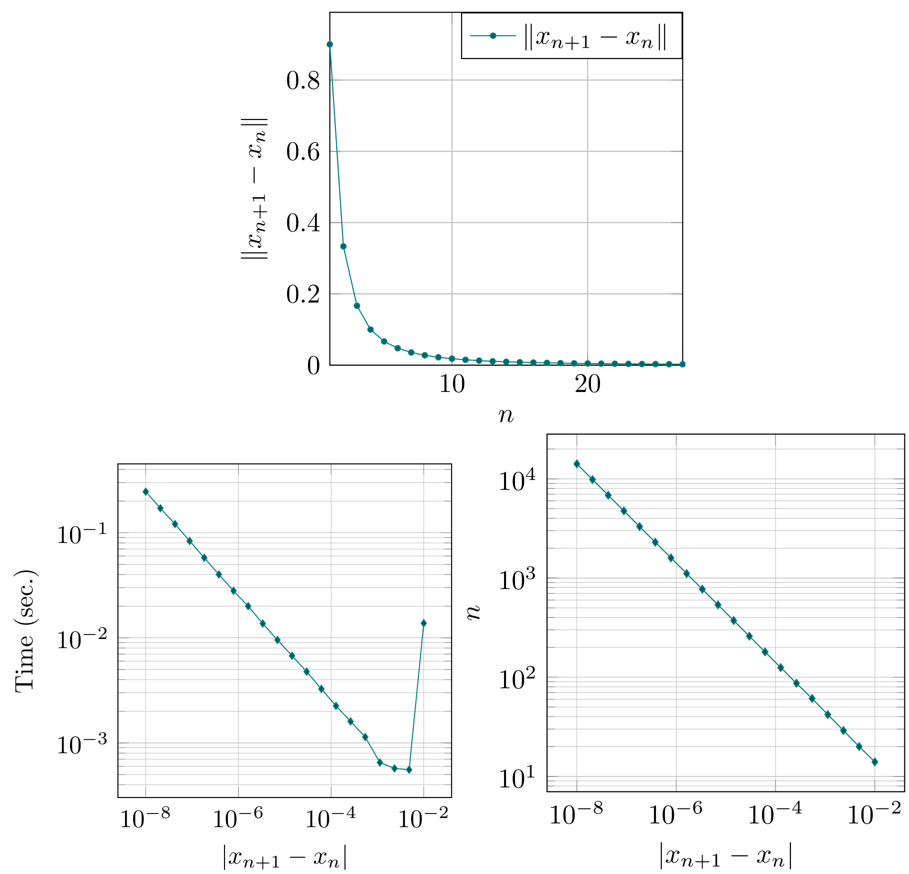


Figure 5.2: Example 4.1, case 1a: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

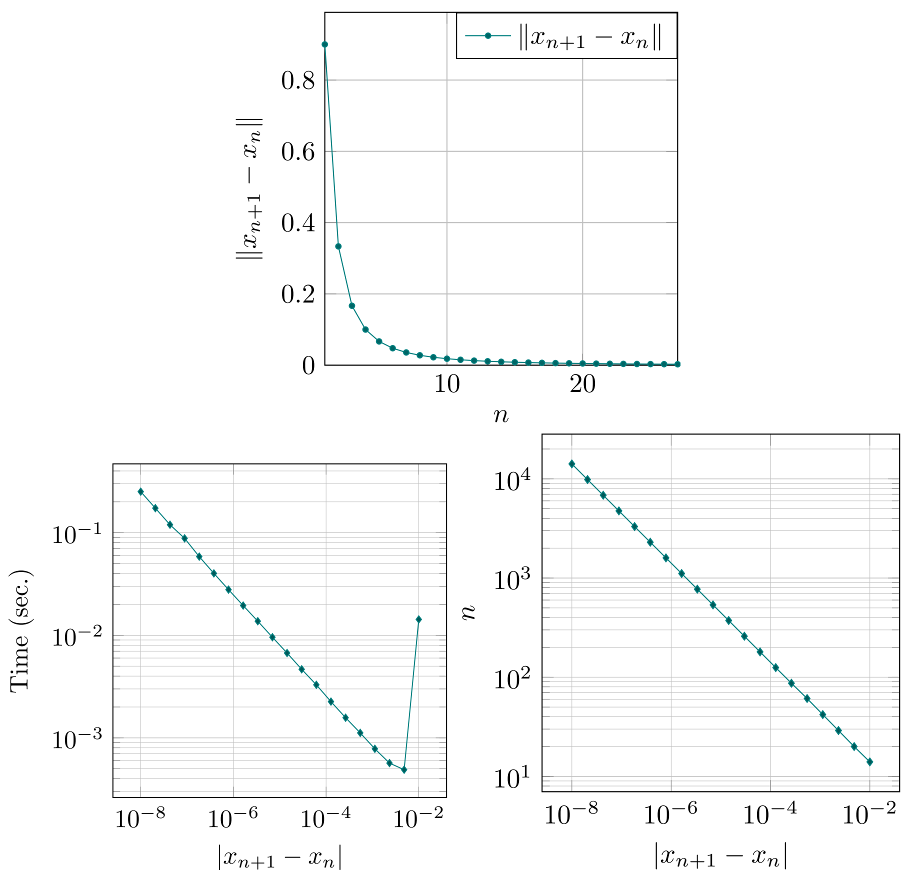


Figure 5.3: Example 4.1, case 1b: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

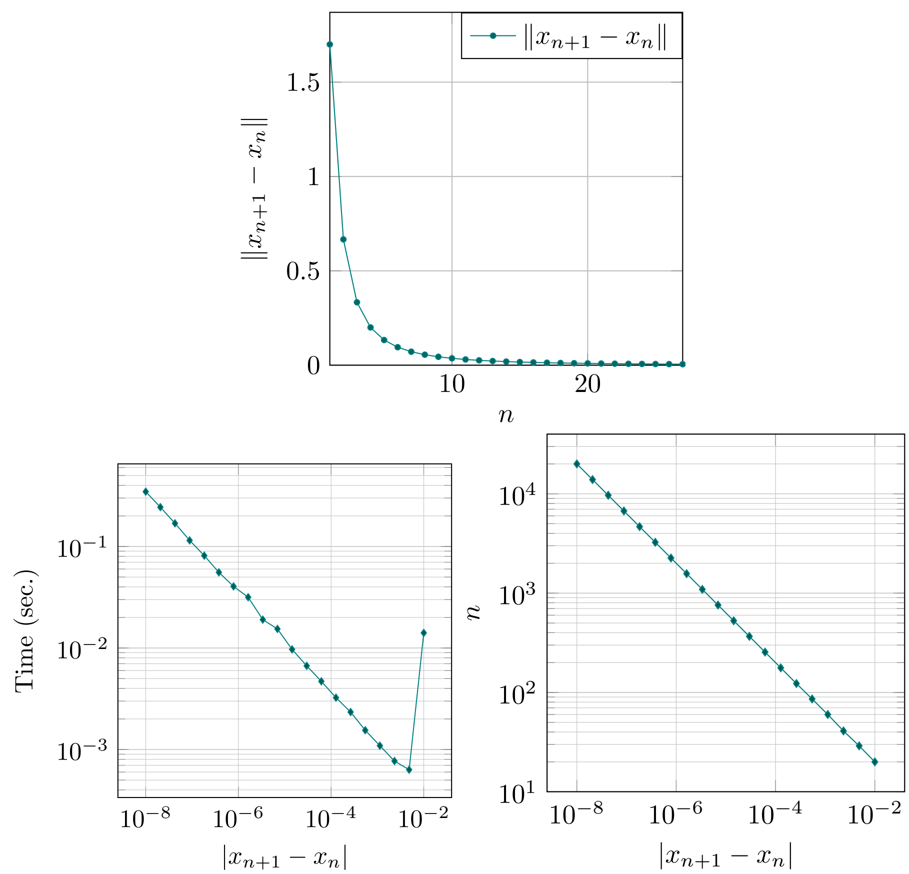


Figure 5.4: Example 4.2, case 1a: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

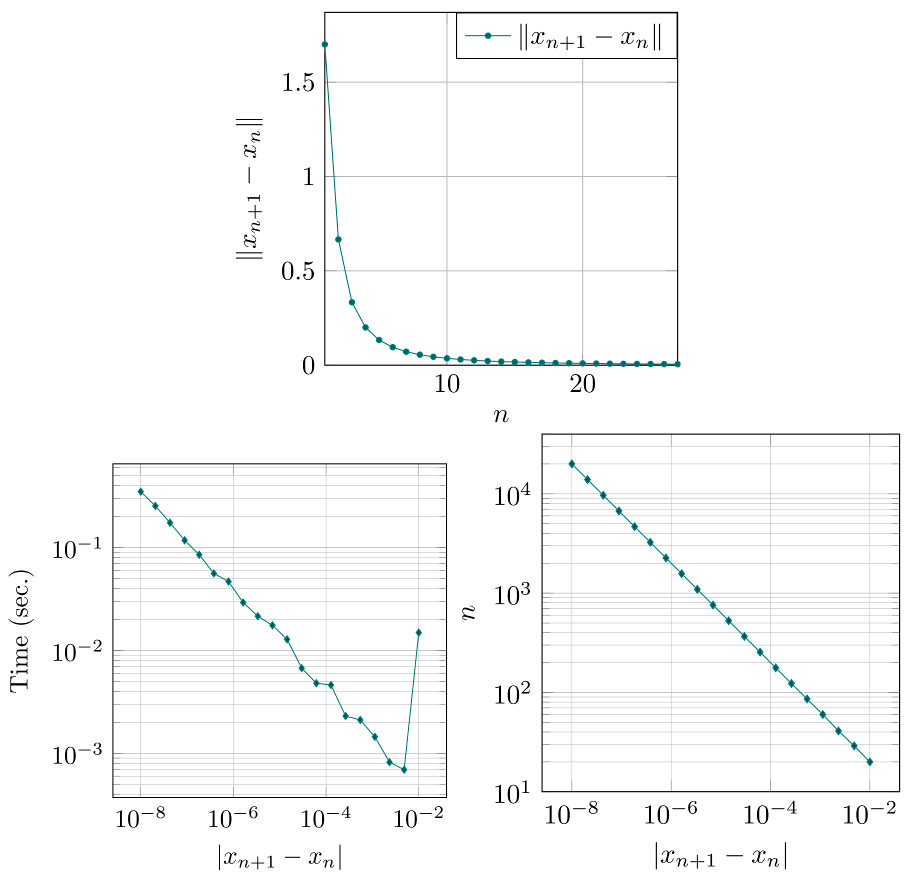


Figure 5.5: Example 4.2, case 1b: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

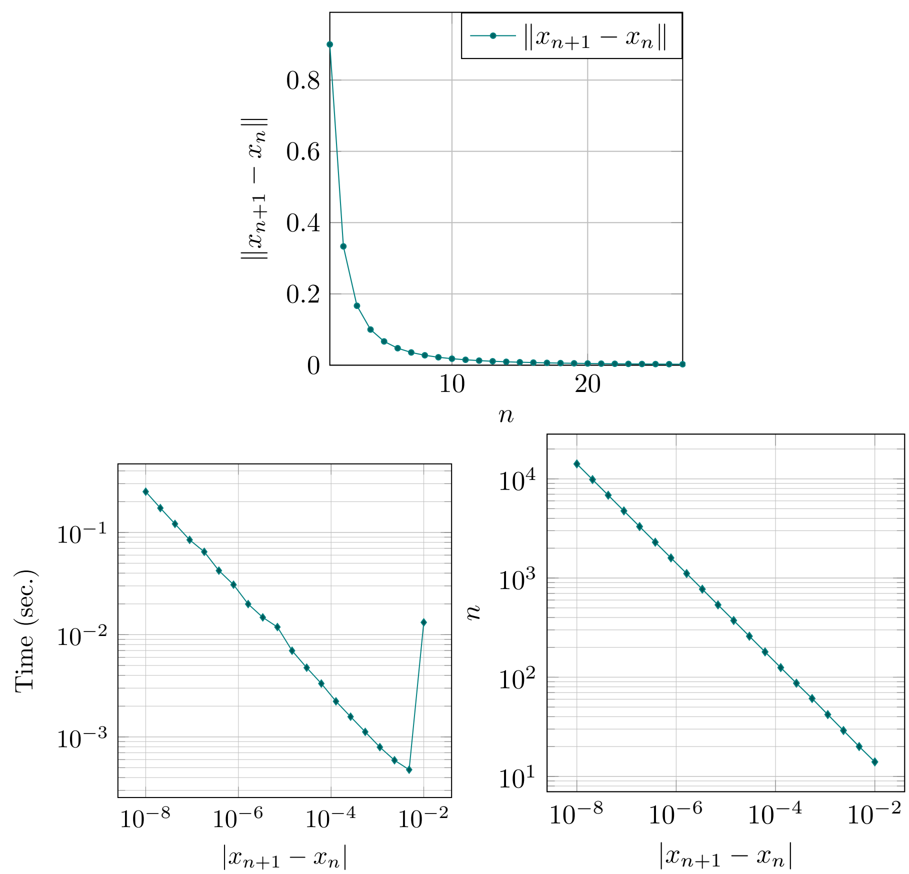


Figure 5.6: Example 4.3, case 1a: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

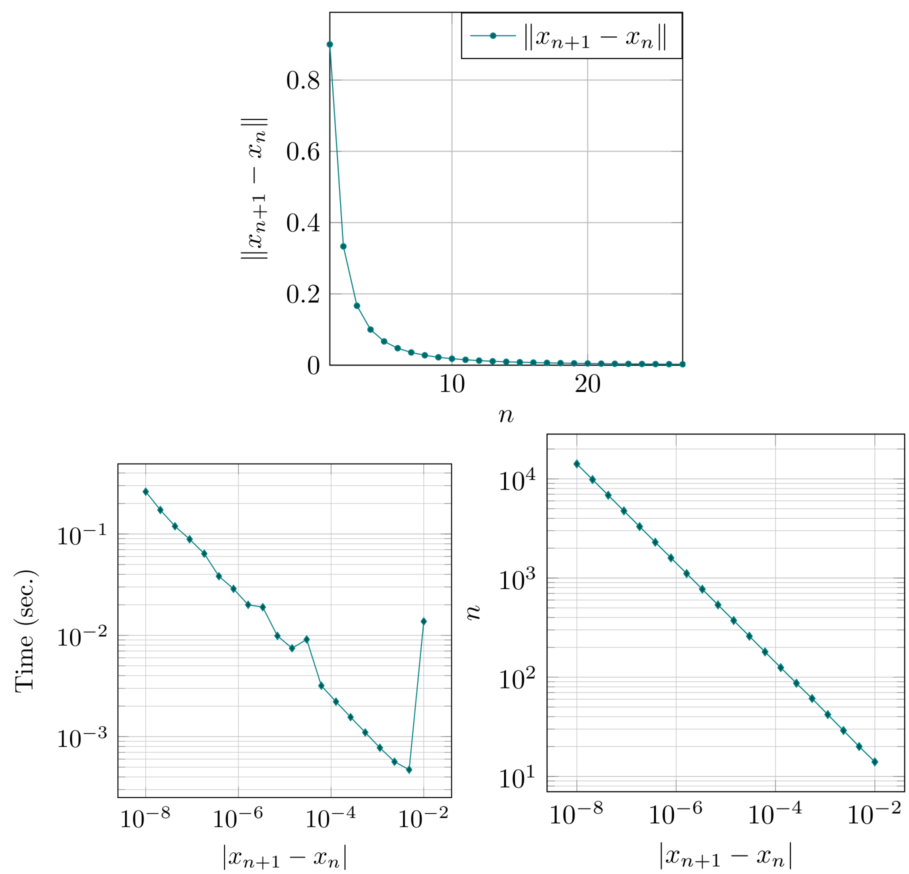


Figure 5.7: Example 4.3, case 1b: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

Chapter 6

Split Variational Inclusion and Fixed Point Problems

6.1 Introduction

Let E_1 and E_2 be real Banach spaces, $f_1 : E_1 \to E_1^*$, $f_2 : E_2 \to E_2^*$ be inverse strongly monotone mappings and $B_1 : E_1 \to 2^{E_1^*}$, $B_2 : E_2 \to 2^{E_2^*}$ be maximal monotone mappings. Let $A : E_1 \to E_2$ be a bounded linear mapping. The Split Monotone Variational Inclusion Problem (SMVIP) is to find $x^* \in E_1$ such that

$$0 \in f_1(x^*) + B_1(x^*) \tag{6.1}$$

and

$$y^* = Ax^* \in E_2 \text{ such that, } 0 \in f_2(y^*) + B_2(y^*). \tag{6.2}$$

The SMVIP (6.1)-(6.2) was introduced in 2011 by Moudafi[160] in the frame work of real Hilbert spaces. The split common fixed point problem, the split variational inequality problem, the split zero problem, and the SFP (see, e.g.,[47, 48, 52, 54, 121, 158, 160, 161]), which have been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning, modelling of inverse problems arising from phase retrieval, and in sensor networks in computerised tomography and data compression [46, 76] are special cases of the SMVIP (6.1)-(6.2).

Suppose in SMVIP (6.1) - (6.2), $f_1 \equiv 0$ and $f_2 \equiv 0$, we obtain in the following Split Variational Inclusion Problem (SVIP): Find $x^* \in E_1$ such that

$$0 \in B_1(x^*) \tag{6.3}$$

and

$$y^* = Ax^* \in H_2 \text{ such that, } 0 \in B_2(y^*). \tag{6.4}$$

Denote Ω_B by the solution set of (6.3)-(6.4).

Let A be a maximal monotone operator, then the resolvent of A denoted by $Res_A^f : E \to 2^E$ is defined as follows (see, [17]):

$$Res_A^f(x) = ((J_p^E + A)^{-1} \circ J_p^E)(x).$$

It is known that $F(Res_A^f) = A^{-1}(0^*)$ and Res_A^f is single valued (see [17]). If f is a Legendre function which is bounded and uniformly Fréchet differetiable on bounded subsets of E, then $\hat{F}(Res_A^f) = F(Res_A^f)$ (see [17]). The Yosida approximation $A_\lambda : E \to E^*, \quad \lambda > 0$ is also defined by

$$A_\lambda(x) = \frac{1}{\lambda}(J_p^E(x) - J_p^E(Res_{\lambda A}^f(x))),$$

for all $x \in E$. From Proposition 2.7 [189], it is known that $(Res_{\lambda A}^f(x)), A_\lambda(x)) \in G(A)$ and $0^* \in Ax$ if and only if $0^* \in A_\lambda x$ for all $x \in E$ and $\lambda > 0$. It is well known that B is maximal if and only if for $(x, u) \in E \times E^*$, $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$.

In a Hilbert space H, the resolvent operator J_λ^B associated with B and λ is the mapping $J_\lambda^B : H \to H$ defined by

$$J_\lambda^B(x) = (I + \lambda B)^{-1}x, \quad x \in H, \lambda > 0. \tag{6.5}$$

Also in a Hilbert space, it is known that the resolvent operator J_λ^B is single valued, nonexpansive and 1-inverse strongly monotone (for example see [31]) and the solution of (4.1) is a fixed point of $J_\lambda^M(I - \lambda f), \quad \forall \lambda > 0$ (see for example [134]). If f is α-inverse strongly monotone mapping with $0 < \lambda < 2\alpha$, then one can easily see that $J_\lambda^M(I - \lambda f)$ is nonexpansive and $I(f, M)$ is closed and convex. If we consider (6.1) and (6.2) separately, we have that (6.1) is a MVIP with its solution set $I(f_1, B_1)$ and (6.2) is a MVIP with solution set $I(f_2, B_2)$.

Byrne *et al.* [47] using the following iterative scheme: for a given $x_0 \in H_1$ the sequence $\{x_n\}$ generated iteratively by;

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \quad \lambda > 0,$$

obtained a weak and strong convergence theorem for solving SVIP (6.3)-(6.4). Inspired by the work of Byrne *et al.*, Kazmi and Rizvi[120], proposed the following algorithm for approximating a solution of SVIP (6.3)-(6.4) which is a fixed point of a nonexpansive mapping S: for a given $x_0 \in H_1$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, n \geq 0, \end{cases} \tag{6.6}$$

and proved that both $\{u_n\}$ and $\{x_n\}$ converge strongly to $z \in F(S) \cap \Omega_B$, where Ω_B is the solution set of SVIP (6.3)-(6.4). In 2015, Wen and Chen [228] introduced a modified general iterative method for a split variational inclusion (6.3)-(6.4) and nonexpansive semigroups, which is defined as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds,$$

where $\gamma, \alpha_n, \beta_n \in [0,1]$ and B is a strongly positive bounded linear operator on H_1 and obtained a strong convergence result.

Also in 2015, Shehu and Ogbuisi [210] proposed the following iterative algorithm for approximating solution of split monotone variational inclusion problem (6.1)-(6.2) with f_1 and f_2 not necessarily 0 which also solves a fixed point problem for strictly pseudocontractive mapping in real Hilbert space:

$$\begin{cases} w_n = (1-\alpha_n)x_n, \\ y_n = J_\lambda^{B_1}(I-\lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \quad \forall n \geq 0, \end{cases} \tag{6.7}$$

where $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A and proved a strong convergence result. For more on variational inclusion problem see [140, 139].

Deepho et. al [87] obtained the following result:

Theorem 6.1.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Let D be strongly positive bounded linear operator on H_1 with a coefficient $\bar{\tau} > 0$. Assume that $T_i : C \to H_1$ $(i = 1, 2, \cdots, N)$ is a family of k_1-strictly pseudo-contraction mappings such that $\cap_{i=1}^N F(T_i) \cap \Omega_B \neq \emptyset$. Suppose that f is a contraction with coefficient $\rho \in (0,1)$ and $\{\eta_i^{(n)}\}_{i=1}^N$ are finite sequences of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \geq 0$. For a given point $x_0 \in C$, $\alpha_n, \beta_n \in (0,1)$ and $0 < \tau < \frac{\bar{\tau}}{\rho}$, let $\{x_n\}$ be a sequence generated in the following:*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^N \eta_{n,i} S_i u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D)y_n, \quad \forall n \geq 1, \end{cases} \tag{6.8}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Suppose the following conditions are satisfied:
(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{i=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$,
(C2) $k_i \leq \beta_n \leq l < 1$, $\lim_{n\to\infty} \beta_n = l$ and $\sum_{n=1}^\infty |\beta_n - \beta_{n-1}| < \infty$,
(C3) $\sum_{n=1}^\infty \sum_{i=1}^N |\eta_i^{(n)} - \eta_i^{n-1}| < \infty$. Then the sequence $\{x_n\}$ generated by (6.8) converges strongly to $q \in \cap_{i=1}^N F(T_i) \cap \Omega_B$ which solves the variational inequality

$$\langle (D - \tau f)q, q - p \rangle \leq 0, \forall p \in \cap_{i=1}^N F(T_i) \cap \Omega_B.$$

Guo et al.[101] considered the split equality variational inclusion problem:

$$\text{find } \ x \in U^{-1}(0) = F(J_{u_n}^U), \ \ y \in V^{-1}(0) = F(J_{u_n}^V) \ \text{ such that } \ Ax = By, \tag{6.9}$$

where H_1, H_2, H_3 are real Hilbert spaces, $U : H_1 \to 2^{H_1}$ and $V : H_2 \to 2^{H_2}$ are set valued maximal monotone mappings, and $A : H_1 \to H_3, \ B : H_2 \to H_3$ are two bounded linear operators. Let Λ be the solution set of (6.9). Precisely Guo et al. [101] stated and proved the following theorem:

Theorem 6.1.2. *([101], Theorem 3.2) Let H_1, H_2, H_3 be real Hilbert spaces and $U : H_1 \to 2^{H_1}$, $V : H_2 \to 2^{H_2}$ be set valued maximal monotone mappings. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators and A^*, B^* the adjoint of A and B respectively. Let $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, where $f_i, i = 1, 2$ are contraction mappings on H_i with constant $k \in (0, 1)$ and $\{S_n\}$ a sequence of nonexpansive mappings on H_1, D a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Assume the solution set of SEVIP (6.9) is nonempty,*

$$J_{u_n}^{(U,V)} = \begin{bmatrix} J_{u_n}^U \\ J_{u_n}^V \end{bmatrix}, \quad G = \begin{bmatrix} A & -B \end{bmatrix}, \quad G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}.$$

Let w_n be generated by

$$\begin{cases} v_n = J_{u_n}^{(U,V)}(I - \gamma G^*G)w_n, \\ w_{n+1} = \alpha_n \sigma f(w_n) + (1 - \alpha_n D) S_n v_n. \end{cases} \tag{6.10}$$

Suppose S_n satisfies the AKKT condition, $F(S) = \cap_{n=1}^{\infty} F(S_n)$ (S is as defined in Lemma 2.3 of Guo et al. [101]). If $F(S) \cap \Lambda$ is nonempty and the following conditions are satisfied:
(i) $\alpha_n \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\sum_{n=0}^{\infty} |u_{n+1} - u_n| < \infty$;
(iv) $0 < \bar{\gamma} < \frac{1}{\alpha_n}$, $0 < \sigma < \frac{\bar{\gamma}}{k}$.

Then the sequence $\{w_n\}$ converges strongly to a point w^*, where
$w^* = P_{F(S)\cap\Lambda}(I - D - \sigma f)(w^*)$ is a unique solution of the variational inequalities

$$\langle (D - \sigma f)w^*, w^* - z \rangle \leq 0, z \in F(S) \cap \Lambda.$$

For Further results on split inclusion problems see [168] and the references therein.

In this Chapter, we obtained an iterative solution of split variational inclusion problems in p-uniformly Banach space which are uniformly convex. We further give iterative algorithms for approximating a common solution of split monotone variational inclusion problems and fixed point problems for two multivalued strictly pseudocontractive type mappings, a common solution of a split monotone variational inclusion problem and a fixed point problem of a finite family of strictly pseudocontractive mappings, which solves a variational inequality problem and a solution of split equality monotone variational inclusion problems which is also a solution of a split equality fixed point problem in real Hilbert spaces.

6.2 Iterative solution of split variational inclusion problem in real Banach spaces

Let E_1 and E_2 be p-uniformly convex real Banach spaces which are also uniformly smooth. Let $B_1 : E_1 \to 2^{E_1^*}$, $B_2 : E_2 \to 2^{E_2^*}$ be maximal monotone mappings. Let $A : E_1 \to E_2$ be a bounded linear mapping. Then in this part, we study the SVIP (6.3)-(6.4), Our result complements other related results in the literature and extends the result of Byrne *et al.* [47] from the frame work of real Hilbert space to the frame work of p-uniformly convex real Banach spaces which are also uniformly smooth.

Theorem 6.2.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Let T and S be the resolvents of multi-valued maximal monotone operators $B_1 : E_1 \to 2^{E_1^*}$ and $B_2 : E_2 \to 2^{E_2^*}$ respectively. Suppose that SVIP (6.3)-(6.4) has a nonempty solution set Ω_B and that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1). For a fixed $u \in E_1$, let the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by $x_0 \in E_1$,*

$$\begin{cases} y_n = J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_2}(I-S)Ax_n], \\ x_{n+1} = J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1} y_n + (1-\beta_n) J_p^{E_1}(Ty_n)\Big)\Big], \end{cases} \tag{6.11}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < a \le \beta_n \le d < 1$;
(iv) $0 < t \le t_n \le k \le \Big(\frac{q}{C_q\|A\|^q}\Big)^{\frac{1}{q-1}}$.
Then, $\{x_n\}$ converges strongly to $\Pi_{\Omega_B} u$.

