Exploring pre-service teachers' mental constructions of matrix algebra concepts: A South African case study

By

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PREFACE

The work done in this thesis was carried out in the School of Education, University of KwaZulu - Natal, from July 2012 to December 2014 under the supervision of Professor D. Brijlall.

This study represents original work done by me and has not been submitted in any form for any degree to any tertiary institution. The work used from others is properly acknowledged.

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ABSTRACT

Pre- service teachers' mental constructions of matrix algebra concepts

At the university where the study was conducted matrix algebra is one of the first advanced mathematics courses that pre-service teachers encounter. The transfer of knowledge from a primarily procedural or algorithmic school approach to formal presentation of concepts is a priority for conceptualisation of matrix algebra concepts. However, it seems to be creating many difficulties for many pre-service teachers. This is due to the fact that many of them are barely coping with procedural aspects of mathematical concepts. The aim of conducting the study was to explore the pre-service teachers' mental constructions when learning matrix algebra. The study was guided by the belief that understanding the mental constructions the pre-service teachers made when learning mathematical concepts leads to improved instructional methods. The study is underpinned by APOS theory (Action, Process, Object and Schema) and uses APOS theory to describe the nature of mental constructions displayed by pre-service teachers when learning matrix algebra concepts. To understand and explain the mental constructions made or not made, the preliminary genetic decompositions for matrix algebra concepts was used to analyse the nature of mental constructions made by these pre-service teachers together with triad mechanism which originates from Piaget's work of reflective abstraction. APOS theory is an extension of reflective abstractions so using these two tools to analyse pre-service teachers' mental constructions strengthen the trustworthiness of this study.

As part of this research project several case studies were conducted where groups of first and second year students were exposed to teaching and learning of some of matrix algebra concepts. These concepts explored are the ones that these students learn under matrix algebra at this university. These concepts were first taught to students and students were expected to express their thinking through solving matrix algebra related problems during tutorials and taking part in the interviews.

Analysis of written work and interviews from ten pre-service teachers provided insight into their mental constructions, revealing ways in which they understood the concepts. In explaining and synthesising the results major themes emerged from which conclusions were drawn about the

mental constructions that were or not made in the learning of matrix concepts. Several themes emerged which were categorised in certain headings in order to identify patterns that emerged from all tasks. What mostly transpired across all tasks was that background knowledge and understanding of notation are important aspects for students to understand in order to conceptualise the concepts in matrix algebra. It was noted that those students who had a weak schema of basic algebra were not able to make the necessary mental constructions or vice versa. Also, it was noted that students often made nonstandard notation and linguistic distinctions. For example, students use A_{11} when referring to entries of a matrix or use $|A|$ while determining the determinant of matrix C. Moreover, evidence from their responses revealed that many pre- service teachers had limited knowledge constructed of the taught concepts. This was observed as they struggle to represent the solutions of a system geometrically, recognise concepts in different registers and unable to link major concepts. Findings from this study revealed that the mental constructions made by pre-service teachers in most cases concur with the preliminary genetic decompositions. In terms of APOS theory students responses revealed that many were mainly operating at an action and process stages, with few pre-service teachers operating at an object stage. Since difficulties with the learning of linear algebra by average students are universally acknowledged, this study provided a modified itemised genetic decomposition which is anticipated to help in the teaching and learning of matrix algebra concepts. The aim of providing the modified genetic decomposition is to contribute in the teaching and learning of advanced mathematics as lectures could use the modified genetic decomposition to analyse the mental constructions of their students when learning matrix algebra concepts. Besides making a contribution to the teaching and learning of some mathematical concepts, the modified genetic decomposition is a contribution to APOS theory as it is shown it can be used in other mathematical concepts in different context.

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CHAPTER ONE

INTRODUCTION

1.1 Overview of this Chapter

In this chapter, the aim is to present the research process undertaken in the study of the nature of students' mental constructions made when learning matrix algebra. This chapter provides an overview of the study, with the background and purpose detailed first. The concept of learning of mathematics; the contribution this study aimed to make in the mathematics education; the research questions that the study addresses; and the key terms in the study; are discussed in detail. At the later stage in this chapter, a summary of successive chapters is provided.

1.1.1 Introduction

Since mathematics education research is concerned with the nature of teaching and learning of mathematics, the better we understand the nature of knowledge and mental constructions, the better we can plan the instructional methods and activities that will enhance the learning of mathematics as a subject. Much of mathematics education research had looked at the means and ways we can improve the teaching and learning of mathematics at school and at tertiary institutions. Research into the teaching and learning of mathematics in general has revealed that students have difficulties in conceptualising mathematical concepts. Naidoo (2007) noted that first year mathematics students rely on rules and algorithms, they do not enjoy mathematics and are demotivated. As a result, they struggle to solve unfamiliar problems. Aziz, Meerah & Tambychic (2010) have argued that the difficulties in the learning of mathematics are manifested in various situations such as poor application of mathematical concepts, poor achievement in mathematics, deficiency in mathematics skills and inefficiency in mathematical problem-solving. They further argue that lack of skills in using cognitive abilities to learn effectively would influence the mathematics skills acquired by students. Although these studies were conducted at different levels amongst high school learners and first year university students respectively, both findings revealed that the problems with the learning and teaching of mathematics are generally the same, and that the difficulties that learners experience with mathematical concepts at high school, if not dealt with, will impact on their learning at the university level.

1.1.2 Advanced Mathematical Thinking

Tall (1992) has stated that "advanced mathematical thinking is characterized [sic] by two important components: precise mathematical definitions (including the statement of axioms in axiomatic theories) and logical deductions of theorems based upon them" (p. 495). Over the past years there had been a growing interest in the research in the learning and teaching of undergraduate mathematics. Many scholars such as (Asiala, Brown, De Vries, Dubinsky, Mathews & Thomas, 2004; Brijlall & Maharaj, 2010, Dubinsky, 1991; Jojo, 2011; Naidoo, 2011; Tall, 2002, 2004) have conducted studies focusing on advanced mathematical thinking and pedagogies in the teaching and learning of mathematics.

Dubinsky (1991) provides a theoretical framework that focuses on mental constructions that can explain the processes involved in the learning of advanced mathematics. Prior to that, Tall & Vinner (1981) provided conceptual framework that explains the construction of knowledge in terms of concept image and concept definition. Subsequent to that, Tall (2004) describes three worlds of mathematical thinking in the teaching and learning of advanced mathematics. The learning of advanced mathematics involves a great deal of abstract mathematical thinking. The body of research around the area of abstract algebra has also grown extensively in recent years. Despite this, little is available in the mathematics education literature regarding the way in which students conceptualise the learning of matrix algebra. Although it is necessary to mention the vast literature in the area of linear algebra, less is known about students' mental construction of concepts in matrix algebra. Furthermore, there is a dearth of this manner of research in South Africa. This study therefore aims to contribute to the empirical and theoretical work in this area of mathematics education by exploring students' mental construction of matrix algebra concepts, and proposes a new genetic decomposition that can be used in the teaching and learning of matrix algebra.

1.2 Background and purpose of the study

At the university where the study was conducted, the first major content module offered to pre-service teachers intending to become high school mathematics teachers is called 'Mathematics for Educators 210'. Matrix algebra is one of the topics covered in this module. In this course the mathematics education students are exposed to the following concepts: matrices, solving the system of linear equations, and matrix inverse and determinants. The concept of linear systems in South Africa is first dealt with in secondary schools from Grade 9, dealing with two equations in two unknowns. At the university, students are expected to generalise their knowledge from systems of two equations with two unknowns, to a system of n equations with n unknowns, given by $n \times n$, i.e. any number of equation with any number of variables. At this level, students are introduced to matrix techniques of solving such systems, employing Gauss elimination and Cramer's rule. The concept of inverse is also taught at school level from Grade 12, dealing with function inverse, whereas the determinant concept is firstly dealt with at the university level. University students are expected to generalise their knowledge of school algebra in understanding matrix inverse. Matrix algebra is not part of the South African school syllabus, however, students' previous knowledge in concepts such as arithmetic algebra and school algebra ought to play a significant role in mastering the matrix algebra concepts.

The researchers' experience of first year mathematics education students in the previous years was that they cope well with routine type problems, but found it difficult to identify and explain the interrelationship between concepts. Naidoo (2007), in his study conducted with first year mathematics students, highlighted that first year mathematics students rely on rules and algorithms. They display what Biggs (2003) referred to as surface learning, as opposed to deep learning. For example, they can compute matrix product of AB when given matrices A=$\begin{bmatrix} 2 & 0 & 1 \\ -1 & 5 & 6 \end{bmatrix}$ and B= $\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, but when given arbitrary matrices and asked to explain whether the product was defined or not, they normally showed difficulty in justifying their answers. This meant that they mainly possess procedural knowledge, rather than conceptual knowledge. This study aimed to explore the mental constructions necessary for students to make in order to develop conceptual understanding of matrix algebra concepts. Ascertaining whether students have the conceptual understanding of a given mathematical concept could prove to be a difficult task for anyone responsible for the teaching of mathematics. Although this might be the case, if teachers or lecturers could understand the way in which students construct knowledge of a particular concept, as well as the cognitive level they are at, this might ultimately aid in determining whether or not students have conceptualised the given concept. Being able to realise the relationship between concepts is a fulfilling experience for anyone working in the discipline of mathematics. Therefore as someone working in mathematics, I believe that it is important to understand the mental constructions that students make, so as to assist in designing the teaching material that will help the student develop a conceptual understanding of matrix algebra concepts.

Students' difficulties in conceptualising linear algebra concepts had been highlighted as the area of great concern in the learning of linear algebra (Bogomolny, 2006, 2007; Carlson, 1993; Dubinsky, 1997; Harel, 2000; Stewart & Thomas, 2007, 2009). Research into this phenomenon had been undertaken extensively in other countries, however not much has been written in the area of linear algebra in South Africa. Bogomolny (2007); Paraguez & Oktac (2009) and Stewart & Thomas (2009) have conducted research on students' mental constructions of concepts such as vectors, linear combination and linear (in)dependency, but none of these studies explored the student's mental constructions in the process of learning of matrix algebra. Findings from these studies are crucial in the teaching and learning of the concepts explored in linear algebra. However, these might not be applicable to a South African context, or to students' conceptualisation of matrix algebraic concepts.

Numerous studies (Bogomolny, 2007; Paraguez & Oktac, 2010; Stewart & Thomas, 2009) have examined students' understanding of linear algebra concepts and the difficulties displayed by students during the process of learning linear algebra. They used APOS theory to explore the mental constructions displayed by students when learning linear algebra. However, the current study differs from the others that come before it, because it focuses on students' understanding of matrix algebra. It describes the nature of mental constructions made by students when constructing knowledge in the learning of matrix algebra. Secondly, it aims to identify and explain the processes students undergo to build their knowledge when learning matrix algebra concepts. Thirdly, the current study discusses the possible reasons that might lead to constructions of those mental processes. Furthermore, it introduces the new itemised genetic decomposition for matrix algebra concepts, which might assist mathematics lecturers to analyse and understand the mental constructions displayed by their students when learning matrix algebra. Findings from this study are anticipated to lead to more effective teaching practices in matrix algebra classes, as mathematics lecturers will know the level of knowledge the students have constructed of the concepts. Therefore, it is necessary to design the teaching in a manner that will help students to conceptually understand the relevant matrix algebra concepts. If the current study is successful in these aspects, then much is owed to Dubinsky (1991), who introduces APOS theory, which explains with clarity the mental constructions involved in the learning of advanced mathematics. Dubinsky & McDonald (2001) have suggested that APOS theory as a tool can be used objectively to explain students' difficulties with a broad range of mathematical concepts, and have recommended ways in which students can learn these concepts. They have further argued that this theory can point us towards pedagogical strategies that could lead to marked improvement in: (a) student learning of complex or abstract mathematical concepts; and (b) students use of these concepts to prove theorems, provide examples, and solve problems. Research shows that to learn and understand mathematics, students need to construct their own knowledge of particular concepts, and to view mathematics as the living subject that seeks to understand patterns that permeate both the world around us, as well as the mind within us (Schoenfeld, 1992). This is emphasised by Maoto & Wallace (2010), who note that knowledge is constructed as the students' strive to organise his/her experiences in terms of pre-existing mental structures or schemas.

1.3 Rationale

The rationale for the study can be identified in three aspects, namely: (1) addressing the gap in this field of research; (2) bringing new knowledge to the mathematics community since matrix algebra is one of the modules that pre-service teachers have to learn, and therefore it has implications for teacher education more broadly; and (3) researchers' experience with the teaching and learning of matrix algebra

1.3.1 Addressing the gap

Regardless of the amount of research being undertaken in undergraduate mathematics in the teaching and learning of linear algebra, research literature in the area of matrix algebra is limited, especially in the context of South Africa. Perhaps this is so because matrix algebra is not part of the school curriculum. Much of the body of research in South Africa in the past and present in mathematics education focuses on improving and developing the teaching and learning of mathematics at the school level. Surveying different journals and other collections on abstract algebra both nationally and internationally between 2000 and 2014, there has been more than 500 research articles in the teaching and learning of undergraduate mathematics. However, it has been discovered by the researcher that much of this research has been done internationally. It seemed that the body of research in the area of linear algebra in South Africa, with particular reference to matrix algebra is very limited or non-existence at all. Although internationally there is substantial research work in the teaching and learning of linear algebra, little research on matrix algebra and students' mental construction has been carried out thus far.

1.3.2 Bringing new knowledge

Dubinsky (1991), in his research on advanced mathematical thinking, proposed the influential APOS theory. He suggested that before mathematical concepts are taught there ought to be an analysis of the students' mental construction of the taught concepts. This needs to be done with the aim of improving instructional methods. For a particular concept, the genetic decompositions ought to be designed such that they might be able to be used to analyse the mental constructions. Literature reviewed in linear algebra indicated that students experience difficulties with the teaching and learning of linear algebraic concepts. With reference to this study, the researcher designed a preliminary genetic decomposition that could be used in the teaching and learning of matrix algebra to analyse the level of mental construction displayed by students. This will help when designing the instructional method that will help enhance the learning of matrix algebra concepts. This is discussed in greater detail in Chapter 3.

1.3.3 The researchers' experiences

For the past twelve years from 2001 to 2012, the researcher has been a mathematics teacher for the GET and FET phase. During the course of these years, being involved with teaching of mathematics, failure by learners to conceptualise mathematical concepts has been observed to be a consistent problem. Towards the end of 2012, the researcher began teaching mathematics at the university level, and in 2013, taught mathematics to first year pre-service teachers intending to teach mathematics in the FET Phase. This is the first content module that university students are required to undertake in order to pursue their careers as mathematics teachers. Students registered for the module were a combination of first and second year students. The minimum requirements to enrol for this module is a 60% pass in mathematics at Grade 12, which is Level 4, and those students who achieved Level 4 (50%-59%), but still want to major in mathematics, need to do a foundational module and require a minimum of 60 percent in the module. Therefore, the second years were those students who had completed a foundational module in their first year for six months. The numbers of students enrolled for this module were 110, taught by three lecturers, among them the researcher. Each lecturer taught certain sections and the researcher taught the section on matrix algebra, which falls within linear algebra. Whilst teaching and from assessment administered, the researcher observed that students tended to ask for formulas to be used in solving mathematical problems, and when using those formulas, there appeared to be a discontinuity between the use of rules and the understanding of concepts. For example, given matrix A = $\begin{bmatrix} 3 & -5 \\ 2 & 4 \end{bmatrix}$ and asked to find the determinant of A, they would use a formula $|A| = ad - bc$; some would write the formula down, but used it incorrectly e.g. $|A| = 3 \times -5 - (2 \times 4)$ This showed that the students did not understand the underlying concepts of the rule they were attempting to apply. Also, when finding determinants of a $3\ x\ 3$ matrix e.g. $\begin{bmatrix} 2 & 5 & 1 \\ 0 & 3 & -2 \\ -1 & 4 & -6 \end{bmatrix}$, students would apply the formula for a $2\ x\ 2$ matrix incorrectly, where they would find the difference of the product of the diagonals, i.e. $a_{11}\, a_{22}a_{33} - a_{31}a_{22}a_{13}$. This displayed an erroneous conception of the determinant formula. The application of determinants is the underlying concept in understanding other concepts in matrix algebra, such as in finding an inverse of a matrix, solving the system of equations using Cramer’s rule, and in analytic geometry, such as finding area, volume and equations of lines and planes.

The observation of the way in which students struggled with these concepts motivated the researcher to undertake a study with the aim to explore the underlying structures students used in constructing knowledge when learning the concepts in matrix algebra, in addition to finding that research concerning this phenomenon is limited in the South African context. Conceptual knowledge, as defined by Hapasaalo (2004), denotes knowledge as skilful drive along particular networks, elements of which can be concepts, rules (for example algorithms and procedures), and even problems given in various representational forms. The researcher’s observation of the students’ results in various assessments of these topics in the previous years revealed that students tended to do well. However, when analysing the nature of assessment used, it becomes apparent that it mainly focused on procedural knowledge, where focus was placed on students getting the correct answer. When some of these tasks were used with this particular group of students, they likewise performed well. However, when the problem was rephrased and required them to explain their thinking and discuss the problem without applying the rule, students showed difficulties. Even though they had correct answers, their explanation displayed a lack of conceptualisation of the learnt concepts. For example, students could find the inverse using the taught rules, but when given matrix A=$\begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix}$ and asked if the matrix is singular or non-singular, students had to first work out the problem step by step before they could answer the question. This meant the concept of finding determinants is not yet interiorised into a process.

1.4 Understanding in mathematics

Michener's (1978) accounts a mathematician's perspective of what it means to *understand* as follows:

> When a mathematician says he *understands* a mathematical theory, he possesses much more knowledge than that which concerns the deductive aspects of theorems and proofs. He knows about examples and heuristics and how they are related. He has a sense of what to use and when to use it, and what is worth remembering. He has an intuitive feeling for the subject, how it hangs together, and how it relates to other theories. He knows how not to be swamped by details, but also to reference them when he needs them [emphasis added] (p. 361).

There are many connotations associated with the word understanding in our daily experiences and it is used to describe a particular kind of conceptual impact. We speak of understanding between two people or a group; we speak of someone understanding a phenomenon, or a given concept. We generally positively associate understanding with progression. The question is: What is understanding specific to a mathematical context? Generally, when we refer to understanding of a concept in mathematics, we refer to capability in presenting concepts in different ways, the ability to logically explain a concept's relationship to other concepts, as well as the ability to apply the concept. It refers also to the ability to display mathematical fluency in problem solving. According to Piaget (as cited in Sierpinska, 1994), understanding belongs to the realm of reason: it must be based on conceptualisation, and such connections between these conceptualisations that are implicative and not causal (p.5). This is what Hasenbank (2006) termed *conceptual understanding*, and what Schoenfeld (1992) termed *relational understanding*. This is the form of understanding that is not mainly associated with an understanding of how to make something or how to solve a mathematical problem; however this form of understanding is focused on why the particular method works or does not work. Sierpinska (1994) distinguishes three main approaches to the question of understanding in mathematics education. One focuses on developing teaching material that would help students understand better. Another concentrates on diagnosing the understanding in students. A third is interested in the more theoretical issue of building models of understanding. These approaches are related, because one might be interested in diagnosing students understanding with the aim of developing teaching material that will enhance the understanding of students. This is within the scope of this study.

The area on understanding in mathematics has been well researched. Here it is not necessary to add to this already dense body of work. Instead, this will be applied to what known researchers such as Sierpinska, (1994; 2000), Skemp (1971) and Usiskin (2012) have discussed about understanding mathematical concepts, so as to build an argument for the current study. This study is not about understanding of mathematical concepts, rather its focuses on the nature of mental constructions that an individual might make when learning a particular concept, in this case, matrix algebra concepts. However, mental constructions of a particular concept do, to a certain extent, reveal the nature of understanding that an individual has of the particular concept. Understanding mathematics is a delicate and deep process (Stewart & Thomas, 2008). Research has indicated that learning based on understanding is more enduring, more psychological, more satisfying and more useful in practice than rote learning. Skemp (1971) differentiate between two types of understanding, namely *instrumental understanding* and *relational understanding.*

Generally speaking, aspects of knowledge and skills can be understood in various ways. Usiskin (2012) discusses four dimensions of understanding: (1) skills – algorithm; (2) property-proof; (3) use application understanding; (4) representation metaphor understanding. The first refers to procedural understanding of mathematical concepts. He emphasised that this type of understanding is important in mathematics, because it gives an individual the ability to make all sorts of decisions, such as: which method is suitable to use; how many ways there are of getting the answer; and in performing skills such as manipulation of symbols to obtain the correct answer. This is what Van de Walle (2007) called *computational fluency*. However, Van de Walle did not consider this type of understanding to be procedural, since to develop computational fluency requires an individual with problem solving skills. The researcher concurs with Van de Walle where he notes that having the ability to think about different heuristics of solving the problems means that one has conceptualised the concepts, and can therefore draw from other concepts to solve a particular problem.

The second dimension refers to conceptual understanding. In mathematics, conceptual understanding concerns the ability to identify the mathematical properties that underlie and describe why a particular way of obtaining the answer was successful (Usiskin, 2012). The third dimension refers to modelling. Here Usiskin argues that this type of understanding is different to skill-algorithm or property proof, because in many cases, the mathematics learnt at school does not assist most people to identify its use in their own daily lives. Therefore, the use must be taught before most students realise what to do. It is in a fact the case that mathematical modelling is important in mathematics. However, here it is argued that mathematical modelling is different to conceptual understanding. Conceptual understanding encompasses both the ability to identify relationship between concepts and the application of concepts within and outside the context. De Villiers (2007) argued that the use of modelling would help in developing higher order thinking ability among students, and that it develops conceptual understanding. However, Usiskin (2012) seems to disagree with this line of argument, as he pointed out that application involved different kinds of thinking, but not necessarily higher or more difficult kinds.

The fourth dimension refers to the ability of students to represent the concept in different ways, such as by means of pictorial representation or metaphor. Usiskin (2012) believes that this type of understanding precede acquisition of the other types, because if students are brought carefully to understand in the representational sense what they are doing, they will ultimately be better with representation skills. Weber (2004) endorses this view, as he advocates building an intuitive understanding of a mathematical concept before giving it a definition. Although the researcher agrees with Usiskin, it is the researcher’s view that conceptual understanding in mathematics encompasses the last three dimensions.