Proof. For any $z \in \Omega_B$ it follows from (6.11) that

$$\begin{aligned} \Delta_p(y_n, z) &= \Delta_p\Big(J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_2}(I-S)Ax_n], z\Big) \\ &= \frac{1}{q}||J_p^{E_1}(x_n) - t_n A^* J_p^{E_2}(I-S)Ax_n||^q - \langle J_p^{E_1}(x_n), z\rangle \\ &\quad + t_n\langle J_p^{E_2}(I-S)Ax_n, Az\rangle + \frac{1}{p}||z||^p \\ &\le \frac{1}{q}||J_p^{E_1}(x_n)||^q - t_n\langle J_p^{E_2}(I-S)Ax_n, Ax_n\rangle \\ &\quad + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_2}(I-S)Ax_n||^q - \langle J_p^{E_1}(x_n), z\rangle \\ &\quad + t_n\langle J_p^{E_2}(I-S)Ax_n, Az\rangle + \frac{1}{p}||z||^p \\ &= \frac{1}{q}||x_n||^p - \langle J_p^{E_1}(x_n), z\rangle + \frac{1}{p}||z||^p + t_n\langle J_p^{E_2}(I-S)Ax_n, Az - Ax_n\rangle \\ &\quad + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_2}(I-S)Ax_n||^q \\ &= \Delta_p(x_n, z) + t_n\langle J_p^{E_2}(I-S)Ax_n, Az - Ax_n\rangle \\ &\quad + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_2}(I-S)Ax_n||^q. \end{aligned} \tag{6.12}$$

But

$$\begin{aligned} \langle J_p^{E_2}(I-S)Ax_n, Ax_n - Az\rangle &= ||Ax_n - S(Ax_n)||^P + \langle J_p^{E_2}(I-S)Ax_n, S(Ax_n) - Az\rangle \\ &\ge ||Ax_n - S(Ax_n)||^P. \end{aligned} \tag{6.13}$$

Thus from (6.13), we have

$$-||Ax_n - S(Ax_n)||^p \quad \geq \quad \langle J_p^{E_2}(I-S)Ax_n, Az - Ax_n\rangle. \tag{6.14}$$

Therefore, from (6.12) we obtain

$$\begin{aligned}
\Delta_p(y_n, z) &\leq \Delta_p(x_n, z) + t_n\langle J_p^{E_2}(I-S)Ax_n, Az - Ax_n\rangle + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_2}(I-S)Ax_n||^p \\
&\leq \Delta_p(x_n, z) - t_n||(I-S)Ax_n||^p + \frac{C_q(t_n||A||)^q}{q}||(I-S)Ax_n||^p \\
&\leq \Delta_p(x_n, z) - \Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big)||(I-S)Ax_n||^p. \qquad (6.15)
\end{aligned}$$

Thus from condition (iv), we have

$$\Delta_p(y_n, z) \quad \leq \quad \Delta_p(x_n, z).$$

Furthermore, from (6.11), we get

$$\begin{aligned}
\Delta_p(x_{n+1}, z) &= \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}y_n + (1-\beta_n)J_p^{E_1}(Ty_n)\Big)\Big], z) \\
&\leq \alpha_n\Delta_p(u, z) + (1-\alpha_n)\beta_n\Delta_p(y_n, z) + (1-\alpha_n)(1-\beta_n)\Delta_p(T(y_n), z) \\
&\leq \alpha_n\Delta_p(u, z) + (1-\alpha_n)\beta_n\Delta_p(y_n, z) + (1-\alpha_n)(1-\beta_n)\Delta_p(y_n, z) \\
&= \alpha_n\Delta_p(u, z) + (1-\alpha_n)\Delta_p(y_n, z) \\
&\leq \alpha_n\Delta_p(u, z) + (1-\alpha_n)\Delta_p(x_n, z) \\
&\leq \max\{\Delta_p(u, z), \Delta_p(x_n, z)\} \\
&\ \ \vdots \\
&\leq \max\{\Delta_p(u, z), \Delta_p(x_1, z)\}. \qquad (6.16)
\end{aligned}$$

Therefore, $\{\Delta_p(x_n, z)\}$ is bounded and consequently we have that $\{x_n\}$, $\{y_n\}$ and $\{T(y_n)\}$ are all bounded. Now,

$$\begin{aligned}
\Delta_p(y_{n+1}, z) &\leq \Delta_p(x_{n+1}, z) \\
&= \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}y_n + (1-\beta_n)J_p^{E_1}(Ty_n)\Big)\Big], z) \\
&= V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}y_n + (1-\beta_n)J_p^{E_1}(Ty_n)\Big), z) \\
&\leq V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}y_n + (1-\beta_n)J_p^{E_1}(Ty_n)\Big) - \alpha_n(J_p^{E_1}(u) - J_p^{E_1}(z), z\Big) \\
&\quad -\langle -\alpha_n(J_p^{E_1}(u) - J_p^{E_1}(z)), J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) \\
&\quad +(1-\alpha_n)\Big(\beta_n J_p^{E_1}y_n + (1-\beta_n)J_p^{E_1}(Ty_n)\Big)\Big] - z\rangle \\
&= V_p\Big(\alpha_n J_p^{E_1}(z) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}y_n + (1-\beta_n)J_p^{E_1}(Ty_n)\Big), z) \\
&\quad +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), x_{n+1} - z\rangle \\
&= \Delta_p\Big(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(z) + (1-\alpha_n)\Big(\beta_n J_p^{E_1}y_n + (1-\beta_n)J_p^{E_1}(Ty_n)\Big)\Big], z\Big) \\
&\quad +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), x_{n+1} - z\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\Delta_p(z,z) + (1-\alpha_n)\beta_n\Delta_p(y_n,z) + (1-\alpha_n)(1-\beta_n)\Delta_p(T(y_n),z) \\
&\quad +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), x_{n+1}-z\rangle \\
&\leq (1-\alpha_n)\Delta_p(y_n,z) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), x_{n+1}-z\rangle. \qquad (6.17)
\end{aligned}$$

We now consider two cases to prove the strong convergence.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Delta_p(y_n,z)\}$ is monotonically nonincreasing. Then obviously $\{\Delta_p(y_n,z)\}$ converges and

$$\Delta_p(y_{n+1},z) - \Delta_p(y_n,z) \to 0, n \to \infty. \qquad (6.18)$$

Let $s_n := J_q^{E_1^*}(\beta_n J_p^{E_1} y_n + (1-\beta_n)Ty_n)$.
Then,

$$\begin{aligned}
\Delta_p(s_n,z) &= \Delta_p(J_q^{E_1^*}(\beta_n J_p^{E_1} y_n + (1-\beta_n)Ty_n), z) \\
&\leq \beta_n\Delta_p(y_n,z) + (1-\beta_n)\Delta_p(Ty_n,z) \\
&\leq \beta_n\Delta_p(y_n,z) + (1-\beta_n)\Delta_p(y_n,z) \\
&= \Delta_p(y_n,z). \qquad (6.19)
\end{aligned}$$

Thus

$$\begin{aligned}
0 &\leq \Delta_p(y_n,z) - \Delta_p(s_n,z) \\
&= \Delta_p(y_n,z) - \Delta_p(y_{n+1},z) + \Delta_p(y_{n+1},z) - \Delta_p(s_n,z) \\
&\leq \Delta_p(y_n,z) - \Delta_p(y_{n+1},z) + \Delta_p(x_{n+1},z) - \Delta_p(s_n,z) \\
&\leq \Delta_p(y_n,z) - \Delta_p(y_{n+1},z) + \alpha_n\Delta_p(u,z) + (1-\alpha_n)\Delta_p(s_n,z) - \Delta_p(s_n,z) \\
&= \Delta_p(y_n,z) - \Delta_p(y_{n+1},z) + \alpha_n(\Delta_p(u,z) - \Delta_p(s_n,z)) \to 0, \quad n \to \infty. \qquad (6.20)
\end{aligned}$$

Again

$$\begin{aligned}
\Delta_p(s_n,z) &\leq \beta_n\Delta_p(y_n,z) + (1-\beta_n)\Delta_p(Ty_n,z) \\
&= \Delta_p(y_n,z) - (1-\beta_n)\Delta_p(y_n,z) + (1-\beta_n)\Delta_p(Ty_n,z) \\
&= \Delta_p(y_n,z) + (1-\beta_n)\Big(\Delta_p(Ty_n,z) - \Delta_p(y_n,z)\Big). \qquad (6.21)
\end{aligned}$$

Hence, by condition (iii)

$$(1-\beta_n)(\Delta_p(x_n,z) - \Delta_p(Ty_n,z)) \leq \Big(\Delta_p(y_n,z) - \Delta_p(Tx_n,z)\Big) \to 0, n \to \infty. \qquad (6.22)$$

Therefore, since T is the resolvent of a maximal monotone operator and hence left Bregman strongly nonexpansive, we have that

$$\lim_{n\to\infty} \Delta_p(Ty_n, y_n) = 0,$$

which implies that

$$\lim_{n\to\infty} ||Ty_n - y_n|| = 0. \qquad (6.23)$$

Since $\{y_n\}$ is bounded and E is reflexive, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ that converges weakly to $x^* \in E_1$. From (6.23), it follows that $x^* \in F(T)$ since $F(T) = \hat{F}(T)$. That is, $0 \in B_1(x^*)$.

Next, we show that $Ax^* \in F(S)$, that is, $0 \in B_2(Ax^*)$. Now from (6.15), we obtain

$$\begin{aligned}\Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big)||Ax_n - S(Ax_n)||^P &\leq \Delta_p(x_n, z) - \Delta_p(y_n, z)\\ &\leq \alpha_{n-1}(\Delta_p(x_{n-1}, z) + \langle J_p^{E_1}(u) - J_p^{E_1}(z), x_n - z\rangle)\\ &\quad +\Delta_p(y_{n-1}, z) - \Delta_p(y_n, z) \to 0, \quad n \to \infty, \qquad (6.24)\end{aligned}$$

and since

$$0 < t\Big(1 - \frac{C_q k^{q-1}(||A||)^q}{q}\Big) \leq \Big(t_n - \frac{C_q(t_n||A||)^q}{q}\Big),$$

we have that $||Ax_n - S(Ax_n)||^p \to 0, n \to \infty$. Hence, we obtain from the definition of y_n that

$$\begin{aligned} 0 &\leq ||J_p^{E_1} y_n - J_p^{E_1} x_n||\\ &\leq t_n||A^*||||J_p^{E_2}(Ax_n - S(Ax_n))||\\ &\leq \Big(\frac{q}{C_q||A||^q}\Big)^{\frac{1}{q-1}}||A^*||||Ax_n - S(Ax_n)|| \to 0, n \to \infty.\end{aligned}$$

Since $J_q^{E_1^*}$ is norm to norm uniformly continuous on bounded subsets of E_1^*, we have that

$$\lim_{n\to\infty}||y_n - x_n|| = \lim_{n\to\infty}||J_q^{E_1^*} J_p^{E_1} y_n - J_q^{E_1^*} J_p^{E_1} u_n|| \to 0, n \to \infty. \qquad (6.25)$$

Now,

$$\begin{aligned}||(I - S)Ax^*||^p &= \langle J_p^{E_2}(Ax^* - SAx^*), Ax^* - SAx^*\rangle\\ &= \langle J_p^{E_2}(Ax^* - SAx^*), Ax^* - Ax_{n_j}\rangle + \langle J_p^{E_2}(Ax^* - SAx^*), Ax_{n_j} - SAx_{n_j}\rangle\\ &\quad +\langle J_p^{E_2}(Ax^* - SAx^*), SAx_{n_j} - SAx^*\rangle\\ &\leq \langle J_p^{E_2}(Ax^* - SAx^*), Ax^* - Ax_{n_j}\rangle + \langle J_p^{E_2}(Ax^* - SAx^*), Ax_{n_j} - SAx_{n_j}\rangle.\end{aligned}$$

By the continuity of A and $||y_n - x_n|| \to 0, n \to \infty$, we have that $Ax_{n_j} \rightharpoonup Ax^*, j \to \infty$. Hence, letting $j \to \infty$, we have that

$$||Ax^* - SAx^*|| = 0.$$

Therefore, $Ax^* = SAx^*$, that is $Ax^* \in F(S)$ which implies $0 \in B_2(Ax^*)$. Hence, we have that $x^* \in \Omega_B$.

Next, we show that $\{x_n\}$ converges strongly to $\Pi_{\Omega_B} u$.
Now, we observe that

$$\begin{aligned}\Delta_p(x_{n+1}, y_n) &= \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + (1 - \alpha_n)\Big(\beta_n J_p^{E_1} y_n + (1 - \beta_n)J_p^{E_1}(Ty_n)\Big)\Big], y_n)\\ &\leq \alpha_n\Delta_p(u, x_n) + (1 - \alpha_n)\beta_n\Delta_p(y_n, y_n)\\ &\quad +(1 - \alpha_n)(1 - \beta_n)\Delta_p(Ty_n, y_n) \to 0, n \to \infty.\end{aligned}$$

Hence,

$$||x_{n+1} - y_n|| \to 0, \quad n \to \infty.$$

Thus

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - y_n|| + ||y_n - x_n|| \to 0, n \to \infty.$$

Let $\bar{x} = \Pi_{\Omega_B} u$. Then from (6.17), we have

$$\begin{aligned} \Delta_p(x_{n+1}, \bar{x}) &\leq (1-\alpha_n)\Delta_p(y_n, \bar{x}) + \alpha_n \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{n+1} - \bar{x}\rangle \\ &\leq (1-\alpha_n)\Delta_p(x_n, \bar{x}) + \alpha_n \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{n+1} - \bar{x}\rangle. \end{aligned}$$

We now choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_n - \bar{x}\rangle = \lim_{j\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{n_j} - \bar{x}\rangle.$$

Then from (2.25), we have

$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_n - \bar{x}\rangle = \lim_{j\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{n_j} - \bar{x}\rangle = \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x^* - \bar{x}\rangle \leq 0.$$

Hence, since $||x_n - x_{n+1}|| \to 0, n \to \infty$, we have

$$\limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{n+1} - \bar{x}\rangle = \limsup_{n\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_n - \bar{x}\rangle \leq 0.$$

Therefore, by Lemma 2.3.12, we conclude that $\Delta_p(x_n, \bar{x}) \to 0, \quad n \to \infty$, that is $||x_n - \bar{x}|| \to 0, \quad n \to \infty$. Therefore, $x_n \to \bar{x}$.

Case 2. Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\Delta_p(y_{n_i,z}) < \Delta_p(y_{n_i+1}, z)$ for all $i \in \mathbb{N}$. Then, by Lemma 2.3.13 there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$.

$$\Delta_p(y_{m_k}, z) \leq \Delta_p(y_{m_k+1}, z),$$

and

$$\Delta_p(y_k, z) \leq \Delta_p(y_{m_k+1}, z).$$

Using the same line of arguments as in (6.19),(6.20),(6.21),(6.22) and noting that $\Delta_p(y_{m_k}, z) \leq \Delta_p(y_{m_k+1}, z)$, we can show that

$$\lim_{k\to\infty} ||Ty_{m_k} - y_{m_k}|| = 0.$$

By same arguments as in case 1, we obtain that

$$\limsup_{k\to\infty} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{m_k+1} - \bar{x}\rangle \leq 0.$$

Again from (6.17), we have

$$\Delta_p(y_{m_k+1}, \bar{x}) \leq (1-\alpha_{m_k})\Delta_p(y_{m_k}, \bar{x}) + \alpha_{m_k} \langle J_p^{E_1} u - J_p^{E_1}\bar{x}, x_{m_k+1} - \bar{x}\rangle,$$

which implies that

$$\alpha_{m_k}\Delta_p(y_{m_k},\bar{x}) \leq \Delta_p(y_{m_k},\bar{x}) - \Delta_p(y_{m_k+1},\bar{x}) + \alpha_{m_k}\langle J_p^{E_1}u - J_p^{E_1}\bar{x}, x_{m_k+1}-\bar{x}\rangle.$$

That is,

$$\Delta_p(y_{m_k},\bar{x}) \leq \langle J_p^{E_1}u - J_p^{E_1}\bar{x}, x_{m_k+1}-\bar{x}\rangle.$$

Therefore

$$\lim_{k\to\infty}\Delta_p(y_{m_k},\bar{x}) = 0$$

and since

$$\Delta_p(y_k,\bar{x}) \leq \Delta_p(y_{m_k+1},\bar{x})$$

for all $k \in \mathbb{N}$, we conclude that

$$y_k \to \bar{x}, \quad k \to \infty.$$

Since $||y_n - x_n|| \to 0$, then $x_k \to \bar{x}$. □

6.2.1 Applications

In this subsection we present two applications to highlight the importance of the theory we have developed in Theorem 6.2.1.

Application to split minimisation problem
In many practical problems in science and engineering, there is always the need to find minimum-norm solution of given problems. In an abstract way, we may formulate such problems as finding a point x^* with the property

$$x^* \in H, \quad ||x^*|| = \min\{||x|| : x \in H\}, \tag{6.26}$$

where H is a Hilbert space. It is well known that (6.26) is equivalent to the following particular variational inequality problem:

$$x^* \in H, \quad \text{such that} \quad \langle x^*, x^* - x\rangle \leq 0, \quad \forall\, x \in H. \tag{6.27}$$

Now, let $E_1 = L^2([a,b]) = E_2$. Suppose that $A : L^2([a,b]) \to L^2([a,b])$ is defined by

$$Ax(s) = \int_a^b V(s,t)x(t)dt, \quad \forall\, x \in L^2([a,b]),$$

$V : [a,b]\times[a,b] \to \mathbb{R}$ is continuous. It can be easily shown that A is a bounded linear operator and the adjoint A^* of A is defined by

$$A^*x(s) = \int_a^b V(t,s)x(t)dt, \quad \forall\, x \in L^2([a,b]).$$

Let $||.||_{L^2} : L^2([a,b]) \to \mathbb{R}, \quad C = \{x \in L^2 : \langle a, x\rangle = b\}$, for some $a \in L^2 - \{0\}$ and $Q = \{x \in L^2 : \langle a, x\rangle \geq b\}$, for some $a \in L^2 - \{0\}, b \in \mathbb{R}$. Then x^* minimises $||.||_{L^2} + \delta_C$ if and only if $0 \in \partial(||.||_{L^2} + \delta_C)(x^*)$ and Ax^* minimises $||.|| + \delta_Q$ if and only if $0 \in$

$\partial(||.||_{L^2} + \delta_Q)(Ax^*)$, where δ_C [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise] and δ_Q stand for the indicator function of C and Q respectively and $\partial\phi$ for the subdifferential of ϕ, $[\partial\phi(x) := \{u \in E : \phi(y) \geq \phi(x) + \langle u, y - x\rangle, \forall y \in E\}]$. If in (6.3) - (6.4), we set $B_1 = \partial(||.||_{L^2} + \delta_C), B_2 = \partial(||.||_{L^2} + \delta_Q)$, then we obtain the following Split Minimization Problem (SMP):

$$\text{find} \;\; x^* \in C \;\; \text{such that} \;\; x^* = \operatorname{argmin}\{||x||_{L^2} : x \in C\}, \tag{6.28}$$

and such that

$$y^* = Ax^* \in Q \;\; \text{solves} \;\; y^* = \operatorname{argmin}\{||x||_{L^2} : x \in Q\}. \tag{6.29}$$

Let Ω_M be a solution set of (6.28)- (6.29) and $\Omega_M \neq \emptyset$. We remark here that the solution to problem (6.28)- (6.29) is a minimum-norm solution. From this example, we see that the problem considered in Theorem 6.2.1 generalizes the Split Minimization Problem considered by several authors (see, e.g., [160]).

Application to split feasibility problem

Furthermore, we consider a split feasibility problem which is a special case of our problem (6.3)-(6.4) in real Hilbert spaces. Let C and Q be a nonempty closed and convex subset of H_1 and H_2 respectively. Suppose that $A : H_1 \to H_2$ be a bounded linear operator. Suppose that in (6.3)-(6.4), we set $B_1 = N_C, B_2 = N_Q$, where N_C and N_Q are the normal cones of C and Q respectively. Then problem (6.3)-(6.4) reduces to:

$$\begin{aligned} &\text{find} \;\; x^* \in H_1 \;\; \text{such that} \;\; x^* = (I + \lambda N_C)^{-1}x^*, \\ &\qquad \text{and} \;\; Ax^* = (I + \lambda N_Q)^{-1}Ax^*, \quad \lambda > 0. \end{aligned} \tag{6.30}$$

We know that

$$y = (I + \lambda N_C)^{-1}x \Leftrightarrow x \in y + \lambda N_C y \Leftrightarrow y = P_C x.$$

Problem (6.30) then reduces to:

$$\text{find} \;\; x^* \in H_1 \;\; \text{such that} \;\; x^* = P_C x^*, \;\; \text{and} \;\; Ax^* = P_Q(Ax^*), \tag{6.31}$$

which in turn reduces to

$$\text{find} \;\; x^* \in C \quad \text{such that} \;\; Ax^* \in Q. \tag{6.32}$$

Thus, problem (6.32), which is the split feasibility problem considered by many authors (see, for example, [45, 180, 231, 237, 239, 243] and references therein), is a special case of our problem (6.3)-(6.4).

6.3 Solving split monotone variational inclusion problem and fixed point problem for certain multivalued maps in Hilbert spaces.

In this section, we introduce an iterative scheme and obtain a strong convergence result for approximating a solution of the SMVIP (6.1) - (6.2) (f_1 and f_2 not necessarily zero) which is also a common solution of fixed point problems for two multivalued strictly pseudocontractive mappings in the sense of Isiogugu [111].

Theorem 6.3.1. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω_{fB} be a solution set of (6.1) - (6.2). Let $S, T : H_1 \to P(H_1)$ be two strictly pseudocontractive-type mappings with contractive coefficients κ_1 and κ_2 such that $F_s(S) \cap F_s(T) \cap \Omega_{fB} \neq \emptyset$. Let $\{x_n\}$ be the sequence generated for $x_0, u \in H_1$ by*

$$\left\{\begin{array}{l} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)[\rho_n v_n + (1-\rho_n)u_n], \quad \forall n \geq 0, \end{array}\right. \tag{6.33}$$

where $v_n \in Ty_n$ and $u_n \in Sy_n$, $0 < \lambda < 2\mu, 2\nu$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Suppose $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\beta_n \geq \max\{\kappa_1, \kappa_2\} \quad \forall n \geq 0$,
(iii) $\liminf_{n\to\infty}(1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1) > 0$,
(iv) $\liminf_{n\to\infty}(1-\beta_n)(\beta_n - \kappa_2)\rho_n > 0$.
Then $\{x_n\}$ converges strongly to $q = P_{F_s(S)\cap F_s(T)\cap\Omega_{fB}} u$.

Proof. Let $p \in F_s(S) \cap F_s(T) \cap \Omega_{fB}$ and let $z_n = \rho_n v_n + (1-\rho_n)u_n$, then

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||[\beta_n y_n + (1-\beta_n)[\rho_n v_n + (1-\beta_n)u_n]] - p||^2 \\ &= ||[\beta_n y_n + (1-\beta_n)z_n] - p||^2 \\ &= \beta_n||y_n - p||^2 + (1-\beta_n)||z_n - p||^2 - \beta_n(1-\beta_n)||y_n - z_n||^2, \end{aligned} \tag{6.34}$$

and

$$\begin{aligned} ||z_n - p||^2 &= ||\rho_n v_n + (1-\rho_n)u_n - p||^2 \\ &= \rho_n||v_n - p||^2 + (1-\beta_n)||u_n - p||^2 \\ &\quad -\rho_n(1-\rho_n)||v_n - u_n||^2. \end{aligned} \tag{6.35}$$

From (6.34) and (6.35), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= \beta_n||y_n - p||^2 + (1-\beta_n)\rho_n||v_n - p||^2 + (1-\beta_n)(1-\rho_n)||u_n - p||^2 \\ &\quad -(1-\beta_n)\rho_n(1-\rho_n)||v_n - u_n||^2 - \beta_n(1-\beta_n)||y_n - z_n||^2 \\ &\leq \beta_n||y_n - p||^2 + (1-\beta_n)\rho_n H^2(Ty_n, Tp) + (1-\beta_n)(1-\rho_n)H^2(Sy_n, Sp) \\ &\quad -(1-\beta_n)\rho_n(1-\beta_n)||v_n - u_n||^2 - \beta_n(1-\beta_n)||y_n - z_n||^2 \\ &\leq \beta_n||y_n - p||^2 + (1-\beta_n)\rho_n[||y_n - p||^2 + \kappa_2||y_n - v_n||^2] \\ &\quad +(1-\beta_n)(1-\rho_n)[||y_n - p||^2 + \kappa_1||y_n - u_n||^2] \\ &\quad -(1-\beta_n)\rho_n(1-\beta_n)||v_n - u_n||^2 - \beta_n(1-\beta_n)||y_n - z_n||^2. \end{aligned} \tag{6.36}$$

Again

$$\begin{aligned} ||y_n - z_n||^2 &= ||y_n - [\rho_n v_n + (1-\rho_n)u_n]||^2 \\ &= \rho_n||y_n - v_n||^2 + (1-\rho_n)||y_n - u_n||^2 - \rho_n(1-\rho_n)||v_n - u_n||^2. \end{aligned} \tag{6.37}$$

Inserting (6.37) into (6.36), we obtain

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq [\beta_n+(1-\beta_n)\rho_n+(1-\beta_n)(1-\rho_n)||y_n-p||^2 \\
&\quad +[(1-\beta_n)\rho_n\kappa_2-\beta_n(1-\beta_n)\rho_n]||y_n-v_n||^2 \\
&\quad +[(1-\beta_n)(1-\rho_n\kappa_1-\beta_n(1-\beta_n)(1-\rho_n)]||y_n-v_n||^2 \\
&\quad +[(1-\beta_n)(1-\rho_n)\rho_n\beta_n-(1-\beta_n)(1-\rho_n)\rho_n||v_n-u_n||^2 \\
&= ||y_n-p||^2-\rho_n(1-\beta_n)(\beta_n-\kappa_2)||y_n-v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)||y_n-u_n||^2 \\
&\quad -(1-\beta_n)^2(1-\rho_n)\rho_n||v_n-u_n||^2 \\
&\leq ||y_n-p||^2. \qquad (6.38)
\end{aligned}$$

But

$$\begin{aligned}
||y_n-p||^2 &= ||J_\lambda^{B_1}(I-\lambda f_1)(w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)-p||^2 \\
&\leq ||w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p||^2 \\
&= ||w_n-p||^2+\gamma^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
&\quad +2\gamma\langle w_n-p, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle, \qquad (6.39)
\end{aligned}$$

and

$$\begin{aligned}
\gamma^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 &= \gamma^2\langle(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n, AA^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle \\
&\leq L\gamma^2\langle(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n, (J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle \\
&= L\gamma^2||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2. \qquad (6.40)
\end{aligned}$$

Let $\Upsilon_n=2\gamma\langle w_n-p, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle$, then from (6.40), we have

$$\begin{aligned}
\Upsilon_n &= 2\gamma\langle w_n-p, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle \\
&= 2\gamma\langle A(w_n-p)+(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n, (J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle \\
&= 2\gamma[\langle J_\lambda^{B_2}(I-\lambda f_2)Aw_n-Ap, J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle-||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2] \qquad (6.41) \\
&\leq 2\gamma\Big[\frac{1}{2}||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\Big] \\
&= -\gamma||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2.
\end{aligned}$$

From (6.39), (6.40) and (6.41), we have

$$\begin{aligned}
||y_n-p||^2 &\leq ||w_n-p||^2+L\gamma^2||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-\gamma||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
&= ||w_n-p||^2+\gamma(L\gamma-1)||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \qquad (6.42) \\
&\leq ||w_n-p||^2.
\end{aligned}$$

By (6.38) and (6.42)

$$\begin{aligned}
||x_{n+1}-p|| &\leq ||w_n-p|| \\
&= ||(1-\alpha_n)x_n+\alpha_n u-p|| \\
&= ||(1-\alpha_n)(x_n-p)-\alpha_n(p-u)|| \\
&\leq (1-\alpha_n)||x_n-p||+\alpha_n||p-u|| \\
&\leq \max\{||x_n-p||, ||p-u||\} \\
&\vdots \\
&\leq \max\{||x_0-p||, ||p-u||\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded and consequently $\{y_n\}$, $\{Sy_n\}$, $\{Ty_n\}$ and $\{w_n\}$ are bounded.

We divide into two cases to establish the strong convergence of $\{x_n\}$ to p.
Case 1. Assume that $\{||x_n - p||\}$ is a monotonically decreasing sequence. Then $\{||x_n - p||\}$ is convergent and clearly

$$\lim_{n\to\infty} ||x_n - p|| = \lim_{n\to\infty} ||x_{n+1} - p||.$$

Now,

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq ||y_n - p||^2 - \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1)||y_n - u_n||^2 - (1-\beta_n)^2(1-\rho_n)\rho_n||v_n - u_n||^2 \\
&\leq ||w_n - p||^2 - \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1)||y_n - u_n||^2 - (1-\beta_n)^2(1-\rho_n)\rho_n||v_n - u_n||^2 \\
&\leq ||(1-\alpha_n)x_n + \alpha_n u - p||^2 - \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1)||y_n - u_n||^2 - (1-\beta_n)^2(1-\rho_n)\rho_n||v_n - u_n||^2 \\
&\leq ||x_n - p||^2 + \alpha_n^2||x_n - u||^2 - 2\alpha_n\langle x_n - p, x_n - u\rangle - \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1)||y_n - u_n||^2 \\
&\quad -(1-\beta_n)^2(1-\rho_n)\rho_n||v_n - u_n||^2. \qquad (6.43)
\end{aligned}$$

Let

$$\begin{aligned}
D_n &= \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad +(1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1)||y_n - u_n||^2.
\end{aligned}$$

Thus, from (6.43), we have

$$\begin{aligned}
D_n &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 \\
&\quad +\alpha_n^2||x_n - u||^2 - 2\alpha_n\langle x_n - p, x_n - u\rangle \to 0, \quad as \quad n \to \infty. \qquad (6.44)
\end{aligned}$$

Thus, by conditions (iii) and (iv) and (6.44), we have

$$||y_n - v_n|| \to 0, \quad as \quad n \to \infty, \qquad (6.45)$$

and

$$||y_n - u_n|| \to 0, \quad as \quad n \to \infty. \qquad (6.46)$$

From (6.33), we have

$$||w_n - x_n|| = \alpha_n||x_n - u|| \to 0, \quad as \quad n \to \infty. \qquad (6.47)$$

Again from (6.38)

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq ||y_n - p||^2 \\
&= ||J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n) - p||^2 \\
&\leq ||w_n - p||^2 + \gamma(L\gamma - 1)||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2 \\
&\leq (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||p - u||^2 - 2\alpha_n(1-\alpha_n)\langle x_n - p, p - u\rangle \\
&\quad +\gamma(L\gamma - 1)||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2. \qquad (6.48)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \gamma(1-L\gamma)||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
\leq\ & ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||p-u||^2 \\
& -2\alpha_n(1-\alpha_n)\langle x_n-p,p-u\rangle \to 0, \quad as \quad n\to\infty. \qquad (6.49)
\end{aligned}$$

Hence,

$$||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n|| \to 0, \quad as \quad n\to\infty. \qquad (6.50)$$

From (6.42), we have

$$\begin{aligned}
||y_n-p||^2 &= ||J_\lambda^{B_1}(I-\lambda f_1)(w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)-p||^2 \\
&\leq \langle y_n-p, w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p\rangle \\
&= \frac{1}{2}[||y_n-p||^2+||w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p||^2 \\
&\quad -||y_n-p-(w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p)||^2] \\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2+\gamma(L\gamma-I)||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
&\quad -||y_n-w_n-\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p)||^2] \\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2-(||y_n-w_n||^2+\gamma^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
&\quad -2\gamma\langle y_n-w_n, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle)] \\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2-||y_n-w_n||^2 \\
&\quad +2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||]. \qquad (6.51)
\end{aligned}$$

That is,

$$\begin{aligned}
||y_n-p||^2 \leq\ & ||w_n-p||^2-||y_n-w_n||^2 \\
& +2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||. \qquad (6.52)
\end{aligned}$$

It then follows from (6.38) and (6.52) that

$$\begin{aligned}
||x_{n+1}-p||^2 \leq\ & ||w_n-p||^2-||y_n-w_n||^2 \\
& +2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||, \qquad (6.53)
\end{aligned}$$

which implies that

$$\begin{aligned}
||y_n-w_n||^2 &\leq ||w_n-p||^2-||x_{n+1}-p||^2+2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n|| \\
&= ||(1-\alpha_n)x_n-p||^2-||x_{n+1}-p||^2+2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n|| \\
&\leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||p-u||^2+2\alpha_n(1-\alpha_n)\langle x_n-p,p-u\rangle \\
&\quad +2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n|| \to 0, \quad as \quad n\to\infty. \qquad (6.54)
\end{aligned}$$

Therefore,

$$||y_n-w_n|| \to 0, \quad as \quad n\to\infty. \qquad (6.55)$$

From (6.47),

$$||x_n - y_n|| \leq ||x_n - w_n|| + ||w_n - y_n|| \to 0. \tag{6.56}$$

Let $\theta_n = w_n + \gamma A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n$, then

$$||\theta_n - w_n||^2 = L\gamma^2||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2 \to 0. \tag{6.57}$$

Combining (6.55) and (6.57), we have

$$||y_n - \theta_n|| \leq ||y_n - w_n|| + ||w_n - \theta_n|| \to 0. \tag{6.58}$$

It follows from (6.45) and (6.46) that there exists a subsequence y_{n_j} of $\{y_n\}$ which converges weakly to a point $x^* \in F_s(S) \cap F_s(T)$ and so do $\{x_{n_j}\}$ and $\{w_{n_j}\}$ converge weakly to x^*. We now show that $x^* \in I(f_1, B_1)$. Since f_1 is $\frac{1}{\mu}$-Lipschitz monotone mapping and the domain of f_1 is H_1 then by Lemma 2.3.12, we conclude that $B_1 + f_1$ is maximally monotone. Let $(v, z) \in G(B_1 + f_1)$, that is $z - f_1 v \in B_1(v)$.
Since $y_{n_j} = J_\lambda^{B_1}(I - \lambda f_1)\theta_{n_j}$, we obtain

$$(I - \lambda f_1)\theta_{n_j} \in (I + \lambda B_1)y_{n_j}.$$

That is,

$$\frac{1}{\lambda}(\theta_{n_j} - \lambda f_1 \theta_{n_j} - y_{n_j}) \in B_1(y_{n_j}).$$

Using the maximal monotonicity of $(B_1 + f_1)$, we have

$$\langle v - y_{n_j}, z - f_1 v - \frac{1}{\lambda}(\theta_{n_j} - \lambda f_1 \theta_{n_j} - y_{n_j})\rangle \geq 0.$$

Therefore,

$$\begin{aligned}
\langle v - y_{n_j}, z\rangle &\geq \langle v - y_{n_j}, f_1 v + \frac{1}{\lambda}(\theta_{n_j} - \lambda f_1 \theta_{n_j} - y_{n_j})\rangle \\
&= \langle v - y_{n_j}, f_1 v - f_1 y_{n_j} + f_1 y_{n_j} - f_1 \theta_{n_j} + \frac{1}{\lambda}(\theta_{n_j} - y_{n_j})\rangle \\
&\geq 0 + \langle v - y_{n_j}, f_1 y_{n_j} - f_1 \theta_{n_j}\rangle + \langle v - y_{n_j}, \frac{1}{\lambda}(\theta_{n_j} - y_{n_j})\rangle. \qquad (6.59)
\end{aligned}$$

By (6.58), we obtain

$$\lim_{n\to\infty} ||f_1 y_{n_j} - f_1 \theta_{n_j}|| = 0.$$

Also, since $y_{n_j} \rightharpoonup x^*$, we have

$$\lim_{j\to\infty} \langle v - y_{n_j}, z\rangle = \langle v - x^*, z\rangle.$$

Thus from (6.59)

$$\langle v - x^*, z\rangle \geq 0.$$

Since $B_1 + f_1$ is maximally monotone, we have $0 \in (B_1 + f_1)x^*$ which implies that

$$x^* \in I(f_1, B_1).$$

Moreover, since $||w_n - y_n|| \to 0$, we have that Aw_{n_j} converges weakly to Ax^*. Thus, by (6.50) and the fact that $J_\lambda^{B_2}(I - \lambda f_2)$ is nonexpansive, it follows from Lemma 2.3.1 that

$$0 \in f_2 Ap + B_2(Ax^*).$$

That is, $Ax^* \in I(f_2, B_2)$. Hence, $x^* \in F_s(S) \cap F_s(T) \cap \Omega_{fB}$.
We now show that $\{x_n\}$ converges strongly to $q = P_{F_s(S) \cap F_s(T) \cap \Omega_{fB}} u$.

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq ||y_n - p||^2 \\
&\leq ||w_n - q||^2 \\
&= ||(1-\alpha_n)x_n + \alpha_n u - q||^2 \\
&= ||(1-\alpha_n)(x_n - q) - \alpha_n(q - u)||^2 \\
&= (1-\alpha_n)^2||x_n - q||^2 + \alpha_n^2||q-u||^2 \\
&\quad -2\alpha_n(1-\alpha_n)\langle x_n - q, q - u\rangle \\
&\leq (1-\alpha_n)||x_n - q||^2 + \alpha_n[\alpha_n||q-u||^2 \\
&\quad -2(1-\alpha_n)\langle x_n - q, q-u\rangle]. \qquad (6.60)
\end{aligned}$$

Choose subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - q, u - q\rangle = \lim_{j\to\infty}\langle x_{n_j} - q, u - q\rangle,$$

Since $x_{n_j} \rightharpoonup q$ then it follows from Proposition2.1.3

$$\begin{aligned}
\limsup_{n\to\infty}\langle x_n - q, u-q\rangle &= \lim_{j\to\infty}\langle x_{n_j} - q, u - q\rangle \\
&= \langle x^* - q, u - q\rangle \leq 0. \qquad (6.61)
\end{aligned}$$

Therefore, by (6.60) and Lemma 2.3.12, we obtain $x_n \to q$, $n \to \infty$.

Case 2. Assume that $\{||x_n - p||\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$(for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \geq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for $n \geq n_0$.
It follows from (6.43) that

$$\begin{aligned}
0 &\leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \\
&\leq \alpha_{\tau(n)}^2||x_{\tau(n)} - u||^2 - 2\alpha_{\tau(n)}\langle x_{\tau(n)} - p, x_{\tau(n)} - u\rangle \\
&\quad -\rho_{\tau(n)}(1-\beta_{\tau(n)})(\beta_{\tau(n)} - \kappa_2)||y_{\tau(n)} - v_{\tau(n)}||^2 \\
&\quad -(1-\beta_{\tau(n)})(1-\rho_{\tau(n)})(\beta_{\tau(n)} - \kappa_1)||y_{\tau(n)} - u_{\tau(n)}||^2 \\
&\quad -(1-\beta_{\tau(n)})^2(1-\rho_{\tau(n)})\rho_{\tau(n)}||v_{\tau(n)} - u_{\tau(n)}||^2.
\end{aligned}$$

Let

$$\begin{aligned} D_{\tau(n)} &= \rho_{\tau(n)}(1-\beta_{\tau(n)})(\beta_{\tau(n)}-\kappa_2)||y_{\tau(n)}-v_{\tau(n)}||^2 \\ &\quad +(1-\beta_{\tau(n)})(1-\rho_{\tau(n)})(\beta_{\tau(n)}-\kappa_1)||y_{\tau(n)}-u_{\tau(n)}||^2. \end{aligned}$$

Then,

$$D_{\tau(n)} \leq \alpha_{\tau(n)}^2||x_{\tau(n)}||^2 - 2\alpha_{\tau(n)}\langle x_{\tau(n)}-p, x_{\tau(n)}-u\rangle \to 0, \quad as \quad n\to\infty.$$

Thus, by conditions (iii) and (iv) and (6.44), we have

$$||y_{\tau(n)}-v_{\tau(n)}|| \to 0, \quad as \quad n\to\infty,$$

and

$$||y_{\tau(n)}-u_{\tau(n)}|| \to 0, \quad as \quad n\to\infty.$$

By the same argument as in case 1, we conclude that there exists a subsequence of $\{x_{\tau(n)}\}$, also denoted $\{x_{\tau(n)}\}$ which converges weakly to $x^* \in F_s(S)\cap F_s(T)\cap \Omega_{fB}$ and

$$\limsup_{n\to\infty}\langle x_{\tau(n)}-q, u-q\rangle \leq 0.$$

Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1}-q||^2 - ||x_{\tau(n)}-q||^2 \\ &\leq (1-\alpha_{\tau(n)})||x_{\tau(n)}-q||^2 + \alpha_{\tau(n)}^2||q-u||^2 \\ &\quad -2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)}-q, q-u\rangle - ||x_{\tau(n)}-q||^2 \\ &= \alpha_{\tau(n)}[\alpha_{\tau(n)}||q-u||^2 - 2(1-\alpha_{\tau(n)})\langle x_{\tau(n)}-q, q-u\rangle - ||x_{\tau(n)}-p||^2]. \end{aligned}$$

Therefore,

$$||x_{\tau(n)}-q||^2 \leq \alpha_{\tau(n)}||q-u||^2 - 2(1-\alpha_{\tau(n)})\langle x_{\tau(n)}-q, q-u\rangle \to 0.$$

Thus,

$$\lim_{n\to\infty}||x_{\tau(n)}-q||^2 = 0,$$

and so

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1}.$$

Furthermore, for $n \geq n_0$, it is observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$(that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Consequently, for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)}+1\} = \Gamma_{\tau(n)}+1.$$

So $\lim_{n\to\infty}\Gamma_n = 0$, that is $\{x_n\}$,$\{y_n\}$ and $\{w_n\}$ converge strongly to $q \in P_{F_s(S)\cap F_s(T)\cap\Omega_{fB}}u$. $\square$

Corollary 6.3.2. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω_{fB} be a solution set of* (6.1) - (6.2). *Let $S, T : H_1 \to P(H_1)$ be two multivalued nonexpansive mappings such that $F_s(S) \cap F_s(T) \cap \Omega_{fB} \neq \emptyset$. Let $\{x_n\}$ be the sequence generated for $x_0, u \in H_1$ by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = J_\lambda^{B_1}(I-\lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)[\rho_n v_n + (1-\rho_n)u_n], \quad \forall n \geq 0, \end{cases} \tag{6.62}$$

where $v_n \in Ty_n$ and $u_n \in Sy_n$, $0 < \lambda < 2\mu, 2\nu$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Suppose $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ satisfying the following conditions
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\liminf\limits_{n\to\infty}(1-\beta_n)(1-\rho_n)\beta_n > 0$,
(iii) $\liminf\limits_{n\to\infty}(1-\beta_n)\beta_n\rho_n > 0$.
Then $\{x_n\}$ converges strongly to $q = P_{F_s(S)\cap F_s(T)\cap\Omega_{fB}}u$.

Corollary 6.3.3. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω_B be a solution set of* (6.3) - (6.4). *Let $S, T : H_1 \to P(H_1)$ be two strictly pseudocontractive-type mappings with contractive coefficients κ_1 and κ_2 such that $F_s(S) \cap F_s(T) \cap \Omega_B \neq \emptyset$. Let $\{x_n\}$ be the sequence generated for $x_0, u \in H_1$ by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = J_\lambda^{B_1}(w_n + \gamma A^*(J_\lambda^{B_2} - I)Aw_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)[\rho_n v_n + (1-\rho_n)u_n], \quad \forall n \geq 0, \end{cases}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Suppose $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ satisfying the following conditions
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\beta_n \geq \max\{\kappa_1, \kappa_2\} \quad \forall n \geq 0$,
(iii) $\liminf\limits_{n\to\infty}(1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1) > 0$,
(iv) $\liminf\limits_{n\to\infty}(1-\beta_n)(\beta_n - \kappa_2)\rho_n > 0$.
Then $\{x_n\}$ converges strongly to $q = P_{F_s(S)\cap F_s(T)\cap\Omega_B}u$.

Corollary 6.3.4. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω_{fB} be a solution set of* (6.1) - (6.2). *Let $S, T : H_1 \to P(H_1)$ be two strictly pseudocontractive-type mappings with contractive coefficients κ_1 and κ_2 such that $F_s(S) \cap F_s(T) \cap \Omega_{fB} \neq \emptyset$. Let $\{x_n\}$ be the sequence generated for $x_0, u \in H_1$ by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = J_\lambda^{B_1}(I-\lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)[\rho_n v_n + (1-\rho_n)u_n], \quad \forall n \geq 0, \end{cases} \tag{6.63}$$

where $v_n \in P_T y_n$ and $u_n \in P_S y_n$, $0 < \lambda < 2\mu, 2\nu$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Suppose $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ satisfying the following conditions
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\beta_n \geq \max\{\kappa_1, \kappa_2\} \quad \forall n \geq 0$,
(iii) $\liminf\limits_{n\to\infty} (1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1) > 0$,
(iv) $\liminf\limits_{n\to\infty} (1-\beta_n)(\beta_n - \kappa_2)\rho_n > 0$.
Then $\{x_n\}$ converges strongly to $q = P_{F_s(S)\cap F_s(T)\cap \Omega_{fB}} u$.

6.3.1 Applications

Application to split minimization problem
Consider the following split minimization problem: find $x^* \in H_1$ such that

$$x^* = \min_{x \in H_1} (\varphi_1(x) + \psi_1(x)), \tag{6.64}$$

and $y^* = Ax^* \in H_2$ is such that

$$y^* = \min_{y \in H_2} (\varphi_2(y) + \psi_2(y)), \tag{6.65}$$

where $\varphi_1, \psi_1 : H_1 \to \mathbb{R}$ and $\varphi_2, \psi_2 : H_2 \to \mathbb{R}$ are convex lower semicontinuous functions. Moreover φ_1 and φ_2 are assumed to be differentiable, $A : H_1 \to H_2$ is a bounded linear operator. Denote the solution set of (6.64)-(6.65) by Λ_m.
Recall that the subdifferential of a function $h : H \to \mathbb{R}$ at x is the set-valued operator on H defined by

$$\partial h(x) := \{z \in H : h(\bar{x}) \geq h(x) + \langle z, \bar{x} - x\rangle \;\; \forall \;\; \bar{x} \in H\}.$$

It is well known that $\partial\psi_1$ and $\partial\psi_2$ are maximal monotone operators. Also we know that $J_\lambda^{\partial\psi_i} = prox_{\lambda\psi_i} (i = 1, 2)$. The proximinal operators $prox_{\lambda\psi_i} (i = 1, 2)$ of ψ_i with parameter $\lambda > 0$ is defined by

$$prox_{\lambda\psi_i}(x) = \arg\min_{u \in H_i} \{\psi_i(u) + \frac{1}{2\lambda}||x - u||\}.$$

Lemma 6.3.5. *([1] Lemma 1.5, [12] Corollary 10) Let $\varphi : H \to \mathbb{R}$ be a differentiable convex function and let $L > 0$. Suppose that $\nabla\varphi$ is L-Lipschitz continuous. Then $\nabla\varphi$ is L^{-1}-inverse strongly monotone. In [1], the word cocoercive is used for inverse strongly monotone.*

Theorem 6.3.6. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $\varphi_1 : H_1 \to \mathbb{R}$ be a differentiable convex function with $\frac{1}{\mu}$-Lipschitz continuous gradient and $\varphi_2 : H_2 \to \mathbb{R}$ be a differentiable convex function with $\frac{1}{\nu}$-Lipschitz continuous gradient. Let $\psi_1 : H_1 \to \mathbb{R}$ and $\psi_2 : H_2 \to \mathbb{R}$ be convex lower semicontinuous functions. Let $S, T : H_1 \to P(H_1)$ be two strictly pseudocontractive-type mappings with* *contractive coefficients* κ_1 *and* κ_2 *such that* $F_s(S) \cap F_s(T) \cap \Lambda_m \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated for* $x_0, u \in H_1$ *by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = prox_{\lambda\psi_1}(I - \lambda\nabla\varphi_1)(w_n + \gamma A^*(prox_{\lambda\psi_2}(I - \lambda\nabla\varphi_2) - I)Aw_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)[\rho_n v_n + (1-\rho_n)u_n], \quad \forall n \geq 0, \end{cases} \tag{6.66}$$

where $v_n \in Ty_n$ *and* $u_n \in Sy_n$, $0 < \lambda < 2\mu, 2\nu$ *and* $\gamma \in (0, \frac{1}{L})$, *L is the spectral radius of the operator* AA^* *and* A^* *is the adjoint of A. Suppose* $\{\alpha_n\}$, $\{\rho_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[0,1]$ *satisfying the following conditions*
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\beta_n \geq \max\{\kappa_1, \kappa_2\} \quad \forall n \geq 0$,
(iii) $\liminf\limits_{n\to\infty}(1-\beta_n)(1-\rho_n)(\beta_n - \kappa_1) > 0$,
(iv) $\liminf\limits_{n\to\infty}(1-\beta_n)(\beta_n - \kappa_2)\rho_n > 0$.
Then $\{x_n\}$ *converges strongly to* $q = P_{F_s(S)\cap F_s(T)\cap\Lambda_m}u$.

Proof: Let $f_1 = \nabla\varphi_1$, $f_2 = \nabla\varphi_2$, $B_1 = \partial\psi_1$ and $B_2 = \partial\psi_2$. Then the conclusion follows from Theorem 6.3.1.

Application to split variational inequality problem
Let $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ be operators and $A : H_1 \to H_2$ a bounded linear operator. Suppose C and Q are nonempty, closed and convex subsets of H_1 and H_2 respectively. The split variational inequality problem is:

$$\text{find a point } x^* \in C \text{ such that } \langle B_1(x^*), x - x^* \rangle \geq 0 \ \ \forall \ x \in C, \tag{6.67}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } \langle B_2(y^*), y - y^* \rangle \geq 0 \ \ \forall \ y \in Q. \tag{6.68}$$

Let Θ denote the solution set of (6.67)-(6.68).
If considered alone, (6.67) is the classical variational inequality problem with solution set $VI(C, B_1)$.

Let D be a nonempty, closed and convex subset of a real Hilbert space H. The normal cone of D at the point $x \in D$ is defined by

$$N_D(x) := \{d \in H : \langle d, y - x \rangle \leq 0, \forall y \in D\}. \tag{6.69}$$

Let h be an α-inverse strongly monotone operator on $D \subseteq H$, let $N_D(x)$ be the normal cone of D at a point $x \in D$ and define the following set valued operator:

$$G(x) := \begin{cases} h(x) + N_D(x), & x \in D \\ \emptyset, & x \notin D. \end{cases} \tag{6.70}$$

From [197], Theorem 3, we have that G is a maximal monotone operator. Furthermore $0 \in G(x)$ if and only if $x \in VI(D, G)$.

Let C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Suppose $N_C(x)$ is the normal cone of C at a point $x \in C$ and $N_Q(y)$ is the normal cone of Q at a point $y \in Q$. Let $h_1 : C \to H_1$ be a ρ-inverse strongly monotone operator on C and $h_2 : Q \to H_2$ be a ϱ-inverse strongly monotone operator on Q. Define

$$G_1(x) := \begin{cases} h_1(x) + N_C(x), & x \in C \\ \emptyset, & x \notin C, \end{cases} \tag{6.71}$$

and

$$G_2(y) := \begin{cases} h_2(y) + N_Q(y), & y \in Q \\ \emptyset, & y \notin Q. \end{cases} \tag{6.72}$$

If we let $B_1 = G_1$ and $B_2 = G_2$, then we obtain from Corollary 6.3.3, a strong convergence theorem for approximating a point $x \in F_s(S) \cap F_s(T) \cap \Theta$.

6.4 Operator norm independent solution of split monotone variational inclusion problem in Hilbert spaces

In this section, we present a general algorithm which does not require prior knowledge of the operator norm for solving split monotone variational inclusion problem, fixed point problem for a finite family of strictly pseudocontractive mappings and certain variational inequality problem.

Theorem 6.4.1. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω_{fB} be a solution set of (6.1) - (6.2), $S_i : H_1 \to H_1$ (i=1,2,...,N) be κ_i-strictly pseudocontractive mappings and $\mathbb{F} \cap \Omega_{fB} \neq \emptyset$ where $\mathbb{F} = \cap_{i=1}^N F(S_i)$. Let D be a strongly positive bounded linear operator on H_1 with a coefficient $\bar{\delta} > 0$, G a ρ contraction on H_1, $0 < \delta < \dfrac{\bar{\delta}}{\rho}$ and $\{\eta_{n,i}\}_{i-1}^N \subset (0,1)$ are such that $\sum_{i=1}^N \eta_{n,i} = 1$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \dfrac{||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2}{||A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2} - \epsilon\right)$ for $J_\lambda^{B_2}(I - \lambda f_2)Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in H_1$ by*

$$\begin{cases} w_n = (I - \alpha_n D)x_n + \alpha_n \delta G(x_n), \\ y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n \sum_{i=1}^N \eta_{n,i} S_i y_n, \quad \forall n \geq 0, \end{cases} \tag{6.73}$$

converges strongly to a point $p \in \Omega_{fB} \cap \mathbb{F}$ which is also a solution of the variational inequality

$$\langle (D - \delta G)p, p - q \rangle \leq 0, \quad \forall q \in \Omega_{fB} \cap \mathbb{F},$$

where $\lambda > 0$ is such that where $0 < \lambda < 2\mu, 2\nu$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$, $\; 0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$.