According to Ernest (1991), mathematics is a symbolic language in which problem-situations and solutions are found and expressed. This is also echoed by Usiskin (2012) where he notes that mathematics is, among its many other attributes, a language of discourse. It is both a written language, and a spoken language. Therefore, in learning and understanding of mathematical concepts, one needs to be able to comprehend the language used. For example, to conceptually understand computation of fractions, one needs to have the correct concept definition of what a fraction is. According to Dubinsky (1991), understanding a concept has to do with relations between the mental constructs, together with interconnections that an individual uses or fails to use in a problem. This means that the types of mental constructs that an individual has or has not made reveals the level of understanding of the individual of a particular learnt concept. For example, in matrix algebra, when an individual is able to identify the order of a matrix and explain it clearly, this shows that he or she has a working action conception of matrices.

1.4.1 Procedural understanding and conceptual understanding

In South Africa, great concern is expressed over the way in which students perform in mathematics. It is often possible to hear lecturers complaining about students' poor results in mathematics, complaining that they are ill-prepared for advanced mathematics. The problem appears to be rooted in the kind of understanding the students seem to have of particular mathematical concepts. In most cases it seems that the focus of students is on getting the answer, rather than on understanding the method used. In the study conducted by Ndlovu (2012). The findings showed that learners tend to memorise rules and to apply them without first constructing their meaning. Instead of learning for the sake of understanding, students devise coping skills that will sustain them in the case of a lack of understanding. Tall (1997) has noted that students develop coping strategies such as computation and manipulative skills that will get them through the next test or examination when faced with conceptual difficulties. These coping skills, such as manipulative approaches and drills, help students to pass through the examination without engaging in problems that involve insight and understanding (Jojo, 2010). Students develop conceptual understanding as a result of responding to problem situations by making mental constructions of mathematical objects and processes, and using them to make sense out of the problem and trying to solve it (Dubinsky, 1997). Naidoo (2007) suggests that to change the way students learn mathematics, there is a need for alternate methods of instruction so as to enhance teaching and learning for understanding. The researcher has observed that in the learning of concepts in matrix algebra, students can effectively apply the procedural techniques, but lack the conceptual understanding of the learnt concepts. Biggs and Tang (2007) refer to this as surface learning, as opposed to deep leaning.

The ever-present goal of mathematics instruction is to help students develop a well-organised collection of versatile mathematical procedures that students can call upon to solve problems in a variety of situations (Carpenter and Lehrer, 1999). Unfortunately, many students that reach university have not yet developed such skills. Instead, they have mastered a collection of rules and algorithms which, in most cases, becomes a barrier in their learning as they encounter difficulties in sifting through the cluttered memorised techniques and procedures to select the appropriate procedure to use for a particular problem. For example, my first year students were asked to find the transpose of matrix A; some students used elementary row operation to find it. This makes it clear that students had memorised certain algorithms, but did not understand the goal, the meaning and appropriate use of such algorithms. Also, students knew that 1 is the identity element of multiplication, but had difficulty in relating that knowledge to finding the inverse of a matrix. In many instances, the researcher observed that students may seem quite confident in using a procedure in a familiar context, but have difficulty in seeing how the same procedure can be used in another. For example, students could effectively determine the determinants of matrix, but failed to apply the knowledge gained in determining the inverse of a matrix. This is also highlighted by Hasenbank (2006) as he pointed out that students becomes quite proficient at using a procedure in the context in which they were drilled to do so, but have difficulty seeing how the procedure can be extended to other contexts. In some cases, students may be quite proficient in a procedure, but otherwise lack the understanding of the underlying structures of the concept; after which they apply it incorrectly to other concepts or contexts. For example, in arithmetic, students have the knowledge that multiplication of numbers is commutative; and because they have learnt that scalars are real numbers, they automatically apply that knowledge to multiplication of matrices, ignoring the order of the matrices that are being multiplied. Thus, students can compute with ease, but lack the ability to see the deep connection between concepts.

In South Africa, complaints of poor matric results in Mathematics have been attributed to the learners' lack of conceptual understanding of mathematical concepts, and the university students emerging from the schooling system still have the preference of procedural knowledge over conceptual knowledge. Hobden (2006) defines conceptual understanding as an integrated and functional grasp of mathematics, resulting in the ability to see connections between ideas and a bigger picture of procedures. Thus, when students have conceptualised a concept, this would mean that they could identify the link between concepts, adopt different heuristics that could be useful to solve a particular problem in a particular context, and be able to link the facts in mathematics. In mathematics, knowing procedures alone is not enough for students to become progressive mathematics teachers, therefore another important goal in mathematics instruction is to help students develop the connection between procedural knowledge and conceptual knowledge.

In many instances when students produce correct answers to problems, it is assumed that they understand the concept, but research has shown that it is not always the case, where these procedures are often learnt by rote. According to Hasenbank (2006), students do not often develop deep knowledge of procedures, even when they learn to compute with them efficiently. The researcher seem to share his view on this aspect, because in many cases, when students are asked to apply the procedures in other settings, they seem not to understand the relationship between the procedures. In the study conducted by Brijlall & Ndlovu (2013) with matric students regarding the construction of knowledge in optimisations, in cubic functions learners could find the maximum/minimum value, but when the same question was posed in a quadratic function, they couldn't provide the minimum value. The learners' understanding of the derivative and the minimum value was only subjected to a cubic function. Also, during interviews, when asked why the derivative was made zero, many could not explain, but in every problem concerning derivative that they solve, they know that they have to equate the derivative to zero and solve for the value of x. Though they could efficiently compute the derivative of a function, it was apparent that the derivative schema they had developed was an impoverished one. This also seems to be the case in matrix algebra, where in a pilot study, we found that students could evaluate determinants; but when required to do the application of determinants to other concepts like finding an inverse, they seemed to struggle, as they could not see the interrelationship between the operational concepts. It is possible for students to develop well-automatised procedural knowledge that is not strongly connected to any conceptual network (Hasenbank, 2006). Hasenbank (2006) further argued that students who lack understanding of the learnt concept may be able to perform quite well on tests of familiar procedural skills, but their knowledge is fragile, inflexible and soon forgotten. Hiebert

& Lefevre (1996) assert that procedures learnt with meaning are those linked to conceptual knowledge, namely concept image and definition, which refers to everything associated in a person's mind by means of mental pictures, properties, mental representation, contexts of application and even statements. The above statements clearly emphasises that merely knowing rules and algorithms proves insufficient knowledge to do mathematics. The growth of mathematical knowledge involves a process of taking new concepts and making connections between different pieces of knowledge that have already been internally presented (Berry, Lapp & Nyman, 2010).

Both procedural knowledge and conceptual knowledge are important in the learning of mathematics. Conceptual and procedural knowledge develop iteratively, with advances in one type of knowledge supporting the advances in the other type, in a hand-over-hand fashion (Rittle-Johnson, Siegler & Alibali, 2001). Dubinsky (1997) suggests that before pedagogical strategies are considered, the particular concepts that give students difficulty in linear algebra need to be analysed by means of research, so as to determine the specific mental constructions that students might make in order to understand these concepts. This study aims to explore such in the case of matrix algebra so that the pedagogical strategies used in the teaching of matrix algebra can lead to students making effective constructions for the solution of related problems.

1.5 Beliefs about mathematics

In the teaching and learning of mathematical concepts, individual beliefs about the nature of mathematics play a vital role in the conceptualisation of mathematical concepts. In most cases, especially at school level, mathematics is presented as a set of rules that needs to be memorised. This evidence can be seen in the mathematics textbook. This set up of textbooks creates a view among students that mathematics is an external body of knowledge. This view is not totally wrong, but it does not provide students with the opportunity to formulate larger ideas about essence of mathematics.

Mathematics is a universal body of knowledge. It encompasses knowledge that is centred on concepts such as quantity, structure, space and change (Naidoo, 2011). It can be integrated with many fields such as Natural Sciences, Engineering, Medicine and Social Sciences. Naidoo (2011) raises another dimension, noting that mathematics is immersed in the science of patterns and that these patterns are not only found in numbers, but also in space, science, computers and imaginary abstractions. For this reason, it is our duty as mathematicians to explore these patterns so as to formulate conjectures and establish truths. It is at this point where mathematics students will be able to understand how mathematical concepts are linked. As a result, they will see mathematics beyond just the application of rules. According to Samson (2007), thoughts about patterns are seen as the essence of mathematics, and the language in which it is expressed, and therefore it is important that everyone involved in the discipline has the skills and knowledge to understand the science of patterns in mathematics that form its language. The language of mathematics plays a vital role in the conceptualisation of mathematics. Sahin & Soylu (2011) reiterate this, noting that mathematics is a universal language, which has been formed as a result of the studies of scientists, which have unique rules, and provide communication between all people in the world, regardless of whether they practice in the field of mathematics or not.

1.5.1 Teacher beliefs

The importance of mathematics is undisputed. Many researchers have discussed the importance of mathematics and the way in which individual beliefs have an impact in the teaching and the learning of the subject. It has been widely acknowledged, as with other subjects, that what teachers believe about mathematics influences their teaching of the subject. Dossey (2007) has argued that the conceptions of mathematics influences the ways in which society views it and this may well influence the teaching of mathematics. The individual's understanding of the value of mathematics will guide the way in which one views and practice of mathematics. Presmeg (2002) has argued that beliefs about the nature of mathematics either enable or constrain the bridging process between everyday practices and school mathematics. This view is argued for in the current study, because I have been able to observe as a teacher that, if one views mathematics as a way of thinking in order to solve problems, problem solving itself will be used as the premise and teaching methodology for helping students learn mathematics as a system of thought. Seeing mathematics as a set of interrelated system mechanisms enables teachers to help students to see the link between everyday practices and school mathematics more clearly, and learn to connect these ideas through modeling (Ollerton, 2006).

Mathematics teachers hold different beliefs systems, but these beliefs are not isolated from one another, but are interrelated (Beswick, 2006). He describes three kinds of belief system that teachers have about mathematics. The first, the centrality of a belief, is a function of the strength of the number of its connections with other beliefs. This means other beliefs are generally the consequence of the central belief and are dependent on the central belief. For example, the teachers' central belief of mathematics is that of an external view, as described by Dossey (2007), which sees mathematics as a fixed body of knowledge that is presented to students, where the teacher's focus is on directing students towards obtaining correct answers. The teacher might have other beliefs, such as discovery learning, but the emphasis will always be on accomplishing the procedural techniques correctly. Although other beliefs can be adopted, the central belief will always suffice in the teaching and organisation of the classroom, because central beliefs are difficulty to change. The second kind of belief, according to Beswick (2006), is that of clustering, where beliefs are held in groups. Lastly, Beswick discussed the basis on which these beliefs are held, where some are evidentially held and some are non-evidentially held. The latter shows a resistance to change. It is important that teachers are aware of their beliefs system, so as to be able to reflect on the way in which they impact on the students' conceptualisation of mathematics.

1.5.2 Student-beliefs

Belief is seen as an individual psychological state regarding the truth of particular proposition or personal premise for specific objects. It is the result of reflecting on actions that may or may not be knowledge-based and intellectually verified (Liu, 2010). Based on the number of research studies done, it is has been the common knowledge that individual mathematical beliefs play a major role in the mathematical thinking and behaviour. The study conducted by Kardash & Howell (2002) indicates that the way in which students view the process of learning as clear cut and unambiguous, showed that they tended to believe firstly, that memorisation plays a major role in learning of mathematics, and secondly, that knowledge can be attributed certainty. This differs to the study conducted by Unlu & Aktas (2013), where students believed that mathematics increased practical intelligence, and therefore according to these type of students mathematics is concerned with intelligence, as oppose to learning mathematics by heart. What is evident here is that both these studies show that the way students view mathematics is likely to have a huge impact in their learning and understanding of mathematical concepts. Students and teachers' beliefs impact largely on the teaching and learning of mathematics. When teachers and students have contradicting views about the nature of mathematics, it is possible that this might cause a barrier in the learning and understanding of mathematics. If the students believe that in mathematics the important aspect is to get the techniques right, the students will not be concerned about seeing the interrelationship between concepts. At all times, the students will want the teacher to give them formulas they can use to solve a given problem, regardless of how the teacher is trying to encourage them construct knowledge for themselves, and vice versa.

The studies by (Yong-Loveridge, Sharma, Taylor, Hawea, 2006; Celluci, 2013; Liu, 2010; Unlu & Aktas, 2013) on students and teacher perspectives on the nature of mathematics emphasised that in most cases, student beliefs about mathematics are a result of their experiences in mathematical classes. This means that teachers' beliefs have either a direct or indirect influence on student beliefs. The teachers' conception of mathematics will determine the degree to which he/she communicates the language of mathematics to students, and this will impact on how students experience mathematics. An individual's emotional experience is one of the most important factors in the information of belief (Unlu & Aktas, 2013). This means that the way students experience mathematics in their lives, whether in their everyday lives, or in the classroom environment, will impact on their formation of beliefs about the nature of mathematics.

1.6 Significance of the study to mathematics education

The literature on the teaching and learning of matrix algebra reveals that this phenomenon is relatively unexamined in the South African context. This study, through the use of APOS theory, draws attention to the genetic decomposition that mathematics teachers can use to analyse the level of knowledge constructed by students about or for the learnt concept, and therefore, to plan their teaching material and pedagogy in a way that will help in enhancing learning and development of necessary schemas. The study is expected to bring about understanding to of mental structures that students have of the learnt concepts in matrix algebra in the South African context. This research study explores the complexity and the nature of mental structures students are able to construct when learning concepts in matrix algebra through the use of interpretive paradigm within this qualitative study. The theoretical work of researchers such as Asiala et al (2004), Dubinsky (1991), Piaget (1978), Tall (2008), and Tall & Vinner (1981) within the field of constructivism was explored. The study highlights the usefulness of such theories, especially the framework for research and curriculum development linked to APOS theory by Dubinsky, which is the extension of Piaget theory on reflective abstraction in the conceptualisation of mathematical concepts. At the same time, it broadens the scope of how these theories overlap and shape the learning of advanced mathematics.

1.7 Terminology and concepts

The terms and concepts that are fundamental in this study are outlined and discussed in more details as the study unfolds. Each term and the concepts used in the study are important in its development.

Systems of Linear Equations

A linear equation in the variables $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots \dots, \boldsymbol{x_n}$ is an equation of the form

$$\boldsymbol{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

Where $\boldsymbol{a_1, a_2, \dots\ a_n}$ ***and b*** are real or complex numbers.

Matrices

It is a set of scalar quantities arranged in a rectangular array, containing *m* rows and *n* columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

This matrix is of the order $m \times n$.

Determinant

The *determinant* of a *square* matrix **A** = [a_{im}] is a number denoted by |**A**| or det *(***A***)*, through which important properties such as singularity can be briefly characterised. This number is defined as the following function of the matrix elements: |**A**| = det *(***A***)*.

APOS

APOS stands for *Action- Process-object-Schema*. These are mental structures that reveal the level of conceptualisation of the learnt concepts by an individual. This is discussed intensely in Chapter Three.

Action: refers to a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step by step instructions on how to perform the operation. For example, in determining whether the system has a solution or not, a student operating at an action stage will have to firstly do the calculation and explicitly find the solution and substitute it back before he/she can decide whether the system has a solution or not. Action is based on rules and algorithms, where a rule is practised repeatedly until it becomes routine, and takes place with not much thinking. In the action stage, the manipulation of entities is thought of as external, and the student only knows how to perform an operation from memory or a clearly given instruction (Weyer, 2010). The action based student cannot see the whole picture, and has not made sufficient connections to predict the results (Stewart, 2008). This means that student with action conception of a concept have limited understanding of the learnt concept. By saying at an action stage that limited knowledge is constructed does not imply it is not important, because this is where learning starts to develop. Stewart (2008) has argued that actions play a crucial role in the early stages of understanding a concept.

Process: describes when an action is repeated and the individual reflects upon it; an action that takes place entirely in the mind is internal, and may be interiorised as the process. This allows the students to undertake transformation without external cues. An individual who has a process conception "can think of performing a process without actually doing it, and therefore can think about reversing it and compositing with other processes" (Dubinsky & McDonald, 2001, p. 276). According to Stewart (2008), at this stage, the student is able to predict an outcome, invent shortcuts, and even describe the action verbally. This is just done by thinking only without performing any calculation. A student can think of a solution in more than one way, use a single method to find the solution, and verify the results using another method. That means the student can easily generalise the actions. In a system of equations, a student can decide if the system has a solution or not, or if it has infinity of solutions, by analysing the augmented matrix given only.

Object: An object is perceived as an entity upon which action and processes can be undertaken. The student can reflect on operations applied to a particular process, becomes aware of the process as a totality, realises those transformations, and is able to construct such transformations. Then we say the process has been encapsulated into an object. For example, a student operating at an object stage will be able to represent the solution using different representation, can construct the system from one, infinitely many or no solution, respectively, and can clearly explain why the system $A_x = b$, A is $m\ x\ n$ cannot not be solved using Cramer's rule or explaining how the system can change from being consistent to inconsistent, or vice versa. Furthermore, based on APOS, to perform an action or process on an object, in order to use its properties, it is necessary to de-encapsulate the object back to the process from where it came (Asiala, et al., 2004). The idea of process and object has been extended by Gray & Tall (2001) to formulate a notion of precepts. Dubinsky (2001) noted this about learning a given theory, namely that it must be applicable to other phenomena. This notion of procept is about amalgamation of three components: a process that produces a mathematical object, and a symbol that represents either the process or the object. This definition allows the symbol to evoke either a process or the object, so that a symbol such as 2+3 can be seen to evoke either the process of addition of two numbers, or the concept of sum (Grey et al., 1994). Although this notion seems similar to APOS, Tall (1999) asserts that it has many differences, such that no one is able to build their cognitive structures from embodied notions alone, nor are they able to do so from encapsulating a mathematical process. Although he agrees that the focus should be on cognitive structure, this does not imply that the mathematical process involved must first be given and encapsulated before any understanding of the concept can be derived (Tall, 1999). In other words, not all mathematical knowledge can be constructed from action-process-object strand. Some students can construct their mathematical thinking from properties.

Schema: it is an individual's collection of actions, processes, objects, and other schemas, which are linked by certain general principles, to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept. Asiala et al. (2004) asserts that an individual's schema is the totality of knowledge which for him/her is connected consciously or unconsciously to a particular mathematical topic. For example, an individual may have an inverse schema, determinant schema or matrix schema. The schemas themselves can be treated as objects and for a larger schema, and in this case, the schema has been thematised (Stewart, 2008). It is worth noting that the idea of schema is similar to Tall and Vinner's (1981) idea of concept image. Having constructed the concept image, an individual is able to understand the structure of a concept, and see the connections to other concepts. The APOS stages are considered to be hierarchical in the sense that an individual is expected to pass through action, process and object stages in order to develop a schema for mathematical concepts (Siyepu, 2013).

Genetic Decomposition

Genetic decomposition refers to the structured set of mental constructs which might describe how the concept can develop in the mind of an individual (Brijlall, Jojo & Maharaj, 2013). The genetic decomposition designed in this study is based on the researchers' experiences of the particular concept and it does not necessary represent how trained mathematicians understand the concepts. This also discussed further in Chapter Three.

Learners - School going learners

Students - People who are studying at any institution of higher learning

Teachers - referring to educators teaching at school level

Lecturers - referring to educators teaching at any institution of higher learning

Pre-service teachers - students studying towards becoming school teachers. For this study the term students and pre-service teachers will be used interchangeably.

1.8 Research questions and objectives of the study

The main aim of the study was to explore and describe the nature of mental constructions that are necessary for pre-service teachers to make for successful accumulation of knowledge in matrix algebra. The following objectives were explored with the aim to achieve the focus of the study. These were the objectives of the study:

- To explore the level of mental constructions that pre-service teachers make while learning matrix algebra, and ascertain how these link to the proposed genetic decomposition.
- To explore at which level pre-service teachers are able to make the necessary mental constructions in order for them to develop conceptual understanding of the concept.
- To identify the difficulties that pre-service teachers experiences in learning of matrix algebra concepts which can become a barrier in making the required mental constructions.
- To explore the impact of concept images in assisting pre-service teachers to make the necessary mental construction that will lead to conceptual understanding of the concept.

In order to address these objectives the study serves to answer the following critical question.

How do pre-service teachers' **mental constructions** of concept in matrix algebra concur with a preliminary genetic decomposition?

To unpack this critical question, the following sub-questions were asked:

- *What levels of conceptualisation of action, process and objects are reflected by pre-service teachers' mental constructions of matrix algebra?*
- *What difficulties do the students experience in their effort to construct the necessary mental constructions in matrix algebra?*
- *To what extent do the students' mental constructions of action, process and object link with the preliminary genetic decompositions?*
- *What characteristics of the schema displayed by the pre service teachers are adaptable to a genetic decomposition of matrix algebra?*

The study adopted the use of activity sheets in collected data and the pre-service teachers responses were the main source of data collection, together with semi-structured interviews. Video and audio recordings were used to strengthen the reliability of the study. A coding system was implemented in order to analyse and discuss data collected.

1.9 The scope of the study

Linear algebra is one of the topics introduced to students in their first year of university study. It is not part of the South African school's curriculum, but it has a link to some of the concepts taught in the school algebra, such as the solution of linear systems of equations. Knowledge of linear systems of equations is needed throughout the whole course of linear algebra, but international literature such as Cutz (2005) has indicated that many students have difficulties understanding this concept, as well as with linear algebra as a whole. The study on this phenomena has not been done in the South African context. Therefore, this in-depth study has been undertaken in order to explore pre-service teachers' mental constructions of concepts in matrix algebra, and the application of these concepts in solving the system of equations using Cramer's rule.

This research study was limited to a group of students who has enrolled at the university and met the minimum requirements of 60% matric pass in mathematics to register as FET mathematics teachers. This is the first module out of the six modules that they are required to do to qualify as FET mathematics teachers. Matrix algebra is one of the topics that students learn in this module. Some of the topics have relevance to what they have learnt in the schooling system, but they will be encountering the matrices and determinants for the first time. A group of 31 students of mixed abilities, mixed race, mix gender and mixed culture participated in this study. Most of the participants here are in their first year of study, with a few students in the second year of study. Those participants in their second year of study are those whose matric results were below the minimum requirements, and therefore were required to first do the foundational module in order to meet the minimum requirements, as well as those who did not pass the module in their first year of study.

1.10 Overview of this study

In determining the appropriate approach to the theses, the following structure has been used. The thesis comprises seven chapters, a bibliography and its appendices. The chapters are as follows:

Chapter One introduces the background and purpose of the study as well as motivation for doing this research. In addition to this, the chapter presents some relevant literature on the nature of understanding a mathematical concept, as well as the impact of beliefs in the teaching and learning of mathematics. Furthermore, it introduces the objectives of the study and the critical research questions. Moreover, the terms and concepts are introduced, together with the overview of the whole study.

Chapter Two presents the relevant literature reviewed based on the area of exploration. The nature of linear algebra, conceptual perspectives on knowledge constructions, and students' difficulties with the concepts processes and pedagogical instructions that could be applied in linear algebra classes, are described. Studies that used APOS theory as the theoretical framework in the learning of linear algebra are summarised. These studies motivated the use of APOS theory for this study.

Chapter Three presents the theoretical framework for this study. The theory that impacts on this study is discussed. More specifically the APOS theory, as an extension of reflective abstraction, together with the framework for research and curriculum development, is discussed. The three components of framework for research and curriculum development are presented and discussed. The relevance of APOS theory in the study is clearly explained, and the proposed genetic decomposition is indicated, where it is shown how this links to the framework for research and curriculum development.

Chapter Four presents the research design, research methodology and procedures undertaken for this study. This chapter outlines and summarises the research design and research instruments employed. The preliminary process involved with respect to the pilot study, research paradigm and how this fits within the study is then presented. The data collection and analysis together with sampling and location of the study are discussed. The limitation of each data collection instrument and limitation of the study as well as the reliability of the study is presented.

Chapter Five focuses on the validation of the research instrument used in Phase 1 of the study. The findings on this preliminary study are outlined in detail and how this affected the initial genetic decomposition as indicated in this chapter.

Chapter Six presents the analysis of students' responses from activity sheets and interviews. Furthermore, this chapter explores the findings and implications of this research study. This chapter also aims to explore and respond to critical questions of the study and the analysis of the video clips.

Chapter Seven presents the discussion and conclusions that were drawn based on the overall study. The limitations and recommendations made in the study are also presented in this chapter.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In the previous chapter, the background, purpose, rationale and overview of the study were presented. In addition, literature on the nature of mathematics, teaching and learning, as well as the understanding of mathematics was detailed and discussed. In this chapter, the literature informing the study and the implication of literature reviewed is discussed under three distinct sections. The first section focuses on abstract algebra and the nature of linear algebra. This includes research on the teaching and learning of linear algebra. Furthermore, studies on matrix algebra concepts such as matrices, determinants, and systems of linear equations are cited. This literature is reviewed in an effort to identify students' conceptualisation of linear algebra concepts and the difficulties experienced in the learning of linear algebra. Research into the teaching and learning of mathematical concepts in school algebra, calculus and geometry is relatively broad in South Africa. However, the area of linear algebra specifically is still relatively underdeveloped, both internationally and particularly in South Africa. Although this body of literature is limited, internationally it covers a wide spectrum of topics. These range from the historical development of linear algebra; discussions about the abstract, formal and unifying nature of linear algebra; various modes of representation and modes of thinking in linear algebra; students' difficulties in the conceptualisation of linear algebra concepts; and various recommendations of pedagogical strategies. While so much has been done and is of utmost importance in the enrichment of the field of linear algebra, relatively little has been done to explore students' mental constructions of concepts such as determinants and system of equations. Thus, only literature considered relevant to the study is reviewed in this chapter.

The second section focuses on knowledge constructions in mathematics. This literature is reviewed to describe the mental processes an individual undergoes in the learning of mathematics. The aforementioned develops the argument around mental constructions in matrix algebra within the scope of the study. Finally, in the third section, I conclude by providing the summary of recent studies investigating students' mental constructions in linear algebra.

2.2 Abstract algebra and the way in which it can enhance the learning of mathematics

Abstraction is a complex concept, which has many facets, and this is especially true in the context of mathematics education (Hazzan, 1999). According to Leron & Dubinsky (1994), abstract algebra is the first undergraduate mathematical course in which students are required to go beyond learning "imitative behavior patterns" for mimicking the solution of a large number of variations (p. 268). These authors further argued that is in abstract algebra course that students are asked to deal with concepts defined and presented by their properties, and by an examination of which facts can be determined just from the properties alone. Students are therefore expected to have developed mental strategies that enable them to mentally cope with the new approach, as well as with the new kind of mathematical objects. Scholars and psychologists have different perspectives about what abstract algebra is (Berth & Piaget, 1966; Frorer, Hazzan & Manes, 1997; Noss & Hoyles, 1996; Tall, 1991). However there is an agreement that abstract algebra is a difficult concept to teach, and that students have difficulties with the acquisition of abstract algebra. There is also broad agreement that the level of abstraction differs. In reality, certain students of mathematical concepts find them more difficult to conceptualise than the other concepts they are required to learn. For this reason, the difficulties experienced with the teaching and learning of abstract algebra will differ between respective students.

Abstract algebra is considered to be the generalisation of school algebra, where variables can represent various mathematical objects, such as matrices, numbers etc. (Findell, 2006). Arnawa, Baskoro, Kartasasmita & Surmano (2007) concur with Findell, and have further stated that in abstract algebra, expression and equations are formed through operations that make sense for particular objects. It also consists of axiomatic theories that provide opportunities to consider many different mathematical systems as special cases of the same abstract structure. Theorems and axioms provide opportunities for student engagement with formal aspect of mathematics, such as constructing, evaluating proofs, making conjectures through inductive and deductive reasoning (Findell, 2006). In learning abstract algebra, students make use of all the mathematical systems used in their previous learning. This can either be in arithmetic algebra, or school algebra. The common features held between the identity element of real numbers, identity element of a matrix and the identity of a function were being explored further and conceptualised through abstract algebra. Therefore, the course in abstract algebra is the place where students might extract common features from many mathematical systems that they have used in previous mathematics courses, such as calculus, and school algebra (Arnawa et al., 2007). Through abstract algebra, students learn the importance of precise language in mathematics (Findell, 2006). Learning abstract algebra, especially for future teachers, might have a positive impact in the teaching of mathematics at the school level. Teachers will be able to connect the ideas learnt in advanced mathematics to those learnt in school algebra.

2.3 Nature of linear algebra

According to Tucker (1993), linear algebra was regarded to be the first mathematics course in undergraduate mathematical curriculum, because its theory is so well structured and comprehensive. It is that branch of modern algebra concerned with the abstract system called vector space, and originates from the solutions of systems of linear equations (Ahmet, Cihan, Sabri & 2003). Dorier, Robert, and Rogalski (2000) developed the notion of a unifying and generalising concept. Regarding this notion, they state that:

> A unifying and generalising concept (or theory) is characterised by the fact that it did not emerge essentially to solve a new type of problems in mathematics (like the derivative or the integral for instance). Its creation and its use by mathematicians were motivated rather by the necessity to unify and generalise methods, objects and tools, which had been independently developed in various fields. Therefore, the formalism attached to a unifying and generalising concept is constitutive of its existence and creation. In other words, formalism cannot be avoided when learning linear algebra. This does not mean that unifying and generalising concepts have no intuitive background. In fact they have several such backgrounds which result from an abstraction of the common characteristics of various objects of a less formal nature" (p. 186).

The above indicates that, by nature, linear algebra consists of formal and unifying concepts. That is, the notion of connecting ideas together is inherent in the way in which the modern field took form (Wawro, 2011). Rabin, Sweeney & Wawro (2011) regard linear algebra as a useful field of mathematical study, because of its unifying power within discipline, and due to its applicability to areas outside pure mathematics" such as engineering, the physical sciences, statistics and also of its power to model real life situations (p. 2). It provides formal structure to analytic geometry as well as solutions of the 2×2 system of linear equations learned in high school (Turker, 1993). Therefore, it is a broad field, dealing with concepts like matrices, determinants, eigenvalues, eigenvectors, vectors spaces, vectors and transformation, and involves many theorems. Its real life application can be evident in teaching of vectors, matrices and application to volumes and areas etc. According to Hillel (2000), linear algebra is often the first mathematics course that students see that presents a mathematical theory, systematically built and reliant upon definitions, explicit assumptions, justifications, and formal proofs. Linear algebra studied in its own right can be highly abstract and formal, which in most cases, stands in contrast to students' previously computational-oriented course work, and which proves to be challenging for them (Nehme 2011). It is considered a useful field of study because it takes a students' background in Euclidean space, and formalises it with vector space theory, building on algebra and the geometric intuition developed in high schools (Turker, 1993). It also joins together methods and insights of geometry, algebra and analysis. Although some students might not have background knowledge in Euclidean geometry, but having studying linear algebra, they come to understand the importance of proof and its construction. As stated earlier, students are expected to develop mental strategies to deal with abstract nature of certain mathematical concepts. Linear algebra is one of such courses.

Berry, Lapp & Nyman (2010) discuss another dimension of what linear algebra entails. They argue that Linear Algebra is a course that deals with multi-representation. Hillel (2000) claimed that there exists three different modes of representation in linear algebra: (1) the abstract modes which utilises language of generalised theory including terms such as dimension, span, linear combination and subspace; (2) the algebraic mode uses concepts more particular to the vector space $\mathbf{R}^{\mathbf{n}}$, such as matrices, rank, and a system of linear equations; (3) the geometric mode uses language that is familiar from our own lived experiences, such as point, line, plane and geometric transformations. The learning of linear algebra provides students with the opportunity to understand different representations of mathematics. In the study conducted by Hong, Thomas & Kwon (2000) the findings revealed that at the beginning of the course, students did not fully understand the multi representation of the solution of the system of linear equations. The solution of the system of equation was presented algebraically, graphically and in tabular form. However, many students struggled to see that this is actually an alternative representation of the same structure. Understanding representations allows one to go back and forth between these modes of representation. It seems so important that these forms of representations are taught to students. However, Bogomolny (2007) has argued that these different representations are the source of errors and confusion for many students.

2.4 Teaching and learning of linear algebra

The teaching of linear algebraic concepts differs with contexts and universities. The decision about which concepts ought to be taught at which level is commonly decided by the university following the curriculum of a particular country, and recommendations from its Council of Higher Education. What is common though, is that all undergraduate students in their first or second year course of study of advanced mathematics come to learn linear algebra. The literature shows that the teaching of Linear Algebra had always been a challenge for many lecturers, because for many students, it is the first course that they encounter that is based on mathematical theory (Dikovic, 2007). High school mathematics is generally concrete, and the emphasis is on procedural techniques. Although the study of geometry is also abstract, at schools, teachers try to focus more on concrete aspects, rather than on abstract aspects. In many cases, learners are shown strategies which they have to imitate to solve riders, without constructing and evaluating proofs. When students reach university, it seems like a new world of mathematics, where theorems, axioms and definitions form the basis of understanding (Harel, 2000). Over the years the teaching of linear algebra had followed the traditional route of 'teacher talk'. Although linear algebra is a topic embedded in theorems and their equivalence, the teaching consisted of telling and showing students. Dubinsky (1997) points out that this does not account for meaningful teaching, and will not yield meaningful learning. It ignores the cognitive level, and the degree of development of each individual student (Dikovic, 2007).

In the 20th century, there has been a revolution in the teaching of linear algebra. The emphasis has been on developing conceptual understanding, learning for meaning instead of learning to carry out procedures with no meaning. This led to the development of Linear Algebra Curriculum Study Group (LACSG), which has aimed at making recommendations about the teaching and learning of linear algebra. According to Day & Kalman (1999) no one really knows the best way of teaching linear algebra. In reality, sometimes there is no single way of teaching the subject, where instead, using different teaching strategies and principles yields the required results. Harel (2000) described three principles for the teaching of linear algebra concepts: (1) concreteness principle; (2) the necessity principle; and (3) the generalisability principle. The concreteness principle states that for students to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the student's eyes. The necessity principle states that for students to learn, they must see an intellectual need for what they are to be taught. Students need to understand and see how the taught concepts will impact in their cognitive growth, and this needs to be meaningful to them. As Usiskin (2012) pointed out, for students to understand the mathematics taught, it should be applicable to their real life. The generalisability principle is concerned more with didactic decisions regarding the choice of teaching material than with the process of learning itself. Harel (2000) has argued that when instruction is concerned with a concrete model, the instructional activities within this model ought to allow – and encourage – generalisability of concepts. Therefore, developing the genetic decomposition of matrix algebra concepts will assist in making the useful decision about the teaching material to use, one that will hopefully assist students in making the necessary mental constructions of the taught concepts.

2.4.1 Alternative teaching strategies

The growing concern in the 19th and 20th century that linear algebra concepts do not meet the needs of the students, led to the formation of a Linear Algebra Curriculum Study Group (LACSG). This group was formed by mathematics lecturers/teachers, researchers and curriculum developers. Their first suggestion was that the curriculum ought to change altogether. Carlson, Johnson, Lay & Porter (1993) suggested that a Linear Algebra course must respond to the needs of client disciplines, and topics should be recognised as a service course and should closely relate application. This meant that the linear algebra curriculum should consists of topics and concepts needed by students, such as matrix operations; a system of linear equations; determinant properties of R^n; eigenvalues and eigenvectors, orthogonality, abstract vector space, linear transformation etc. In the university, where the researcher is employed, and where this study was conducted, the first three topics are taught with the inclusion of matrix inverses. Dubinsky (1997) agreed with the LACSG group proposal on the change of curriculum, but he added geometric interpretation of linear transformation.

In addition to that, the group suggested that a department should seriously consider making their first course in Linear Algebra a matrix-oriented course. Their argument was that the more practical matrix-oriented course would meet the needs of not only mathematics students, but also of students from various client disciplines. Stewart (2008) supported these recommendations. However, he highlight that the LACSG recommendation does not clearly explain how making the course matrix-oriented might benefit the students, which needs specifically might be met, and how? Dubinsky (1997) also raised concerns about this recommendation, as he felt that it would perpetuate what is already done in schools, giving students more computational procedures to imitate. He further asked how introducing Formalism as a second course, instead of as the first course, would help to deal with the students' difficulties with the subject. This study argues for the merit of the LACSG recommendations, while acknowledging Dubinsky's reservations. Matrices are less abstract, and the teaching of this can be related to real life situations. Therefore students are more likely to understand matrices better than the other concepts that are more abstract. Also, students' knowledge of school algebra and physical science can be used to generalise the learning of these concepts. Therefore, students will be able to use their prior knowledge to make sense of new knowledge; unlike formalism, which most students have no prior knowledge which to draw from. Although this is true, the difficulty is at what point one should decide to introduce Formalism. In most cases, like in the university, where the study was conducted, Linear Algebra is a semester-long course; where there might be not enough time to introduce Formalism. The researcher feels that at university level, the teaching and learning ought to be aimed at preparing students to reach the formal world of mathematical thinking, which goes beyond simply knowing the procedures. According to Dubinsky (1997), students develop conceptual understanding as a result of responding to problem situations by making mental constructions of mathematical processes and objects, and using them to make sense out of the problem and trying to solve it.

In ensuring meaningful teaching, Dubinsky (1997) proposes a shift from traditional way of teaching. He suggests that before any instructional strategy is designed, research is needed to determine the specific mental constructions that a student might make in order to understand concepts. The mental constructions made will reveal the level of understanding and the knowledge constructed about the particular topic. Therefore, it is imperative that at first, we understand the students mental constructions, how they make those constructions, and what makes them construct/fail to construct the necessary mental constructions. Then, the teaching strategies should be aimed at helping students make the necessary mental constructions. Klasa (2010) concurs, and indicates that there is a necessity for lecturers to proceed to a genetic decomposition of every mathematical concept in linear algebra before conceiving a pedagogic scenario that will bring students from “action”, to the more elaborated state of “process”, and probably make them reach the most abstract level of “objects”. This is one of the reasons that prompted the researcher in this study to explore the specific mental constructions that pre-service teachers might make when learning matrix algebra concepts.

Following Dubinsky’s (1997) suggestion, a new genetic decomposition of matrix algebra concepts was designed; one which was used to analyse specific mental constructions made by students in their learning of matrix algebra. Once these mental constructions were analysed, instructional strategies that aimed at helping students conceptualise matrix algebra concepts could be developed. The findings from the study by Arnawa et al. (2007) have indicated that students who were taught how to proof using the APOS approach perform better than those taught using a traditional method. One of the advantages of using APOS to inform instruction, as indicated by the authors, was that in APOS, topics are designed around the steps in mental construction, where students are actively involved in learning; while in the traditional approach, topics are not designed specifically but rather follow the ‘textbook approach’, in which students receive information passively. The findings confirm that using the APOS approach generally yields understanding among students. This study employed the same approach in the concepts of matrix algebra, not claiming that it would yield the same results, but gauging the applicability of these findings to a South African context, in concepts that are less abstract.

Bogomolny (2007) concurs with Dubinsky (1997) where the authors note that there should be pedagogical strategies that allow students a chance to construct their own ideas about concepts in linear algebra. In her 2007 study, Bogomolny used example generation tasks to teach the concepts in linear algebra. Students were asked to construct examples of linear algebraic concepts such as linear transformations, basis, linear (in) dependence, null space or column space. According to the author, the findings showed that example-generation tasks are a useful tool to explore and discover students' understanding of the linear algebraic concepts. It revealed student appreciation of the structure of the concepts involved, the connections students make between different concepts, students level of understanding according to the APOS theoretical framework, and students' existing concept images. Bogomolny (2007) further proposed that the example-generation task be used as a teaching strategy, because it allows students to construct their knowledge and understanding of the concepts. This, according to Dubinsky (1997), is missing in the teaching of linear algebra. He also argued that allowing students generate their own example is a valuable addition to undergraduate mathematics education. Through constructing examples, students get an opportunity to reflect on their own knowledge. The researcher concurs with Bogomolny (2007) because when students generate their own examples of the concepts, they are actually engaging with different structures of these concepts. That might help them conceptualise the concept, start realising that they could construct and deconstruct a mathematical problem, instead of copying examples from the textbooks. Implementing such pedagogy might assist students in acquiring important skills and knowledge that learning for understanding produces long term satisfaction, and one needs to keep on working on a problem time after time, until it makes sense in a broader context. Although this might be a valuable teaching strategy, it could prove difficult to implement in large classes. For this strategy to work effectively, cooperative learning should be used, which might be problematic to manage in large classes.

According to Hamdan (2005) the abstract nature of linear algebra can also be dealt with by emphasising the importance of writing in mathematics. He argued that writing puts students in control of personally structuring the meaning, and of establishing the necessary links between different representations of the concept. Expressing thoughts about the concepts in words can be powerful in developing students' conceptual understanding of the concept. The language of mathematics can be difficult to comprehend, because certain words have a different meaning in mathematics than they do in spoken language, such as the term 'limits'. Studies have shown that incorrect use of terminology sometimes causes barriers to learning (Brijlall & Maharaj, 2009a; Ndlovu, 2012; Tall, 1997). Asking students to explain their answers in writing helps them to be able to communicate their thought processes using the language of mathematics. All the above authors are echoing the same thing, noting that the teaching of mathematical concepts ought to be aimed at helping students make meaning of the taught concepts.

Researchers (such as Aydin, 2009; Day & Kalman, 1999; Klasa, 2010; Stewart & Thomas, 2008) have done extensive research regarding the way in which technology can be used to enhance the teaching and learning of linear algebra. Increasingly nowadays, many researchers in mathematics education are emphasising the important use of technology in the teaching and learning of mathematics. De Villiers (2007) points out that technology provides useful modelling tools, and notes that modelling is very important in the teaching and learning of mathematics. Since technology seems useful in the teaching of other mathematical concepts like geometry, calculus and algebra, then it could be used in the teaching of linear algebra. According to Aydin (2009) there are several roles that technology can play as a methodology for teaching. Like eliminating computational drudgery in application, providing environments for actively exploring the properties of mathematical concepts and structures. He further argued that computer programmes provide students a means to instantly and effortlessly perform linear algebra. It also frees them to concentrate on what computations mean, and when and why to perform them. This means that it encourages computational fluency among students. For example, when learning to evaluate determinants, the 2 x 2 and 3 x 3 determinants hand calculations can be manageable, but when solving determinants of an order > 3, the calculations can become messy and tedious. This caused students to spend most of their time trying to correct errors instead of making sense of the problem. Day & Kalman (1999) echoed the same by pointing out that computers are used for computation in meaningful application for visualisation, and that they provide environment for active exploration of mathematical structures. In addition to this, they further ascertain that through the use of technology in teaching and learning, teachers will be able to provide students with the first-hand experience, with real applications in realistic settings, and students will be able to explore beyond what is included in the textbooks. This would then make mathematics more meaningful to them, as they would see its relation to the world of mathematics, rather than simply as stumbling block in achieving their dreams.

Dreyfus, Hillel & Sierpinska (2000), as well as Klasa (2010) discovered that the use of technology was advantageous to students, where they seem to achieve more understanding of linear algebra when using it. According to the authors, this is normally not the case in the ordinary classroom.

Klasa (2010) further argued that technology makes tough mathematics concepts easier, and helps students become expert in their discipline. Using technology can be a way of dealing with some of the difficulties that most students may tend to have in the traditional set up, such as weak mathematical background, and poor algebraic and analytic skills (Hillel, 2000). De Villiers (2007) has emphasised that computing technology strongly challenges the traditional approach, which emphasises computational and manipulative skills before applications; but the lecturers and teachers need to be cautious that technology does not replace common sense. Hillel (2002) concurs, and points out that though computers are helpful with manipulating of matrices and solving system of equations, they do not provide an obvious means of helping students with understanding the abstract constructs of the general theory of vector spaces. Although computers are a valuable teaching tool, they cannot alone help students learn theorems and axioms. Seemingly, it can work well following the recommendations of LACSG made earlier. Also, Stewart (2008) emphasised that computers are a valuable tool and help students to visualise concepts, but are no substitute for thinking logically through theory. They further stated that the main aim of using technology should be to increase students' conceptual understanding, by investigating different ways of looking at the problem. Although technology does enhance learning in mathematics, the success of its use depends on the lecturers' knowledge of how to use it effectively to maximise students' understanding of linear algebra. It is very enticing to be told and to know the importance of teaching strategy, but it is fruitless if one doesn't know how to implement such a strategy. For its effective use, both students and lecturers need to be computer literate, or technology literate. If this is not possible, it can become a barrier to learning. Instead of focusing on the taught concepts, students might end up spending time learning to use technology rather than conceptualising the concepts.

Ahmet, Cihan & Sabri (2003) have suggested that visualisation can be an alternative method and powerful resource for students doing mathematics. It encourages visual thinking, and one basic aspect of visual thinking is the ability to move back and forth between the graphical and analytical representation of a problem (Naidoo, 2011). Exposing students to both graphical and algebraic representation of mathematical structures would allow students to develop different ways of thinking about it. Ahmet, Cihan & Sabri (2003) conducted a study investigating the role of visualisation in the teaching of vector spaces. In the study, students were divided into two groups. In both groups, the concept of vector space was presented geometrically and algebraically. One group had one hour taught geometrically and two hours taught algebraically, and vice versa, for four weeks. The findings revealed that students that had two hours taught geometrically had the ability to establish the connection between concrete structures and abstract nature of axiomatic language of vector space theory. These students could easily describe the concepts from geometric point of view, as well as from an algebraic point of view. The authors argued that including visualisation into the teaching process increases students' motivation towards the course.