Proof. For any $x, y \in H$, we have

$$\begin{aligned} ||P_{\Omega\cap\mathbb{F}}(I - D + \delta G)(x) - P_{\Omega\cap\mathbb{F}}(I - D + \delta G)(y)| &\leq ||(I - D + \delta G)(x) - (I - D + \delta G)(y)|| \\ &\leq ||(I - D)x - (I - D)y|| + \delta||Gx - Gy|| \\ &\leq (1 - \bar{\delta})||x - y|| + \delta\rho||x - y|| \\ &\leq (1 - (\bar{\delta} - \delta\rho)||x - y||. \end{aligned}$$

Thus, $P_{\Omega_{fB}\cap\mathbb{F}}(I - D + \delta G)$ is a contraction and by the Banach contraction mapping principle, we conclude that there exists $p \in H$ such that $p = P_{\Omega_{fB}\cap\mathbb{F}}(I - D + \delta G)p$.
Next, we show that $\{x_n\}$ is bounded.

$$\begin{aligned} ||w_n - p|| &= ||(I - \alpha_n D)(x_n - p) + \alpha_n(\delta G(x_n) - Dp)|| \\ &\leq (1 - \alpha_n\bar{\delta})||x_n - p|| + \alpha_n||\delta G(x_n) - Dp|| \\ &\leq (1 - \alpha_n\bar{\delta})||x_n - p|| + \alpha_n||\delta G(x_n) - \delta G(p)|| + \alpha_n||\delta G(p) - Dp)|| \\ &\leq [1 - (\bar{\delta} - \delta\rho)\alpha_n]||x_n - p|| + \alpha_n||\delta G(p) - Dp)||. \end{aligned} \tag{6.74}$$

But

$$\begin{aligned} ||y_n - p||^2 &= ||J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n) - p||^2 \\ &\leq ||w_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n - p||^2 \\ &= ||w_n - p||^2 + \gamma_n^2||A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2 \\ &\quad +2\gamma_n\langle w_n - p, A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n\rangle. \end{aligned} \tag{6.75}$$

Now by Lemma 2.3.7 and (2.40), we have

$$\begin{aligned} 2\gamma_n\langle w_n - p, A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n\rangle &= 2\gamma_n\langle A(w_n - p), (J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n\rangle \\ &= 2\gamma_n[\langle J_\lambda^{B_2}(I - \lambda f_2)Aw_n - Ap, (J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n\rangle \\ &\quad -||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2] \\ &\leq 2\gamma_n[\frac{1}{2}||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2 \\ &\quad -||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2] \\ &= -\gamma_n||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2. \end{aligned} \tag{6.76}$$

Thus from (6.75) and (6.76), we have

$$\begin{aligned} ||y_n - p||^2 &\leq ||w_n - p||^2 + \gamma_n^2||A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2 - \gamma_n||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2 \\ &= ||w_n - p||^2 + \gamma_n[\gamma_n||A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2 \\ &\quad -||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2]. \end{aligned} \tag{6.77}$$

Hence from the condition $\gamma_n \in \Big(\epsilon, \frac{||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2}{||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2} - \epsilon\Big)$, we obtain

$$||y_n - p||^2 \le ||w_n - p||^2. \tag{6.78}$$

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\beta_n)y_n + \beta_n \sum_{i=1}^{N} \eta_{n,i} S_i y_n - p||^2 \\
&= (1-\beta_n)||y_n - p||^2 + \beta_n || \sum_{i=1}^{N} \eta_{n,i} S_i y_n - p||^2 - \beta_n(1-\beta_n)||y_n - \sum_{i=1}^{N} \eta_{n,i} S_i y_n||^2 \\
&\le (1-\beta_n)||y_n - p||^2 + \beta_n[||y_n - p||^2 + \kappa||y_n - \sum_{i=1}^{N} \eta_{n,i} S_i y_n||^2] \\
&\quad -\beta_n(1-\beta_n)||y_n - \sum_{i=1}^{N} \eta_{n,i} S_i y_n||^2 \\
&= ||y_n - p||^2 - \beta_n(1-\beta_n-\kappa)||y_n - \sum_{i=1}^{N} \eta_{n,i} S_i y_n||^2 \\
&\le ||y_n - p||^2.
\end{aligned} \tag{6.79}$$

Therefore, from (6.74), we have

$$\begin{aligned}
||x_{n+1} - p|| &\le ||w_n - p|| \\
&\le [1 - (\bar{\delta} - \delta\rho)\alpha_n]||x_n - p|| + \alpha_n ||\delta G(p) - Dp)|| \\
&= [1 - (\bar{\delta} - \delta\rho)\alpha_n]||x_n - p|| + (\bar{\delta} - \delta\rho)\alpha_n \frac{1}{(\bar{\delta} - \delta\rho)} ||\delta G(p) - Dp)|| \\
&\le \max\{||x_n - p||, \frac{1}{(\bar{\delta} - \delta\rho)} ||\delta G(p) - Dp)||\} \\
&\vdots \\
&\le \max\{||x_0 - p||, \frac{1}{(\bar{\delta} - \delta\rho)} ||\delta G(p) - Dp)||\}.
\end{aligned} \tag{6.80}$$

We then conclude that $\{x_n\}$ is bounded.
Again

$$\begin{aligned}
||w_n - x_n|| &= ||(I - \alpha_n D)x_n + \alpha_n \delta G(x_n) - x_n|| \\
&= \alpha_n ||Dx_n - \delta G(x_n)|| \to 0, n \to \infty.
\end{aligned} \tag{6.81}$$

Moreover,

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq ||y_n-p||^2-\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&\leq ||w_n-p||^2-\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&= ||(I-\alpha_nD)x_n+\alpha_n\delta G(x_n)-p||^2-\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&\leq ||x_n-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2-2\alpha_n\langle x_n-p, Dx_n-\delta G(x_n)\rangle\\
&\quad -\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2. \qquad (6.82)
\end{aligned}$$

We divide into two cases to obtain strong convergence.
Case 1. Assume that $\{||x_n-p||^2\}$ is a monotonically decreasing sequence. It then follows that $\{||x_n-p||^2\}$ is convergent and

$$||x_n-p||-||x_{n+1}-p||\to 0, n\to\infty. \qquad (6.83)$$

Therefore, from (6.82), we have

$$\begin{aligned}
\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2 &\leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2\\
&\quad -2\alpha_n\langle x_n-p, Dx_n-\delta G(x_n)\rangle\to 0, n\to\infty.
\end{aligned}$$

That is

$$||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||\to 0, n\to\infty. \qquad (6.84)$$

Also,

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq ||y_n-p||^2\\
&\leq ||w_n-p||^2+\gamma_n[\gamma_n||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2]\\
&\leq ||x_n-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2-2\alpha_n\langle x_n-p, Dx_n-\delta G(x_n)\rangle\\
&\quad +\gamma_n[\gamma_n||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2]. \qquad (6.85)
\end{aligned}$$

It then follows from (6.85) and the condition

$$\gamma_n\in\Big(\epsilon, \frac{||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2}{||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2}-\epsilon\Big),$$

that

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq ||x_n-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2-2\alpha_n\langle x_n-p, Dx_n-\delta G(x_n)\rangle\\
&\quad -\epsilon^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2, \qquad (6.86)
\end{aligned}$$

which implies

$$\epsilon^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2 -2\alpha_n\langle x_n-p, Dx_n-\delta G(x_n)\rangle \to 0, n\to\infty.$$

Therefore,

$$\lim_{n\to\infty}||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||=0. \tag{6.87}$$

Also from (6.85), we have

$$\begin{aligned}
&\gamma_n||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\\
\leq\ &||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2\\
&-2\alpha_n\langle x_n-p, Dx_n-\delta G(x_n)\rangle\\
&+\gamma_n^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\to 0, n\to\infty.
\end{aligned}\tag{6.88}$$

Further,

$$\begin{aligned}
||y_n-p||^2 &= ||J_\lambda^{B_1}(I-\lambda f_1)(w_n+\gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)-p||^2\\
&\leq \langle y_n-p, w_n+\gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p\rangle\\
&= \frac{1}{2}[||y_n-p||^2+||w_n+\gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p||^2\\
&\quad -||y_n-p-(w_n+\gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)-p)||^2]\\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2\\
&\quad +\gamma_n(\gamma_n||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2)\\
&\quad -||y_n-p-(w_n+\gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p)||^2]\\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2-(||y_n-w_n||^2+\gamma_n^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\\
&\quad -2\gamma_n\langle y_n-w_n, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle)]\\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2-||y_n-w_n||^2\\
&\quad +2\gamma_n||y_n-w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||].
\end{aligned}\tag{6.89}$$

That is,

$$\begin{aligned}
||y_n-p||^2 \leq\ &||w_n-p||^2-||y_n-w_n||^2\\
&+2\gamma_n||y_n-w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||.
\end{aligned}\tag{6.90}$$

Thus, it follows from (6.79) and (6.90) that

$$\begin{aligned}
||x_{n+1}-p||^2 \leq\ &||w_n-p||^2-||y_n-w_n||^2\\
&+2\gamma_n||y_n-w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||.
\end{aligned}\tag{6.91}$$

Hence

$$\begin{aligned}
||y_n-w_n||^2 &\leq ||w_n-p||^2-||x_{n+1}-p||^2+2\gamma_n||y_n-w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||\\
&\leq ||x_n-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2\\
&\quad -2\alpha_n\langle x_n-p, Dx_n-\delta G(x_n)\rangle-||x_{n+1}-p||^2\\
&\quad +2\gamma_n||y_n-w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||\to 0, n\to\infty.
\end{aligned}\tag{6.92}$$

Furthermore

$$||x_n - y_n|| \leq ||x_n - w_n|| + ||w_n - y_n|| \to 0, n \to \infty. \tag{6.93}$$

$$||x_{n+1} - y_n|| = \beta_n ||y_n - \sum_{i=1}^{N} \eta_{n,i} S_i y_n|| \to 0, n \to \infty, \tag{6.94}$$

and hence

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - y_n|| + ||x_n - y_n|| \to 0, \to \infty. \tag{6.95}$$

Let $u_n = w_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n$, then

$$\|u_n - w_n\|^2 = L\gamma_n^2 \|(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n\|^2 \to 0. \tag{6.96}$$

Combining (6.92) and (6.96), we have that

$$\|y_n - u_n\| \leq \|y_n - w_n\| + \|w_n - u_n\| \to 0. \tag{6.97}$$

Next, we show that

$$\limsup_{n\to\infty} \langle (D - \delta G)q, q - x_n \rangle \leq 0.$$

We choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle (D - \delta G)q, q - x_{n_i} \rangle = \limsup_{n\to\infty} \langle (D - \delta G)q, q - x_n \rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ also denoted as $\{x_{n_i}\}$ that converges weakly to some $q \in H$ and consequently we have $\{y_{n_i}\}$ and $\{w_{n_i}\}$ converge weakly to q. From Lemma 2.3.1 , Lemma 4.1.3, Lemma 2.3.4 and (6.84), we conclude $q \in \mathbb{F}$.

We now show that $q \in I(f_1, B_1)$. Since f_1 is a $\frac{1}{\mu}$-Lipschitz monotone mapping and the domain of f_1 is H_1, then by Lemma 2.3.12, we conclude that $B_1 + f_1$ is maximally monotone. Let $(v, z) \in G(B_1 + f_1)$, that is $z - f_1 v \in B_1(v)$.
Since $y_{n_i} = J_\lambda^{B_1}(I - \lambda f_1)u_{n_i}$, we obtain that

$$(I - \lambda f_1)u_{n_i} \in (I + \lambda B_1)y_{n_i}.$$

That is,

$$\frac{1}{\lambda}(u_{n_i} - \lambda f_1 u_{n_i} - y_{n_i}) \in B_1(y_{n_i}).$$

Using the maximal monotonicity of $(B_1 + f_1)$, we have

$$\langle v - y_{n_i}, z - f_1 v - \frac{1}{\lambda}(u_{n_i} - \lambda f_1 u_{n_i} - y_{n_i}) \rangle \geq 0.$$

Therefore,

$$\begin{aligned}\langle v-y_{n_i},z\rangle &\geq \langle v-y_{n_i}, f_1v+\frac{1}{\lambda}(u_{n_i}-\lambda f_1u_{n_i}-y_{n_i})\rangle\\ &= \langle v-y_{n_i}, f_1v-f_1y_{n_i}+f_1y_{n_i}-f_1u_{n_i}+\frac{1}{\lambda}(u_{n_i}-y_{n_i})\rangle\\ &\geq 0+\langle v-y_{n_i}, f_1y_{n_i}-f_1u_{n_i}\rangle+\langle v-y_{n_i},\frac{1}{\lambda}(u_{n_i}-y_{n_i})\rangle. \qquad (6.98)\end{aligned}$$

By (6.97), we obtain that

$$\lim_{i\to\infty}\|f_1y_{n_i}-f_1u_{n_i}\|=0.$$

Also, since $y_{n_i}\rightharpoonup q$, we have

$$\lim_{i\to\infty}\langle v-y_{n_i},z\rangle=\langle v-q,z\rangle.$$

Thus, from (6.98)

$$\langle v-q,z\rangle\geq 0.$$

Since B_1+f_1 is maximally monotone, we have $0\in(B_1+f_1)q$ which implies that

$$q\in I(f_1,B_1).$$

Moreover, we have Aw_{n_i} converges weakly to Aq, thus by (6.88) and the fact that $J_\lambda^{B_2}(I-\lambda f_2)$ is nonexpansive, then by Lemma 2.3.1, we get that

$$0\in f_2Aq+B_2(Aq).$$

That is, $Aq\in I(f_2,B_2)$. Hence $q\in\Omega_{fB}\cap\mathbb{F}$.
Since $p=P_{\Omega_{fB}\cap\mathbb{F}}(I-D+\delta G)p$ and $q\in\Omega_{fB}\cap\mathbb{F}$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(D-\delta G)p,p-x_n\rangle &= \lim_{i\to\infty}\langle(D-\delta G)p,p-x_{n_i}\rangle\\ &= \langle(D-\delta G)p,p-q\rangle\leq 0. \qquad (6.99)\end{aligned}$$

We now show that $\{x_n\}$ converges strongly to p.

$$\begin{aligned}||x_{n+1}-p||^2 &\leq ||y_n-p||^2\\ &\leq ||w_n-p||^2\\ &= ||(I-\alpha_nD)x_n+\alpha_n\delta G(x_n)-p||^2\\ &\leq (1-\alpha_n\bar{\delta})^2||x_n-p||^2+\alpha_n^2||\delta G(x_n)-Dp||^2\\ &\quad +2\alpha_n\langle(I-\alpha_nD)(x_n-p),\delta G(x_n)-Dp\rangle\\ &\leq (1-\alpha_n\bar{\delta})^2||x_n-p||^2+\alpha_n^2||\delta G(x_n)-Dp||^2\\ &\quad +2\alpha_n\langle x_n-p,\delta G(x_n)-Dp\rangle-2\alpha_n^2\langle Dx_n-Dp,\delta G(x_n)-Dp\rangle\\ &\leq (1-\alpha_n\bar{\delta})^2||x_n-p||^2+\alpha_n^2||\delta G(x_n)-Dp||^2+2\alpha_n\delta\langle x_n-p,G(x_n)-G(p)\rangle\\ &\quad +2\alpha_n\langle x_n-p,\delta G(p)-Dp\rangle-2\alpha_n^2\langle Dx_n-Dp,\delta G(x_n)-Dp\rangle\\ &\leq (1-\alpha_n\bar{\delta})^2||x_n-p||^2+\alpha_n^2||\delta G(x_n)-Dp||^2+2\alpha_n\rho\delta||x_n-p||^2\\ &\quad +2\alpha_n\langle x_n-p,\delta G(p)-Dp\rangle-2\alpha_n^2\langle Dx_n-Dp,\delta G(x_n)-Dp\rangle\\ &= (1-2\alpha_n(\bar{\delta}-\rho\delta)+\alpha_n^2\bar{\delta}^2)||x_n-p||^2+\alpha_n^2||\delta G(x_n)-Dp||^2\\ &\quad +2\alpha_n\langle x_n-p,\delta G(p)-Dp\rangle-2\alpha_n^2\langle Dx_n-Dp,\delta G(x_n)-Dp\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}||x_{n+1}-P||^2 &\leq (1-\alpha_n(\bar{\delta}-\rho\delta))||x_n-p||^2+\alpha_n(\bar{\delta}-\rho\delta)[\frac{1}{(\bar{\delta}-\rho\delta)}(\alpha_n\bar{\delta}^2||x_n-p||^2 \\ &\quad +\alpha_n||\delta G(x_n)-Dp||^2+2\langle x_n-p,\delta G(p)-Dp\rangle \\ &\quad -2\alpha_n\langle Dx_n-Dp,\delta G(x_n)-Dp\rangle)].\end{aligned}$$

Thus by Lemma 2.3.12, we have $x_n \to p$.

Case 2. Assume that $\{\|x_n-p\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n-p\|^2$ and let $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) := \max\{k\in\mathbb{N} : k\geq n, \Gamma_k\leq\Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$, for $n\geq n_0$.

Then from (6.82), we have

$$\begin{aligned}0 &\leq ||x_{\tau(n)+1}-p||^2-||x_{\tau(n)}-p||^2 \\ &\leq \alpha^2_{\tau(n)}||Dx_{\tau(n)}-\delta G(x_{\tau(n)})||^2-2\alpha_{\tau(n)}\langle x_{\tau(n)}-p,Dx_{\tau(n)}-\delta G(x_{\tau(n)})\rangle \\ &\quad -\beta_{\tau(n)}(1-\beta_{\tau(n)}-\kappa)||y_{\tau(n)}-\sum_{i=1}^{N}\eta_{\tau(n),i}S_iy_{\tau(n)}||^2,\end{aligned}$$

which implies

$$\begin{aligned}\beta_{\tau(n)}(1-\beta_{\tau(n)}-\kappa)||y_{\tau(n)}-\sum_{i=1}^{N}\eta_{\tau(n),i}S_iy_{\tau(n)}||^2 &\leq \alpha^2_{\tau(n)}||Dx_{\tau(n)}-\delta G(x_{\tau(n)})||^2 \\ &\quad -2\alpha_{\tau(n)}\langle x_{\tau(n)}-p,Dx_{\tau(n)}-\delta G(x_{\tau(n)})\rangle\to 0.\end{aligned}$$

By the same argument as in (6.84)-(6.99) in case 1, we conclude that

$$\limsup_{n\to\infty}\langle D-\delta G)q,q-x_{\tau(n)}\rangle\leq 0.$$

Hence for all $n\geq n_0$,

$$\begin{aligned}0 &\leq ||x_{\tau(n)+1}-p||^2-||x_{\tau(n)}-p||^2 \\ &\leq (1-\alpha_{\tau(n)}(\bar{\delta}-\rho\delta))||x_{\tau(n)}-p||^2 \\ &\quad +\alpha_{\tau(n)}(\bar{\delta}-\rho\delta)[\frac{1}{(\bar{\delta}-\rho\delta)}(\alpha_{\tau(n)}\bar{\delta}^2||x_{\tau(n)}-p||^2+\alpha_{\tau(n)}||\delta G(x_{\tau(n)})-Dp||^2 \\ &\quad +2\langle x_{\tau(n)}-p,\delta G(p)-Dp\rangle-2\alpha_{\tau(n)}\langle Dx_{\tau(n)}-Dp,\delta G(x_{\tau(n)})-Dp\rangle)]-||x_{\tau(n)}-p||^2 \\ &= \alpha_{\tau(n)}(\bar{\delta}-\rho\delta)[\frac{1}{(\bar{\delta}-\rho\delta)}(\alpha_{\tau(n)}\bar{\delta}^2||x_{\tau(n)}-p||^2+\alpha_{\tau(n)}||\delta G(x_{\tau(n)})-Dp||^2 \\ &\quad +2\langle x_{\tau(n)}-p,\delta G(p)-Dp\rangle-2\alpha_{\tau(n)}\langle Dx_{\tau(n)}-Dp,\delta G(x_{\tau(n)})-Dp\rangle)-||x_{\tau(n)}-p||^2].\end{aligned}$$

That is

$$\begin{aligned}||x_{\tau(n)}-p||^2 &\leq \frac{1}{(\bar{\delta}-\rho\delta)}(\alpha_{\tau(n)}\bar{\delta}^2||x_{\tau(n)}-p||^2+\alpha_{\tau(n)}||\delta G(x_{\tau(n)})-Dp||^2 \\ &\quad +2\langle x_{\tau(n)}-p,\delta G(p)-Dp\rangle \\ &\quad -2\alpha_{\tau(n)}\langle Dx_{\tau(n)}-Dp,\delta G(x_{\tau(n)})-Dp\rangle)\to 0, n\to\infty. \qquad (6.100)\end{aligned}$$

Therefore,

$$\|x_{\tau(n)} - p\|^2 \leq \alpha_{\tau(n)}\|p\|^2 - 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - p, p\rangle \to 0.$$

Thus,

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\|^2 = 0,$$

and hence

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1}.$$

Furthermore, for $n \geq n_0$, it is observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$(that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

So $\lim\limits_{n\to\infty} \Gamma_n = 0$, that is $\{x_n\}$,$\{y_n\}$ and $\{w_n\}$ converge strongly to $p \in \Omega_{fB} \cap \mathbb{F}$ which is also a solution of the variational inequality

$$\langle (D - \delta G)p, p - q\rangle \leq 0, \quad \forall q \in \Omega_{fB} \cap \mathbb{F}.$$

Corollary 6.4.2. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let $S_i : H_1 \to H_1$ (i=1,2,...,N) be κ_i-strictly pseudocontractive mappings and $\mathbb{F} \cap \Omega_B \neq \emptyset$ where $\mathbb{F} = \cap_{i=1}^N F(S_i)$. Let D be a strongly positive bounded linear operator on H_1 with a coefficient $\bar{\delta} > 0$, G a ρ contraction on H_1, $0 < \delta < \dfrac{\bar{\delta}}{\rho}$ and $\{\eta_{n,i}\}_{i-1}^N \subset (0,1)$ are such that $\sum_{i=1}^N \eta_{n,i} = 1$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \dfrac{||(J_\lambda^{B_2} - I)Aw_n||^2}{||A^*(J_\lambda^{B_2} - I)Aw_n||^2} - \epsilon\right)$ for $J_\lambda^{B_2}Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in H_1$ by*

$$\begin{cases} w_n = (I - \alpha_n D)x_n + \alpha_n \delta G(x_n), \\ y_n = J_\lambda^{B_1}(w_n + \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{i=1}^N \eta_{n,i} S_i y_n, \quad \forall n \geq 0, \end{cases} \tag{6.101}$$

converges strongly to a point $p \in \Omega_B \cap \mathbb{F}$ which is also a solution of the variational inequality

$$\langle (D - \delta G)p, p - q\rangle \leq 0, \quad \forall q \in \Omega_B \cap \mathbb{F},$$

where $\lambda > 0$ is a positive real number and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$, $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$.

Corollary 6.4.3. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω_{fB} be a solution set of* (6.1) - (6.2)*, $S_i : H_1 \to H_1$ (i=1,2,...,N) be nonexpansive mappings and $\mathbb{F} \cap \Omega_{fB} \neq \emptyset$ where $\mathbb{F} = \cap_{i=1}^{N} F(S_i)$. Let D be a strongly positive bounded linear operator on H_1 with a coefficient $\bar{\delta} > 0$, G a ρ contraction on H_1, $0 < \delta < \dfrac{\bar{\delta}}{\rho}$ and $\{\eta_{n,i}\}_{i-1}^{N} \subset (0,1)$ are such that $\sum_{i=1}^{N} \eta_{n,i} = 1$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \dfrac{||(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n||^2}{||A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n||^2} - \epsilon\right)$ for $J_\lambda^{B_2}(I-\lambda f_2)Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in H_1$ by*

$$\begin{cases} w_n = (I - \alpha_n D)x_n + \alpha_n \delta G(x_n), \\ y_n = J_\lambda^{B_1}(I-\lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{i=1}^{N} \eta_{n,i} S_i y_n, \quad \forall n \geq 0, \end{cases} \tag{6.102}$$

converges strongly to a point $p \in \Omega_{fB} \cap \mathbb{F}$ which is also a solution of the variational inequality

$$\langle (D - \delta G)p, p - q \rangle \leq 0, \quad \forall q \in \Omega_{fB} \cap \mathbb{F},$$

where $\lambda > 0$ is such that where $0 < \lambda < 2\mu, 2\nu$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ *satisfying the following conditions*
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.

6.5 An iterative technique for split equality monotone variational inclusion and fixed point problems

In this section, we consider the following problem: Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be inverse strongly monotone mappings and $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ maximal monotone mappings. Find $x^* \in F(S_1)$ and $y^* \in F(S_2)$ such that

$$0 \in f_1(x^*) + T_1(x^*), \tag{6.103}$$

$$0 \in f_2(y^*) + T_2(y^*), \quad \text{and} \quad Ax^* = By^*, \tag{6.104}$$

where $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ are two strictly pseudocontractive mappings. In what follows, we will denote by Υ the solution set of problem (6.103)-(6.104). Furthermore, we propose an iterative scheme and using the iterative scheme, we state and prove a strong convergence result for the approximation of a solution of problem (6.103)-(6.104).

Theorem 6.5.1. *Let H_1, H_2 and H_3 be three real Hilbert spaces, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ be multivalued maximal monotone mappings. Let $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ be κ_1-strictly pseudocontractive and κ_2-strictly pseudocontractive mappings respectively. Suppose $\Upsilon \neq \emptyset$ and*

$$\gamma_n \in \Big(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\Big), n \in \Omega$$

otherwise, $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$. Let $\{(x_n, y_n)\}$ be the sequence generated for $x_0 \in H_1$ and $y_0 \in H_2$ by

$$\begin{cases} w_n = (1-\alpha_n)x_n, \\ z_n = (1-\alpha_n)y_n, \\ u_n = J_\lambda^{T_1}(I - \lambda f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = J_\lambda^{T_2}(I - \lambda f_2)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S_1 u_n, \quad \forall n \geq 0, \\ y_{n+1} = (1-\delta_n)v_n + \delta_n S_2 v_n, \quad \forall n \geq 0, \end{cases} \tag{6.105}$$

where $0 < \lambda < 2\mu, 2\nu$ and A^*, B^* are the adjoint of A and B respectively. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < a \leq \beta_n \leq b < 1 - \kappa_1$,
(iii) $0 < c \leq \delta_n \leq d < 1 - \kappa_2$,
then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Upsilon$.

Proof. Clearly γ_n is well defined, since for any $(x, y) \in \Upsilon$, we have

$$\langle A^*(Aw_n - Bz_n), w_n - x\rangle = \langle Aw_n - Bz_n, Aw_n - Ax\rangle \tag{6.106}$$

and

$$\langle B^*(Aw_n - Bz_n), y - z_n\rangle = \langle Aw_n - Bz_n, By - Bz_n\rangle. \tag{6.107}$$

Adding (6.106) and (6.107) and taking into account the fact $Ax = By$, we obtain $\forall\, n \in \Omega$,

$$\begin{aligned} ||Aw_n - Bz_n||^2 &= \langle A^*(Aw_n - Bz_n), w_n - x\rangle + \langle B^*(Aw_n - Bz_n), y - z_n\rangle \\ &\leq ||A^*(Aw_n - Bz_n)||||w_n - x|| + ||B^*(Aw_n - Bz_n)||||y - z_n||. \end{aligned}$$

Therefore, for $n \in \Omega$, that is, $||Aw_n - Bz_n|| > 0$, we have $||A^*(Aw_n - Bz_n)|| \neq 0$ or $||B^*(Aw_n - Bz_n)|| \neq 0$. Thus γ_n is well defined.

Let $(x^*, y^*) \in \Upsilon$, then

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||(1-\beta_n)u_n + \beta_n S_1 u_n - x^*||^2 \\
&= ||(1-\beta_n)(u_n - x^*) + \beta_n(S_1 u_n - x^*)||^2 \\
&= (1-\beta_n)^2||u_n - x^*||^2 + \beta_n^2||S_1 u_n - x^*||^2 + 2\beta_n(1-\beta_n)\langle u_n - x^*, S_1 u_n - x^*\rangle \\
&\leq (1-\beta_n)^2||u_n - x^*||^2 + \beta_n^2[||u_n - x^*||^2 + \kappa_1||u_n - S_1 u_n||^2] \\
&\quad +2\beta_n(1-\beta_n)[||u_n - x^*||^2 - \frac{1-\kappa_1}{2}||u_n - S_1 u_n||^2] \\
&= (1 - 2\beta_n + \beta_n^2)||u_n - x^*||^2 + \beta_n^2[||u_n - x^*||^2 + \kappa_1||u_n - S_1 u_n||^2] \\
&\quad +2\beta_n(1-\beta_n)||u_n - x^*||^2 - \beta_n(1-\beta_n)(1-\kappa_1)||u_n - S_1 u_n||^2 \\
&= ||u_n - x^*||^2 + [\beta_n^2\kappa_1 - \beta_n(1-\beta_n)(1-\kappa_1)]||u_n - S_1 u_n||^2 \\
&= ||u_n - x^*||^2 + \beta_n[\kappa_1 + \beta_n - 1]||u_n - S_1 u_n||^2 \\
&\leq ||u_n - x^*||^2
\end{aligned} \tag{6.108}$$

and

$$\begin{aligned}
||u_n - x^*||^2 &= ||J_\lambda^{T_1}(I - \lambda f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*||^2 \\
&\leq ||w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*||^2 \\
&= ||w_n - x^*||^2 - 2\gamma_n\langle w_n - x^*, A^*(Aw_n - Bz_n)\rangle \\
&\quad +\gamma_n^2||A^*(Aw_n - Bz_n)||^2.
\end{aligned} \tag{6.109}$$

Also, from Lemma 2.3.8, we have

$$\begin{aligned}
-2\langle w_n - x^*, A^*(Aw_n - Bz_n)\rangle &= -2\langle Aw_n - Ax^*, Aw_n - Bz_n\rangle \\
&= -||Aw_n - Ax^*||^2 - ||Aw_n - Bz_n||^2 \\
&\quad +||Bz_n - Ax^*||^2.
\end{aligned} \tag{6.110}$$

Substituting (6.110) into (6.109), we obtain

$$\begin{aligned}
||u_n - x^*||^2 &\leq ||w_n - x^*||^2 - \gamma_n||Aw_n - Ax^*||^2 - \gamma_n||Aw_n - Bz_n||^2 \\
&\quad +\gamma_n||Bz_n - Ax^*||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2.
\end{aligned} \tag{6.111}$$

It then follows from (6.108) and (6.111) that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||w_n - x^*||^2 - \gamma_n||Aw_n - Ax^*||^2 - \gamma_n||Aw_n - Bz_n||^2 \\
&\quad +\gamma_n||Bz_n - Ax^*||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2.
\end{aligned} \tag{6.112}$$

Following similar steps as in (6.108)-(6.112), we have

$$\begin{aligned}
||y_{n+1} - y^*||^2 &\leq ||z_n - y^*||^2 - \gamma_n||Bz_n - By^*||^2 - \gamma_n||Aw_n - Bz_n||^2 \\
&\quad +\gamma_n||Aw_n - By^*||^2 + \gamma_n^2||B^*(Aw_n - Bz_n)||^2.
\end{aligned} \tag{6.113}$$

Adding inequalities (6.112) and (6.113) and noting that $Ax^* = By^*$, we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 &\leq ||w_n - x^*||^2 + ||z_n - y^*||^2 - \gamma_n[2||Aw_n - Bz_n||^2 \\
&\quad -\gamma_n(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2)] \qquad (6.114) \\
&\leq ||w_n - x^*||^2 + ||z_n - y^*||^2 \\
&= ||(1-\alpha_n)x_n - x^*||^2 + ||(1-\alpha_n)y_n - y^*||^2 \\
&= ||(1-\alpha_n)(x_n - x^*) - \alpha_n x^*||^2 + ||(1-\alpha_n)(y_n - y^*) - \alpha_n y^*||^2 \\
&\leq (1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2 + (1-\alpha_n)||y_n - y^*||^2 + \alpha_n||y^*||^2 \\
&= (1-\alpha_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] + \alpha_n[||x^*||^2 + ||y^*||^2] \\
&\leq \max\{||x_n - x^*||^2 + ||y_n - y^*||^2, ||x^*||^2 + ||y^*||^2\} \\
&\;\;\vdots \\
&\leq \max\{||x_0 - x^*||^2 + ||y_0 - y^*||^2], [||x^*||^2 + ||y^*||^2\}. \qquad (6.115)
\end{aligned}$$

Thus $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is bounded and consequently $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{w_n\}$ and $\{z_n\}$ are all bounded.

Now from (6.114), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 &\leq ||w_n - x^*||^2 + ||z_n - y^*||^2 - \gamma_n[2||Aw_n - Bz_n||^2 \\
&\quad -\gamma_n(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2)] \\
&\leq (1-\alpha_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] + \alpha_n[||x^*||^2 + ||y^*||^2] \\
&\quad -\gamma_n[2||Aw_n - Bz_n||^2 - \gamma_n(||A^*(Aw_n - Bz_n)||^2 \\
&\quad +||B^*(Aw_n - Bz_n||^2)]. \qquad (6.116)
\end{aligned}$$

We now divide into two cases to obtain the strong convergence of $\{(x_n, y_n)\}$ to $(\bar{x}, \bar{y}) \in \Upsilon$.

Case 1. Assume $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is monotonically decreasing. Clearly

$$(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2) - (||x_n - x^*||^2 + ||y_n - y^*||^2) \to 0, n \to \infty.$$

Let $P_n = ||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2$ and $n \in \Omega$, then it follows from (6.116) that

$$\begin{aligned}
\gamma_n[2||Aw_n - Bz_n||^2 - \gamma_n P_n] &\leq [||x_n - x^*||^2 + ||y_n - y^*||^2] - [||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2] \\
&\quad +\alpha_n[||x^*||^2 + ||y^*||^2], \qquad (6.117)
\end{aligned}$$

which implies

$$\begin{aligned}
\epsilon P_n &\leq [||x_n - x^*||^2 + ||y_n - y^*||^2] - [||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2] \\
&\quad +\alpha_n[||x^*||^2 + ||y^*||^2]. \qquad (6.118)
\end{aligned}$$

From (6.118), we have

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} ||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2 = 0.$$

Observe that $Aw_n - Bz_n = 0$, if $n \notin \Omega$.
Hence,

$$\lim_{n\to\infty} ||A^*(Aw_n - Bz_n)||^2 = 0, \tag{6.119}$$

and

$$\lim_{n\to\infty} ||B^*(Aw_n - Bz_n)||^2 = 0. \tag{6.120}$$

From (6.105), we have

$$\lim_{n\to\infty} ||w_n - x_n||^2 = \alpha_n||x_n||^2 \to 0, n \to \infty, \tag{6.121}$$

and

$$\lim_{n\to\infty} ||z_n - y_n||^2 = \alpha_n||y_n||^2 \to 0, n \to \infty. \tag{6.122}$$

Again

$$\begin{aligned}
||u_n - x^*||^2 &= ||J_\lambda^{T_1}(I - \lambda f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*||^2 \\
&\leq \langle u_n - x^*, w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*\rangle \\
&= \frac{1}{2}[||u_n - x^*||^2 + ||w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*||^2 \\
&\quad -||u_n - x^* - (w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*)||^2] \\
&\leq \frac{1}{2}[||u_n - x^*||^2 + ||w_n - x^*||^2 \\
&\quad +\gamma_n^2||A^*(Aw_n - Bz_n)||^2 + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -(||u_n - w_n||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2 - 2\gamma_n\langle u_n - w_n, A^*(Aw_n - Bz_n)\rangle] \\
&= \frac{1}{2}[||u_n - x^*||^2 + ||w_n - x^*||^2 + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n\langle u_n - w_n, A^*(Aw_n - Bz_n)\rangle] \\
&\leq \frac{1}{2}[||u_n - x^*||^2 + ||w_n - x^*||^2 + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||] \\
&\leq \frac{1}{2}[||u_n - x^*||^2 + ||x_n - x^*||^2 + \alpha_n^2||x_n||^2 - 2\alpha_n\langle x_n - x^*, x_n\rangle \\
&\quad +2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| - ||u_n - w_n||^2 \\
&\quad +2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||].
\end{aligned} \tag{6.123}$$

That is

$$\begin{aligned}
||u_n - x^*||^2 &\leq ||x_n - x^*||^2 + \alpha_n^2||x_n||^2 - 2\alpha_n\langle x_n - x^*, x_n\rangle + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||.
\end{aligned} \tag{6.124}$$

It follows from (6.108) and (6.124) that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 + \alpha_n^2||x_n||^2 - 2\alpha_n\langle x_n - x^*, x_n\rangle + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||,
\end{aligned} \tag{6.125}$$

which implies

$$\begin{aligned}||u_n - w_n||^2 &\leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \alpha_n^2||x_n||^2 - 2\alpha_n\langle x_n - x^*, x_n\rangle \\ &+2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| - ||u_n - w_n||^2 \\ &+2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)|| \to 0, n \to \infty. \end{aligned} \tag{6.126}$$

By using similar argument as in (6.123) to (6.126), we obtain

$$||v_n - z_n||^2 \to 0, n \to \infty. \tag{6.127}$$

Observe that since S_1 is κ_1-strictly pseudocontractive and $x^* \in F(S_1)$.

$$\begin{aligned}&||S_1x - x^*||^2 \leq ||x - x^*||^2 + \kappa_1||x - S_1x||^2 \\ &\Rightarrow \langle S_1x - x^*, S_1x - x^*\rangle \leq \langle x - x^*, x - x^*\rangle + \kappa_1||x - S_1x||^2 \\ &\Rightarrow \langle S_1x - x^*, S_1x - x\rangle + \langle S_1x - x^*, x - x^*\rangle \leq \langle x - x^*, x - x^*\rangle + \kappa_1||x - Sx||^2 \\ &\Rightarrow \langle S_1x - x^*, S_1x - x\rangle \leq \langle x - S_1x, x - x^*\rangle + \kappa_1||x - S_1x||^2 \\ &\Rightarrow \langle S_1x - x, S_1x - x\rangle + \langle x - x^*, S_1x - x\rangle \leq \langle x - S_1x, x - x^*\rangle + \kappa_1||x - S_1x||^2 \\ &\Rightarrow ||S_1x - x||^2 \leq \langle x - x^*, x - S_1x\rangle - \langle x - x^*, S_1x - x\rangle + +\kappa_1||x - S_1x||^2 \\ &\Rightarrow (1 - \kappa_1)||S_1x - x||^2 \leq 2\langle x - x^*, x - S_1x\rangle. \end{aligned} \tag{6.128}$$

Thus, from (6.105), we have

$$\begin{aligned}||x_{n+1} - x^*||^2 &= ||(1 - \beta_n)u_n + \beta_nS_1u_n - x^*||^2 \\ &= ||(u_n - x^*) + \beta_n(S_1u_n - u_n)||^2 \\ &= ||u_n - x^*||^2 + \beta_n^2||S_1u_n - u_n||^2 - 2\beta_n\langle u_n - x^*, u_n - S_1u_n\rangle \\ &\leq ||u_n - x^*||^2 + \beta_n(\beta_n - (1 - \kappa_1))||u_n - S_1u_n||^2 \\ &\leq ||w_n - x^*||^2 - \gamma_n||Aw_n - Ax^*||^2 - \gamma_n||Aw_n - Bz_n||^2 \\ &+\gamma_n||Bz_n - Ax^*||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2 \\ &+\beta_n(\beta_n - (1 - \kappa_1))||u_n - S_1u_n||^2. \end{aligned} \tag{6.129}$$

Similarly,

$$\begin{aligned}||y_{n+1} - y^*||^2 &\leq ||z_n - y^*||^2 - \gamma_n||Bz_n - By^*||^2 - \gamma_n||Aw_n - Bz_n||^2 \\ &+\gamma_n||Aw_n - By^*||^2 + \gamma_n^2||B^*(Aw_n - Bz_n)||^2 \\ &+\delta_n(\delta_n - (1 - \kappa_2))||v_n - S_2v_n||^2. \end{aligned} \tag{6.130}$$

Adding (6.129) and (6.130), we have

$$\begin{aligned}||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 &\leq ||w_n - x^*||^2 + ||z_n - y^*||^2 + \beta_n(\beta_n - (1 - \kappa_1))||u_n - S_1u_n||^2 \\ &+\delta_n(\delta_n - (1 - \kappa_2))||v_n - S_2v_n||^2 \\ &+\gamma_n^2[||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2]. \end{aligned} \tag{6.131}$$

Let $K_n = \beta_n(\beta_n - (1-\kappa_1))||u_n - S_1u_n||^2 + \delta_n(\delta_n - (1-\kappa_2))||v_n - S_2v_n||^2$.
Then from (6.131), we obtain

$$\begin{aligned} K_n &\leq (||w_n - x^*||^2 + ||z_n - y^*||^2) - (||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2) \\ &\quad + \gamma_n^2[||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2] \\ &= (||(1-\alpha_n)x_n - x^*||^2 + ||(1-\alpha_n)y_n - y^*||^2) - (||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2) \\ &\quad + \gamma_n^2[||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2] \\ &\leq (||x_n - x^*||^2 + ||y_n - y^*||^2) - (||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2) \\ &\quad + \alpha_n^2(||x_n||^2 + ||y_n||^2) - 2\alpha_n(\langle x_n - x^*, x_n\rangle + \langle y_n - y^*, y_n\rangle) \\ &\quad + \gamma_n^2[||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2] \to 0, n \to \infty. \end{aligned} \tag{6.132}$$

Therefore,

$$||S_1u_n - u_n|| \to 0, n \to \infty, \tag{6.133}$$

and

$$||S_2v_n - v_n|| \to 0, n \to \infty. \tag{6.134}$$

Also,

$$||x_{n+1} - u_n|| = \beta_n||u_n - S_1u_n|| \to 0, n \to \infty, \tag{6.135}$$

and

$$||y_{n+1} - v_n|| = \beta_n||v_n - S_2v_n|| \to 0, n \to \infty. \tag{6.136}$$

Furthermore,

$$||x_n - u_n|| \leq ||x_n - w_n|| + ||w_n - u_n|| \to 0, n \to \infty, \tag{6.137}$$

and

$$||y_n - v_n|| \leq ||y_n - z_n|| + ||z_n - v_n|| \to 0, n \to \infty. \tag{6.138}$$

From (6.135) and (6.137)

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - u_n|| + ||u_n - x_n|| \to 0, n \to \infty,$$

and from (6.136) and (6.138)

$$||y_{n+1} - y_n|| \leq ||y_{n+1} - v_n|| + ||v_n - y_n|| \to 0, n \to \infty.$$

It follows from (6.133) and Lemma 2.3.1 that $w_w(u_n) \subset F(S_1)$, where $w_w(u_n) := \{x : \exists u_{n_i} \rightharpoonup x\}$ is the weak $w-limit$ set of $\{u_n\}$. But $w_w(x_n) = w_w(u_n) = w_w(w_n)$, therefore, $w_w(u_n) \subset F(S_1)$. Thus, since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ that converges weakly to a fixed point $\bar{x}$ of S_1.
It also follows from (6.134) and Lemma 2.3.1 that $w_w(v_n) \subset F(S_2)$, where $w_w(v_n) := \{x : \exists v_{n_i} \rightharpoonup x\}$ is the weak $w-limit$ set of $\{v_n\}$. But $w_w(y_n) = w_w(v_n) = w_w(z_n)$, therefore,

$w_w(v_n) \subset F(S_2)$, thus since $\{u_n\}$ is bounded, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ that converges weakly to a fixed point $\bar{y}$ of S_2.
Also, $Aw_{n_i} \rightharpoonup A\bar{x}$ and $Bz_{n_i} \rightharpoonup B\bar{y}$.
But

$$\begin{aligned}
||A\bar{x} - B\bar{y}||^2 &= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y} - Aw_{n_i} + Aw_{n_i} - Bz_{n_i} + Bz_{n_i}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - Aw_{n_i}\rangle + \langle A\bar{x} - B\bar{y}, Aw_{n_i} - Bz_{n_i}\rangle \\
&\quad + \langle A\bar{x} - B\bar{y}, Bz_{n_i} - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - Aw_{n_i}\rangle + \langle A\bar{x}, Aw_{n_i} - Bz_{n_i}\rangle \\
&\quad - \langle B\bar{y}, Aw_{n_i} - Bz_{n_i}\rangle + \langle A\bar{x} - B\bar{y}, Bz_{n_i} - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - Aw_{n_i}\rangle + \langle \bar{x}, A^*(Aw_{n_i} - Bz_{n_i})\rangle \\
&\quad - \langle \bar{y}, B^*(Aw_{n_i} - Bz_{n_i})\rangle + \langle A\bar{x} - B\bar{y}, Bz_{n_i} - B\bar{y}\rangle \\
&\leq \langle A\bar{x} - B\bar{y}, A\bar{x} - Aw_{n_i}\rangle + ||\bar{x}||||A^*(Aw_{n_i} - Bz_{n_i})|| \\
&\quad + ||\bar{y}||||B^*(Aw_{n_i} - Bz_{n_i})|| + \langle A\bar{x} - B\bar{y}, Bz_{n_i} - B\bar{y}\rangle \to 0, n \to \infty.
\end{aligned}$$

That is $||A\bar{x} - B\bar{y}|| = 0$, which implies $A\bar{x} = B\bar{y}$.

We now show that $\bar{x} \in I(f_1, T_1)$ and $\bar{y} \in I(f_2, T_2)$.
Since f_1 is $\frac{1}{\mu}$-Lipschitz monotone mapping and the domain of f_1 is H_1, then by Lemma 2.3.12, we conclude that $T_1 + f_1$ is maximal monotone. Let $(u, w) \in G(T_1 + f_1)$, that is $w - f_1 u \in T_1(u)$.
Let $\Theta_{n_i} = (w_{n_i} - \gamma_{n_i} A^*(Aw_{n_i} - Bz_{n_i}))$. Then $u_{n_i} = J_\lambda^{T_1}(I - \lambda f_1)\Theta_{n_i}$, which implies

$$(I - \lambda f_1)\Theta_{n_i} \in (I + \lambda T_1)u_{n_i}.$$

That is,

$$\frac{1}{\lambda}(\Theta_{n_i} - \lambda f_1 \Theta_{n_i} - u_{n_i}) \in T_1(u_{n_i}).$$

Applying the maximal monotonicity of $(T_1 + f_1)$, we have

$$\langle u - u_{n_i}, w - f_1 u - \frac{1}{\lambda}(\Theta_{n_i} - \lambda f_1 \Theta_{n_i} - u_{n_i})\rangle \geq 0.$$

Therefore,

$$\begin{aligned}
\langle u - u_{n_i}, w\rangle &\geq \langle u - u_{n_i}, f_1 u + \frac{1}{\lambda}(\Theta_{n_i} - \lambda f_1 \Theta_n - u_{n_i})\rangle \\
&= \langle u - u_{n_i}, f_1 u - f_1 u_{n_i} + f_1 u_{n_i} - f_1 \Theta_{n_i} + \frac{1}{\lambda}(\Theta_n - u_{n_i}\rangle \\
&\geq 0 + \langle u - u_{n_i}, f_1 u_{n_i} - f_1 \Theta_{n_i}\rangle + \langle u - u_{n_i}, \frac{1}{\lambda}(\Theta_n - u_{n_i})\rangle. \qquad (6.139)
\end{aligned}$$

But,

$$||\Theta_{n_i} - w_{n_i}|| = \gamma_{n_i} ||A^*(Aw_{n_i} - Bz_{n_i})|| \to 0,$$

and

$$||u_{n_i} - \Theta_{n_i}|| \leq ||u_{n_i} - w_{n_i}|| + ||w_{n_i} - \Theta_{n_i}|| \to 0, n \to \infty.$$

Thus, it follows that

$$\lim_{n\to\infty} ||f_1 u_{n_i} - f_1 \Theta_{n_i}|| = 0.$$

Also, since $u_{n_i} \rightharpoonup \bar{x}$, we have

$$\lim_{n\to\infty} \langle u - u_{n_i}, w\rangle = \langle u - \bar{x}, w\rangle.$$

Thus from (6.139), we have

$$\langle u - \bar{x}, w\rangle \geq 0.$$

Since $T_1 + f_1$ is maximal monotone, we have $0 \in (T_1 + f_1)\bar{x}$ which implies that $\bar{x} \in I(f_1, T_1)$. By a similar argument, we also have $\bar{y} \in I(f_2, T_2)$.
Thus $(\bar{x}, \bar{y}) \in \Upsilon$.

Next, we show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

$$\begin{aligned}
||x_{n+1} - \bar{x}||^2 + ||y_{n+1} - \bar{y}||^2 &\leq ||w_n - \bar{x}||^2 + ||z_n - \bar{y}||^2 \\
&= ||(1-\alpha_n)x_n - \bar{x}||^2 + ||(1-\alpha_n)y_n - \bar{y}||^2 \\
&= (1-\alpha_n)^2||x_n - \bar{x}||^2 + \alpha_n^2||\bar{x}||^2 - 2\alpha_n(1-\alpha_n)\langle x_n - \bar{x}, \bar{x}\rangle \\
&\quad + (1-\alpha_n)^2||y_n - \bar{y}||^2 + \alpha_n^2||\bar{y}||^2 - 2\alpha_n(1-\alpha_n)\langle y_n - \bar{y}, \bar{y}\rangle \\
&\leq (1-\alpha_n)[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2] \\
&\quad + \alpha_n[\alpha_n(||\bar{x}||^2 + ||\bar{y}||^2) \\
&\quad - 2(1-\alpha_n)(\langle x_n - \bar{x}, \bar{x}\rangle + \langle y_n - \bar{y}, \bar{y}\rangle)]. \qquad (6.140)
\end{aligned}$$

Applying Lemma 2.3.12 to (6.140), we conclude $\{(x_n, y_n)\} \to (\bar{x}, \bar{y})$.

Case 2. Assume $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is not monotonically decreasing. Set $\Gamma_n = ||x_n - x^*||^2 + ||y_n - y^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Obviously, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.$$

Let $P_{\tau(n)} = ||A^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 + ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2$ and $\tau(n) \in \Omega$, then it follows from (6.116) that

$$\begin{aligned}
\gamma_{\tau(n)}[2||Aw_{\tau(n)} - Bz_{\tau(n)}||^2 - \gamma_{\tau(n)}P_{\tau(n)}] &\leq [||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2] \\
&\quad - [||x_{\tau(n)+1} - x^*||^2 + ||y_{\tau(n)+1} - y^*||^2] \\
&\quad + \alpha_{\tau(n)}[||x^*||^2 + ||y^*||^2], \qquad (6.141)
\end{aligned}$$

which implies

$$\begin{aligned}
\epsilon^2 P_{\tau(n)} &\leq [||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2] - [||x_{\tau(n)+1} - x^*||^2 + ||y_{\tau(n)+1} - y^*||^2] \\
&\quad + \alpha_{\tau(n)}[||x^*||^2 + ||y^*||^2] \\
&\leq \alpha_{\tau(n)}[||x^*||^2 + ||y^*||^2]. \qquad (6.142)
\end{aligned}$$

From (6.142), we have

$$\lim_{n\to\infty} P_{\tau(n)} = \lim_{n\to\infty} ||A^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 + ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 = 0.$$

Observe that $Aw_{\tau(n)} - Bz_{\tau(n)} = 0$, if $\tau(n) \notin \Omega$.
Hence,

$$\lim_{n\to\infty} ||A^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 = 0, \tag{6.143}$$

and

$$\lim_{n\to\infty} ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 = 0. \tag{6.144}$$

By the same argument as in case 1, we conclude that there exists a subsequence of $\{(x_{\tau(n)}, y_{\tau(n)})\}$, still denoted as $\{(x_{\tau(n)}, y_{\tau(n)})\}$ which converges weakly to $(\bar{x}, \bar{y}) \in \Upsilon$.
Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq [||x_{\tau(n)+1} - x^*||^2 + ||y_{\tau(n)+1} - y^*||^2] - [||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2] \\ &\leq (1-\alpha_{\tau(n)})[||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2] - [||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2] \\ &\quad +\alpha_{\tau(n)}[\alpha_{\tau(n)}[||\bar{x}||^2 + ||\bar{y}||^2 - 2(1-\alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, \bar{x}\rangle + \langle y_{\tau(n)} - \bar{y}, \bar{y}\rangle)], \end{aligned}$$

which implies

$$\begin{aligned} ||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2 &\leq \alpha_{\tau(n)}[||\bar{x}||^2 + ||\bar{y}||^2 \\ &\quad -2(1-\alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, \bar{x}\rangle + \langle y_{\tau(n)} - \bar{y}, \bar{y}\rangle) \to 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty}(||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2) = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, it is easily observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus, $\lim \Gamma_n = 0$. That is $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$. □

Corollary 6.5.2. *Let H_1, H_2 and H_3 be three real Hilbert spaces, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ be multivalued maximal monotone mappings. Let $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ be κ_1-strictly pseudocontractive and κ_2-strictly pseudocontractive mappings respectively. Suppose $\Upsilon \neq \emptyset$ and*

$$\gamma_n \in \Big(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\Big), n \in \Omega$$

otherwise, $\gamma_n = \gamma$ *(*γ *being any nonegative value), where the set of indexes* $\Omega = \{n : Aw_n - Bz_n \neq 0\}$. *Let* $\{(x_n, y_n)\}$ *be the sequence generated for* $x_0 \in H_1$ *and* $y_0 \in H_2$ *by*

$$\begin{cases} w_n = (1-\alpha_n)x_n, \\ z_n = (1-\alpha_n)y_n, \\ u_n = J_\lambda^{T_1}(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = J_\lambda^{T_2}(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S_1 u_n, \quad \forall n \geq 0, \\ y_{n+1} = (1-\delta_n)v_n + \delta_n S_2 v_n, \quad \forall n \geq 0, \end{cases} \tag{6.145}$$

where A^*, B^* *are the adjoints of* A *and* B *respectively. Suppose* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\delta_n\}$ *are real sequences in* $(0,1)$ *satisfying the following conditions*
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < a \leq \beta_n \leq b < 1 - \kappa_1$,
(iii) $0 < c \leq \delta_n \leq d < 1 - \kappa_2$,
then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Upsilon$.

Corollary 6.5.3. *Let* H_1, H_2 *and* H_3 *be three real Hilbert spaces, and* $A : H_1 \to H_3$, $B : H_2 \to H_3$ *be two bounded linear operators. Let* $f_1 : H_1 \to H_1$ *be* μ*-inverse strongly monotone mapping and* $f_2 : H_2 \to H_2$ *be* ν*-inverse strongly monotone mapping. Let* $T_1 : H_1 \to 2^{H_1}$ *and* $T_2 : H_2 \to 2^{H_2}$ *be multivalued maximal monotone mappings. Let* $S_1 : H_1 \to H_1$ *and* $S_2 : H_2 \to H_2$ *be nonexpansive mappings. Suppose* $\Upsilon \neq \emptyset$ *and*

$$\gamma_n \in \Big(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\Big), n \in \Omega$$

otherwise, $\gamma_n = \gamma$ *(*γ *being any nonegative value), where the set of indexes* $\Omega = \{n : Aw_n - Bz_n \neq 0\}$. *Let* $\{(x_n, y_n)\}$ *be the sequence generated for* $x_0 \in H_1$ *and* $y_0 \in H_2$ *by*

$$\begin{cases} w_n = (1-\alpha_n)x_n, \\ z_n = (1-\alpha_n)y_n, \\ u_n = J_\lambda^{T_1}(I - \lambda f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = J_\lambda^{T_2}(I - \lambda f_2)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S_1 u_n, \quad \forall n \geq 0, \\ y_{n+1} = (1-\delta_n)v_n + \delta_n S_2 v_n, \quad \forall n \geq 0, \end{cases} \tag{6.146}$$

where $0 < \lambda < 2\mu, 2\nu$ *and* A^*, B^* *are the adjoints of* A *and* B *respectively. Suppose* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\delta_n\}$ *are real sequences in* $(0,1)$ *satisfying the following conditions*
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < a \leq \beta_n \leq b < 1$,
(iii) $0 < c \leq \delta_n \leq d < 1$,
then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Upsilon$.

6.5.1 Application and numerical example

Split equality convex minimization problem Let $F : H \to \mathbb{R}$ be a convex and differentiable function and $G : H \to \mathbb{R}$ is a convex function. It has been established (see

[12]) that if ∇F is $\frac{1}{L}$-Lipschitz continuous, then it is L-inverse strongly monotone. It is also known (see [196]) that ∂G is maximal monotone. Moreover,

$$F(x^*) + G(x^*) = \min_{x \in H} F(x) + G(x) \Leftrightarrow 0 \in \nabla F(x^*) + \partial G(x^*).$$

Consider the following fixed point and split equality convex minimization problem: find $x^* \in F(S_1), y^* \in F(S_2)$ and

$$F_1(x^*) + G_1(x^*) = \min_{x \in H_1} F_1(x) + G_1(x), \tag{6.147}$$

$$F_2(y^*) + G_2(y^*) = \min_{y \in H_2} F_2(y) + G_2(y) \tag{6.148}$$

such that $Ax^* = By^*$, where H_1, H_2 and H_3 are real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators, $F_i : H_i \to \mathbb{R}$ (i=1,2) are convex and differentiable functions and $G_i : H_i \to \mathbb{R}$ (i=1,2) convex function. $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ are strictly pseudocontractive mappings. Let the set of solutions of problem (6.147)-(6.148) be $\mho$, then putting $T_i = \partial G_i$ and $f_i = \nabla F_i \quad (i = 1, 2)$, we obtain the following result.