2.5 Matrix algebra

Matrix algebra is the standard language of applied mathematics (Tucker, 1993). It is the basic topic in linear algebra. According to Tucker (1993), a vector space is the natural choice for a first algebraic system for students to study formally, because its properties are all part of student's knowledge of analytic geometry (p. 4). In the university where this study was conducted, topics such as matrices, singular and non-singular matrices, determinants, and system of equations are taught. Unlike that which Tucker (1993) has mentioned above for this university students have no choice of what subtopic of matrix algebra to learn. He pointed out that the properties of taught topics are all part of students' knowledge of school algebra, with the exception of determinants. This is a semester course in which matrix algebra and other mathematical topics are taught to first year mathematics students. This means that students have only a few weeks to conceptualise all the concepts covered under matrix algebra. In the learning of matrix algebra, students' background knowledge of arithmetic and school algebra plays a vital role. The concepts taught are mostly embedded in language such as geometric language of lines and planes, algebraic language of equations and matrices, and the graphical language of the tabular and the symbolic, which students need to comprehend in order to understand the concepts.

In South African high schools, learners learn to solve the system of two equations in two unknowns. At university the knowledge constructed of solving system of equations with two unknowns and two equations is generalised to solving the system of 'n' equations in 'm' unknowns or "n" equations in "n" unknowns. Also, computation properties, identities, etc. are generalised in the learning of matrix algebra. The knowledge gained at school level should help students to develop a deeper understanding of the concepts taught in matrix algebra. Sierpinska (2005) distinguishes two modes of thinking viz. practical and theoretical thinking. Practical thinking is goal-oriented and rooted in physical action, while theoretical thinking is more focused on objects of reflection. When learning matrix algebra, a student who works productively will focus on solving the problems presented. A student with theoretical thinking will attempt to find a strategy to solve all problems related to matrices. To be successful in the learning of matrix algebra, concepts need to be effective in both ways of thinking.

2.5.1 Matrices

Matrices occur in all parts of mathematics and its applications, and everyone working in the mathematical sciences and related areas need to be able to, for instance, "diagonalise a real symmetric matrix" (Stewart & Thomas, 2007, p. 202). Matrices are applied in every aspect of linear algebra, and remain closely associated with linear transformation and vector space. When solving the system of equations using Cramer's rule, the system needs to be represented in a matrix form. For example given: $2x + 3y = 5$ and $x + y = -4$.

In a matrix form A=$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$. Then, once it is represented in a matrix form, the determinant can be determined, and could lead to the solution of this system of equations using Cramer's rule. Also, using the same matrix, invertible matrices can be explored. In the teaching of matrices students need be made aware of the interrelationship between school algebra and matrix algebra. The computation of matrices in many cases uses the same properties of arithmetic algebra.

The understanding of matrices can be explained in five strands, as discussed by Berman, Koichu, Shvartsman (n.d.**),** which are: (1) formal understanding in which a student displays the capabilities to recall definitions and axioms needed to solve given problem; (2) instrumental understanding in which a student is able to use rules of computation of matrices; (3) representational understanding, in which a student is able to represents the system of equation in a matrix format and use the matrix structure to simplify the problem to be solved; (4) relational understanding, in which a student is able to relate matrices to other concepts; and (5) application understanding, in which a student can identify problems in which matrices can be used to solve the problem. Therefore, in the teaching and learning of matrices, it is useful to develop these five strands of understanding.

2.5.2 Determinants and their applications

Determinants arose from the recognition of special patterns that occur in the solution of system of linear equations. Each square matrix A is associated with a special number, called a determinant of order n, given by det (A) or simply $|A|$. The use of the letter depends on the matrix given. In the teaching and learning of matrix algebra, a determinant has many applications. Like matrices, it is used to solve the system of equation, and to determine invertible matrices. For example, A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, by using the adjoint, the determinants of order > 2 can be easily evaluated and used to solve other related problems. What is evident here is that the understanding of matrices and determinants will help students make sense of the other concepts. If the determinant concept is not fully understood, it will create barriers in the learning and conceptualisation of an invertible matrix, and in the system of an equation. Durkay, Senel, Ocal, Kaplan, Aksu & Konyaliogu (2011), in their study on pre-service teachers' multiple representation of competences about the determinant concept, revealed that students were aware of the related multiple representations of the determinants, thereby showing the basic understanding of the determinants concepts.

2.5.3 System of linear equations

Knowledge about a linear system of equations is used, and needed, throughout the whole course of linear algebra in the solution of different kinds of problems, and in understanding of most of the concepts related to this subject (Trigueros, Oktac & Manzanero, 2007). In the learning of system of equations, students are expected to generalise their knowledge of two equations in two unknowns, to a system with more than two equations and two unknowns. The methods learnt at school are suitable to solving the system of two equations. In matrix algebra courses, students are introduced to the use of matrix techniques to solve problems relating to system of equations of the order even $>\ 2$. The solution of the system of equations can be represented in different ways viz. in a tabular form, algebraically, or geometrically. Tucker (1993) has pointed out that linear algebra gives a formal structure to solutions of system of equations learnt at school. When students reach university, it becomes imperative that students are exposed in all these kinds of techniques, construct appropriate meaning for them to develop deep understanding of the concepts. To build rich relational schemas containing internal representations of the external one, students should experience the links and the sub-concepts of one-on-one, independent and dependent variables in each representation (Hong, Thomas & Kwon, 2000). In finding the solution of a system of $n \times m$ and $n\ \times n$ system of equations, Gauss elimination rule and Cramer's rule are used. Gauss elimination can be used for any system of equation, while Cramer's rule can be used for $n \times n$

equations, which can be considered as the limitation of Cramer's rule. Gauss elimination is not part of the scope of this study, so it will not be elaborated further here.

2.5.4 Cramer's rule

In a system of 'n' linear equation in 'n' unknown, where $Ax = b$, such that $|A| \neq 0$, the given system, will have a unique solution. The solution is given by:

$x_1 = \frac{|A_1|}{|A|}$; $x_2 = \frac{|A_2|}{|A|}$; $x_n = \frac{|A_n|}{|A|}$. For example given: $2x_1 + x_2 = 5$ and $x_1 + x_2 = 3$, where x_1 and x_2 are the unknowns. Therefore, to solve for x_1, the new matrix is used, which is obtained by replacing the entries in the 1st column of A, by the constants and determining the new determinants A_1. The solution for $x_1$is the quotient $\frac{|A_1|}{|A|}$ where $|A|$ is the determinant of the co efficient. The same applies to solving for x_2, where Cramer's rule can only be used when the system of equation has a unique solution. In comparison to other methods, there has been a criticism of Cramer's rule that it fails to provide accurate solutions. In response to this criticism, Habgood & Arel (2012) have argued that the perceived inaccuracies do not originate from Cramer's rule, but rather from the methods utilised for obtaining determinants. Another concern with Cramer's rule, as discussed by these authors however, was that when working with large matrices, Cramer's rule is generally impractical. The complexity of Cramer's rule depends exclusively on determinant calculations. Furthermore, these authors argued that Cramer's rule is unsatisfactory even for 2×2 systems, mainly because of rounding error difficulties; although they pointed out that if an accurate method for evaluating determinants is used, then Cramer's rule can in fact, be numerically stable. The proofs of eligibility and accuracy of Cramer's rule is beyond the scope of this study, however these limitations were highlighted because Cramer's rule is the method used in the data collection of the study, so as to make the reader aware of these discussions around this method. The arguments made by Habgood & Arel (2012) seemed to be valid in that the accurate and correct use of Cramer's rule depending on the methods used to evaluate the determinants. The incorrect or correct solution of the system cannot be entirely attributed to the use of the Cramer's rule, since this rule depends on the application of determinants

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2.6 Students difficulties in linear algebra

The teaching and learning of linear algebra concepts has raised many issues. One of these is the difficulties students have with conceptualising linear algebra concepts. In the next section, the literature reviewed on students' various difficulties is discussed. Research into student understanding of linear algebra concepts indicates that many students experiences difficulties when learning linear algebra. Many of studies have been conducted to unveil the difficulties that students have with linear algebra. It has been argued that the abstract nature of linear algebra seemed to be the source of students' difficulties. Like in other mathematical concepts such as geometry and calculus, research has shown that the use of formalism in mathematics causes barriers to learning. In his study, Hamdam (2005) pointed out that the main criticism students make of linear algebra course is the extensive use of formalism, as well as the overwhelming amounts of new definitions. The same is echoed by Britton & Henderson (2009) as they point out that students' inexperience with proofs, logic, and set theory is the source of students' difficulties with linear algebra. For many students, it is their first time to come across formalism in mathematics. As a result, they find it difficult to understand. Constructing and evaluating a proof is an important skill, and an important component of mathematics education. It promotes a deep understanding of mathematics (Arnawa, Baskoro, Kartasamita & Surnamo, 2007), because it helps students to learn how to think logically, thus making sense of what they learnt. However, at times, students find it too difficult to comprehend, and become too overwhelmed with definition and properties.

Harel (1997) has agreed with this contention, but has further argued that when it comes to certain concepts, the students' difficulties are not necessarily dependent on their ability to comprehend definition, but rather, arise when trying to use these concepts to solve other related problems like in linear (in)dependence. The same point is emphasised by Wawro (2011), who notes that the difficulties that students have with the subject is as a result not only of their relative inexperience with formal mathematics, but also of the formal and theoretical value of linear algebra content itself. Although the above arguments may be valid, it is nonetheless a reality that linear algebra is too abstract for students that they need to be able comprehend it before they can make sense of other related concepts. Dealing with concrete mathematics for all their high school life would impact their understanding of mathematical concepts that are too abstract, such as linear algebra. If students have never constructed or evaluated a proof in their learning of other mathematical concepts, and have never required exposure to concepts where they need to make conjectures, they will surely experience problems with concepts that require them to do that. Tall (2008) contends that when students learn a new mathematical concept, they use their prior knowledge to construct new knowledge. This means that students' prior knowledge and experiences plays a vital role in understanding and comprehending new concepts. For students to be successful in linear algebra, the knowledge of using the formal proof to solve other problems is essential. Their inexperience in working with proofs will make it difficult for them to even comprehend other related problems. Dorier (2002) pointed out that linear algebra is a formal theory and simplifies the solving of many problems. However, this simplification is only visible to the specialist who can anticipate the advantage of generalisation, because they already know many contexts in which a new theory can be useful. This confirms that one needs to be experienced in working with formalism in order to conceptualise concepts that are embedded in formal theory.

Dorier (2002) argued that not only is the abstract nature of linear algebra problematic for students, but also that the issue of the variety of languages, semiotic registers of representations, points of view and settings through which the objects of linear algebra are presented, seemed to cause lot of difficulty for linear algebra students. Cutz (2005) concurs, as he argued that in the learning of system of equations, students experienced difficulties with representing their solution geometrically, interpreting graphs and trying to pass from one representation to another. Duval (2006) defines semiotic representation as productions made by the use of signs belonging to a system of representation that has its own constraints of meaning and functioning. He emphasises that these representations are absolutely necessary in mathematical activity, because its objects cannot be directly perceived and must therefore be represented. Moreover, semiotic representation plays an essential role in developing mental representations, in accomplishing different cognitive functions, as well as in producing knowledge (Dorier, 2002). Pavlopoulou (1993) distinguished between three registers of semiotic representation of vectors: (1) the graphical registers (arrows); (2) the table register (column of coordinates); and (3) symbolic register (axiomatic theory of vector spaces). In the teaching of linear algebra and even in textbooks, translations between these registers are not taken into account (Pavlopoulou, 1993). Dogan-Dunlap's (2006, 2010) findings in their study of students' mistakes in linear algebra, revealed that lack previous of knowledge in mathematical structures and set theory to construct new concept, cause problems for many students. In addition, Carrizales (2011) further argued that because of different representation employed in linear algebra, students often struggle to understand, explain and relate the theory learned. Linear Algebra is a formal theoretical course, therefore it uses formal mathematical language. That means students need to both have knowledge of this language, as well as be able to comprehend it, so as to make meaning of it. To understand a theorem, students need to understand the language used.

A study by Alves & Artique (1995) showed that the tasks offered to students in textbooks and in classes are noticeably limited in terms of flexibility. Many mathematical tasks assigned to students focused mainly on one strand, namely accuracy. This does not give students enough room to cognitively think about relationship between concepts, because they focus instead on memorising algorithms. Their focus is on getting an accurate answer, even if they do not understand its meaning. Van de Walle (2007) emphasises the importance of computational fluency in mathematics, which consists of flexibility, accuracy and efficiency. The point raised by Alves & Artique (1995) is highly significant. The solution provided in many textbooks does not show different strategies that could be employed in solving tasks. This gives students the impression that the problems can be solved in only one way. As a result, students struggle to represent their solution in different ways. In his study, Stewart (2008) investigated the difficulties in understanding certain linear algebra concepts, and proposed potential paths for preventing them. They discovered that many students lack the ability to recognise concepts in different representations, and they struggle to move between various presentations, which show a lack of computational fluency among the students. That is one of the reasons that Dubinsky (1997) calls for change in the pedagogical instructions, in order to alleviate the students' difficulties in linear algebra. The discrepancies between the way students viewed mathematics and classroom instruction, which is based on formal structure, indeed contribute to students' difficulties in advanced mathematics (Tall, 1991). Bogomolny (2007) indicated that student's difficulties in linear algebra are linked to pedagogical approaches. Students are told about mathematics and shown how it works. The problem is that in most cases algorithms work even if their meaning is not understood. Students can multiply matrices but sometimes lack the understanding of the need for the respective order of the matrices.

Sierpienska (2002), in her experimental design study, gives some explanation as to the types of thinking that students display in learning linear algebra concepts, which also seem to contribute to the difficulties that student's experience. In her study, she talked of practical thinking which students display which she recognised as one of the sources of their difficulties. She claimed that students tend to think practical than theoretical. She stated that students who display features of practical thinking were unable to go beyond the appearance of the graphical and dynamic representation of what they were observing and manipulating. This meant that they could not make conjectures, and could manipulate the figure on the plane, but could not visualise it in the space. She further pointed out that these types of students have the tendency to base their understanding of an abstract concept on a prototypical example, rather than on its definition. That means they haven't internalised their thinking processes. They could produce solutions for routine exercises, but struggled to see the application of the learnt concept to other concepts or contexts. In his study, Stewart (2008) revealed that students were mainly thinking and representing their understanding in an embodied-symbolic world, in terms of the three worlds of thinking as described in Tall (2002). Their findings concur with Sierpinska (2002) regarding the level of thinking displayed by certain linear algebra students. Both authors pointed out that students were able to do relatively easy procedures, but that they had difficulty in solving non-routine types of problems. This was also evident in the study by Ndlovu (2012) with Grade 12 learners exploring their mental construction when solving optimisation problem. The author pointed out that when learners were required to explain their reasoning, they seemed to struggle, and instead just carried out procedures.

Harel (2000) noted that a substantial range of mental processes must be encapsulated into conceptual objects by the time students get to study linear algebra. For example, the mental processes of finding an inverse of a function must be encapsulated into an object so that students can develop the inverse function schema which then helps them to develop the understanding of matrix inverse. When students are unable to encapsulate these mental processes, they will be unable to reach the trans-object stage of thinking, as explained by Piaget & Garcia (1989). Instead, they will develop coping mechanisms to cope with examinations and tests, but not to conceptualise the concepts introduced.

Linear algebra is a volatile compound of languages, and this ought to be embraced by a mathematics student (Berry, Lapp & Nyman, 2009). In linear algebra, there are a number of theorems that students are required to understand in order to be able to comprehend the questions they work on and apply these theorems in solving problems in matrix algebra. The theorems are embedded in the language that, in most cases, is difficult for students to comprehend, where students end up feeling lost in the mathematical language. Thus, for the majority of students, linear algebra is nothing more than a catalogue of very abstract notions that they will never able to fully imagine (Parraguez & Oktac, 2010). Stewart (2008) discussed some of students' difficulties with linear algebra, one of which is that students were found to struggle to comprehend the linear algebra language used. Students tended to pick some familiar words, and then to use this to interpret the question, instead of comprehending the whole sentence or question. For example, when seeing the word linear, they quickly associated it with linear equations, and as result, they could not clearly define linear combination. As a result, what they learnt before seems to cause great difficulties in understanding new concepts. Tall (2008) refers to this as 'met before'. Another point that Stewart noted (2008) was the confusion displayed by students between English definition of the terms and mathematical definitions of similar terms. For example, in English, discipline the prefix "sub" means (1) part of, (2) below. In defining the term subspace, many students referred to a "portion of space or part of space". The mismatch in the use of language is not only a problem applicable with linear algebraic concepts. The study conducted by Brijlall and Maharaj (2009a), about pre-service teachers' understanding of single-valued-limit, indicated that students interpret the English word limit as meaning something beyond which nothing can go, and that this causes difficulty when trying to explain the term in mathematical context. This was also echoed by Tall (2002).

There seems to be an agreement that the main problem associated with the learning of linear algebra concerns the issue of formalism. Although there are other reasons for students' difficulties, the main problem seem to be the abstract nature of linear algebra, from which all other difficulties originate. One of the recommendations of LACSG was that concrete concepts such as matrices, which are considered less abstract, should be taught first. Now the question is: does that mean matrices are easy to understand? I, for one, seem to disagree. The truth of the matter is that students tend to do well, because there are algorithms to follow. Does that constitute understanding? In general, the main concern with the teaching and learning of mathematics is the development of conceptual understanding. Students need to conceptualise the mathematics learnt in order to understand its application to other contexts and to the real world. Dubinsky (1997) pointed out that we should be careful not to fall into the same trap of promoting imitative behaviour in the learning of advanced mathematics, while claiming to promote the application of mathematical concepts. Even when it comes to those concepts that are considered to be less abstract, research is still needed to explore how students construct knowledge for those particular concepts. Instead of generalising, Dubinsky (1997) made an important proposal, noting that before a concept is taught, the mental constructions of students about that particular concept ought to be investigated. This would assist in preparing teaching material that will help students make the necessary mental constructions, dealing with the identified difficulties, whether this be formalism or multi-representations, . This study follows Dubinsky's proposal, and is aimed at investigating the mental constructions made in the learning of linear algebra.

2.7 Students' misconceptions in linear algebra

Misconceptions are a cause of learning barriers in mathematics. Misconceptions are sometimes generated from within a students' concept image. Goris & Dyrenfurth (2010) have noted that misconception originates from prior learning. This means that before students learnt any mathematical concept, they have already constructed some ideas. The difficulty arises when the conceptions held are different from the accepted knowledge about the concept. Misconceptions are beliefs, ideas, and concepts that are in conflict with currently-held scientific thought (Mestre, 2004). He describes two contemporary views about misconception, namely: (1) as personal theories; (2) as the activation of knowledge pieces in response to context. Based on the first view, he ascertains that a student could possess coherent theories about mathematics and science. These cognitive entities are activated, and applied as bundled units and are remarkably consistent. For example, in arithmetic, algebra students construct knowledge that division makes smaller. This works well with whole numbers, but not with rational numbers. The constructed knowledge then becomes a barrier to learning, as students' understanding of division has developed some flaws, and is not applicable to the entire real number system. Such misconception results in the students' failure to conceptualise the concept, instead develop coping mechanisms for dealing with the division of numbers. The second view highlighted the fact that students possess pieces of knowledge that are acquired through experience or formal schooling, and that these are activated in response to a context. The change in the context thereby results in the new set of knowledge pieces being activated. By context, I refer to the familiarity/ unfamiliarity with previous questions or concepts. In arithmetic, algebra students learn that the multiplication of numbers is commutative. Since entries in matrices are real numbers, they might end up constructing the knowledge that multiplication of matrices is always commutative. According to Swam (2000), misconception is not wrong thinking, but is a concept in embryo, or in local generalisation, made by a student.

Many researchers in the field of linear algebra have identified misconceptions about the taught concepts in the subject, as one of the main reasons that lead to students' poor performance. Uygor & Ozdag (2012) conducted a study about the misconceptions displayed by students when solving examples in matrices and determinants. In their findings, they indicated that students confused matrix operation with determinant operation. Students were asked to show that det (B) = -det (A), when two rows of matrix A are interchanged to give matrix B. Instead of interchanging the rows of matrix A, students change the sign of matrix A. Generally, the students displayed a lack of understanding with regard to the relationship between the matrices and the determinants. In the teaching of matrix algebra concepts in this university, where the study was conducted, I have observed similar misconception among the students. Students' responses display the misconceptions about the use of notations, confusing $A^T with\ A^{-1}$. This is discussed further in the phase one findings.

Another misconception that Uygor & Ozdag (2012) identified was that students were asked to transform matrix A to matrix B by multiplying the single column or row by scalar k such that det (B) =k det (A); however, the students confused this with matrix operation, and multiplied all the elements of the determinant with k. These findings showed that the students had memorised the rules of operation in matrices, and that when similar contexts were presented, these were activated and incorrectly applied. This indicated that when students answer a mathematical problem, particular knowledge pieces are activated by the problem (Scherr, 2007). If the knowledge was not fully constructed, then it becomes a misconception. When the knowledge is constructed as pieces that are not coherent, the ideas seem to fluctuate, and are pliable, since they are independent of one another. However, Scherr (2007) considers these not to be permanent, as students do not stay committed to their conclusions.

Soylu & Sahin (2011) conducted another study with elementary school children on mistakes and misconceptions about the concept of variable. The authors discovered that the concept of a variable seemed to be difficult for learners. They tended to overlook the variable, reducing it to a constant, and not being able to find a connection between verbal expressions and variables, without using parenthesis. The authors considered these to be mistakes or misconceptions. In their findings, they did not explain which of the learners' solutions were considered to be mistakes or misconceptions. This leaves the impression that the misconceptions and mistakes are one and the same thing. However, the literature has differentiated between the phenomena of mistakes and misconception. Mistakes are just slips that can be easily corrected, while misconceptions are resistant to change Matz, (as cited in Siyepu, 2013). Uygur & Ozdag (2012) refer to mistakes as misunderstanding. Maznichenko (2002) has argued that misconceiving and misunderstanding have common features: both lead to an inadequate perception of the reality, but there are a few distinctions between them. Mistakes and misunderstanding are caused only by personal insight and sensitivity of the learner. Misunderstanding is casual, but misconceiving is not; the occurrence of misconceptions follows certain rules. What they all agreed on is that mistakes are not resistant to change while misconceptions are. Therefore not using parenthesis can be regarded as a mistake. In a similar study by Egodawatte (2011) conducted with high school learners, the author noted certain misconceptions regarding the use of a variable in the solution of system of equations. She indicated that learners do not interpret the variable as a number, but judge the magnitude of a variable by its coefficient, and prove unable to translate word problems using symbols. Failure to translate word problem into symbols can be attributed to many things other than a misconception per se. This could be a problem of language, or simply the failure to comprehend the statement.

In their study, Appova & Berezovski (n.d.) investigated common misconception about vectors. Their findings revealed that the misconceptions displayed by students were related to the reasoning and spatial sense about vector operation and projections. The authors pointed out that 34% of the students demonstrated a fundamental misunderstanding of the meaning of vectors and scalars. Students generally confused vectors and scalars, and performed arithmetic operations as they will do with numbers. Although the misconceptions discussed by these authors focus on the misunderstanding between vectors and scalars, the issue regarding misunderstanding and confusing concepts is not only a problem in linear algebraic concepts, but in mathematics as a whole. In the study conducted by Ndlovu (2012) regarding learners' construction of knowledge in optimisation, learners displayed misunderstanding of algebraic equations and algebraic expressions. In many cases, these are a result of pedagogical instruction. Teachers and lecturers move between concepts without explicit explanation for students, and with limited time given to allow learners/students to construct meaning of the taught concepts.

Thompson & Logue (2006) investigated common student misconceptions in science. They discovered that students' intuition is based on direct experience when determining the solution. This roughly correct conception could be easily become confused by what they are subsequently taught. This statement concurs with what Nogueira de Lima & Tall (2008), as they have pointed out, that the new experiences could negatively or positively affect our knowledge construction. The new knowledge learnt, if not assimilated into existing schemas, would not help in expanding the existing schema. Instead, two or more pieces of knowledge will stand alone causing cognitive conflict. Thompson & Logue (2006) further argued that students become fixated on a concept even if the context of the problem doesn't allow the use of it, and they are reluctant to modify the identified misconception. This can be attributable to what Goris & Dyrenfurth (2010) stated as the speculated, where students have a pre-conception and hold naive theories in their mind about the new or experiential concept. Therefore, if a student does not recognise, through instruction, that his/her initial concept is wrong, then new information is assimilated into the flawed mental world (Thompson & Logue, 2006).

2.8 Construction of knowledge and mathematical thinking

Individuals construct mathematical knowledge in different ways, but the entire thinking processes takes place cognitively. To explain how the process evolves, we will draw from Tall & Vinner (1981), in which the authors explain what they term concept image and concept definition. We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all mental pictures and associated properties and processes. It is built up over the years, through experiences of all kinds, changing as the individual meets new stimuli, and matures. Concept definition is a form of words used to specify that given concept.

Concept images consist of examples and non-examples, representations, definitions and alternative characterisations, properties, results, processes and objects, contexts and impressions from previous experiences (Findell, 2006). Mathematics learning involves the solving of mathematical problems. When solving a mathematical problem, one needs to recall, reconstruct examples cognitively, and to represent them in a way that will make sense mathematically. The constructivist view about learning describes that students actively construct knowledge, based on their own experiences. This implies that students come to mathematics classes with some already constructed schema. These can either be in line with what is considered to be correct in terms of the discipline, or it can be in conflict. This is what Tall & Vinner (1981) referred to as evoking concept image. When students encounters a mathematical problem, some strategies that they think are suitable to solve the problem are evoked, based on what has been cognitively constructed. These do not follow any structural set. In some cases, they are context dependent. The way the problem is presented might in the mind of the student evoke some images because of its similarity to previous questions.

During the process of learning, students do not make connections in the same way. Depending on the conception they have of the particular topic, and previous knowledge, they can make connections more quickly with one concept than the other, and therefore, the concept image constructed will also vary, based on the examples and non-examples student have in their cognition. That means concept images are not static. According to Findell (2006), concept images can be dominated by particular examples. Some students might focus on the computation of matrices, whereas successful students might conceive of matrix structures as objects. Thus, the concept images constructed can either be a barrier to learning, or lead to successful conceptualisation of the concepts.

In the thinking process, there are many processes that take place cognitively to give us insight in a problem solving process. When mathematical concepts are introduced to students, they can be given by means of a symbol, or a name, which then automatically, through the cognitive structures, can be communicated to the mind. While the brain is busy recalling and manipulating these symbols, it brings in different images to associate with the concept, and new concept images can be formed. As the learner starts to formulate the concept image, he/she uses some form or words to define the images that he/she is constructing (Tall & Vinner, 1981). This definition may be learned through rote learning, for example theorems. Through memorising the definition, the student might start to form mental pictures and concepts relating to the theorem. At some point, a student may have a strong mental picture, while his/her concept definition image is weak. The student may understand statements of theorems, but he/she probably cannot follow the way in which the proof came about. In that instance, a student may reconstruct the definition to fit his/her image of a particular concept (Ndlovu, 2012). According to Tall & Vinner (1981), this definition is referred to as a personal concept definition and it can be different to formal concept definition. The formal concept definition is the one that is accepted by mathematics community at large (Pillay, 2008). In most cases, the concept definition that students employ are not formally defined for them, but it is expected that through experience they will learn to recognise and to use them when the need arises (Ndlovu, 2012). The constructed concept definition might not be wrong, but the words used to define it could carry a misconception, which may then cause a conflict in the students' mind when a similar concept is met.

The model of concept image and concept definition allows for an analysis of the cognitive processes influencing the learning of mathematics, by simultaneously considering mathematical characteristics (Rosken & Rolka, 2007). It is through understanding the new information that new aspects of the concept are formed, and are integrated into previously-existing knowledge structures. Students ought to be able to develop their own ideas in order to remember the explanations given of a particular concept (Ndlovu, 2012). Learning a new concept requires forming a comprehensive concept image, but one should keep in mind that potentially important aspects of a mathematical concept are not adequately presented (Rosken & Rolka, 2007).

Pinto & Tall (2001) have argued that students mainly construct knowledge by trying to incorporate the new knowledge into their existing schema. These authors surmised that an individual imagines objects and investigates their properties and relationships to form a wider schema. It is the student concept image that will determine the way in which a student responds to the task. If the image is built on experiences that conflict with the formal definition, this can lead to responses which vary from the formal theory (Tall, 1988). For instance, when asked to sketch $f(x) = \frac{1}{x}$, a student might know that is a hyperbola, but when given $f(x) = x$ and asked to sketch its reciprocal, the student might experience difficulties indicating that the concept image constructed of a hyperbola is built on experiences that are in conflict with formal theory. This was also evident in the study by Tall (1988), conducted with learners on tangents. Learners were asked to draw a tangent to the curve $y = x^3$ at the origin. It was discovered that many drew a line to one side, which did not pass through the curve. This resulted from the fact that in most cases, tangent are taught in relation to circles. The learners' understanding was correct in relation to the circle, however their distorted concept image of a tangent would now cause cognitive conflict. In a nutshell, this meant that the learners constructed knowledge based on the concept image made by the previously learnt concept.

Pinto & Tall (2001) explored how students construct meaning. In their study, they describe two types of students in the classroom, viz. formal and natural thinkers. Both these types of thinkers construct knowledge cognitively, but they use different ways to make sense of the learnt knowledge, respectively. Formal thinkers attempt to base their work on definitions, often focusing on manipulating the symbols and inequalities rather than logic, whereas natural thinkers construct new knowledge from their concept image (Pinto & Tall, 2001). Natural thinkers are somehow creative in their thinking, as they reconstruct their concept image to develop the formal theory which is in line with what they have encountered in the course. Students who follow the formal route of learning were largely unable to coordinate processes. In most cases, they followed rules without understanding, there is no logic evident in their working, and what they wrote down is disjointed in relation to their images (Ndlovu, 2012). Tossavainen, Attorps & Vaisanen (2012) examined South African mathematics teachers' concept image of the equation concept. The results showed that the participants' understanding of equation concept was mainly based on their concept image, not on concept definition. The authors attributed this to the fact that the concept definition may be a recollection of definition from textbooks. This has meant that a student might not construct logical thinking from a concept definition only.

Tomita (2008) pointed out that students' prior experiences can have a positive or a negative effect on their learning, depending on how the previously constructed knowledge is used for future learning. Scholars have agreed that students' prior knowledge in the learning of mathematics has a huge impact. This is what Tall (2008) refers to as "met before", which he defines "to be the current mental facility based on specific prior experiences of the individual" (p. 6). In the study by Nogueira de Lima & Tall (2008) the results showed how met before could affect students thinking abilities. They pointed out that the arithmetic algebra played a dominant role in the solution of the system of equations. An individual knows that $3 + 2 = 5$. Later in algebraic expression, when asked to solve $3 + 2x$, students will just add the numbers and attach x, since he/she does not know what to do with it. Most of the research confirms that an individual's prior knowledge is the primary cause for knowledge construction, whether positive or negative.

Nogueira de Lima & Tall (2008) highlighted that as much as 'met before' affects new learning, new experiences may affect the way we conceive of old knowledge. They refer to this as 'met after' For example, students might have an understanding of the meaning of unit of measurement such as "m", "cm" etc. However, when they are taught the meaning of a variable in an algebraic expression, they might end up confusing the meaning of a variable with the letter used. Perhaps they may be given 2m to convert into cm, as a result of what has been taught that a variable represents any number in the expression the student might allocate "m" any given value.

Students build their knowledge from prior experiences. It is the previous experiences which form connections in the brain that affect how we make sense of new situations (Tall, 2008). The above statement implies that the knowledge construction has to do with cognitive growth. The mental constructions made will impact on how we conceptualise the particular concept. For example given $2x + 3 = y$ and asked to sketch it. Using a table of values will reveal that knowledge constructed is at the action level. Tall (1997) conducted a study with 192 learners. They were divided into four categories, 'very capable', 'capable,' 'average' and 'incapable'. It was discovered that the very capable learners remembered general strategies, curtailed their solutions to focus on essentials, and were able to provide alternative solutions. Average learners remembered specific details, shortened their solutions only after practice involving several of the same type, and generally offered single solution to a problem. Incapable learners remembered only incidental, often irrelevant detail; had lengthy solutions, often including errors, repetitions and redundancies; and were unable to begin to think of alternatives. Relating this to the study conducted by Pinto & Tall (2001), the very capable and capable could be regarded as natural thinkers, while the average and incapable could be regarded as formal thinkers.

A similar study by Gray, Pinto, Pitta & Tall (1999) was conducted with high school learners. Learners were divided into two categories, viz. high achievers and low achievers. It appeared that low achievers tended to translate symbols into numerical processes supported by the use of imaginistic objects, that possess shape, and in many instances colour. Their object of thought was based on analogues of perceptual items that seem to force them to carry out procedures in the mind. The high achievers tended to focus on abstraction, which enabled them to make choices, where they were able to link concepts moving from one focus to another. These learners had the ability to filter out information, to operate with the symbol as an object, and to connect with an action schema in order to perform any required computation (Gray et al., 1999). These studies showed that students in the same context of learning learn the same concept construct knowledge differently. Their mental constructions do not develop iteratively, each student constructs his/her own meanings of the taught knowledge, even if common components of these meanings are shared in the class group (Tiberghien, 2007). It is true that the knowledge construction process takes place cognitively. For conceptual understanding to develop, the new knowledge constructed ought to be assimilated into existing schemas.

2.8.1 Importance of language in the construction of knowledge

Parker (2010) conducted a study on how intuition and language used relate to students' understanding of span and linear independence. In his findings, he noted that students with stronger language skills generally exhibited a better understanding of span and linear independence. This means that having the ability to communicate mathematically using appropriate mathematical terminologies might help students conceptualise the concepts much better. This has been confirmed by other literature in the other fields of mathematics, like calculus and geometry. Tall (2008) has argued that language and the related use of symbols enable us to focus on important ideas, to name them, and to talk about them. Perception of figures is at the foundation of geometry, but it takes the power of language to make a hierarchical classification between them (Gray et al., 1999). Language is a fundamental aspect to learning. Physical and mental pictures supported by linguistic description may become conceived in a more pure, imaginative way (Gray et al, 1999). Abstract algebra, once embedded in definitions and axioms, can then become imperative for students to understand the language used in order to understand the concept learnt.

Mathematics has its own language, with its own vocabulary and syntax. For example, when dealing with the concept of limit and continuity, a problem may arise with information translation of some of the sophisticated ideas, such as 'tend to' or 'close to'. For many learners, 'close' means 'near', not 'coincident'. So, in the learning of the limit of $f(x)$ *as* $x \rightarrow a$ with the use of informal translation, x can be 'close to', but not 'equal to' a (Brijlall & Maharaj, 2009b). Ferriri-Mundy & Graham (1991) have stated that learners' difficulties may be inherent in the colloquial meaning of the terms used in this information translation. The same was echoed by Tall (1990), as he argued that mathematics contains concepts such as limit and infinity, which carry complex meanings that may be interpreted in inconsistent ways, and that the message may be framed in a language that evokes inappropriate ideas, which may be presented in a sequence that is inappropriate for cognitive development.

2.8.2 Correct use of notation

So far I have discussed the processes that students might employ to develop their thinking when learning mathematics. I have highlighted the way in which concept image and concept definition could impact the learning of mathematics. It is true that understanding the concept is the important aspect of mathematics learning, but the concept requires understanding of other aspects like notation. Notation and symbols are important aspects of mathematics. However studies have shown that in many cases, students use notation incorrectly (Jojo, 2013; Ndlovu, 2012; Siyepu, 2013). Mathematical learning requires not only the construction of concepts, but also learning the standard names and notations for those concepts, and the appropriate verbal and mathematical syntax for referring to those concepts in mathematical discourse (Findell, 2006). In this study, the understanding of notation is important, because using wrong notation may give an incorrect meaning to the solution. Concepts in matrix algebra are made of notations. Being able to express your mathematical thinking in words is very important, so the correct use of notation is equally important in the learning of mathematics. This can be represented in structural form.

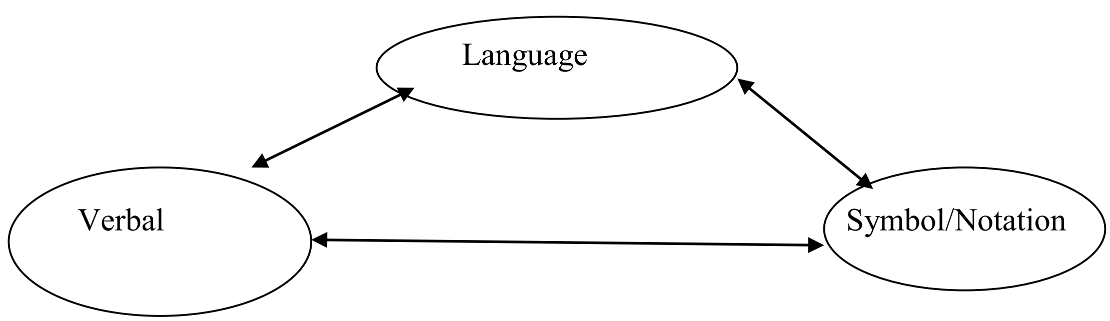


Figure 1: Conceptualising Notation

This can be interpreted in a triangular form, and the arrows going to and fro, meaning that given the mathematical problem, the students ought to be able translate from one form to the other. Ndlovu (2012) pointed out that students do not conceptualise the meaning of the notation, it is rather, memorised. For example, in calculus for the use of Leibniz notation students used $\frac{dy}{dx}$ regardless of the variables given in the expression. Maharaj (2014) has suggested that teaching should focus on the need for students to interrogate what they write. This could help them to reorganise their mental structures and schema accordingly.

2.9 Studies using APOS

In this section, I will discuss the studies that use APOS theory as the theoretical framework. The theoretical framework used in this study was motivated by these studies. There are more studies that used APOS theory in the field of calculus. However, I will only present the summary of those done in the field of linear algebra.

2.9.1 APOS studies done in linear algebra

Trigueros, Oktac & Manzanero (2007) used APOS theory to explore students' understanding of system of equations in linear algebra. The authors argued that the study is important because linear algebra makes use of the concepts involved in the solution of linear systems of equations. The participants of the study were six students taking a course in linear algebra. The data were collected through interviews at the beginning of the course, and at the end of the course, to study the viability of the proposed genetic decomposition, their difficulties, their reasoning pattern and an evolution of their schema. The authors discovered that the construction of schema for variables that includes the interpretation and differentiation between different uses of variables, that is an unknown, general number, and variables in functional relationship, and the understanding of solution of an equation as an object, constitute the necessary conditions for students to make the constructions that are needed in order to construct the systems of equations schema. The lack of evidence of those previous notions with the students interviewed seems to interfere severely with their possibility to make the constructions needed to understand new abstract concepts. The understanding of the solution of an equation as an object constitutes a necessary condition for students to make the constructions that are needed in order to construct the systems of equations schema. Trigueros et al (2007) further pointed out that some conceptual evolution is possible when students follow pedagogy where they are required to carry out actions on mathematical objects.

Stewart & Thomas (2007, 2008, 2009) investigated the difficulties in understanding some linear algebra concepts, and proposed potential paths for preventing them. Their research proposes applying APOS theory, in conjunction with Tall's three worlds of embodied, symbolic and formal mathematics, to create a framework that could be used to examine the learning of a variety of linear algebra concepts by groups of first and second year university students. The hypothesis made is that the transfer from a primarily procedural or algorithmic school approach to an abstract and formal presentation of concepts through concrete definitions seems to be creating difficulty for many students who are barely coping with the procedural aspects of the subject.

As part of that research project, several case studies were conducted, where groups of first and second year students were exposed to teaching and learning of certain introductory linear algebra concepts. The results suggested that the students had limited understanding of the concepts, they struggled to recognise the concepts in different registers, and their lack of ability in linking the major concepts became apparent. However, the results revealed that those with more representational diversity possessed more overall understanding of the concepts. In particular, the embodied introduction of the concept proved a valuable adjunct to their thinking. Since difficulties with learning linear algebra by average students are universally acknowledged, it is anticipated that this study may provide suggestions with the potential for widespread positive consequences for learning.

Bogomolny (2007) conducted a study to explore students' understanding of the key concepts of linear algebra and the difficulties students experience when engaged in the tasks. This study also explored the use of example-generation tasks in revealing students understanding of linear algebra. Through the study, some of the difficulties experienced by students when learning key concepts like vectors, linear dependence and independence, linear transformation and basis, were identified. The participants of the study were 113 students enrolled in Elementary Linear Algebra course at a Canadian University. Later in the study, interviews were conducted with six students who volunteered to be interviewed. The data were from students written responses, and from the interviews. The results showed that students had constructed the schema of linear (in) dependence tasks. However, geometric and algebraic representations of concepts seemed to be completely detached. The researcher recommends the use of example-generation tasks as a pedagogical tool to alleviate some of the difficulties students had. Constructing examples helped students develop their intuition of what linear (in) dependence really means (Bogomolny, 2007).

De Vries & Arnon (2004) used APOS theory to explore students' understanding of the concept of a solution of a system of equations. The participants of the study were 12 students at a Teachers' college, who took one semester course in linear algebra. Data were collected through interviews, which were done with individual participants. The students had to answer a questionnaire, which the participants had responded to. Thereafter, the students discussed their work with the interviewer. The purpose of the interview was to get to know students' ideas about solution, and to get started with a first version of a genetic decomposition of the concept. The results showed that the questionnaire used provided insufficient insight into the research questions. Little information about the constructions that students made in their understanding of the concepts could be revealed with this instrument. Also, the instrument provided little possibility of distinguishing between the action and process level.

Parraguez & Oktac (2009) used APOS theory to propose a possible method that students might follow in order to construct the vector space concept. In the study, they identified the mental mechanisms and constructions that might take place when students are learning this concept. The participants of the study were 10 undergraduate mathematics students. The data were collected via questionnaires and interviews. The findings of the study revealed that when students lack the prerequisite constructions, it becomes very difficult for them to develop a sufficiently strong schema of the vector concept. The authors suggested that there ought to be special emphasis on the construction of the binary operation schema, giving students the opportunity to experiment with different kinds of sets and binary operations, so that they develop flexibility in thinking about structures other than the ones containing the usual operations.

Cooley, Loch, Martin Dexter & Vidakovic (2010) used APOS to investigate the way in which the parallel study of learning theories and advanced mathematics influences the development of thinking of high school mathematics teachers in both domains. The findings of the study showed that prospective and practicing teachers were able to make connections between their concurrent study of linear algebra, and of learning theories relating to mathematics education, specifically APOS theoretical framework. Based on the findings, these authors developed a new strategy that embeds the learning theory in the linear algebra course, through the use of carefully designed modules that combine important linear algebraic concepts, related student intuitions and experiences.

Arnawa, Surmano, Kartasasmita & Baskoro (2007) used APOS to analyse students' achievement in proofs. It focuses on the effect of instruction based upon APOS theory to improve students' ability to prove in abstract algebra. The participants were 180 students from two different universities, with two classes from the department of mathematics, and two classes from the department of mathematics education. The findings showed that the ability to prove of students who had been involved in APOS theory instruction to be significantly greater than the ability to prove of those students who were involved in only teacher-centered traditional learning. These findings suggested that the use of APOS theory instruction may benefit the Indonesian students who participated in the development of abstract algebra concepts in particular, and mathematical concepts in general.

2.10 Implications of literature review

The concept of matrix algebra is important to mathematics, especially with its application to other disciplines and outside of schooling system. However, such importance is not much understood by our students, and perhaps some of the curriculum developers. The studies that the researcher reviewed provided evidence of the importance of linear algebra on student understanding of concepts in linear algebra such as vectors, span and set theory, linear (in) dependent. However, little was found with particular regard to students' understanding of matrix algebra, and no studies were found, even on students' mental constructions in linear algebra in South Africa.

2.11 Conclusion

In this chapter, an overview has been provided of literature concerned with the nature of linear algebra, knowledge constructions, students' difficulties, and concept image. From what has been gathered, it has become evident that the concept of matrix algebra is not well researched, and there seem to be a view that among other forms of linear algebra which are considered to be more abstract, matrix algebra is regarded as concrete, and therefore students are expected to experience less difficulty with it. However, there is no actual evidence to support this. Following this discussion, it became clear that new knowledge ought to be incorporated into the existing literature. A key aspect relating to matrix algebra, such as matrices, determinants and system of equation, is explored in this Chapter Three focuses on the theoretical framework for this study.

CHAPTER THREE

THEORETICAL CONSIDERATIONS

3.1. Introduction

This study has explored the mental constructions of pre-service teachers when learning matrix algebra concepts. In the previous chapter, literature informing this study was introduced and discussed. In this chapter, the theoretical framework within which this study is located will be discussed. This chapter begins with a discussion of the meaning of mental construction and presents the theoretical orientation of this research study, with the intention of creating the context for the theoretical framework that is established thereafter.

Subsequently, I provide a discussion on the origins and development of APOS theory. This is achieved by examining the background of the theory. Next, a description of the heart of the theoretical framework used in this research study, namely, APOS theory and framework for research and curriculum development is presented together. Finally, because APOS theory and Piaget's triad mechanism provides the theoretical framework for the presentation and analysis of data in Chapters Five and Six, respectively, a detailed account of how APOS theory has emerged in this study is presented.