Theorem 6.5.4. *Let H_1, H_2 and H_3 be three real Hilbert spaces, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $F_1 : H_1 \to \mathbb{R}$ be a convex and differentiable function with $\dfrac{1}{\mu}$-Lipschitz continuous gradient ∇F_1 and let $F_2 : H_2 \to \mathbb{R}$ be a convex and differentiable function with $\dfrac{1}{\nu}$-Lipschitz continuous gradient ∇F_2. Let $G_1 : H_1 \to \mathbb{R}$ and $G_2 : H_2 \to \mathbb{R}$ be convex and lower semi-continuous functions. Let $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ be κ_1-strictly pseudocontractive and κ_2-strictly pseudocontractive mappings respectively. Suppose $\Upsilon \neq \emptyset$ and*

$$\gamma_n \in \Big(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\Big), n \in \Omega$$

otherwise, $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$. Let $\{(x_n, y_n)\}$ be the sequence generated for $x_0 \in H_1$ and $y_0 \in H_2$ by

$$\begin{cases} w_n = (1 - \alpha_n)x_n, \\ z_n = (1 - \alpha_n)y_n, \\ u_n = J_\lambda^{\partial G_1}(I - \lambda \nabla F_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = J_\lambda^{\partial G_2}(I - \lambda \nabla F_2)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n S_1 u_n, \quad \forall n \geq 0, \\ y_{n+1} = (1 - \delta_n)v_n + \delta_n S_2 v_n, \quad \forall n \geq 0, \end{cases} \tag{6.149}$$

where $0 < \lambda < 2\mu, 2\nu$ and A^*, B^* are the adjoints of A and B respectively. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < a \leq \beta_n \leq b < 1 - \kappa_1$,
(iii) $0 < c \leq \delta_n \leq d < 1 - \kappa_2$,
then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \mho$.

Numerical example

We give a numerical example in $(\mathbb{R}^2, ||.||_2)$ of the problem considered in Theorem 6.5.1 in this subsection. Now take $H_1 = H_2 = H_3 = \mathbb{R}^2$. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be given as

$$A(\acute{x}) := \begin{pmatrix} 3 & -5 \\ 1 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and $B : \mathbb{R}^2 \to \mathbb{R}^2$ be given as

$$B(\acute{x}) := \begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Define $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ by:

$$T_1(\acute{x}) = T_1(x_1, x_2) = (x_2, -x_1),$$

Then

$$J_\lambda^{T_1}(\acute{x}) = J_\lambda^{T_1}(x_1, x_2) = \Big(\frac{x_1 - \lambda x_2}{1 + \lambda^2}, \frac{x_2 + \lambda x_1}{1 + \lambda^2}\Big).$$

Again define $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by:

$$T_2(\acute{x}) = T_2(x_1, x_2) = (x_1 + x_2, x_2 - x_1).$$

Then

$$J_\lambda^{T_2}(\acute{x}) = J_\lambda^{T_2}(x_1, x_2) = \Big(\frac{(1 + \lambda)x_1 - \lambda x_2}{1 + 2\lambda + 2\lambda^2}, \frac{\lambda x_1 + (1 + \lambda)x_2}{1 + 2\lambda + 2\lambda^2}\Big).$$

Let $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping $f_1(\acute{x}) = f_1(x_1, x_2) = (3x_1, 3x_2)$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ the mapping $f_2(\acute{x}) = f_2(x_1, x_2) := (4x_1 - 1, 4x_2 - 1)$. Also, we define $S_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $S_2 : \mathbb{R}^2 \to \mathbb{R}^2$ as $S_1(\acute{x}) = -2\acute{x}$ and $S_2(\acute{x}) = \acute{x}$, where $\acute{x} = (x_1, x_2) \in \mathbb{R}^2$. Let $\alpha_n = \dfrac{1}{2(n+1)}$, $\beta_n = \dfrac{1 - \frac{1}{n}}{6(1 + \frac{1}{n})}$ and $\delta_n = \dfrac{2}{3(1 + \frac{1}{n})}$. Then our iterative scheme (6.105) becomes: for $x_0 \in H_1$ and $y_0 \in H_2$,

$$\begin{cases} w_n = (1 - \dfrac{1}{2(n+1)})x_n, \\ z_n = (1 - \dfrac{1}{2(n+1)})y_n, \\ u_n = J_\lambda^{T_1}(I - \lambda f_1)(w_n - \gamma_n A^T(Aw_n - Bz_n)), \\ v_n = J_\lambda^{T_2}(I - \lambda f_2)(z_n + \gamma_n B^T(Aw_n - Bz_n)), \\ x_{n+1} = (1 - \dfrac{1 - \frac{1}{n}}{6(1 + \frac{1}{n})})u_n + \dfrac{1 - \frac{1}{n}}{6(1 + \frac{1}{n})}S_1 u_n, \quad \forall n \geq 0, \\ y_{n+1} = (1 - \dfrac{2}{3(1 + \frac{1}{n})})v_n + \dfrac{2}{3(1 + \frac{1}{n})}S_2 v_n, \quad \forall n \geq 0, \end{cases} \tag{6.150}$$

A^T and B^T stand for the transpose of A and B respectively and

$$\gamma_n \in \Big(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\Big), n \in \Omega$$

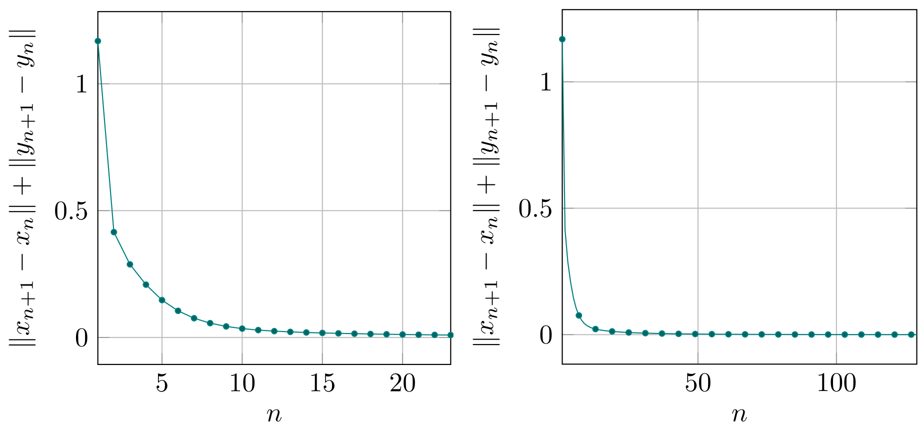


Figure 6.1: Errors: Case I(a), $\varepsilon = 10^{-2}$ (left) and $\varepsilon = 10^{-4}$ (right).

otherwise, $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.

Case I.

(a) Take $x_0 = (1, \frac{1}{2})^T$, $y_0 = (-1, 1)^T$ and $\lambda = 0.002$.

(b)Take $x_0 = (1, \frac{1}{2})^T$, $y_0 = (-1, 1)^T$ and $\lambda = 0.0000001$.

Case II.

(a) Take $x_0 = (0.02, 0.03)^T$, $y_0 = (0.3, 0.05)^T$ and $\lambda = 0.001$.

(b) Take $x_0 = (0.1, 0.2)^T$, $y_0 = (1, 0.2)^T$ and $\lambda = 0.001$.

The Mathlab version used is R2014a and the execution times and number of iterations are as follows:

1. (case 1a, $\varepsilon = 10^{-2}$) 0.063765 seconds, Number of iterations: 23.
2. (case 1a, $\varepsilon = 10^{-4}$) 0.079304 seconds, Number of iterations: 129.
3. (case 1b, $\varepsilon = 10^{-2}$) 0.054949 seconds, Number of iterations: 23.
4. (case 1b, $\varepsilon = 10^{-4}$) 0.081124 seconds, Number of iterations: 166.
5. (case 2a, $\varepsilon = 10^{-2}$) 0.051959 seconds, Number of iterations: 7.
6. (case 2a, $\varepsilon = 10^{-4}$) 0.067724 seconds, Number of iterations: 93.
7. (case 2b, $\varepsilon = 10^{-2}$) 0.058681 seconds, Number of iterations: 13.
8. (case 2b, $\varepsilon = 10^{-4}$) 0.074799 seconds, Number of iterations: 130.

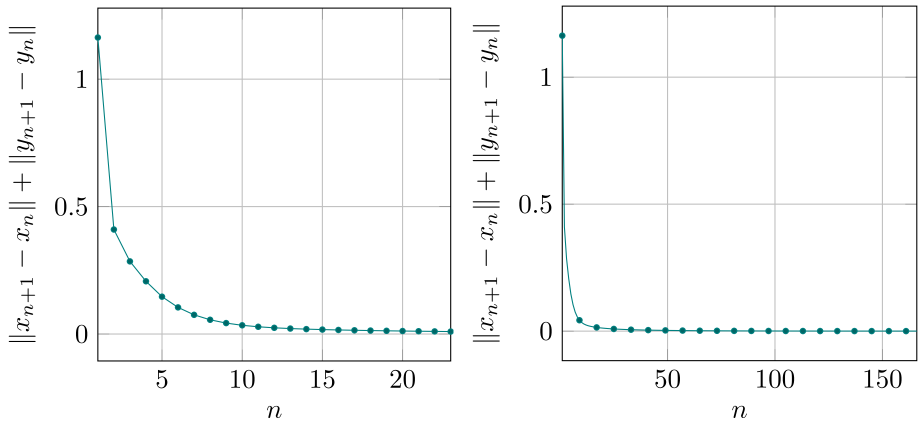


Figure 6.2: Errors: Case I(b), $\varepsilon = 10^{-2}$ (left) and $\varepsilon = 10^{-4}$ (right).

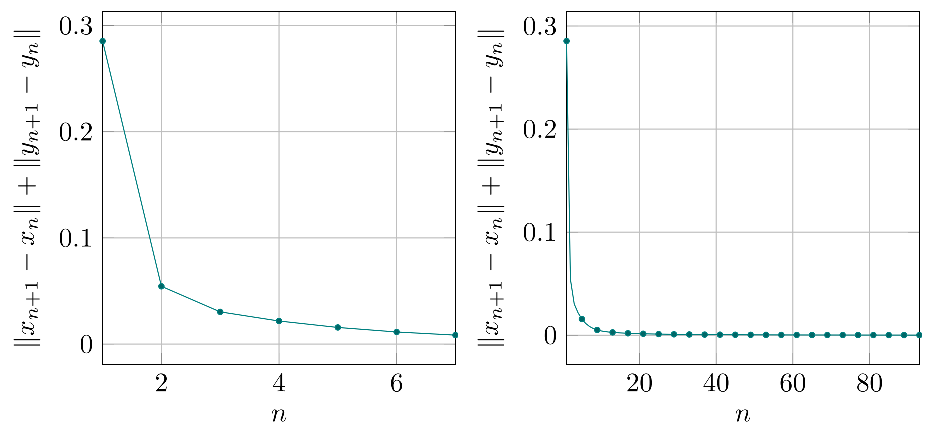


Figure 6.3: Errors: Case II(a), $\varepsilon = 10^{-2}$ (left) and $\varepsilon = 10^{-4}$ (right).

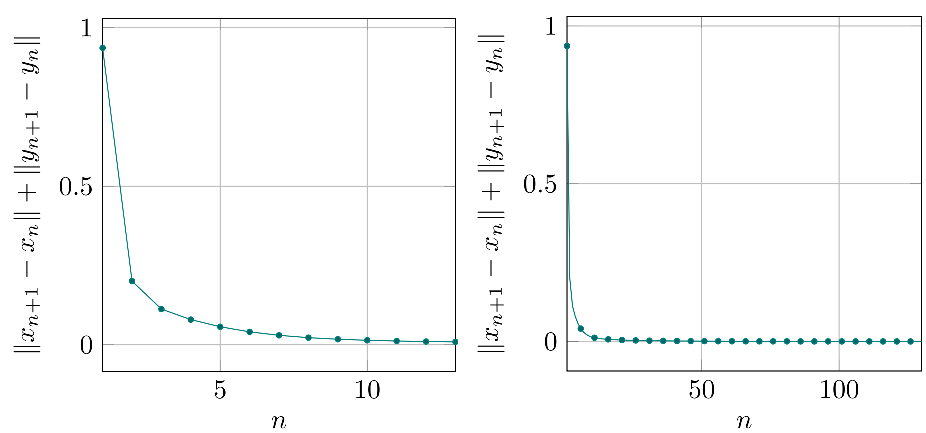


Figure 6.4: Errors: Case II(b), $\varepsilon = 10^{-2}$ (left) and $\varepsilon = 10^{-4}$ (right).

Chapter 7

Variational Inequality and Minimisation Problems

In this chapter, we introduce an iterative algorithm to approximate the solution of split hierarchical variational inequality problems and further introduce an iterative scheme that converges strongly to a minimizer of the sum of two convex functions in real Hilbert spaces.

7.1 Strong convergence result for solving split hierarchical variational inequality problem for demicontractive mappings.

Recently, Ansari et al. [10] introduced the Split Hierarchical Variational Inequality Problem (SHVIP) which is given as follows: Let H_1 and H_2 be real Hilbert spaces, $f, T : H_1 \to H_1$ be operators such that $F(T) \neq \emptyset$ and $h, S : H_2 \to H_2$ with $F(S) \neq \emptyset$. Let $A : H_1 \to H_2$ be an operator with $R(A) \cap F(S) \neq \emptyset$, where $R(A)$ denotes the range of A. The SHVIP is to find $x^* \in F(T)$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{7.1}$$

and such that $Ax^* \in F(S)$ satisfies

$$\langle h(Ax^*), y - Ax^* \rangle \geq 0, \quad \forall y \in F(S). \tag{7.2}$$

We shall denote by Ω_{SH} the solution set of SHVIP (7.1)-(7.2).

Another problem which is closely related to SHVIP is the following Split Hierarchical Minty Variational Inequality Problem (SHMVIP): Find $x^* \in F(T)$ such that

$$\langle f(x), x - x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{7.3}$$

and such that $Ax^* \in F(S)$ satisfies

$$\langle h(y), y - Ax^* \rangle \geq 0, \quad \forall y \in F(S). \tag{7.4}$$

It can be seen by the Minty lemma [159], Lemma 1 that if $F(T)$ and $F(S)$ are nonempty, closed and convex, f and h are monotone and continuous, then the SHVIP (7.1)-(7.2) and SHMVIP (7.3)-(7.4) are equivalent. For further details on hierarchical variational inequality problems, see [9].

Ansari *et al.* [10] showed that many problems, namely split convex minimization problem, split variational inequality problem over the solution set of monotone variational inclusion problem, split variational inequality problem over the solution set of equilibrium problem are particular cases of SHVIP.

Ansari *et al.* [10] proposed an iterative scheme for solving SHVIP and got a weak convergence result. In 2015, Ansari *et al.* [11] give a common solution method for finding a fixed point of a nonexpansive operator and a solution of split hierarchical variational inequality problems, stated and proved a weak convergence result.

Motivated by the recent results in this direction, we study SHVIP in Hilbert spaces. We further propose an iterative algorithm for approximating solution of SHVIP and stated and proved a strong convergence result.

Theorem 7.1.1. *Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be κ-demicontractive and λ-demicontractive mappings respectively, with $C = F(S) \neq \emptyset$ and $Q = F(T) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Assume that $\Omega_{SH} \neq \emptyset$ and suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\gamma > 0, \tau > 0$ real numbers and the following conditions are satisfied:*
(1) $\beta_n \in (a, (1-\kappa))$ for some $a > 0$,
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(3) $\gamma \in \left(0, \dfrac{1-\lambda}{\|A\|^2}\right)$,
(4) $S - I$ and $T - I$ are demiclosed at the origin.
Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0, u \in H_1, \\ w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = w_n + \gamma A^*(T-I)Aw_n, \\ u_n = P_C(I-\rho f_1)(y_n + \tau A^*(P_Q(I-\rho f_2)-I)Ay_n), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S u_n, \quad n \geq 0, \end{cases} \tag{7.5}$$

converges strongly to $q = P_{\Omega_{SH}} u$, where $0 < \rho < \mu, \nu$ and $\tau \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A.

Proof. Let $p \in \Omega_{SH}$, then by Lemma 2.3.8, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(u_n - p) + \beta_n(Su_n - p)\|^2 \\
&= (1-\beta_n)^2\|u_n - p\|^2 + \beta_n^2\|Su_n - p\|^2 \\
&\quad +2(1-\beta_n)\beta_n\langle Su_n - p, u_n - p\rangle \\
&= (1-\beta_n)^2\|u_n - p\|^2 + \beta_n^2\|Su_n - p\|^2 \\
&\quad +2(1-\beta_n)\beta_n\langle Su_n - u_n + u_n - p, u_n - p\rangle \\
&= (1-\beta_n)^2\|u_n - p\|^2 + \beta_n^2\|Su_n - p\|^2 \\
&\quad +2(1-\beta_n)\beta_n[\langle Su_n - u_n, u_n - p\rangle + \langle u_n - p, u_n - p\rangle] \\
&\leq (1-\beta_n)^2\|u_n - p\|^2 + \beta_n^2[\|u_n - p\|^2 + \kappa\|Su_n - u_n\|^2] \\
&\quad +(\kappa - 1)(1-\beta_n)\beta_n\|Su_n - u_n\|^2 + 2(1-\beta_n)\beta_n\|u_n - p\|^2 \\
&= [(1-\beta_n)^2 + \beta_n^2 + 2(1-\beta_n)\beta_n]\|u_n - p\|^2 \\
&\quad +[\kappa\beta_n^2 + (\kappa-1)(\beta_n - \beta_n^2)]\|Su_n - u_n\|^2 \\
&= \|u_n - p\|^2 + \beta_n(\beta_n + \kappa - 1)\|Su_n - u_n\|^2 \\
&\leq \|u_n - p\|^2. \qquad (7.6)
\end{aligned}$$

But,

$$\begin{aligned}
||u_n - p||^2 &= ||P_C(I - \rho f_1)(y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n) - p||^2 \\
&\leq ||y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n - p||^2 \\
&= ||y_n - p||^2 + \tau^2||A^*(P_Q(I - \rho f_2) - I)Ay_n||^2 \\
&\quad +2\tau\langle y_n - p, A^*(P_Q(I - \rho f_2) - I)Ay_n\rangle, \qquad (7.7)
\end{aligned}$$

and

$$\begin{aligned}
\tau^2||A^*(P_Q(I - \rho f_2) - I)Ay_n||^2 &= \tau^2\langle(P_Q(I - \rho f_2) - I)Ay_n, AA^*(P_Q(I - \rho f_2) - I)Ay_n\rangle \\
&\leq L\tau^2\langle(P_Q(I - \rho f_2) - I)Ay_n, (P_Q(I - \rho f_2) - I)Ay_n\rangle \\
&= L\tau^2||A^*(P_Q(I - \rho f_2) - I)Ay_n||^2. \qquad (7.8)
\end{aligned}$$

Also, we get

$$\begin{aligned}
&2\tau\langle y_n - p, A^*(P_Q(I - \rho f_2) - I)Ay_n\rangle \\
&= 2\tau\langle A(y_n - p) + (P_Q(I - \rho f_2) - I)Ay_n - (P_Q(I - \rho f_2) - I)Ay_n, (P_Q(I - \rho f_2) - I)Ay_n\rangle \\
&= 2\tau[\langle P_Q(I - \rho f_2)Ay_n - Ap, (P_Q(I - \rho f_2) - I)Ay_n\rangle - ||(P_Q(I - \rho f_2) - I)Ay_n||^2] \\
&\leq 2\tau[\frac{1}{2}||(P_Q(I - \rho f_2) - I)Ay_n||^2 - ||(P_Q(I - \rho f_2) - I)Ay_n||^2] \\
&= -\tau||(P_Q(I - \rho f_2) - I)Ay_n||. \qquad (7.9)
\end{aligned}$$

From (7.7), (7.8) and (7.9), we have

$$\begin{aligned}
||u_n - p||^2 &\leq ||y_n - p||^2 + L\tau^2||(P_Q(I - \rho f_2) - I)Ay_n||^2 - \tau||(P_Q(I - \rho f_2) - I)Ay_n||^2 \\
&= ||y_n - p||^2 + \tau(L\tau - 1)||(P_Q(I - \rho f_2) - I)Ay_n||^2 \\
&\leq ||y_n - p||^2. \qquad (7.10)
\end{aligned}$$

Furthermore,

$$\begin{aligned}\|y_n - p\|^2 &= \|w_n - p + \gamma A^*(T-I)Aw_n\|^2 \\ &= \|w_n - p\|^2 + \gamma^2\|A^*(T-I)Aw_n\|^2 \\ &\quad +2\gamma\langle w_n - p, A^*(T-I)Aw_n\rangle. \end{aligned} \tag{7.11}$$

Now

$$\begin{aligned}\gamma^2\|A^*(T-I)Aw_n\|^2 &= \gamma^2\langle A^*(T-I)Aw_n, A^*(T-I)Aw_n\rangle \\ &= \gamma^2\langle AA^*(T-I)Aw_n, (T-I)Aw_n\rangle \\ &\leq \gamma^2\|A\|^2\|TAw_n - Aw_n\|^2, \end{aligned} \tag{7.12}$$

and

$$\begin{aligned}2\gamma\langle w_n - p, A^*(T-I)Aw\rangle &= 2\gamma\langle A(w_n - p), (T-I)Aw_n\rangle \\ &= 2\gamma[\langle Aw_n - Ap, TAw_n - Ap\rangle + \langle Aw_n - Ap, Ap - Aw_n\rangle \\ &= 2\gamma[\langle Aw_n - Ap, TAw_n - Ap\rangle - \|Ap - Aw_n\|^2] \\ &\leq \gamma[(\lambda-1)\|TAw_n - Aw_n\|^2 - 2\|Ap - Aw_n\|^2] \\ &\leq \gamma(\lambda-1)\|TAw_n - Aw_n\|^2. \end{aligned} \tag{7.13}$$

Substituting (7.12) and (7.13) in (7.11), we obtain

$$\begin{aligned}\|y_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma\|A\|^2\|TAw_n - Aw_n\|^2 + \gamma(\lambda-1)\|TAw_n - Aw_n\|^2 \\ &\leq ||w_n - p||^2. \end{aligned} \tag{7.14}$$

Therefore, from (7.6),(7.7),(7.10) and (7.14), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \|u_n - p\| \\ &\leq ||y_n - p|| \\ &\leq ||w_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n||p - u|| \\ &\leq \max\{||x_n - p||, ||p - u||\}. \end{aligned}$$

Thus, by induction we have

$$||x_n - p|| \leq \max\{||x_0 - p||, ||p - u||\}.$$

Hence $\{||x_n - p||\}$ is bounded and consequently $\{x_n\}$, $\{w_n\}$, $\{u_n\}$ and $\{y_n\}$ are bounded. We divide into two cases to establish strong convergence.
Case 1. Suppose $\{||x_n - p||\}$ is monotonically decreasing, then obviously

$$||x_{n+1} - p|| - ||x_n - p|| \to 0. \tag{7.15}$$

From (7.6), we have

$$\begin{aligned}||x_{n+1} - p||^2 &\leq ||u_n - p||^2 + \beta_n(\beta_n + \kappa - 1)\|Su_n - u_n\|^2 \\ &\leq ||y_n - p||^2 + \beta_n(\beta_n + \kappa - 1)\|Su_n - u_n\|^2 \\ &\leq ||w_n - p||^2 + \gamma^2\|A\|^2\|TAw_n - Aw_n\|^2 + \gamma(\lambda-1)\|TAw_n - Aw_n\|^2 \\ &\quad +\beta_n(\beta_n + \kappa - 1)\|Su_n - u_n\|^2 \\ &= ||x_n - p||^2 + \alpha_n^2\|x_n - u\|^2 - 2\alpha_n\langle x_n - p, x_n - u\rangle + \gamma^2\|A\|^2\|TAw_n - Aw_n\|^2 \\ &\quad +\gamma(\lambda-1)\|TAw_n - Aw_n\|^2 + \beta_n(\beta_n + \kappa - 1)\|Su_n - u_n\|^2. \end{aligned} \tag{7.16}$$

Therefore,

$$\begin{aligned}\alpha_n[\alpha_n\|x_n-u\|^2-2\langle x_n-p,x_n-u\rangle] &\geq \|x_{n+1}-p\|^2-\|x_n-p\|^2\\ &-[\gamma^2\|A\|^2+\gamma(\lambda-1)]\|TAw_n-Aw_n\|^2\\ &-\beta_n(\beta_n+\kappa-1)\|Su_n-u_n\|^2. \qquad (7.17)\end{aligned}$$

Since $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{w_n\}$ are bounded, there is an $M>0$ such that

$$\alpha_n\|x_n-u\|^2-2\langle x_n-p,x_n-u\rangle \leq M.$$

$$\begin{aligned}\alpha_n M &\geq \|x_{n+1}-p\|^2-\|x_n-p\|^2-[\gamma^2\|A\|^2+\gamma(\lambda-1)]\|TAw_n-Aw_n\|^2\\ &-\beta_n(\beta_n+\kappa-1)\|Su_n-u_n\|^2. \qquad (7.18)\end{aligned}$$

From (7.15), (7.18), the condition $\alpha_n\to 0$ and since $\gamma^2\|A\|^2+\gamma(\lambda-1)\leq 0$, we have

$$\|Su_n-u_n\|\to 0, \qquad (7.19)$$

and

$$\|TAw_n-Aw_n\|\to 0. \qquad (7.20)$$

From (7.19),

$$\|u_n-x_{n+1}\|=\beta_n\|Su_n-u_n\|\to 0.$$

Also from (7.10), we have

$$\begin{aligned}||x_{n+1}-p||^2 &\leq ||u_n-p||^2\\ &= ||P_C(I-\rho f_1)(y_n+\tau A^*(P_Q(I-\rho f_2)-I)Ay_n)-p||^2\\ &\leq ||y_n-p||^2+\tau(L\tau-1)||(P_Q(I-\rho f_2)-I)Ay_n||^2\\ &\leq ||w_n-p||^2+\tau(L\tau-1)||(P_Q(I-\rho f_2)-I)Ay_n||^2\\ &\leq (1-\alpha_n)^2||x_n-p||^2+\alpha_n^2||p-u||^2-2\alpha_n(1-\alpha_n)\langle x_n-p,p-u\rangle\\ &+\tau(L\tau-1)||(P_Q(I-\rho f_2)-I)Ay_n||^2. \qquad (7.21)\end{aligned}$$

Thus,

$$\begin{aligned}\tau(1-L\tau)||(P_Q(I-\rho f_2)-I)Ay_n||^2 &\leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||p-u||^2\\ &-2\alpha_n(1-\alpha_n)\langle x_n-p,p-u\rangle\to 0. \qquad (7.22)\end{aligned}$$

Hence,

$$||(P_Q(I-\rho f_2)-I)Ay_n||\to 0, n\to\infty. \qquad (7.23)$$

Again from (7.5) and (7.20)

$$\begin{aligned}\|y_n-w_n\| &= \|\gamma A^*(T-I)Aw_n\|\\ &\leq \gamma\|A\|\|TAw_n-Aw_n\|\to 0.\end{aligned}$$