3.2 Mental constructions

According to Tall (1997), characteristics of recent developments have been a focus of attention not only on the mathematics to be taught, but also the mental processes by which it is conceived and learnt. The learning of mathematics is a constructive process in which the student attempts to make sense of information by evaluating, connecting and organising it relative to prior experiences and existing knowledge structures (Ferriri-Mundy and Graham, 1991). Mental constructions might have different connotations. Some research discusses cognitive constructions while other refer to metacognition. All these phenomena focus on a particular aspect however, namely the construction of knowledge in learning mathematics. Whether focused on problem solving or on the learning of a particular concept, all have to do with cognitive development to deal with and conceptualise mathematical concepts. For this study, when referring to mental constructions, we are referring to processes that an individual undergoes in dealing with perceived mathematical problems. This study uses mental constructions as described in the APOS theory, which was explained in Chapter One. It is these mental constructions that help to explain the level of knowledge construction made in the learning of mathematics. Originally, this theory aimed at understanding students' mental construction in undergraduate level. However, it has recently been used with learners at school level as well. In this study, the explanation of mental construction was about one's knowledge, and strategies used to make sense of the learnt concept. These strategies can be physical or mental, but the most important thing is how an individual communicates his/her way in these activities, so as to make sense of the learnt knowledge.

3.2.1 Mental construction and metacognition

The mental constructions an individual possesses are activated during the problem solving process. Problem solving requires more than just arithmetic and calculations skills. In problem solving, the flexibility of the thought processes and adaptability of the previous knowledge in the new context shows cognitive growth. For conceptualising a mathematical concept, both cognitive growth and metacognition are essential. Wilson & Clarke (2004) refer to metacognition as an individuals' awareness of their own thinking, their evaluation of that thinking and their regulation of that thinking. This definition is an extension of Flavell's (1976) definition as he referred to metacognition as a form of individual awareness, consideration and control of his/her own cognitive process. There are many definitions of metacognition, but all of these are developed from Flavell (1976).

Mental construction refers to cognitive growth and metacognition refers to the awareness of such mental construction that leads to cognitive growth. The question is: how do cognition and metacognition link, and how do they help in developing understanding of a concept. Dunlosky & Metcalfe (2009) have argued that cognition is a mental process that manifests itself in problem solving, learning memory and reason. They further argued cognition to describe doing, while metacognition describing the awareness of the thought procedures, where metacognition focuses more on planning and choosing what to do in a particular situation during problem solving processes. Given a mathematical problem, an individual will analyse the problem, plan strategies, and choose the most appropriate way suitable to solve the problem. Simultaneously an individual may apply the cognitive strategies like reading, calculation, sketching diagrams, etc. Magiera &

Zawojeski (2011) maintain that during the process of solving a mathematical problem, cognitive and metacognitive processes are parallel and interactive, rather than sequential. Cognition is inherent in metacognitive activity, while the metacognitive may be present in many cognitive activities. While the completion of a mathematical task is basically a cognition activity, utilising cognitive strategies, metacognitive behaviours deal with the selection and use of cognitive strategies (Wilson & Clarke, 2004). According to Kuzle (2013) metacognition helps the student to recognise the presence of a problem that needs to be solved, to discern what exactly the problem is, and to understand how to reach a goal.

In solving a mathematical problem, one applies the necessary metacognitive procedures. While analysing the problem he/she has to read the problem to identify key ideas. Then he/she draws up a plan of how to go about solving the problem. In that manner certain cognitive procedures such as the drawing of sketches or modelling of the problem are incorporated. Once the plan is in place, calculation and manipulation of symbols is implemented. In that process an individual continues monitoring and evaluating whether or not the solution makes sense in relation to problem asked. Continuous interplay of cognitive and metacognitive behaviours is essential for successful problem solving and maximum student involvement (Shahbari, Daher & Rasslan, 2014). Thus, in the learning of mathematics, high levels of metacognitive and cognitive skills for the development of understanding the mathematical concept are required.

According to Wilson and Clarke (2004) metacognitive awareness relates to individual awareness of where an individual finds themselves in the learning of their content-specific knowledge. Evaluation refers to judgements made regarding one's thinking, capacities and limitations. Regulation occurs when an individual make use of their metacognitive skills to direct their knowledge about self and strategies to use, and uses executive skills to optimise the use of their own cognitive resources (p. 3). This means that students reflect and build on their existing thought activities to construct the new knowledge. This can also be linked to Piaget's theory of assimilation and accommodation, that new knowledge can be assimilated into existing knowledge and expand an already-constructed schema. Tall (2008) has indicated that the met-after and met-before can have a positive or negative impact on the conceptualisation of mathematical concepts; and that in a similar way, they can either facilitate or hinder the metacognitive activity of a student (Wilson & Clarke, 2004).

3.2.2 Thinking strategies in mathematical thinking

Sierpinska (2005) distinguishes two types of thinking, viz. the practical and the theoretical. Practical thinking is goal-oriented and rooted in action, while theoretical thinking is more focused on objects of reflection. To distinguish these two modes of thinking, she use five categories: (1) reasons for thinking; (2) object of thinking; (3) means of thought; (4) main concerns; and (5) products of thinking. When it comes to the reason for thinking, practical thinking focuses on solving the problem at hand, while theoretical thinking focuses on devising methods of solving all problems of a certain type. In the situation 'object of thinking', practical thinking focuses on particular things, matters, events and people, while in the situation 'theoretical thinking' the focus is placed on a system of concepts. In the situation 'means of thought', practical thinking focus on calculations of concrete results, using known techniques and unquestioned techniques, while in theoretical thinking, focus is placed on devising better adapted techniques for solving a range of problems. For the most part, practical thinking focuses on realistic reasoning, while theoretical thinking focuses on hypothetical reasoning; and lastly, in the situation 'outcomes of thinking', practical thinking focus on statements of fact, while theoretical thinking focus on hypothetical statements. She concludes that for students to be successful in mathematics, they need to be effective in both kinds of thinking, because purely theoretical thinkers tend not to be able to find smart ways to solve problems, while purely practical thinkers tend not to be able to develop deep conceptual understanding (Sierpinska, 2005).

The first year pre-service mathematics teachers at this university undertake the major content module over the period of six months. Some had spent six month in the previous year doing a foundational module, aimed at preparing them to develop concepts and at preparing them for the first year mathematics module. During the teaching of this first year mathematics module, matrix algebra is not given sufficient time, and the emphasis is more on manipulation of rules rather than application of knowledge, and problem solving. The researcher is of the view that the development of conceptual understanding of matrix algebra can be achieved by understanding the mental constructions that the pre-service teachers have constructed, as well as the metacognitive strategies employed during the problem-solving process. These can be best analysed through APOS theory, which is a theory of conceptual understanding (Dubinsky, 1991). This theory is an extension of a theory of reflective abstraction developed by Piaget (1966). Dubinsky (1991) believed that the theory of reflective abstraction can be a powerful tool to describe the development of the study of advanced mathematical thinking. This study adopted APOS theory to reveal the nature of mental constructions made in the learning of matrix algebra. This was prompted by a statement made by Dubinsky (1997), that research is required into those concepts that give students difficulty, so as to prepare teaching material that will advance the conceptual understanding of such concepts. This theory has been successful in revealing the nature of students' mental constructions in calculus and abstract algebra. Influenced by Piaget's conception (1978) of how actions and operations become thematised objects of thought, Dubinsky (1991) described such objects of thought as entities in which actions and processes can be made. Although Piaget's works mainly focused on the development of mathematical knowledge among children, Dubinsky and others have extended his work so as to come to understand the development of mathematical knowledge in the learning of advanced mathematics with undergraduates. Many lenses by many theorist and scholars have developed an understanding of the way in which students construct mathematical knowledge and mathematical thinking, however, this study chooses to reveal the nature of students' mental construction in matrix algebra through the lens of APOS theory, because while this theory seems to be helpful in describing the mental constructions made, it also serves as an analytic tool that can be used to interpret data.

In this study, the researcher uses APOS theory to describe the mental constructions displayed by pre-service teachers when learning matrix algebra, and to develop instruction that will help students to make the necessary mental constructions in developing the conceptual understanding of matrix algebra concepts. The researcher believes the theory is a useful tool to analyse the nature of mental constructions, it can contribute to understanding the thinking procedures students undergo in the learning of matrix algebra, and suggest ideas of why students may or may not be able to make the necessary mental constructions.

3.3 Research Framework

The theory of constructivism derives from the arguments of Piaget, John Dewey and others. This theory premised on the understanding of learning as an active process of constructing or putting together a conceptual framework (Cobern, 1995). This implies that learning takes place when an individual is active in constructing the meaning of the learnt concept through interacting with their prior ideas and experiences. According to Hobden (2006), the social and the cognitive are based on the premise that we all construct our own perspective of the world, based on individual experiences and schema. Constructivists believe that every student can construct knowledge through a learning process. To develop knowledge, students should be allowed to work on concepts until they are able to process them to form objects, which help to develop schemas that will allow them to accommodate or assimilate new schemas to the existing ones.

Even though constructivism describes the basis of the way in which learners learn, this study is based on the framework for research and curriculum development in mathematics education as advocated by Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas (1997), which focuses on the cognitive growth of the way in which students construct mathematical knowledge. The framework consists of three components, namely: theoretical analysis, instruction design and implementation and observations and assessments. Theoretical analysis produces assertions about mental constructions that can be made in order to learn a particular mathematical topic; instruction endeavours to create situations that can foster making these constructions; observation and assessment endeavours to determine if the constructions appear to have been made, and the extent to which the student actually learned" (Asiala et al., 1997, p.1). In this section, the researcher will consider theoretical analysis with a focus on mental construction. Under theoretical analysis, Dubinsky's APOS theory of learning advanced mathematics is useful as it describes the cognitive structures used by students to construct knowledge through action, process, object and schema. The figure below shows the research framework for research and curriculum development. It is within this research framework that the theoretical framework adopted from Brijlall, Jojo & Maharaj (2013) is used for this study

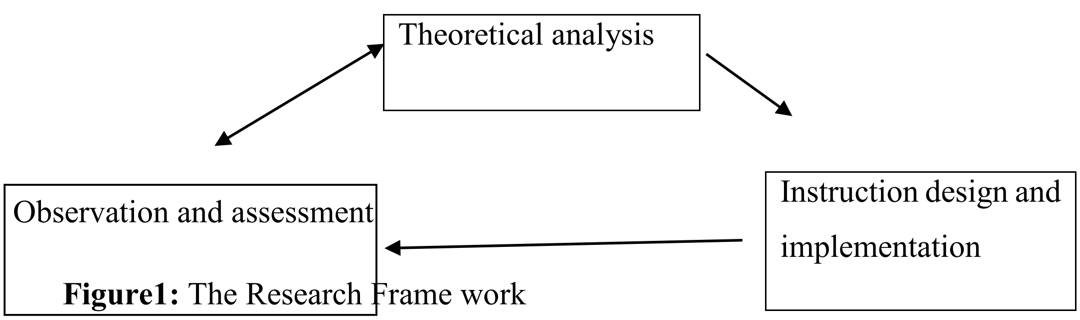


Figure1: The Research Frame work

3.3.1 Theoretical analysis

Under theoretical analysis, this study employed APOS theory to describe and analyse pre-service teachers' knowledge construction of matrix algebra concepts. The theory will be discussed in detail later. The aim of applying APOS theory was to reveal the nature of students' mental constructions, and not to provide statistical comparison of pre-service teachers' performances in mathematical concepts. APOS theory proposes that an individual has to possess the appropriate mental structures relating to action, process, object and schema in order to make sense of a given mathematical concept. These mental structures need to be detected and then learning activities that will be suitable for the development of those mental structures ought to be designed so as to enhance the construction of those mental structures (Maharaj, 2014). This leads to a design of a genetic decomposition relative to specific mental constructions that students make in order to develop his/her understanding of matrix algebra concepts.

In line with the research framework under theoretical analysis, the genetic decomposition for matrix algebra concepts relative to specific mental constructions that a student might make in order to develop an understanding of matrices and systems of equations was designed. In order to ascertain whether the mental constructions indicated in the genetic decomposition are made, the instructional design in the form of an activity sheet that focuses directly on attempting to help students make those mental constructions was designed. The last step was to implement the instruction while collecting data. The theoretical perspective adopted for the study was used in order to analyse the data collected, and observations were made as to which mental constructions were constructed by students. A complete description of the research framework used in the study can be found in Asiala, Brown, De Vries, Dubinsky, Mathews, & Thomas (2004), and Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros & Weller (2014).

3.3.2 The development of matrix algebra schema

A schema is a more or less logically connected collection of objects and processes (Dubinsky, 1991). An individual may have a number of different schemas constructed which may assist in making connections between the perceived problem situation, based on his/her constructed knowledge of similar mathematical concepts. A schema for a certain mathematical concept is an individual's collection of actions, processes, objects and other schemas which are linked by some general principles to form a framework in the individual's mind, which may be brought to bear upon a problem situation involving that concept (Dubinsky & McDonald, 2001). Asiala et al.

(2004) asserts that an individual’s schema for a concept includes his/her version of the concept described by the genetic decomposition, as well as other concepts that are perceived to be linked to the concept in the ‘concept of problem’ situation. A genetic decomposition representing one reasonable way students might construct a particular concept is formed by isolating small portions of the complex structure and giving explicit descriptions of possible relations between schemas (Jojo, 2011). The schema constructed is not a once-off construction however, its existence is continuous, as it becomes constructed and reconstructed as an individual learns new mathematical concepts. For example, an individual may have interiorised the action of using Cramer’s rule to solve the system of linear equations. If the process is fully constructed, an individual might be able to reverse the process by using the solution to determine whether the system of such equation is consistent or inconsistent. A process of understanding a certain mathematical concept using APOS can therefore be attributed to a successful student’s construction of schemas for that concept (Jojo, 2011).

A structured set of mental constructs, which might describe how the concept can develop in the mind of an individual, is referred to as the **genetic decomposition** of that particular concept. The mental constructions that the learner might make include actions, processes, objects and schemas. As part of his/her matrix schema, the student should be able to:

- relate the matrices to real number system;
- manipulate symbols to compute matrices
- recognise matrices as mathematical structures in which actions and processes can be applied to;
- recognise that multiplying a matrix by its identity matrix is the same as “multiplying by 1”;
- use the conditions of computation of matrices to solve related problems; and
- relate matrix subtraction to additive inverse.

As part of his/ her determinant schema, the student should be able to:

- explain the relationship between matrix and its sub matrix, i.e. matrix n and sub matrix $n-1$.
- explain the use of parenthesis in different context;

- explain the relationship between the determinant of a matrix and determinant of its transpose; and
- apply the determinant to solve related problems.

As part of his/her schema for the system of equation, the student should be able to:

- represent the solution of a system algebraically and geometrically;
- use the knowledge of analytic geometry, proportionality and Cramer's rule to determine consistent and inconsistent system of equations;
- recognise the relationship between matrices and system of equations;

As part of his/ her matrix inverse schema, the student should be able to:

- relate the matrix inverse to school algebra;
- realise that finding an adjoint is the same as finding the transpose of a matrix; and

These are summarised in Figure 2 below.

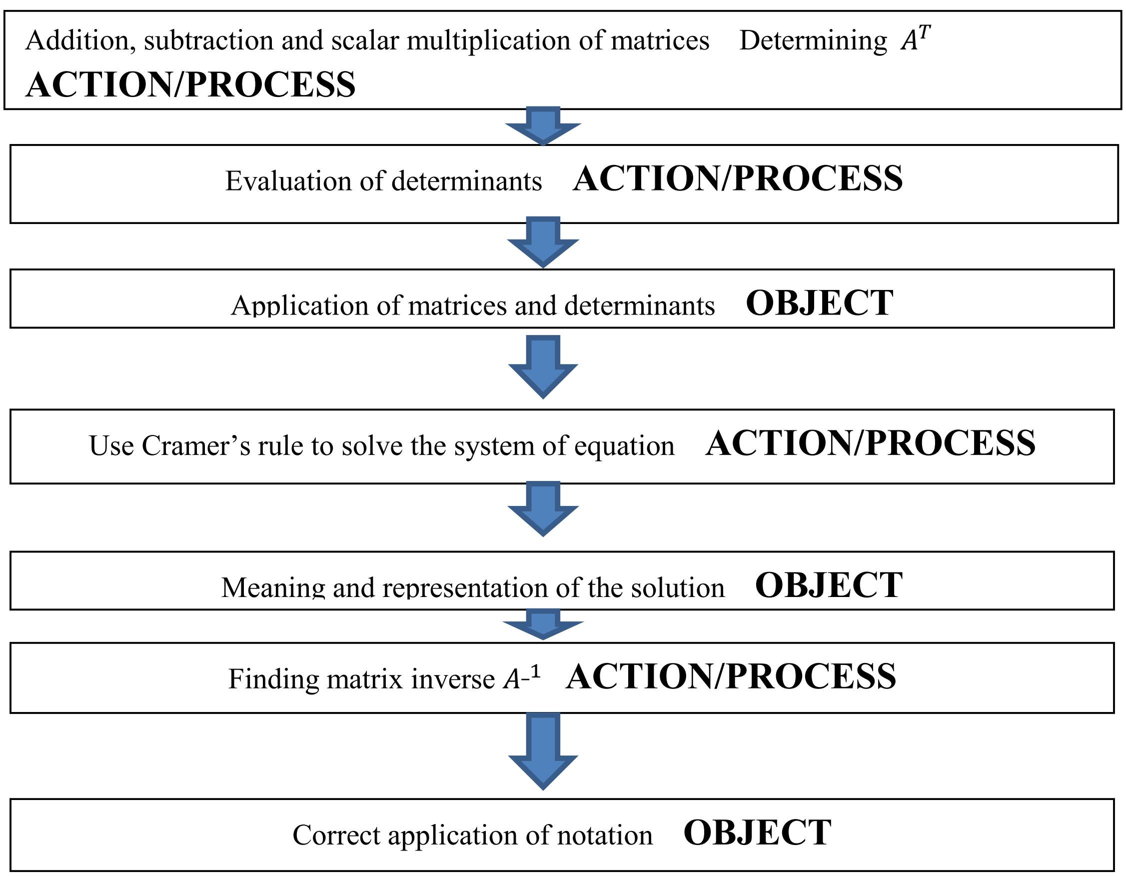


Figure 2: A Schema to solve problems related to matrix algebra

3.3.3 Preliminary genetic decomposition

Theoretical analysis concerns the way in which students cognitively construct knowledge in mathematics. Therefore, it is necessary to design instruction that would allow students to make such constructions. This is called a genetic decomposition. A genetic decomposition refers to the structured set of mental constructs which might describe the way in which the concept can develop in the mind of an individual (Brijlall & Ndlovu, 2013). As Dubinsky (1991) points out, constructing a genetic decomposition of a concept does not mean that the learning of a mathematical concept follows one route and that is the only way it can be learnt. However, it helps with observing of the learning in progress, and it is a guide for one possible way of designing instruction. The genetic decomposition designed is based on the researchers' experiences of a particular concept, and does not necessarily represent how trained mathematicians understand the concepts. Although the genetic decomposition is presented in a linear, structured way, this does not imply that learning takes place in a linear way; but focus is placed on how students cognitively construct knowledge of a particular concept. Based on a theoretical analysis, the structure of the genetic decomposition of matrix algebra was presented. This is not presented here as genetic decomposition, but as Dubinsky (1991) asserted, it represents one reasonable way that students might use to construct matrix algebra concepts. What shall follow in Figure 3 is the proposed genetic decomposition of the concept of matrix algebra.

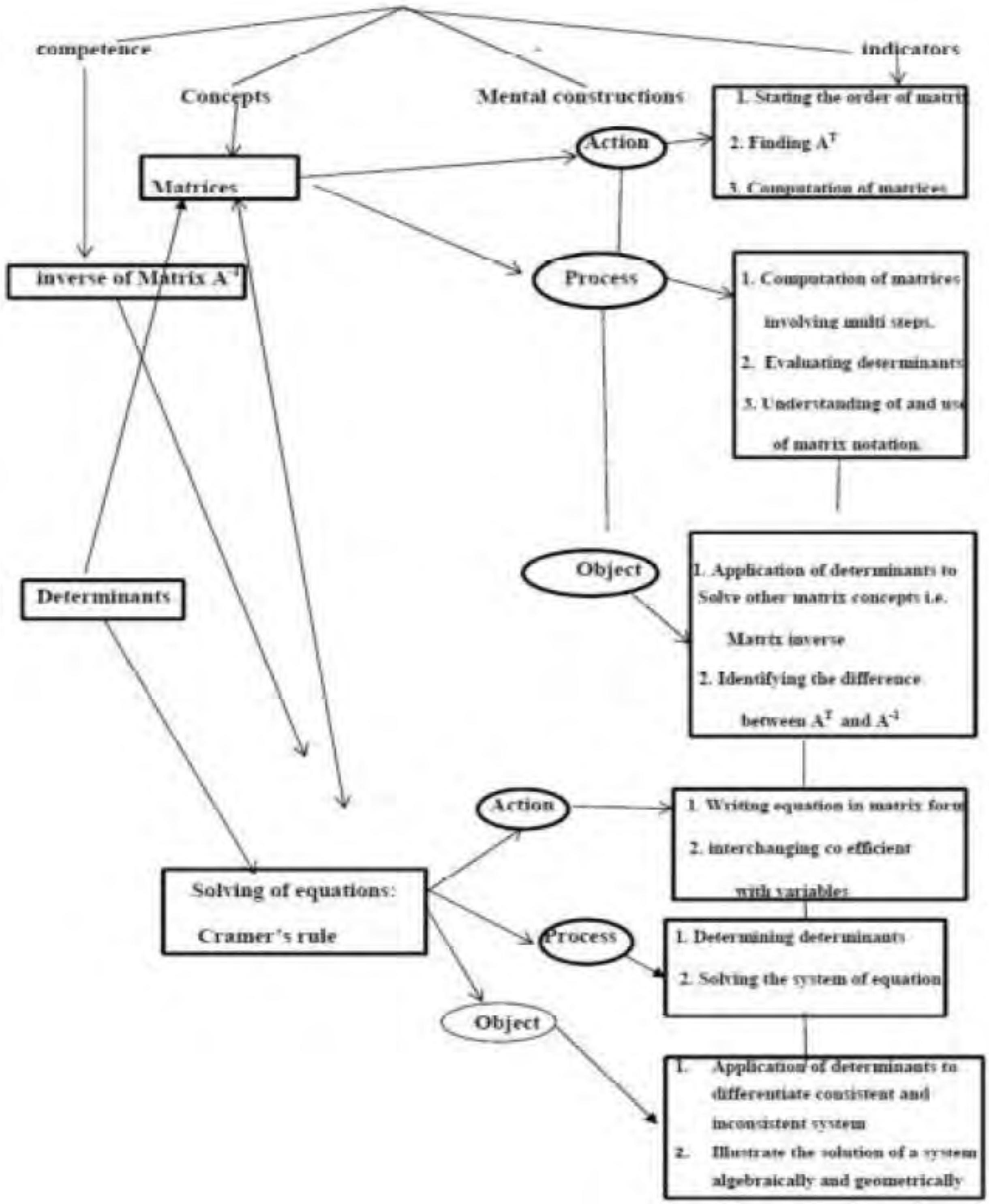


Figure 3: A preliminary genetic decomposition for matrices and solving the system of equations using Cramer's rule.

3.3.4 Instructional design and implementation

The preliminary genetic decomposition for the matrix algebra concept guided the researchers' teaching instruction in class, and the design of activity sheets. The activities were designed to assist students in making the suitable mental constructions as indicated in the itemised genetic decomposition. To re-emphasise, a genetic decomposition refers to the structured set of mental constructs which might describe the way in which a given concept can develop in the mind of an individual (Brijlall, Jojo & Maharaj, 2013). The preliminary genetic decomposition designed was based on the researchers' experiences of the particular concept, and it does not necessarily represent how trained mathematicians understand concepts. In the theoretical analysis, the preliminary genetic decomposition indicating how the knowledge in matrix algebra can be constructed by students was presented. This was followed by preparation of the design activities aimed at getting students to make the constructions. This followed the Activity – Class Exercises' teaching style (ACE). To create an environment for students to make the necessary mental constructions, students met in the tutorial groups where they worked individually to try and solve the mathematical problems, and later worked in groups. By allowing students to work individually, students were given an opportunity to figure out how to solve the problem using the learnt ideas and knowledge already constructed. The activities were prepared in such a way that students will complete them with the 90 minutes. After each task was completed a class discussion took place allowing students to reflect on the work they had done.

To provide students with the opportunity to construct the mental constructions, the activities followed the action-process-object format. It began with activities in which students responded to external stimuli in order to answer the question, or used certain formulas to compute. These were followed by problems where students were expected to build on the actions by providing activities that were intended to prompt them to interiorise their actions as processes. This was then followed by higher order activities, providing students with an opportunity to organise a variety of previously constructed actions, processes and objects into schemas that could be applied to solving problems related to matrix algebra.

3.3.5 Observation and assessment

This allowed the researcher to gather and analyse data. The results were used to: (1) test the itemised genetic decomposition and revise it; and (2) reveal the nature of students' mental construction and the way in which they concur with the itemised genetic decomposition. For a clear description of these three components, the reader is referred to Asiala et al. (2004). For this study, the main concern was to explore the mental constructions students might make and the level of understanding they seem to display in their learning. For this reason, the mental constructions that students appeared to have made were compared to those presented in the genetic decomposition. In this study, the data were gathered through activity sheets. The students' responses from the structured activity sheets were used in designing the interview questions. This was done with the aim of understanding the mental constructions made by an individual. It was not necessary to interview each and every student, but as Asiala et al. (2004) points out that it is better to access a full range of understanding, by including students who gave the correct answer, partially correct, and incorrect answers. In most cases, two students were selected in each category, based on their responses to the tasks. The interviews were audio taped, and transcribed. These were later analysed, together with the transcripts, and emerging themes were coded. This was done in order to explore the mental constructions the students appeared to make, as well as those they failed to make – revealing the limits in the knowledge constructed.

3.4 APOS theory as a theory of learning in mathematics

APOS theory is a theory of learning in mathematics. The main focus of mathematics education community is the development of conceptual understanding in the learning of mathematics. The students' lack of conceptual understanding is a concern raised by many teachers, researchers and scholars alike. For this reason, many learning theories wrestle with precisely characterising the notion of understanding (Hierbert & Carpenter, 1992). There seems to be a view that research into the notion of learning and understanding in mathematics is critical, and therefore structured theories of learning are a necessity. According to Stewart (2008), learning theories must recognise that understanding is a fundamental aspect of learning. In mathematics education, there are many theories related to aspects of learning in mathematics. One of these is APOS theory, introduced by Dubinsky (1991). According to Dubinsky & McDonald (2001), a theory is an attempt to understand how mathematics can be learnt, and what a programme of research can do in order to assist learning. Furthermore, Dubinsky (2001) has argued that the theory should be able to explain specific successes and failures of students in learning mathematics. It must also be applicable to other phenomena beyond the one it was originally developed to explain. It should help in directing the researcher to ask the right type of questions, and assist in the interpretation of results. This is supported by Weyer (2010) as she points out that:

> Learning theory should: a) possess explanatory power; b) be applicable to a broader range of experiences; c) help organize thinking about learning experiences; and d) provide language for communication about learning (p. 9).

This study adopts APOS theory because it is a framework for the process of learning mathematics that pertains specifically to learning more complex mathematical concepts (Weyer, 2010). APOS theory is premised on the hypothesis that mathematical knowledge consists of an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organising them into *schemas* to make sense of situations and to solve problems (Brown, Dubinsky, McDonald, Stenger & Weller, 2004). Therefore, APOS allows for the development of ways of thinking about how abstract mathematics can be assimilated and learned (Cooley, Loch, Martin, Meagher & Vidakovic, 2006). Drawing from Dubinsky's (2001) discussion of features of a learning theory, the researcher believes that using APOS theory will help in revealing the nature of mental constructions made by students.

3.5 APOS theory as an extension of reflective abstraction

The theory of reflective abstraction and the triad suggested by Piaget & Garcia (1989) are important for higher mathematics, as they explain children's logical thinking. In developing a new theory, Dubinsky (1991) extracted certain features of reflective abstraction, reconstructed them, and extended their applicability to the learning of advanced mathematics through forming a coherent theory of mathematical knowledge and construction, called APOS.

3.5.1 Reflective abstraction

Reflective abstraction was introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the course of cognitive development. While analysing the way in which reflective abstraction leads to the construction of logico-mathematical structures, he made two important observations. One was that it has no absolute beginning, but is present at the earliest ages in the coordination of sensori-motor structures (Beth & Piaget, 1966). This meant that one cannot determine the time at which an individual, in this case a child, starts to develop logical thinking. Two, it continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction.

Piaget distinguished three major kinds of abstraction, namely: (1) empirical abstraction; (2) pseudo empirical; and (3) reflective abstraction. He refers to the first as that which derives knowledge from the properties of objects (Beth & Piaget, 1966). Dubinsky (1991) interprets that this is due to experiences that appear to the subject to be external. For example, given the function $y = 2x$, a student might look at this function and try different values to substitute so that he/she can decide how the graph might appear. Mentally, the student might know how the graph with positive slope or negative slope should look. Although this has do with external experiences, the knowledge of these properties is internal. According to Piaget & Garcia (1989), this kind of abstraction leads to the extraction of common properties of objects and extensional generalisation, that is, the passage from the specific to the general. The second abstraction is an intermediate one, between the empirical and the reflective, and it spells out properties that the actions of the subject have introduced into objects (Piaget, 1985). The third one is drawn from what Piaget (1985) called general co-ordination of actions by the subject internally. He further asserts that this kind of abstraction leads to a very different sort of generalisation, which is constructive, and which results in a new synthesis in the midst of which particular laws acquire new meaning (Piaget & Garcia, 1989). This meant that it is more concerned with interrelationships between concepts (Dubinsky, 1991).

In solving determinants problems, when a student is asked to evaluate the determinant of matrix A= $\begin{vmatrix} 1 & 5 & 3 \\ 0 & -1 & 4 \\ 2 & 3 & 0 \end{vmatrix}$, she performs many individual actions to determine the sub-matrices of A. She would then interiorise and coordinate the actions to create new matrices, which she could evaluate and relate the solution to matrix A. These processes coordinated together will then form new objects. It is reflective abstraction that leads to this kind of mathematical thinking, by which form or processes is separated from the content, where processes themselves are converted in the mind of the mathematician to the object of content (Piaget, 1980). According to Piaget, the first part of reflective abstraction consists of drawing properties from mental or physical actions at a particular level of thought (Beth & Piaget, 1966). The distinguishing factor of reflective abstraction is that not only is the subject able to project the structures created by his/her activities on to a new level and reorganise them, but the subject also remains consciously aware of what has been abstracted (Von Glasersfeld, 1995). Goodson (2014) suggested that in order to encourage these reflections, one must place careful emphasis on offering students appropriate mathematical learning tasks.

Piaget' work concentrated on the development of mathematical knowledge at early ages. Dubinsky (1991) claimed that the theory of reflective abstraction can be used to describe any mathematical concept together with its acquisition. He argued for the use of reflective abstractive concepts in helping students develop advanced mathematical thinking. He also advocated an instructional approach that induces students to make specific reflective abstractions (Goodson, 2014).

3.5.2 Piaget's Triad mechanism

The triad mechanism consists of three stages, referred to as '*intra*', '*inter*', and '*trans*' in the development of connections an individual can make between particular constructs within the schema, as well as the coherences of these connections (Dubinsky, 1991). The *intra* stage focuses on a "single object". This implies that everything is constructed as isolated fact. The student has a collection of rules, but at this moment, cannot link them to form coherent thoughts. The *inter* stage focuses on transformations between objects. At this stage, the students can make connections between concepts, but could not explain the underlying features. The student is able to manipulate symbols effectively. The *trans* stage – noted as schema development – is about connection of actions, processes and objects. At this stage, the student is able to construct the structure of a mathematical concept and to explain its underlying features. Clark, Cordeo, Cottril, Czarnocha, De Vries, St. John & Vidakovic (1997) have asserted that at the *trans* stage, the elements of the schema must go beyond being described essential by list, to being described by single rule. For example, a student who displays a coherent understanding of different representation of matrices and system of equations is at the *trans* stage. Dubinsky (1991) believes that an individual at the *trans* stage is able to construct various systems of transformation. Piaget's triad stages have some similarity with Tall's three world of thinking, which is embodied, conceptual and formal, but the triad mechanism links with the first two in the sense that the embodied mode is the fundamental human mode of operation, based on perception and action, and the warrant for truth is that things behave predictable in an expected way (Tall, 2002). So, as in the intra-stage, the student learning is mainly driven by physical and visual objects. The symbolic world relates to both *intra* and *inter* stages, because the symbolic world is a world of mathematical symbol processing (Tall, 2006). In this world, the thinking is not based on a single problem, but based on similar problems that have been solved and considered to be correct. Like the *intra* stage, the modes of thinking are more practical, such as counting, sharing, using manipulatives etc., but through modified instructions and use of symbols, an individual might be able to switch from processes to do mathematics to concepts to think about (Tall, 2006).

3.6 Basic Ideas of APOS theory

APOS theory begins with a statement of what it means to learn and know some aspects of mathematics as presented in Asiala et al. (2004):

> An individual's mathematical knowledge is his/her tendency to respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context and constructing and reconstructing mathematical actions, processes and objects and organising these in schemas to use in dealing with situations (p. 5).

The authors further believed that understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorised to form processes which are then encapsulated to form objects. APOS theory comprises four components, viz. *Action-Process-Object-Schema.* Asiala et al. (2004) describe APOS in relation to Piagetian ideas:

> What we are calling actions are close to Piaget's action schemes, our processes are related to his operations, and an object is one of the terms Piaget uses for that to which actions and processes can be applied. The term scheme is more difficult because Piaget uses several different terms in different places, but with very similar meanings, partly because of the difficulty of translating between two languages

> which both have many terms related to schema with a subtle and not always corresponding distinction (p. 6).

The researcher describes the four concepts in this Chapter that are used in APOS theory for a clear understanding of the genetic decomposition of matrix algebra. The description of action, process, object and schema used in this study, as discussed in Chapter One, are those given by Brown, Dubinsky, McDonald, Stenger and Weller (2004).

Dubinsky (1991) describes five kinds of construction in reflective abstractions, namely: interiorisation, encapsulation, coordination, reversal and generalisation. These can be linked to the four stages of APOS, which are: action, process, object and schema, to develop the mental construction of knowledge of the learnt concept. Interiorisation permits one to become conscious of an action, reflecting on it and combining it with other actions to form new ones (Dubinsky, 1991). Attorps (2006) has argued that it is through interiorisation that a student becomes acquainted with a concept, and performs operations or processes on mathematical objects. This links with Sfard's (1991) theory of reification. In her theory, the different mathematical notions can be conceived, either structurally as objects, or procedurally as processes. According to Sfard (1991), during the interiorisation phase, a student becomes familiar with a process and can carry it out through mental representation.

Coordination refers to using one or more actions to construct a new action or object. Sfard (1994) refers to this as condensation, where there is a gradual quantitative change in which a sequence of mathematical operations is dealt with, in terms of input and output, without necessarily considering its component steps. Encapsulation refers to building new forms that bear on previous forms, include them as contents, and draw from more elementary forms the elements used to construct new forms. This meant that the development of mathematical knowledge is constantly a way of constructing and reconstructing knowledge until new forms are built. These new forms of mathematical knowledge are at a more advanced level than the original forms of knowledge. When a mathematical concept is appropriately understood, a student reflects on the processes use to construct that mathematical knowledge, and used those processes to build new forms of knowledge. Dubinsky (1991) rates this process as the most difficult, because it is not just mere substitution. Generalisation refers to applying an existing schema to a wider collection of phenomena. Then, once a process exists internally, the subject may think of it in a reverse mode –

not undoing it – but as a means of constructing a new process this is reversal (Brijlall & Bansilal, 2010). These structures as discussed in Dubinsky (1991), are crucial for mathematical learning to take place. Figure 4 shows that the process of knowledge construction can be thought of in the learning of any mathematical concept adopted from Dubinsky (1991).

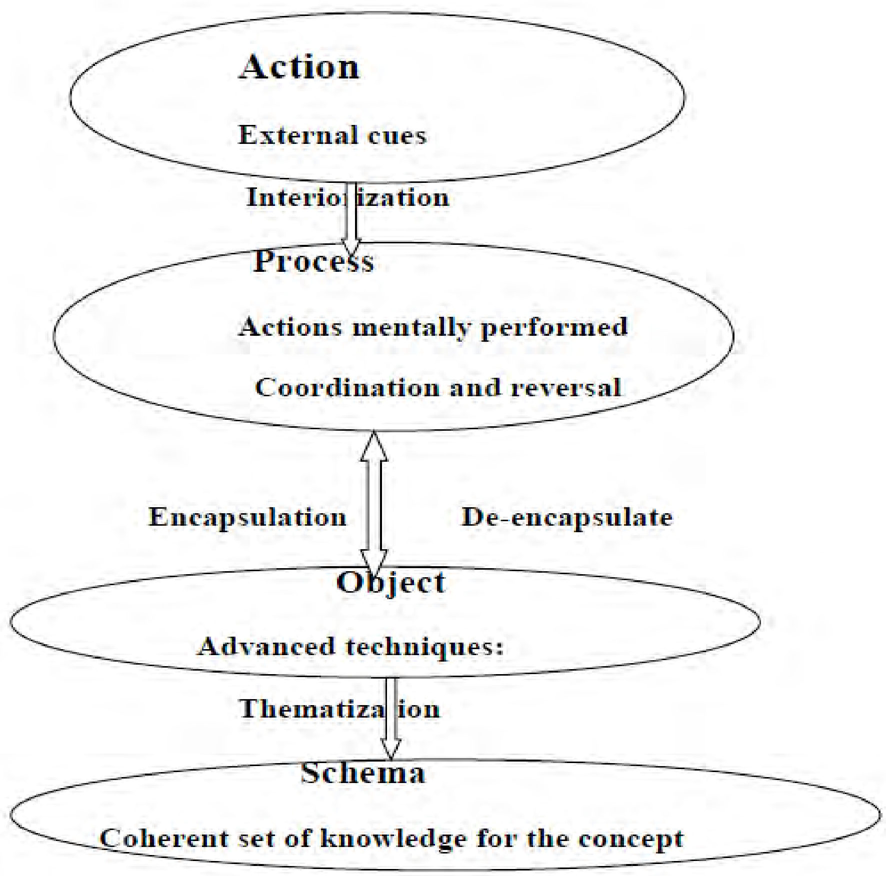


Figure 4: Process of knowledge construction

3.7 Value of APOS theory in the learning of matrix algebra

In this study, the proposed genetic decomposition of matrix algebra concepts presented the mental structures of how these concepts could be conceptualised. In order to analyse whether students were able to make these mental constructions, activity sheets were designed. By engaging with the activities, students would hopefully have gained sufficient experience and knowledge in learning and solving problems related to matrices. In each concept, the action-process-object formation was followed, because according to APOS theory, the cognitive operations must precede the cognitive concepts (Tall, 1999). The activities were done during tutorials, and were collected for analysis. Although marks were allocated, the focus was not on whether the answer was correct or incorrect; instead the focus was on analysing the procedures used to solve the problems. The procedures used would reveal the mental constructions made of the learnt concepts. In addition to the tasks designed to gain a deeper understanding of the mental constructions made, interviews with students were conducted. APOS theory made it possible to analyse the mental constructions displayed by students in the matrix algebra concepts. Dubinsky & McDonald (2001) suggested that APOS theory can serve directly as a tool in the analysis of data. Using genetic decomposition, the researcher is able to organise a hypothesis of how learning of matrix algebra concepts might take place. Therefore, the researcher would be able to differentiate between students who have constructed conceptual understanding of the concepts, and those who have not.

The main aim of this research was to investigate students' mental constructions in the learning of matrix algebra concepts at university level. As the learning of matrix algebra involves actions, processes and objects, APOS theory was deemed to be a suitable tool to explain and reveal the nature of mental constructions made. Revealing this could also assist in explaining the difficulties students experienced with the learning of matrix algebra. These might have caused them not to make the necessary mental constructions required for conceptualising the taught concepts. The genetic decomposition of each particular concept could serve as a lens through which the process of construction of knowledge could be understood. Data analysed constantly contributed to the modification of the preliminary genetic decomposition. Therefore, through revealing the nature of mental constructions the proposed genetic decomposition can be altered and improved, thus providing pedagogical strategies that a student might take in order to improve their understanding of a concept.

The construction process in terms of APOS begins with manipulating previously constructed mental or physical objects to form actions which are then interiorised to form processes. Processes are then encapsulated to form objects, which can then be de-encapsulated back to processes, which would finally be organised into schemas (Asiala et al., 2004). The researcher provides an illustration of knowledge construction in the understanding of computation of matrices in Figure 5a. In the multiplication of matrices, we begin with two matrices, matrix A and matrix B, and transform them into a single function: matrix AB. Although it is not easy to de-encapsulate these back to original matrices, one whose schema has developed would understand the order of the two matrices that forms matrix AB.

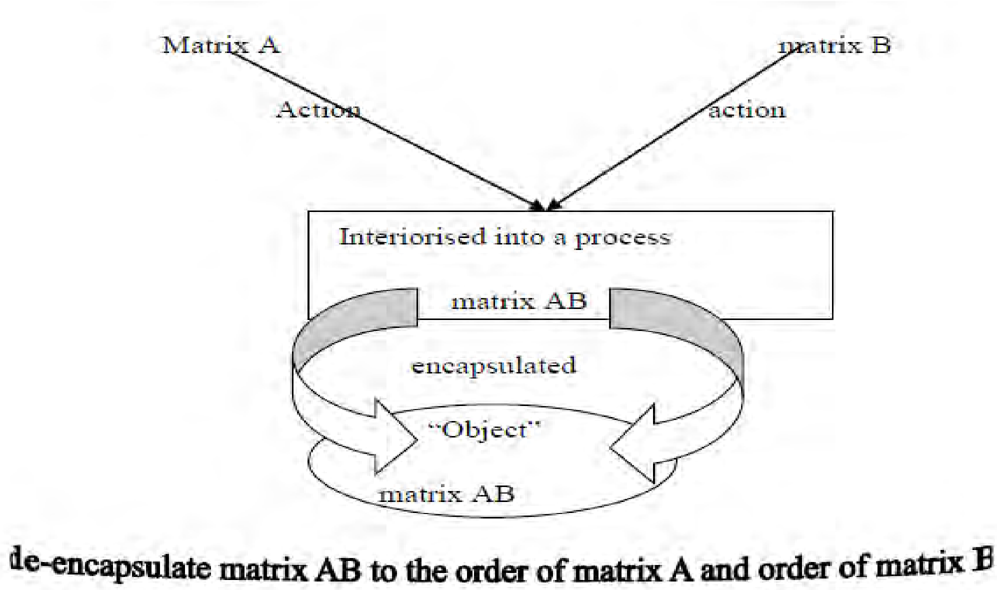


Figure 5a: Illustration of the construction process of matrices

The system of equations can be $n\ x\ m\ or\ n\ x\ n$ equations. For this study, because the solution is determined by Cramer's rule, we focus on $n\ x\ n$ equations. One can begin with any number of equations with the same number of unknowns, e.g., $2\ x\ 2$, apply actions to find the solutions, and then interiorise the solution as process, by explaining the meaning of the solution. By representing the solution in a plane, one has encapsulated the process into an object. From the geometrical representation one can de-encapsulated these into the original equations (see Figure 5b). A student who has the object conception can de-encapsulate the functions to their original equations, and organise these into schemas in which she can clearly explain which functions are consistent or inconsistent, and why. It is also possible to predict what type of solutions exists in the consistent functions.