From (7.5)

$$\|x_n - w_n\| = \alpha_n \|x_n - u\| \to 0.$$

Thus

$$\|x_n - y_n\| \leq \|x_n - w_n\| + \|w_n - y_n\| \to 0.$$

Now, since $P_C(I - \rho f_1)$ is firmly nonexpansive, we have

$$\begin{aligned}
||u_n - p||^2 &= ||P_C(I - \rho f_1)(y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n) - p||^2 \\
&\leq \langle u_n - p, y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n - p\rangle \\
&= \frac{1}{2}[||u_n - p||^2 + ||y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n - p||^2 \\
&\quad -||u_n - p - (y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n - p)||^2] \\
&\leq \frac{1}{2}[||u_n - p||^2 + ||y_n - p||^2 + \tau(L\tau - I)||P_Q(I - \rho f_2) - I)Ay_n||^2 \\
&\quad -||u_n - y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n||^2] \\
&\leq \frac{1}{2}[||u_n - p||^2 + ||y_n - p||^2 - (||u_n - y_n|| + \tau^2||A^*(P_Q(I - \rho f_2) - I)Ay_n||^2 \\
&\quad -2\tau\langle u_n - y_n, A^*(P_Q(I - \rho f_2) - I)Ay_n\rangle)] \\
&\leq \frac{1}{2}[||u_n - p||^2 + ||y_n - p||^2 - ||u_n - y_n||^2 \\
&\quad +2\tau||A(u_n - y_n)||||P_Q(I - \rho f_2) - I)Ay_n||],
\end{aligned} \tag{7.24}$$

that is,

$$\begin{aligned}
||u_n - p||^2 &\leq ||y_n - p||^2 - ||u_n - y_n||^2 \\
&\quad +2\tau||A(u_n - y_n)||||P_Q(I - \rho f_2) - I)Ay_n||.
\end{aligned} \tag{7.25}$$

It then follows from (7.25), that

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq ||y_n - p||^2 - ||u_n - y_n||^2 \\
&\quad +2\tau||A(u_n - y_n)||||P_Q(I - \rho f_2) - I)Ay_n||,
\end{aligned} \tag{7.26}$$

which implies

$$\begin{aligned}
||u_n - y_n||^2 &\leq ||y_n - p||^2 - ||x_{n+1} - p||^2 + 2\tau||A(u_n - y_n)||||P_Q(I - \rho f_2) - I)Ay_n|| \\
&\leq ||w_n - p||^2 - ||x_{n+1} - p||^2 + 2\tau||A(u_n - y_n)||||P_Q(I - \rho f_2) - I)Ay_n|| \\
&\leq ||(1 - \alpha_n)x_n + \alpha_n u - p||^2 - ||x_{n+1} - p||^2 + 2\tau||A(u_n - y_n)||||P_Q(I - \rho f_2) - I)Ay_n|| \\
&\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n^2||p - u||^2 - 2\alpha_n(1 - \alpha_n)\langle x_n - p, p - u\rangle \\
&\quad +2\tau||A(u_n - y_n)||||P_Q(I - \rho f_2) - I)Ay_n|| \to 0, n \to \infty.
\end{aligned} \tag{7.27}$$

Therefore,

$$||u_n - y_n|| \to 0, n \to \infty.$$

Moreover,

$$||u_n - x_n|| \leq ||u_n - y_n|| + ||y_n - x_n|| \to 0, n \to \infty,$$

and

$$||u_n - w_n|| \leq ||u_n - x_n|| + ||x_n - w_n|| \to 0, n \to \infty.$$

Since $I - S$ is demiclosed at 0 and $\{u_n\}$ is bounded, then it follows from (7.19) that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which converges weakly to a fixed point x^* of S and consequently $\{y_{n_i}\}$, $\{w_{n_i}\}$ and $\{x_{n_i}\}$ converge weakly to x^*. Thus, $x^* \in C = F(S)$.
It also follows from (7.20), since A is continuous that $Ax^* \in Q = F(T)$.
We now show that $x^* \in VI(f_1, C)$.
Let

$$Bz = \begin{cases} f_1 z + N_C z, & z \in C \\ \emptyset, & z \notin C. \end{cases} \tag{7.28}$$

Then B is maximal monotone (see [197]). Let $(z, w) \in G(B)$, since $w - f_1 z \in N_C z$ and $u_n \in C$, we have $\langle z - u_n, w - f_1 z\rangle \geq 0$.
Let $v_n = y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n$.
Then

$$||v_n - y_n||^2 = L\tau^2 ||P_Q(I - \rho f_2) - I)Ay_n||^2 \to 0, n \to \infty,$$

and

$$||u_n - v_n|| \leq ||u_n - y_n|| + ||y_n - v_n|| \to 0, n \to \infty.$$

From $u_n = P_C(v_n - \rho f_1 v_n)$, we have

$$\langle z - u_n, u_n - (v_n - \rho f_1 v_n)\rangle \geq 0,$$

that is

$$\langle z - u_n, \frac{u_n - v_n}{\rho} + f_1 v_n\rangle \geq 0.$$

Therefore, we obtain

$$\begin{aligned}
\langle z - u_{n_i}, w\rangle &\geq \langle z - u_{n_i}, f_1 z\rangle \\
&\geq \langle z - u_{n_i}, f_1 z\rangle - \langle z - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\rho} + f_1 v_{n_i}\rangle \\
&= \langle z - u_{n_i}, f_1 z - f_1 v_{n_i} - \frac{u_{n_i} - v_{n_i}}{\rho}\rangle \\
&= \langle z - u_{n_i}, f_1 z - f_1 u_{n_i}\rangle + \langle z - u_{n_i}, f_1 u_{n_i} - f_1 v_{n_i}\rangle - \langle z - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\rho}\rangle \\
&\geq \langle z - u_{n_i}, f_1 u_{n_i} - f_1 v_{n_i}\rangle - \langle z - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\rho}\rangle.
\end{aligned}$$

Noting that f_1 is Lipschitz continuous, we obtain $\langle z - x^*, w\rangle \geq 0$. Since B is maximal monotone, we have $x^* \in B^{-1}(0)$, and so $x^* \in VI(f_1, C)$.
Furthermore, since $||u_n - y_n|| \to 0$, and A is continuous, we conclude that Ay_{n_i} converges weakly to Ax^*. Thus, by (7.23) and the fact that $P_Q(I - \rho f_2)$ is nonexpansive, we have from Lemma 2.3.19 that $x^* = P_Q(I - \rho f_2)x^*$.

Thus $x^* \in VI(f_2, Q)$.
We next prove that $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ converge strongly to q.
From(7.15), we get

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|u_n - q\|^2 \\ &\leq \|y_n - q\|^2 \\ &\leq \|w_n - q\|^2 \\ &= \|(1-\alpha_n)(x_n - q) - \alpha_n(p - u)\|^2 \\ &= (1-\alpha_n)^2\|x_n - q\|^2 + \alpha_n^2\|q - u\|^2 \\ &\quad -2\alpha_n(1-\alpha_n)\langle x_n - q, q - u\rangle \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n[\alpha_n\|q - u\|^2 \\ &\quad -2(1-\alpha_n)\langle x_n - q, q - u\rangle]. \end{aligned} \tag{7.29}$$

Clearly, Choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - q, u - q\rangle = \lim_{i\to\infty}\langle x_{n_i} - q, u - q\rangle,$$

Since $x_{n_i} \rightharpoonup q$ then it follows from Proposition2.1.3

$$\begin{aligned} \limsup_{n\to\infty}\langle x_n - q, u - q\rangle &= \lim_{i\to\infty}\langle x_{n_j} - q, u - q\rangle \\ &= \langle x^* - q, u - q\rangle \leq 0. \end{aligned} \tag{7.30}$$

Thus,

$$\alpha_n\|p\|^2 - 2(1-\alpha_n)\langle x_n - q, q - u\rangle \to 0 \quad n \to \infty,$$

and applying Lemma 2.3.12 to (7.29), we obtain that $\{x_n\}$ converges strongly to q.

Case 2. Assume that $\{\|x_n - p\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n - p\|^2$ and let $\eta : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough)

$$\eta(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, η is a nondecreasing sequence such that $\eta(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\eta(n)} \leq \Gamma_{\eta(n)+1} \quad \text{for} \quad n \geq n_0.$$

It follows from (7.16) that

$$\begin{aligned} 0 &\leq ||x_{\eta(n)+1} - p||^2 - ||x_{\eta(n)} - p||^2 \\ &\leq ||x_{\eta(n)} - p|| - ||x_{\eta(n)} - p|| + \alpha_{\eta(n)}^2||x_{\eta(n)}||^2 - 2\alpha_{\eta(n)}\langle x_{\eta(n)} - p, x_{\eta(n)}\rangle \\ &\quad +\gamma^2||A||^2 + \gamma(\lambda-1)||TAw_{\eta(n)} - Aw_{\eta(n)}||^2 + \beta_{\eta(n)}(\beta_{\eta(n)} + \kappa - 1)||Su_{\eta(n)} - u_{\eta(n)}||^2. \end{aligned}$$

$$\gamma[(\lambda-1) - \gamma\|A\|^2]\|TAw_{\eta(n)} - Aw_{\eta(n)}\|^2 - \beta_{\eta(n)}(\beta_{\eta(n)}^2 + \kappa - 1)\|Su_{\eta(n)} - u_{\eta(n)}\|^2 \leq \alpha_{\eta(n)}M \to 0.$$

Hence,

$$\|u_{\eta(n)} - Su_{\eta(n)}\| \to 0,$$

and

$$\|TAw_{\eta(n)} - Aw_{\eta(n)}\| \to 0.$$

By using the same argument as in case 1, we get that there exists a subsequence of $\{x_{\eta(n)}\}$ also denoted as $\{x_{\eta(n)}\}$ which converges weakly to x^* and

$$\limsup_{n\to\infty} \langle x_{\tau(n)} - q, u - q\rangle \leq 0.$$

Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq \|x_{\eta(n)+1} - q\|^2 - \|x_{\eta(n)} - q\|^2 \\ &\leq \alpha_{\eta(n)}[\alpha_{\eta(n)}\|q-u\|^2 - 2(1-\alpha_{\eta(n)})\langle x_{\eta(n)} - q, q-u\rangle - \|x_{\eta(n)} - p\|^2], \end{aligned}$$

i.e.

$$\|x_{\eta(n)} - p\|^2 \leq \alpha_{\eta(n)}\|q-u\|^2 - 2(1-\alpha_{\eta(n)})\langle x_{\eta(n)} - q, q-u\rangle \to 0.$$

Hence,

$$\lim_{n\to\infty} \|x_{\eta(n)} - q\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\eta(n)} = \lim_{n\to\infty} \Gamma_{\eta(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_{\eta(n)} \leq \Gamma_{\eta(n)+1}$ if $n \neq \eta(n)$ (that is $\eta(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\eta(n)+1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\eta(n)}, \Gamma_{\eta(n)+1}\} = \Gamma_{\eta(n)+1}.$$

So, $\lim_{n\to\infty} \Gamma_n = 0$, that is $\{x_n\}$ as well as $\{y_n\}$, $\{u_n\}$ and $\{w_n\}$ converges strongly to q.

$\square$

Corollary 7.1.2. *Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be κ-strictly pseudocontractive and λ-strictly pseudocontractive mappings respectively,with $C = F(S) \neq \emptyset$ and $Q = F(T) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Assume that $\Omega_{SH} \neq \emptyset$ and suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\gamma > 0, \tau > 0$ real numbers and the following conditions are satisfied:*
(1) $\beta_n \in (a, (1-\kappa))$ for some $a > 0$,
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(3) $\gamma \in \Big(0, \dfrac{1-\lambda}{\|A\|^2}\Big)$.
Then the sequence $\{x_n\}$ generated by

$$\left\{\begin{array}{l} x_0, u \in H_1, \\ w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = w_n + \gamma A^*(T-I)Aw_n, \\ u_n = P_C(I-\rho f_1)(y_n + \tau A^*(P_Q(I-\rho f_2) - I)Ay_n), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S u_n, \quad n \geq 0, \end{array}\right. \tag{7.31}$$

converges strongly to a point $q = P_{\Omega_{SH}}$, where $0 < \rho < \mu, \nu$ and $\tau \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A.

Corollary 7.1.3. *Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be nonexpansive mappings, with $C = F(S) \neq \emptyset$ and $Q = F(T) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Assume that $\Omega_{SH} \neq \emptyset$ and suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\gamma > 0, \tau > 0$ real numbers and the following conditions are satisfied:*
(1) $\beta_n \in (a, 1)$ for some $a > 0$,
(2) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(3) $\gamma \in \Big(0, \dfrac{1}{\|A\|^2}\Big)$.
Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0, u \in H_1, \\ w_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = w_n + \gamma A^*(T - I)Aw_n, \\ u_n = P_C(I - \rho f_1)(y_n + \tau A^*(P_Q(I - \rho f_2) - I)Ay_n), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S u_n, \quad n \geq 0, \end{cases} \tag{7.32}$$

converges strongly to a point $q = P_{\Omega_{SH}}$, where $0 < \rho < \mu, \nu$ and $\tau \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A.

7.2 Strong convergence of regularized algorithm for minimizing sum of two functions in Hilbert spaces

Definition 7.2.1. Let h be a proper, convex and lower semicontinuous function on a real Hilbert space H. The *proximal map* associated with h is the function $\text{prox}_h : H \to H$ defined by

$$\text{prox}_h(x) = arg \min_{y\in H} \Big[h(y) + \frac{1}{2}||x - y||^2\Big].$$

The proximal map defined in Definition 7.2.1, is uniquely defined and generalizes the projection on a closed convex set to convex functions.

Let f and g be two convex and lower semi continuous functions from H to $\mathbb{R} \cup \{+\infty\}$ such that f is differentiable with $L-$Lipschitz continuous gradient, and g is "simple" meaning that its "proximal map"

$$x \to arg \min_{y\in H} \Big[g(y) + \frac{||x - y||^2}{2\tau}\Big]$$

can easily be computed. In this section, we shall consider the following minimization problem

$$\min_{x\in H} F(x) := \min_{x\in H}[f(x) + g(x)] \tag{7.33}$$

and assume that this problem has at least a solution. Let the set of solutions of (7.33) be denoted by Γ_m.

The proximal-gradient method (see, for example, [79, 167]) have been employed in solving (7.33) with a sequence $\{x_n\}$ generated by the algorithm: for an initial $x_1 \in H$,

$$x_{n+1} = (\text{prox}_{\gamma_n g} \circ (I - \gamma_n \nabla f))x_n, \tag{7.34}$$

where ∇f is the gradient of f, and $\{\gamma_n\}$ is a sequence of positive real numbers. If $\Gamma_m \neq \emptyset$ and the following conditions are satisfied

$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||, \quad x, y \in H \tag{7.35}$$

and

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \frac{2}{L}, \tag{7.36}$$

then the sequence $\{x_n\}$, (see, [79, 232]) converges weakly to a point in Γ_m. The proximal gradient algorithm can also be interpreted as a fixed point iteration. It is well known that a point $x^* \in H$ is a solution to the problem (7.33), that is, x^* is a minimizer $f(x) + g(x)$, if and only if $0 \in \nabla f(x^*) + \partial g(x^*)$. For any $\gamma > 0$ this optimality condition holds if and only if the following equivalent statements hold:

$$\begin{aligned}
0 &\in \gamma \nabla f(x^*) + \gamma \partial g(x^*) \\
0 &\in \gamma \nabla f(x^*) - x^* + x^* + \gamma \partial g(x^*) \\
(I + \gamma \partial g)(x^*) &\in (I - \gamma \nabla f)(x^*) \\
x^* &= (I + \gamma \partial g)^{-1}(I - \gamma \nabla f)(x^*) \\
x^* &= \text{prox}_{\gamma g}(x^* - \gamma \nabla f(x^*)).
\end{aligned} \tag{7.37}$$

The last two expressions hold with equality because the proximal operator is single-valued. The final statement says that x^* minimizes $f + g$ if and only if it is a fixed point of the Forward-Backward operator $(I + \partial g)^{-1}(I - \gamma \nabla f)$. The proximal gradient method repeatedly applies this operator to obtain a fixed point and thus a solution to the original problem. The condition $\gamma \in (0, \frac{1}{L}]$, where L is the Lipschitz constant of ∇f guarantees that the Forward-Backward operator is averaged and thus that the iteration converges to a fixed point (when one exists).

Many authors have studied Problem (7.33) using different iterative algorithms. Among these algorithms that have been applied to problem (7.33) is the proximal splitting algorithm, which performs alternating descents in f and in g and is frequently used because of it simplicity and relatively small per-iteration complexity. Other algorithms that have been used to study problem (7.33), include but not limited to; the Forward-Backward [143], the Douglas-Rachford splitting [89], and the alternating direction method of multipliers (ADMM). Problem (7.33) has wide applications in many imaging problems such as denoising, inpainting, deconvolution and colour transfer. For more details on the recent algorithms for solving Problem (7.33) and its applications, kindly see [61, 78, 79].

Recently, Chambolle and Dossal [61] studied problem (7.33) using the Fast Iterative Soft Thresholding Algorithm(FISTA), which is an accelerated variant of the Forward-Backward algorithm proposed by Beck and Teboulle [20]. As a passing comment, the Forward-Backward algorithm of Beck and Teboulle [20] was built upon the ideas of Nesterov [166] and G$\ddot{u}$ler [97]. So, in [61], Chambolle and Dossal proved the following convergence result.

Theorem 7.2.1. *([61]) Let $a > 2$ be a positive real number, and for all $n \in \mathbb{N}$, let $t_n = \dfrac{n+a-1}{a}$. Then the sequence $\{x_n\}, \{y_n\}$ and $\{u_n\}$, generated for $x_1 \in H$ by $y_1 = u_1 = x_1$ and for all $n \geq 2$*

$$\begin{cases} x_n = T(y_{n-1}), \\ y_n = (1 - \dfrac{1}{t_{n+1}})x_n + \dfrac{1}{t_{n+1}}u_n, \\ u_n = x_{n-1} + t_n(x_n - x_{n-1}), \end{cases} \tag{7.38}$$

or

$$\begin{cases} x_n = T(y_{n-1}), \\ y_n = x_n + \alpha_n(x_n - x_{n-1}), \alpha_n := \dfrac{t_n - 1}{t_{n+1}}, \quad n \geq 2, \end{cases} \tag{7.39}$$

converges weakly to a minimizer of F, where $Tx := \operatorname{prox}_{\gamma g}(x - \gamma \nabla f(x))$.

Using the result of Hundal [108], Xu [233] constructed an example to show that the iterative scheme (7.34), in general, has weak convergence only, unless the underlying Hilbert space is finite-dimensional. Therefore, it is of interest to obtain strong convergence result for solving (7.33) by modifying (7.34) appropriately.

Theorem 7.2.2. *Let H be a real Hilbert space and f and g be two convex, lower semi-continuous functions from H to $\mathbb{R} \cup \{+\infty\}$ such that f is differentiable with Lipschitz continuous gradient, and g is simple, be such that Γ_{MI} denotes the set of solution of (7.33) and $\Gamma_{MI} \neq \emptyset$. Let $\{\gamma_n\}$ denote a nonnegative real sequence and L is the Lipschitz constant of ∇f. Let the sequence $\{x_n\}$ be generated for initial $x_1 \in H$ and a fixed but arbitrary $u \in H$ by*

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n \operatorname{prox}_{\gamma_n g}(y_n - \gamma_n \nabla f(y_n)), \quad n \geq 1. \end{cases} \tag{7.40}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are non-negative sequences that satisfy the following conditions:
(H1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(H2) $\lim_{n\to\infty} \alpha_n = 0$;
(H3) $\sum_{n\geq 1} \alpha_n = \infty$;
(H4) $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$;
(H5) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < \dfrac{1}{L}$.
Then $\{x_n\}$ converges strongly to $q := P_{\Gamma_{MI}} u$, where $P_{\Gamma_{MI}}$ is the metric projection from H onto Γ_{MI}.

Proof. First we show that the sequence $\{x_n\}$ given by (7.40) is bounded.
Let $q = P_{\Gamma_{MI}} u$. Since $\operatorname{prox}_{\gamma_n g}(I - \gamma_n \nabla f)$, for all $n \geq 1$, is averaged and hence nonexpansive,

then for any x in the domain of $\operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)$

$$\begin{array}{rcl}
\|\operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x-q\|^2 & \leq & \|x-q\|^2 \\
\Rightarrow \langle \operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x-q, \operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x-q\rangle & \leq & \langle x-q, x-\operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x\rangle \\
 & & +\langle x-q, \operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x-q\rangle \\
\Rightarrow \langle \operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x-q, \operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x-x\rangle & \leq & \langle x-q, x-\operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x\rangle \\
\Rightarrow \langle \operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x-x, \operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x-x\rangle & \leq & \langle x-q, x-\operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x\rangle \\
 & & +\langle x-q, x-\operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x\rangle \\
\Rightarrow \|x-\operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x\|^2 & \leq & 2\langle x-q, x-\operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)x\rangle.
\end{array}$$

Therefore, from (7.40),

$$\begin{array}{rcl}
||x_{n+1}-q||^2 & = & ||(y_n-q)-\beta_n(y_n-\operatorname{prox}_{\gamma_n g}(y_n-\gamma_n\nabla f(y_n))||^2 \\
 & = & ||y_n-q||^2+\beta_n^2||y_n-\operatorname{prox}_{\gamma_n g}(y_n-\gamma_n\nabla f(y_n))||^2 \\
 & & -2\beta_n\langle y_n-q, y_n-\operatorname{prox}_{\gamma_n g}(y_n-\gamma_n\nabla f(y_n))\rangle \\
 & \leq & ||y_n-q||^2-\beta_n(1-\beta_n)||y_n-\operatorname{prox}_{\gamma_n g}(y_n-\gamma_n\nabla f(y_n))||^2. \quad (7.41)
\end{array}$$

But,

$$\operatorname{prox}_{\gamma_n g}(y_n-\gamma_n\nabla f(y_n))-y_n=\frac{1}{\beta_n}(x_{n+1}-y_n).$$

Therefore, from (7.41) we have,

$$||x_{n+1}-q||^2 \leq ||y_n-q||^2-\frac{1}{\beta_n}(1-\beta_n)||x_{n+1}-y_n||^2. \tag{7.42}$$

Using condition (H1) and (7.40) in the last inequality (7.42), we get

$$\begin{array}{rcl}
||x_{n+1}-q|| & \leq & ||y_n-q|| \\
 & \leq & \alpha_n\|u-q\|+(1-\alpha_n)\|x_n-q\| \\
 & \leq & \max\Big\{\|x_n-q\|, \|u-q\|\Big\} \\
 & \vdots & \\
 & \leq & \max\Big\{\|x_1-q\|, \|u-q\|\Big\},
\end{array}$$

which shows that $\{x_n\}$ is bounded.

Observe from (7.40) that

$$\begin{array}{rcl}
\|\operatorname{prox}_{\gamma_n g}(y_n-\gamma_n\nabla f(y_n))-y_n\|^2 & = & \|\frac{1}{\beta_n}(x_{n+1}-y_n)\|^2 \\
 & = & \frac{1}{\beta_n^2}\|x_{n+1}-y_n\|^2 \\
 & = & \frac{\alpha_n}{\beta_n}\Big(\frac{\|x_{n+1}-y_n\|^2}{\alpha_n\beta_n}\Big). \quad (7.43)
\end{array}$$

Using Lemma 2.3.8 in (7.40) (noting that $\alpha_n \in (0,1)$), we have

$$\begin{aligned} \|y_n - q\|^2 &= \|\alpha_n(u-q) + (1-\alpha_n)(x_n - q)\|^2 \\ &= \alpha_n^2\|u-q\|^2 + 2\alpha_n(1-\alpha_n)\langle u-q, x_n - q\rangle + (1-\alpha_n)^2\|x_n - q\|^2 \\ &\leq \alpha_n^2\|u-q\|^2 + 2\alpha_n(1-\alpha_n)\langle u-q, x_n - q\rangle + (1-\alpha_n)\|x_n - q\|^2 \\ &= \alpha_n^2\|u-q\|^2 - 2\alpha_n(1-\alpha_n)\langle u-q, q - x_n\rangle + (1-\alpha_n)\|x_n - q\|^2. \quad (7.44) \end{aligned}$$

Using (7.44) in (7.42), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \alpha_n^2\|u-q\|^2 - 2\alpha_n(1-\alpha_n)\langle u-q, q-x_n\rangle \\ &\quad +(1-\alpha_n)\|x_n - q\|^2 - \frac{1}{\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2 \\ &= (1-\alpha_n)\|x_n - q\|^2 - \alpha_n\Big(- \alpha_n\|u-q\|^2 + 2(1-\alpha_n)\langle u-q, q-x_n\rangle \\ &\quad + \frac{1}{\alpha_n\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2\Big). \quad (7.45) \end{aligned}$$

Let

$$\begin{aligned} \Gamma_n &= -\alpha_n\|u-q\|^2 + 2(1-\alpha_n)\langle u-q, q-x_n\rangle \\ &\quad + \frac{1}{\alpha_n\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2, \quad n \geq 1. \quad (7.46) \end{aligned}$$

Then (7.45) becomes

$$\|x_{n+1} - q\|^2 \leq (1-\alpha_n)\|x_n - q\|^2 - \alpha_n\Gamma_n. \quad (7.47)$$

We know that $\{x_n\}$ is bounded and so it is bounded below. Hence, Γ_n is bounded below. Furthermore, using Lemma 2.3.20 and condition (H3) in (7.47), we have

$$\begin{aligned} \limsup_{n\to\infty}\|x_n - q\|^2 &\leq \limsup_{n\to\infty}(-\Gamma_n) \\ &= -\liminf_{n\to\infty}\Gamma_n. \quad (7.48) \end{aligned}$$

Therefore, $\liminf\limits_{n\to\infty} \Gamma_n$ is a finite real number and by condition (H2), we have from (7.46) that

$$\liminf_{n\to\infty} \Gamma_n = \liminf_{n\to\infty}\Big(2\langle u-q, q-x_n\rangle + \frac{1}{\alpha_n\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2\Big).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in H$ and

$$\liminf_{n\to\infty} \Gamma_n = \lim_{k\to\infty}\Big(2\langle u-q, q-x_{n_k}\rangle + \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})||x_{n_k+1} - y_{n_k}||^2\Big). \quad (7.49)$$

Since $\{x_n\}$ is bounded and $\liminf\limits_{n\to\infty} \Gamma_n$ is finite, we have that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})||x_{n_k+1} - y_{n_k}||^2$ is bounded. Also, by condition (H1), we have that there exists $b \in (0,1)$ such that $\beta_n \leq b < 1$ and this implies that $\frac{1}{\alpha_n\beta_n}(1-\beta_n) \geq \frac{1}{\alpha_n\beta_n}(1-b) > 0$ and so we have that

$\frac{1}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1} - y_{n_k}\|^2$ is bounded. Therefore, we obtain from (7.43) using condition (H2) that

$$\|\text{prox}_{\gamma_{n_k} g}(y_{n_k} - \gamma_{n_k}\nabla f(y_{n_k})) - y_{n_k}\| \to 0, \quad k \to \infty.$$

Similarly from (7.40), we have

$$\|x_{n_k+1} - y_{n_k}\| = \beta_{n_k}\|\text{prox}_{\gamma_{n_k} g}(y_{n_k} - \gamma_{n_k}\nabla f(y_{n_k})) - y_{n_k}\| \to 0, \quad k \to \infty \tag{7.50}$$

and

$$\|y_{n_k} - x_{n_k}\| = \alpha_{n_k}\|u - x_{n_k}\| \to 0, \quad k \to \infty.$$

Hence,

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0, \quad k \to \infty.$$

Observe that $y_{n_k} \rightharpoonup x^* \in H, \quad k \to \infty$ since $y_{n_k} - x_{n_k} \to 0, \quad k \to \infty$ and $x_{n_k} \rightharpoonup x^* \in H, \quad k \to \infty$. By condition (H5), we have that $\gamma_n \geq \delta > 0, \quad \forall n \geq 1$ and by Lemma 2.3.21, we get that

$$\begin{aligned} & ||\text{prox}_{\delta g}(y_{n_k} - \delta\nabla f(y_{n_k})) - y_{n_k}|| \\ \leq \; & 2||\text{prox}_{\gamma_{n_k} g}(y_{n_k} - \gamma_{n_k}\nabla f(y_{n_k})) - y_{n_k}|| \to 0, \quad k \to \infty. \end{aligned} \tag{7.51}$$

Using Lemma 2.3.1 (demiclosedness principle) in (7.51), we have that $x^* \in \Gamma_{MI}$. Now, we obtain from (7.49) and (7.50) that

$$\begin{aligned} \liminf_{n\to\infty}\Gamma_n &= \lim_{k\to\infty}\Big(2\langle u - q, q - x_{n_k}\rangle + \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})||x_{n_k+1} - y_{n_k}||^2\Big) \\ &= 2\lim_{k\to\infty}\langle u - q, q - x_{n_k}\rangle \\ &= 2\langle u - q, q - x^*\rangle \geq 0. \end{aligned} \tag{7.52}$$

Then, we have from (7.48) that

$$\limsup_{n\to\infty}\|x_n - q\|^2 \leq -\liminf_{n\to\infty}\Gamma_n \leq 0.$$

Therefore, $\lim_{n\to\infty}\|x_n - q\| = 0$ and this implies that $\{x_n\}$ converges strongly to q. This completes the proof. □

7.2.1 Applications

1. **Application to split feasibility problems**
   Let C be closed convex subset of a Hilbert space. Recall that the indicator function on C is the function i_C, defined as

$$i_C(x) := \begin{cases} 0, & x \in C, \\ \infty, & \text{otherwise}. \end{cases} \tag{7.53}$$

   It is well known that the proximal mapping of i_C is the metric projection on C; i.e,

$$\begin{aligned} \text{prox}_{i_C}(x) &= \arg\min_{u\in C}||u - x|| \\ &= P_C(x). \end{aligned}$$

Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let P_Q be the projection of H_2 onto nonempty, closed and convex subset Q. Define $f : H_1 \to \mathbb{R}^+$ by $f(x) := \frac{1}{2}||Ax - P_Q Ax||^2$. Obviously f is continuous and the gradient of f which is given as $\nabla f(x) = A^*(I - P_Q)Ax$ is Lipschitz with Lipschitz constant $L = ||A||^2$. Therefore, from Theorem 7.2.2, we obtain the following theorem for solving split feasibility problems:

Theorem 7.2.3. *Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let C and Q be nonempty closed convex subsets of H_1 and H_2 respectively, and $g(x) = i_C(x)$ and $f(x) = \frac{1}{2}||Ax - P_Q Ax||^2$. Let γ_n denote a nonnegative real sequence such that $\gamma_n \in (0, \dfrac{1}{||A||^2})$.Let $\Psi := \{x \in C : Ax \in Q\} \neq \emptyset$. Let the sequence $\{x_n\}$ be generated for initial $x_1 \in H_1$ and a fixed but arbitrary $u \in H_1$ by*

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n P_C(y_n - \gamma_n A^*(I - P_Q)Ay_n), \quad n \geq 1. \end{cases} \tag{7.54}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are non-negative sequences that satisfy the following conditions:
(H1) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;
(H2) $\lim\limits_{n\to\infty} \alpha_n = 0$;
(H3) $\sum_{n\geq 1} \alpha_n = \infty$;
(H4) $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$;
(H5) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < \dfrac{1}{||A||^2}$.
Then $\{x_n\}$ converges strongly to $q := P_\Psi u$, where P_Ψ is the metric projection from H_1 onto Ψ.

2. **Application to LASSO problem**
The l_1-norm regularized least squares model is

$$min_{x\in\mathbb{R}^n} \frac{1}{2}||Ax - b||_2^2 + \lambda||x||_1, \tag{7.55}$$

where $A \in \mathbb{R}^{m\times n}$ is a given matrix, b is a given vector and λ a positive scalar. Let Υ be the solution set of (7.55).
The concept of l_1 regularization has been studied for many years. The least squares problem with l_1 penalty was presented and popularized independently under names; Least Absolute Selection and Shrinkage Operator (LASSO)[238], and Basis Pursuit Denoising [135].

The interest in compressed sensing, is in recovering a solution x to an undetermined system of linear equations $Ax = b$ in the case where $n \gg m$. It is known from linear algebra that this linear system either does not exist or is not unique when the number of unknowns is greater than the number of equations. The system is usually solved by finding the minimum l_2-norm solution, also known as linear least squares. If x is sparse, as is usually the case in applications, then x can be recovered by computing the above l_1-norm regularized least squares model (7.55). This (7.55) model is most often referred to as LASSO. The LASSO problem can be cast as a second order cone programming and solve by standard general algorithms like an interior point method [21], but the computational complexity of such traditional methods is too high to handle large-scale data encountered in many real applications.

Two notable algorithms that take advantage of special structure of LASSO problems are iterative shrinkage thresholding algorithm (ISTA) and its accelerated version fast iterative shrinkage thresholding algorithm (FISTA). The computation of ISTA, which is also known as the proximal gradient method, only involves matrix and vector multiplication,and w has great advantage over standard convex algorithms by avoiding a matrix factorization [171]. Beck and Teboulle [20] put forward an accelerated ISTA named as FISTA, in which a relaxation parameter is chosen. Meanwhile, Nestrov [166] had earlier developed a similar algorithm to Fista. These two algorithms are designed for solving problems containing convex differentiable objectives combined with an l_1 regularization terms as the following problem:

$$\min\{f(x) + g(x) : x \in \mathbb{R}^n\}, \tag{7.56}$$

where f is a smooth convex function and g is continuous function but possibly nonsmooth. Clearly, LASSO problem is a special case of (7.56), formulation with $f(x) = \frac{1}{2}||Ax - b||^2$, $g(x) = \lambda||x||_1$. Its gradient $\nabla f = A^T Ax - A^T b$ is Lipschitz continuous with constant $L(f) = ||A^T A||$. The proximal map with $g(x) = \lambda||x||_1$ is given as $\text{prox}_g(x) = arg\min_u \lambda||x||_1 + \dfrac{1}{2}||u - x||_2^2$, which is separable in indices. Each i^{th} coordinate can be optimized separately and the i^{th} coordinate is given by $\text{prox}_g(x)i = \text{sgn}(x_i)[|x_{i|-\min\{|x_i|}, \lambda\}$. That is

$$\begin{cases} x_i - \lambda, & x_i \geq \lambda, \\ 0, & |x_i| \leq \lambda, \\ x_i + \lambda, & x_i \leq -\lambda. \end{cases} \tag{7.57}$$

Thus we get from Theorem 7.2.2 the following theorem for solving Lasso problem.

Theorem 7.2.4. *Let H be a real Hilbert space and f and g from H to $\mathbb{R}$ such that $f(x) = \frac{1}{2}||Ax - b||^2$, $g(x) = \lambda||x||_1$. Let γ_n denote a nonnegative real sequence such that $\gamma_n \in (0, \dfrac{1}{||A^T A||})$ and let $\Upsilon \neq \emptyset$. Let the sequence $\{x_n\}$ be generated for initial $x_1 \in H$ and a fixed but arbitrary $u \in H$ by*

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n \text{prox}_{\gamma_n g}(y_n - \gamma_n \nabla f(y_n)), \quad n \geq 1, \end{cases} \tag{7.58}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are non-negative sequences that satisfy the following conditions:
(H1) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;
(H2) $\lim\limits_{n\to\infty} \alpha_n = 0$;
(H3) $\sum_{n\geq 1} \alpha_n = \infty$;
(H4) $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$;
(H5) $0 < \liminf\limits_{n\to\infty} \gamma_n < \limsup\limits_{n\to\infty} \gamma_n < \dfrac{1}{||A^T A||}$.
Then $\{x_n\}$ converges strongly to $q := P_\Upsilon u$, where P_Υ is the metric projection from H onto Υ.

7.2.2 Numerical example

In this section, we present some numerical example to illustrate the performance of our algorithm. We consider the linear inverse problem as our test problem.

Example 7.2.5. Let $H_1 = L_2([\alpha,\beta]) = H_2$ and we give a numerical example in $(L_2([\alpha,\beta]), ||.||_{L_2})$ of the problem considered in Theorem 7.2.3 in this section. Now take

$$C := \{x \in L_2([\alpha,\beta]) : \langle a, x\rangle \leq b\},$$

where $0 \neq a \in L_2([\alpha,\beta])$ and $b \in \mathbb{R}$, then (see [50])

$$P_C(x) = \begin{cases} \frac{b-\langle a,x\rangle}{||a||^2_{L_2}}a + x, & \langle a,x\rangle > b \\ x, & \langle a,x\rangle \leq b. \end{cases}$$

Let

$$Q = \{x \in L_2([\alpha,\beta]) : ||x-d||_{L_2} \leq r\}$$

be a closed ball centered at $d \in L_2([\alpha,\beta])$ with radius $r > 0$, then

$$P_Q(x) = \begin{cases} d + r\frac{x-d}{||x-d||}, x \notin Q \\ x, \quad x \in Q. \end{cases}$$

Now, suppose

$$C := \left\{x \in L_2([0,2\pi]) : \int_0^{2\pi} x(t)dt \leq 1\right\}$$

and

$$Q = \left\{x \in L_2([0,2\pi]) : \int_0^{2\pi} |x(t) - \sin(t)|^2 dt \leq 16\right\}$$

and $A : L^2([0,2\pi]) \to L^2([0,2\pi]), \quad (Ax)(s) = x(s), \quad \forall x \in L^2([0,2\pi])$. Then $(A^*x)(s) = x(s)$ and $\|A\| = 1$. Let us consider the following problem:

$$\text{find} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{7.59}$$

Observe that the set of solutions of problem (7.59) is non-empty (since $x(t) = 0, \quad a.e.$ is in the set of solutions). Since there are steering sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ to be chosen in our iterative scheme (7.54), we shall investigate the influence of the different sequences using numerical example. Take $\alpha_n = \frac{1}{n+1}, \quad \forall n \geq 1$ and $x_(t) = u = \sin(t), \quad t \in [0,2\pi]$. Then our iterative scheme (7.54) becomes

$$\begin{cases} y_n = \frac{1}{n+1}\sin(t) + (1 - \frac{1}{n+1})x_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n P_C((1-\gamma_n)y_n + \gamma_n P_Q(y_n)), \quad n \geq 1, \end{cases} \tag{7.60}$$

where

$$P_Q(y_n(t)) = \begin{cases} \sin t + 4\frac{y_n(t)-\sin t}{\sqrt{\int_0^{2\pi}|y_n(t)-\sin(t)|^2dt}}, y_n(t) \notin Q \\ y_n(t), y_n(t) \in Q \end{cases}$$

and

$$P_C(w_n(t)) = \begin{cases} \frac{1-\int_0^{2\pi} w_n(t)}{2\pi} + w_n(t), & w_n(t) \notin C \\ w_n(t), w_n(t) \in C \end{cases}$$

with $w_n(t) := (1-\gamma_n)y_n(t) + \gamma_n P_Q(y_n(t))$.

We investigate the influence of the sequences $\{\beta_n\} \subset (0,1)$ and $\{\gamma_n\} \subset (0,\infty)$ on the iterative scheme (7.60) by choosing different $\beta_n \in (0,1)$ and $\gamma_n \in (0, \frac{1}{4\pi^2})$.

With our choice of $\alpha_n = \frac{1}{n+1}, \quad \forall n \geq 1$, we consider and study these cases:
(1) Take $\beta_n := \frac{1}{2}$ and $\gamma_n := \frac{1}{8\pi^2}$;
(2) Take $\beta_n := \frac{n}{2n+1}$ and $\gamma_n := \frac{n}{8\pi^2 n+1}$;
(3) Take $\beta_n := \frac{n}{4n+1}$ and $\gamma_n := \frac{n}{16\pi^2 n+1}$;
(4) Take $\beta_n := \frac{n}{8n+1}$ and $\gamma_n := \frac{n}{32\pi^2 n+1}$;
(5) Take $\beta_n := \frac{n}{2n+1}$ and $\gamma_n := \frac{n}{32\pi^2 n+1}$.
The Mathlab version used is R2014a and the different tolerance levels ε with the execution times are as follows:

1. (case 1, $\varepsilon = 10^{-2}$), execution time is 8.232 seconds.
2. (case 1, $\varepsilon = 10^{-3}$) , execution time 35.486 seconds.
3. (case 2, $\varepsilon = 10^{-2}$), execution time is 9.2525 seconds.
4. (case 2, $\varepsilon = 10^{-3}$) , execution time 42.0306 seconds.
5. (case 3, $\varepsilon = 10^{-2}$), execution time is 8.9618 seconds.
6. (case 3, $\varepsilon = 10^{-3}$) , execution time 53.9514 seconds.
7. (case 4, $\varepsilon = 10^{-2}$), execution time is 7.3688 seconds.
8. (case 4, $\varepsilon = 10^{-3}$) , execution time 52.9123 seconds.
9. (case 5, $\varepsilon = 10^{-2}$), execution time is 9.983 seconds.
10. (case 5, $\varepsilon = 10^{-3}$) , execution time 45.835 seconds.

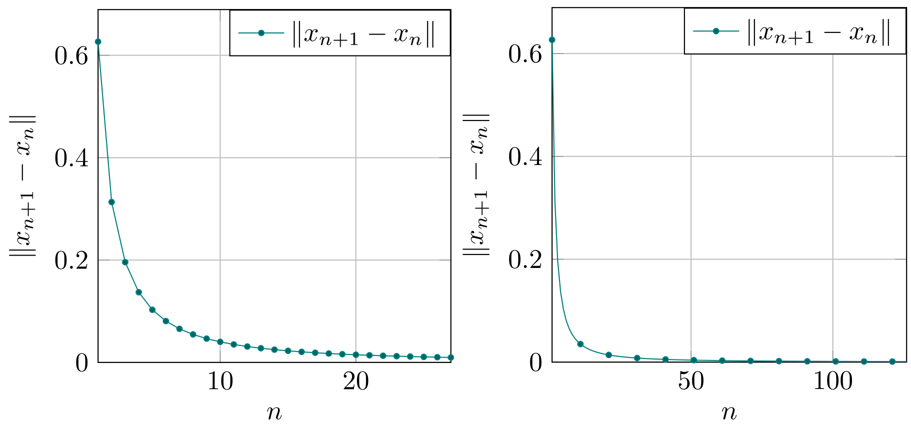


Figure 7.1: Errors: Case (1), $\varepsilon = 10^{-2}$ (left; 8.2324sec), $\varepsilon = 10^{-3}$ (right; 35.4863sec).

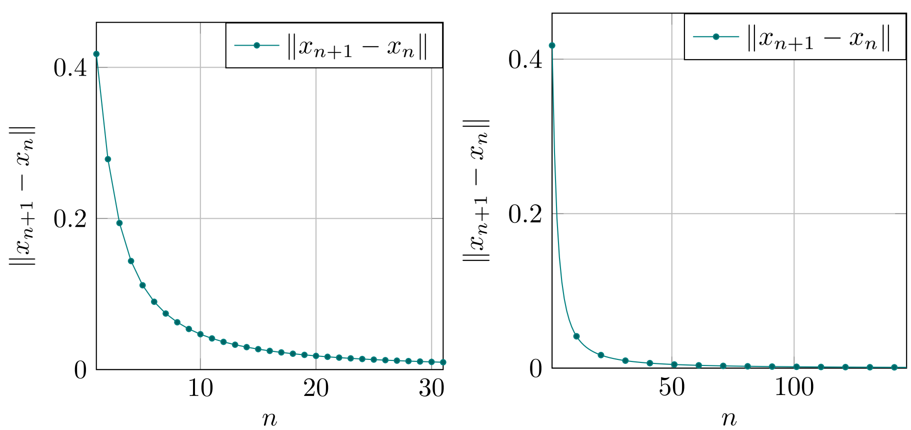


Figure 7.2: Errors: Case (2), $\varepsilon = 10^{-2}$ (left; 9.2525sec), $\varepsilon = 10^{-3}$ (right; 42.0306sec).

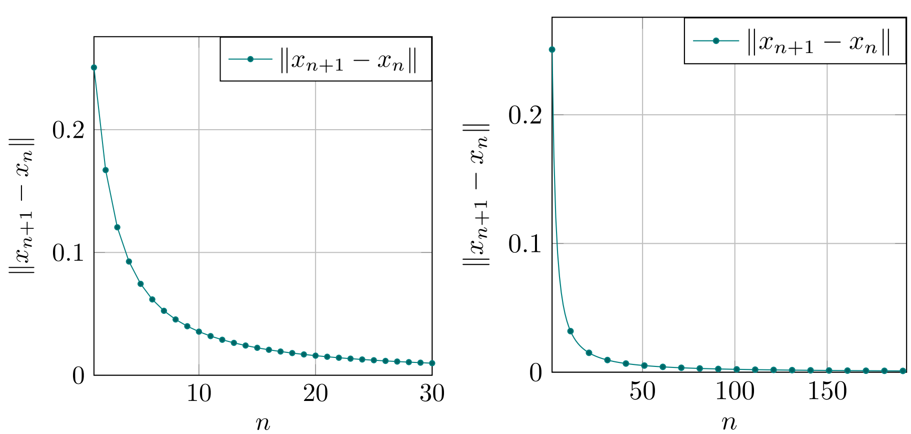


Figure 7.3: Errors: Case (3), $\varepsilon = 10^{-2}$ (left; 8.9618sec), $\varepsilon = 10^{-3}$ (right; 53.9514sec).

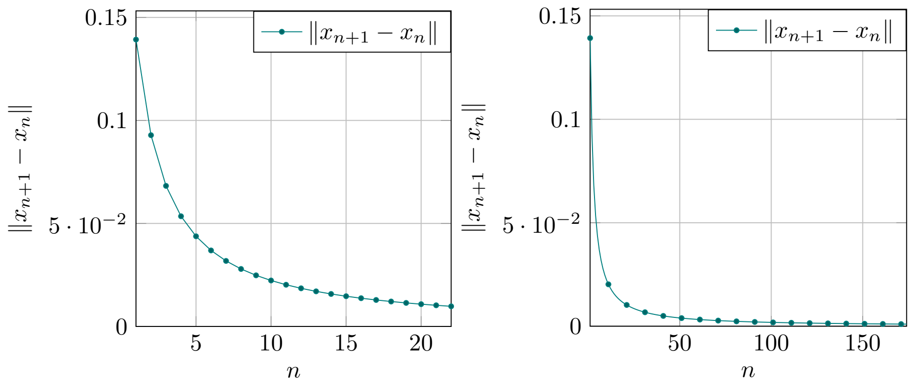


Figure 7.4: Errors: Case (4), $\varepsilon = 10^{-2}$ (left; 7.3688sec), $\varepsilon = 10^{-3}$ (right; 52.9123sec).

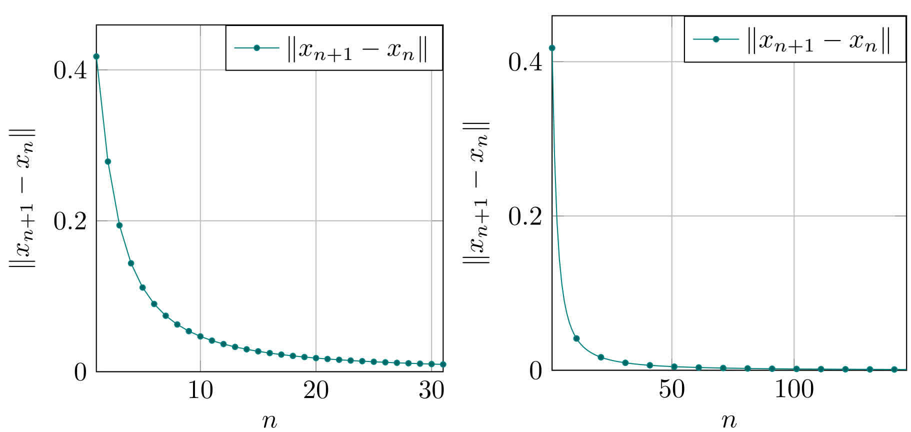


Figure 7.5: Errors: Case (5), $\varepsilon = 10^{-2}$ (left; 9.9838sec), $\varepsilon = 10^{-3}$ (right; 45.8356sec).

Chapter 8

Contribution to Knowledge and Area of Future Research

In this chapter, we give the summary of the results obtained in this work which serve as our contribution to knowledge. Also, possible areas of future research are also identified and discussed.

8.1 Contribution to knowledge

We now highlight our contributions in this work.

Takahashi et al.[217] stated and proved weak convergence results for solving (3.18) for nonexpansive mappings while in section 3.2, we presented a strong convergence theorem for solving (3.18) for demicontractive mappings. Thus, the result of section 3.2 is an improvement on the two results of [217].

Furthermore, the result of section 3.3 extends results on MSFP (3.2) from the frame work of real Hilbert spaces to p-uniformly Banach spaces which are also uniformly smooth.
In [46], a weak convergence result was obtained for split feasibility problems in real Hilbert spaces, while in section 3.4, we obtained a strong convergence result for split feasibility problems and fixed point problems for right Bregman strongly nonexpansive mappings in p-uniformly convex real Banach spaces which are also uniformly smooth. Hence, our result improve and extend the results of Byrne [46] and more applicable than the results of Schöpfer et.al. [201] and Wang [222].

In [246], Zegeye and Shahzad proved strong convergence result for a common fixed point of a finite family of right Bregman strongly nonexpansive mappings in the framework of real reflexive Banach spaces, while in the result of section 3.4, we obtained strong convergence result for approximation of a common solution of split feasibility problem and fixed point problem for a finite family of right Bregman strongly nonexpansive mappings p-uniformly convex real Banach spaces which are also uniformly smooth (see Corollary 3.4.3). Therefore, our result complements the result of Zegeye and Shahzad [246] in p-uniformly convex real Banach spaces which are also uniformly smooth.

In chapter four, we gave an iterative algorithm for finding a common fixed point of an infinite family of left Bregman strongly nonexpansive mappings, a common solution of a finite system of generalised mixed equilibrium problems and a finite system of variational inclusion problems in real reflexive Banach spaces. A strong convergence result was stated and proved using the iterative scheme. The result of chapter six complements some convergence results on equilibrium and variational inclusion problems for example [4] and [71].

The result of Bnouhachem [24] and other related results on split equilibrium in literature depend on the prior knowledge of the operator norm. But J. M. Hendrickx and A. Oleshevsky [106] observed that when $p = \infty$ or $p = 1$ the p-matrix norm is the largest of the row/column sums, and thus may be easily computed exactly. When $p = 2$, this problem reduces to computing an eigenvalue of $A^T A$ and thus can be solved in polynomial time in n, $log\frac{1}{\epsilon}$ and the bit-size of the entries of A.

J. M. Hendrickx and A. Oleshevsky [106] further stated and proved the following theorem:

Theorem 8.1.1. *(J. M. Hendrickx and A. Oleshevsky [106]) Fix a rational $p \in [1, \infty)$ with $p \neq 1, 2$. Unless $P = NP$, there is no algorithm which, given input ϵ and a matrix M with entries in $\{-1, 0, 1\}$, computes $||M||_p$ to relative accuracy ϵ, in time which is polynomial in ϵ^{-1} and the dimensions of the matrix.*

The result of Theorem8.1.1 shows that sometimes it is very difficult if not impossible to calculate or even estimate the operator norm.

In section 5.2, we introduce an iterative scheme which does not require any knowledge of the operator norm and obtain a strong convergence theorem for approximating solution of split generalised mixed equilibrium problem which also solves a fixed point problem for κ-strctly pseudocontractive mapping. Thus overcoming the set back of having to estimate the operator norm which sometimes is a big task.

In section 5.3, we gave a strong convergence theorem for split equality generalised mixed equilibrium problem in p-uniformly Banach space which is uniformly smooth. This is an improvement as previous results on split equality generalised mixed equilibrium were all in the frame work of Hilbert spaces.

Byrne *et al* [47], Kazmi and Rizvi [120] and Wen and Chen [228] obtained convergence results for solving SVIP (6.3)-(6.4) in the frame work of real Hilbert spaces but in section 6.2, we obtained a strong convergence result for approximating a solution of the split variational inclusion problem (6.3)-(6.4) in p-uniformly convex real Banach spaces that are also uniformly smooth, which is an extension of the works of Byrne *et al* [47], Kazmi and Rizvi [120] and Wen and Chen [228] from Hilbert spaces to p-uniformly convex real Banach spaces which are also uniformly smooth.

Shehu and Ogbuisi [210] obtained a strong convergence result for approximating a solution of SMVIP (6.1)-(6.2) which is also a solution of a fixed point problem for single valued strictly pseudocontractive mapping. In section 6.3, we presented a strong convergence result for approximating a solution of SMVIP (6.1)-(6.2) which is also a common solution of fixed point problems for two multivalued strictly pseudocontractive type mappings. Thus the result of section 6.2 extends the result of Shehu and Ogbuisi [210] from single valued to multivalued strictly pseudocontractive type mapping.

In section 6.4, we present a general algorithm which does not require prior knowledge of the operator norm for solving split monotone variational inclusion problem, fixed point problem for a finite family of strictly pseudocontractive mappings and certain variational inequality problem. The result of this Section 6.4 improve on the results of Shehu and Ogbuisi [210] and Deepho *et. al* [87] as follows:

1. The results of Shehu and Ogbuisi [210] and Deepho *et. al* [87] both require the knowledge of the operator norm while the result of Section 6.4 does not require any knowledge of the operator norm.

2. The result of Deepho et. al [87] took f_1 and f_2 to be identically zero but the result of Section 6.4 does not require f_1 and f_2 to be necessarily zero.

3. The result of this section solve a variational inequality problem while the result of Shehu and Ogbuisi [210] did not do so. Furthermore, the split equality monotone variational inclusion studied in Section 6.5 is a obvious generalisation of the SMVIP (6.1)-(6.2). Therefore, result of Section 6.5 generalises the result of Shehu and Ogbuisi [210] and other similar results on split monotone variational inclusion problems.

In 2015, Ansari *et al.* [11] gave a method for finding a fixed point of a nonexpansive operator and a solution of SHVIP (7.1)-(7.2), stated and proved a weak convergence result but in Section 7.1, we stated and proved a strong convergence result for solving SHVIP (7.1)-(7.2) and split fixed point problem for demicontractive mappings. Therefore our result in Section 7.1 extends and complements the result of Ansari *et al.* [11].

In Section 7.2, we improve the existing weak convergence result for solving (7.33) using a combination of relaxation and viscosity terms and obtain strong convergence result. Secondly, we give a more simpler alternative proof to solving (7.33) without recourse to two cases method of proof studied by other authors (see, for example, Wang and Xu [224]) for solving (7.33). Our method of proof here is shorter, easier to read and less technical. We also give some applications of our results and give some numerical example to illustrate the performance of our algorithm.

8.2 Area for future research

The result of section 6.2 give a strong convergence theorem for approximating solution of the split variational inclusion problem (6.3)-(6.4), i.e the split variational inclusion problem (6.1)-(6.2) with $f_1 = f_2 = 0$ in p-uniformly convex real Banach spaces which are also uniformly smooth. It will also be interesting to consider the result of Shehu and Ogbuisi [210] which gave an iterative algorithm and a strong convergence theorem for approximating solution of split monotone variational inclusion problem (6.1)-(6.2) with

f_1 and f_2 not necessarily 0, with a view to extend the result from real Hilbert space to p-uniformly convex real Banach spaces which are also uniformly smooth.

In section 7.2, we propose a proximal algorithm based on the Forward-Backward algorithm with relaxation and a viscosity term which ensure the strong convergence of the iterates of the algorithm in real Hilbert spaces. In implementing our result in this section, one needs to compute or obtain an estimate of the Lipschitz constant of ∇f. This is a draw back since computing the Lipschitz constant of ∇f may be difficult for certain functions f in application. In the future, we shall try to get rid of this strong condition and still obtain strong convergence result which will improve the results of this Section 7.2.

Furthermore, part of our future research is to present accelerated Forward-Backward algorithm with relaxation and a viscosity term and obtain strong convergence result in real Hilbert spaces. This will improve the result of Chambolle and Dossal [61], which is Theorem 7.2.1.

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