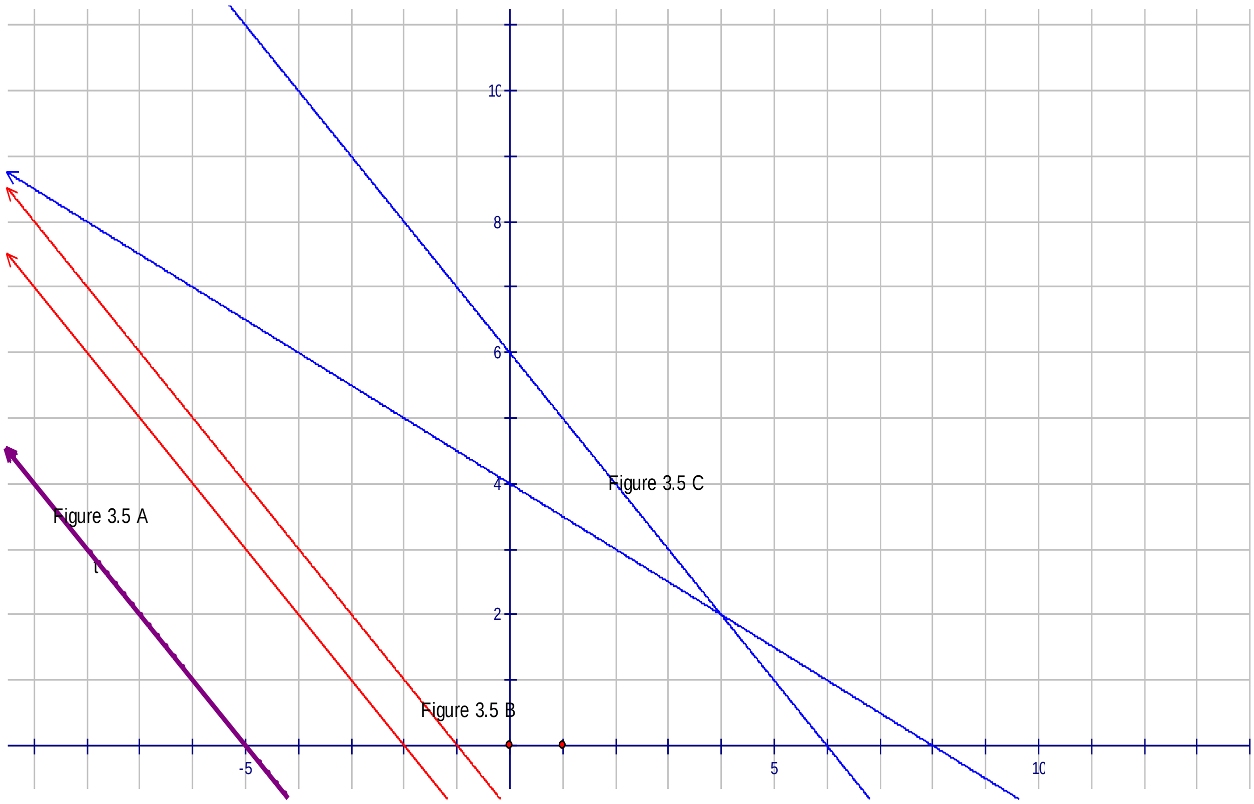


Figure: 5b The graphical illustration of solutions.

In the learning of linear algebra concepts, Sierpinska (2000) talks about three modes of thinking: (1) synthetic geometry; (2) analytic arithmetic; and (3) analytic and structural. Thinking of the possible solution to the system of three linear equations in three unknowns as intersection of planes in R^3 corresponds to the synthetic geometric mode; and if one thinks about the same problem in terms of row reduction of a matrix, one is in the analytic mode and lastly thinking about a solution in terms of the singular or invertible matrix correspond to analytic-structural mode. Each of the three mode leads to different meanings of the notions involved (Bogomony, 2007). This can be seen as an easy process for a mathematician, but research has shown that students have troubles transferring from one mode to another.

Students' understanding of many topics in abstract algebra, calculus, and discrete mathematics has been, in many cases, explained with the help of APOS theory. Tall (1999) agreed that this theoretical perspective is a useful tool in understanding the construction of mathematical knowledge in calculus and algebra, but that it cannot be regarded as a universal tool, as it would not be particularly applicable in the learning of geometry, since in geometry, the starting point is with the object, where the processes involved are more oriented towards gaining knowledge about the object. Although this might be true, since in learning geometry the focus is mostly on developing an understanding of axioms, and solving related problems through using certain processes like drawing, no research has been done to prove that APOS theory cannot be used to explain the construction of knowledge in geometry. Tall (1999) also asserted that although cognitive operations must precede cognitive concepts, it is necessary to take into consideration that the varieties of thinking in professional mathematicians need an expression beyond that of the measured *action-process-object-schema* development. Tall (1999) regards APOS as a major contribution to the understanding of mathematical cognition, but as a valued tool, not a global template.

3.8 Conclusion

This chapter commenced with a discussion of what we consider to be mental constructs. This was followed by the discussion of the framework that underpins the study. The link between the framework for curriculum research and APOS theory was revealed through the comprehensive discussion around this theory, as well as its development. APOS theory is the theoretical framework that underpins this study. In an attempt to show how the matrix algebra concepts can be conceptualised, the genetic decomposition of matrix algebra concepts was discussed, in which the interpretation of the mental constructions made using APOS was made. As mentioned, APOS theory is a theory of learning. Therefore, its focus is on how learning a mathematical concept might take place. Based on this genetic decomposition, a student is said to understand the concepts once he/she is able to explain the relationship between concepts. Following a discussion on how APOS theory can help in understanding the mental constructions made in the learning of mathematical concepts, the theory's link to this study was discussed, and it was shown how useful could be in the learning of matrix algebra concepts.

Furthermore, as part of the study, a large amount of data was collected and analysed. Therefore, while APOS theory proved to be useful as a theoretical framework, at the same time it is proved to be a useful analytic tool together with Piaget's triad mechanism. This will be shown and discussed in more detail in Chapter Six. The next chapter discusses the research design and methodology for this study.

CHAPTER FOUR

RESEARCH METHODOLOGY

4.1. Introduction

Chapter Three provided an argument for the use of APOS theory as the theoretical framework for this study. As discussed in the previous chapter, APOS theory provides an ideal framework for describing the mental constructions made in the learning of mathematical concepts. This theory is relevant to this study, because it reveals the nature of mental constructions that students might make in the learning of matrix algebra concepts. It also acts as an analytic tool that could be used to analyse data by revealing the specific mental constructions discussed in the genetic decompositions under the theoretical analysis. With this in mind, the research questions and research instruments were designed.

In our attempts to come to terms with problems of day-to-day living, we depend heavily on our experiences and authority (Cohen, Manion & Morrison, 2007). Based on our experiences, we develop intuitive senses of the approaches we can use to solve such problems. However, it is through research that we set to discover truth about the nature of phenomena. Research is a combination of both experience and reasoning and must be regarded as the most successful approach to the discovery of truth (Borg as cited in Cohen et al., 2007). The literature reveals that a variety of educational research methods may be used, and certainly they all make their own particular contribution towards educational knowledge. Kaplan (as cited in Cohen et al., 2007) suggests that, "the aim of methodology is to help us to understand, in the broadest possible terms, not the products of scientific inquiry, but the process itself". In this chapter, the researcher will describe the methodology for this research, which was designed specifically to examine the mental constructions which students might make in the learning of matrix algebra concepts. This chapter re-caps on the critical research questions and describes the limitations governing the research. It then goes on to discuss the methodology of the research by describing the paradigm within which the study was located, the research design adopted, the methods employed to conduct the study, and the separate phases (1 and 2) of the study. Additionally, the chapter concludes with a brief summary of the methodology.

4.2. Critical research questions

The study proposed to explore pre-service teachers' mental constructions in the learning of matrix algebra. It sought to identify the mental constructions made and projected, so as to explore the reasons why pre-service teachers were able to, or fail to, make the suitable mental constructions, respectively. There was a further aim to explore the link between the mental constructions made with the proposed genetic decomposition. This study proposed the APOS (Action-Process-Object-Schema) approach in exploring pre-service teachers' conceptual understanding of matrix algebra concepts. The key question the study addressed was:

How do pre-service teachers' **mental constructions** of concepts in matrix algebra concur with a preliminary genetic decomposition?

The main question will be answered through the following sub questions.

- *What levels of conceptualisation of actions, process and objects are reflected by pre-service teachers' mental constructions of matrix algebra?*
- *What difficulties do the students experience in their effort to construct the necessary mental constructions in matrix algebra?*
- *To what extent do the students' mental constructions of action, process and object link with the preliminary genetic decompositions?*
- *What characteristics of the schema displayed by the pre service teachers are adaptable to a genetic decomposition of matrix algebra?*

4.3. Interpretive research paradigm

This study was concerned with knowledge construction in the learning of linear algebra. The main aim was to explore the mental constructions that students make when learning matrix algebra concepts. In line with the aim of the study, the interpretive paradigm is the most suited paradigm underpinning the methodological framework of this study. According to Cohen, Manion & Morrison (2007) the interpretive paradigm is characterised by a concern for the individual, and is used to understand the subjective world of human experience. Interpretive researchers begin with individuals and set out to understand their interpretation of the world around them (Cohen et al., 2007). Interpretive researchers study meaningful social action, not just the external or observable behavior (Neuman, 2006). The same is echoed by Cohen et al. (2011), who note that interpretive researchers are set to examine situations through the eyes of the participants rather than through those of the researcher.

In this study, the data were generated from students' interpretations of mathematical problems, and their solutions to the given problems. It is through analysing the generated data that the proposed genetic decomposition hoped to yield insight and understanding of how the subjects make meaning of the learnt concepts. The mental constructions made are analysed through the set of meanings proposed in the genetic decomposition, thus allowing the development of the theory. Cohen et al. (2011) argue that in interpretive research, theory should be grounded in data generated, and interpretive researchers work directly with experience and understanding to build their theory. According to Schultz (1962), interpretive researchers believe reality is constructed inter-subjectively, through meanings, and that understanding is developed socially and experientially. This means that interpretive researchers are more concerned with how individuals make meaning and interpretive reality in their contexts. Angen (2000) concurs with Schultz, and further claimed that 'interpretive' assumes that researchers' values are inherent in all phases of the interview, and that the truth is negotiated right through the interview process. He outlined the following as characteristics of the interpretive paradigm: (1) interpretive approaches rely heavily on naturalistic methods (interviewing, observations and analysis of existing texts); (2) these methods ensure an adequate dialogue between the researchers and those with whom they interact in order to collaboratively construct a meaningful reality; (3) generally, meanings are emergent from the research process; and (4) qualitative methods are used.

Cohen et al. (2011) argued that interpretive inquiry interprets and discovers the perspectives of the participants in the study, and answers to the enquiry are practically dependent in the context. This study examined the mental construction that student participants might make in the learning of matrix algebra. To obtain the answers to the enquiry, this study was classroom-based, which the researcher considers to be the natural setting of learning for these students. The knowledge constructed is discovered and interpreted in the natural settings. According to an interpretive approach, the researcher presents experiences as they become constructed, and collects multiple stories when planning to group stories around a common theme (Denzin, 2002). Therefore, the interpretive approach is described as the "systematic analysis of socially meaningful action through the direct detailed observations of people in natural settings in order to arrive at understanding and interpretation of how people create and maintain their social world" (Neuman, 2006, p. 88). Denzin (2002) argues that in the interpretive approach, the interpretation should be based on material that comes from the world of lived experiences, and which incorporates prior understanding into the interpretation. According to Neuman (2006), the interpretive approach is idiographic and inductive. This means that interpretive research is more concerned with giving detailed descriptions of the phenomena. It focuses specifically on concepts that require an in-depth understanding of how the participants construct their meaning. According Denzin (2002), in an interpretive paradigm, the researcher uses his or her own life experiences as part of the research and topic of inquiry. He further articulates that the researcher seeks out subjects who have experienced the type of experiences the researcher seeks to understand. In this study, the researcher uses her experience to show how matrix algebra concepts could be conceptually understood by an individual. The study, as mentioned earlier, is taught to students who were planning to be FET mathematics teachers, and therefore these were typical students, whose mental constructions of the concepts could be explored.

4.3.1 How the paradigms fits with this study

The goal of interpretive paradigm is to develop an understanding of social life, and to discover how people construct meaning in a natural setting (Neuman, 2006). In this study, the goal was to reveal the nature of mental constructions, meaning constructed, and to explore how students came to construct their understanding of the learnt concepts in the classroom-based situation. Interpretive enquiry is concerned with the way in which individuals experience the world, the way in which they collaborate, and the settings in which these collaborations occur. This framework is applicable in this study, since it examined the individual conceptual understanding of the concepts. It was of importance in the study that students solved the problems first individually, then working in groups during tutorials where they shared their experiences about solving the tasks, and constructing meaning. In interpretive enquiry, the researcher uses his/her experiences to form part of inquiry. To reiterate these ideologies, this study proposed a genetic decomposition for matrix algebra concepts. The design of this structured set was based on the experiences of the researcher with the learning of these concepts. Students studying this module were the suitable subjects, whose mental constructions can be examined in relation to the proposed genetic decomposition.

Interpretive approaches rely on naturalistic methods, such as interviewing, observation and analysing existing texts (Angen, 2000). He also asserted that these approaches ensure an adequate dialogue between the researcher and those with whom he/she interacts, so as to construct a meaningful reality and to derive meanings from the research process. An interpretive researcher studies a text, such as a conversation, to draw out elusive verbal communications in order to discover embedded meanings (Pillay, 2008). In this study, it was of importance to analyse students' responses to written tasks in order to reveal their mathematical thinking in the context of matrix algebra. It was also of significance to use interviews to understand how the participants constructed the meaning, and to draw out embedded meanings. This was done with the hope that engaging in a dialogue would shed more light in the mental constructions made, in relation to the proposed genetic decomposition. Through an interpretive paradigm, the researcher is able to observe different approaches to solving problems, and use multiple ways to understand how students construct meaning. Interpretive studies examine how problematic, turning point experiences are organised, constructed and given meaning (Denzin, 2002). This implies that the researcher is set to ask and answer crucial questions in the study. The format of research questions in this study indicates the interpretive research designs.

4.4 Research design and methodology

A research methodology describes the selected design and sampling method used in this study. To answer the research questions, a qualitative approach was adopted. The qualitative approach provides multiple ways of understanding the inherent complexity and variability of human behaviour and experience (Grace, Higgs & Horsfall, 2009). Different people in different settings construct their realities based on their perception of reality. Therefore, qualitative research provides an opportunity to understand peoples' perception in their natural settings, and directly speaks to the participants seeing how they behave and act within their context. According to Corbin & Strauss (2005) qualitative researchers have a desire to step beyond the known and enter into the world of participants, to see the world from their perspectives. In doing so, they make discoveries that contribute to the development of empirical knowledge. This study aimed to explore the specific mental constructions that students make in the learning of matrix algebra, and to understanding how they construct knowledge. In doing so, the researcher hopes to bring new knowledge in the teaching and learning of matrix algebra, by providing a structured set called genetic decomposition, which describes mental constructions that students might make in the learning of matrix algebra concepts.

By its nature, a qualitative research methodology allows one to use different research strategies to collect data. Merrian (2002) describes four qualities of qualitative research: (1) qualitative research elicits participation accounts of meanings, experience or perception about concepts; (2) it produces descriptive data; (3) qualitative approaches allows for more diversity in responses as well as capacity to adapt to new development or issues; and (4) in qualitative methods, forms of data collected can include interviews, group discussions, observations, various texts, pictures and other materials. This study makes use of variety of methods to collect data as it used text from students' responses, interviews, audio recorder as well as video recorder. Asiala et al (2004) mentioned two aspects of qualitative approach that needs to be addressed, namely: (1) the theoretical perspective taken by researchers using that approach; and (2) the actual methods by which data are collected and analysed. In addition, Creswell (2009) claims that theory in qualitative study provides (1) an overall lens for the study in question; (2) it becomes an advocacy perspective that shape the types of questions asked; and (3) it informs the way in which data is collected and analysed. In this study, the theoretical framework used informs the qualitative methodological framework that was taken by the researcher. Also, the methods used align with the theoretical framework, which then allows the researcher to use the theoretical framework as the analytic tool. All in all, the theory used informs this study.

In qualitative research, the idea is to discover patterns of behaviour or thoughts in a set of texts (Bernard, 2000). Creswell (2007) concurs with this idea, and has stated that the researcher establishes patterns and searches out correspondence between two or more categories. Since the study was based on a qualitative approach, both inductive and deductive analyses were used. This was done through coding the written responses of all the participants. Thereafter, the categories were determined, and patterns and trends that emerged were further analysed. The patterns that emerged were analysed using the genetic decomposition that was presented under theoretical analysis. The genetic decomposition describes specific mental constructions that a student might make in order to develop conceptual understanding of matrix algebra concepts. The design of this genetic decomposition was influenced by the researcher's own experience in the teaching and learning of matrix algebra concepts, as well as her understanding of these concepts. The theoretical perspectives of this study focused on what it means to learn and understand something in mathematics. This is based on Asiala et al.'s (2004) description of understanding as it was discussed in Chapter Three. Matrix algebra concepts are considered to be less abstract, however to develop conceptual understanding of these concepts goes beyond a mere application of rules. Students' need to be able construct and reconstruct the knowledge learnt in order to move beyond the urge to do mathematics to construct processes leading to thinking about mathematics. This would then assist them in dealing with more abstract concepts in linear algebra.

This study is qualitative in nature, and therefore, it explores pre-service teachers' mental construction of mathematical knowledge, in matrix algebra. According to Cohen, Manion & Morrison (2007), a case study provides a unique example of real people in real situations, enabling the reader to understand the events more clearly than simply presenting them with abstract theories or principles. The pre-service mathematics teachers were encountering the concept of matrix algebra for the first time. Therefore their experiences of learning the concept and the way in which they make meaning, were unique. The data collected generated a new understanding for the mathematics community about how students construct the required knowledge in learning matrix algebra. Patton (2002) asserted that a case study should take the reader into the case situation and experience. It is imperative that the students' experiences of the concept are understood. According to Tellis (1997), a case study is an ideal methodology when holistic, in-depth investigation is needed, and is designed to bring out details from the viewpoint of the participants by using multiple sources of data. This study consisted of two stages of data collection. First, data were collected from students' responses to matrix algebra problems. Once the responses were analysed, the semi-structured interviews were used to verify and clarify students' understanding of matrix algebra concepts. Case studies tend to be selective, focusing on one or two issues that are fundamental to understanding the system being examined (Tellis, 1997). This is supported by Guthrie (2010), as he argued that case studies are not a representation of the entire population, therefore the results are not generalised, but if appropriately selected, findings could be used in other settings. In this study, the researcher did not intend to generalise the findings, and as a result, specific choices were made as to who the participants of the study were, regardless of whether they were representative of the whole population or not. Also, the preliminary genetic decomposition presented is unique to the teaching and learning of matrix algebra concepts

4.4.1 Gaining access

The purpose of the study was to explore mental constructions made by pre-service teachers when learning matrix algebra. For this study, the researcher needed to conduct research at the university. Since the researcher worked at the university and taught matrix algebra to undergraduate students, I decided to conduct the study with a group of students at the university at which the researcher work. The researcher was required to obtain permission from the institution to conduct research as well as to obtain the consent from the students sought out to participate. To gain access to conduct the study, permission needs to be obtained from the Research Office, and the Dean. A copy of the letter from the Dean is attached in Appendix B and ethical clearance certificate no HSS/1470/013D from the Research Office may be found in Appendix C.

4.4.2 Informed consent

When conducting research, ethical consideration is important, therefore the reseracher had to take into consideration the following factors: informed consent; right to withdraw; confidentiality; methodological rigour; and fairness. Before the researcher proceeded with this study, she provided all students enrolled for this module with an introductory letter. This letter discussed and defined informed consent, the right to withdraw, and confidentiality. The letter provided each participant with the reasons and purpose of the study. Each participant was required to provide their signed consent. The researcher also explained the procedures that would be followed during the research process, provided timeframes, and relevant contact details of personnel at the University. A copy of this letter may be found in Appendix B.

4.4.3 Context of the study

The study was conducted in a South African university with a combination of first year and second year students, who were training to become FET mathematics teachers. In this university, linear algebra is taught as a level one, semester one module to students who plan to be FET mathematics teachers. The University has a diverse student body. The group that participated in the study mainly consisted of African and Indians. To major in mathematics, a student must have achieved 60% or above in mathematics for their matric. Any student who achieved level 4 (50% to 59%) in their matric results, but wishes to be a FET mathematics teacher, needs first to do a foundational module in mathematics and achieve 60% or above. The students undertaking this module are thus a combination of first year and second year students in terms of their period of registration at the university. However, to all of them, this is the first major mathematics module they were required to complete. Many of the mathematics concept that are taught in this module are related to what students have done at school level, and they will be encountering matrices and determinants for the first time. However, their previous knowledge in arithmetic algebra proves useful in the computation of matrices.

4.4.4 Lecturer/Researcher

In my interactions with students, the researcher played two roles, as lecturer and researcher, which brought both opportunities and pitfalls. Ball (2000) suggests that such an approach "offers the researcher a role in creating the phenomenon to be investigated, coupled with the capacity to examine it from the inside, to learn that which is less visible" (p. 388). In assuming both roles in this study, the researcher gained considerable inside knowledge that helped in designing problems that yielded the necessary mental constructions. In the whole process of learning of matrix algebra, the researcher was able to guide students as they work individually, and in groups. In this way, the researcher was able to learn about the students' thinking processes, which were of importance in the analysis of their responses. Being a researcher and lecturer at the same time also helped me to get to know students much better, in a way observing them from the back of the class would not have afforded.

Findell (2006) has pointed out that many of the pitfalls of being a teacher/researcher arise when the purpose of the research is to study teaching, and that the main problem is gaining sufficient objectivity to ensure the reliability of observations and the validity of conclusions about one's own thoughts and actions. Since this study focused on learning, such pitfalls were not present. However, any research that focuses on particular cases has certain challenges. It cannot be generalised, but its findings can be extrapolated explored to other similar cases. The most challenging issue in this study was that of power dynamics. Ball (2000) highlights such challenges, where students might have some reservations as to what they should or should not say. Although it was not that evident in this study, the researcher decided to clearly explain the purpose of the study before it commenced, and during the study itself, students were reminded that their solution to these tasks had no bearing on their assessment. The goal of the study was communicated to be exclusively aimed at understanding students thinking processes and how they come to learn these concepts.

4.4.5 Participants of the study

The study included 122 first and second year pre-service teachers majoring in mathematics. For tutorial sessions, students were divided into groups of four, with each group consisting of 30 or more pre-service teachers. Each group had a tutor. The researcher is a tutor of one group of 31 pre-service teachers who are the participants of this study and the lecturer who is also a researcher was also a tutor of one of the group. This group consisted of 31 pre-service teachers. Students were grouped alphabetically for their tutorial groups. Although data were collected among the group of 31 students the researcher was tutoring, it was important that all students knew what the study was about. This was done so that they would not think that they were somehow put at a disadvantage by not being part of the study. One-hundred-and-twenty students gave consent to be part of the study. The study was conducted over four lecture sessions and two tutorial sessions of 90 minutes each, where the researcher first discussed with students what needed to be done, and students had to solve problems individually; then later get into groups, where they discussed their solutions. At the end of each session, individual responses and the group responses were handed in. To ensure that all students were doing similar tasks before each session began, the researcher held a meeting with the other tutors to make sure that they clearly understood what students ought to be doing during tutorials. This was done to ensure that all students, even those not part of the study, were being assisted in developing a conceptual understanding of these concepts.

4.4.6 Purposive Sampling

The study used purposive sampling. In qualitative research, the choice of participants merely depends on relevance to the research topic, rather than on representativeness (Neuman, 2006). Neuman further asserts that qualitative researchers select cases gradually, with specific content of a case. In this study, the researcher does not intend to generalise the findings to other universities or to other mathematics modules, therefore purposive sampling is considered suitable for this study. The choice to use this group of students is due to ease of access to the participants, due to the fact that the researcher works at the institution. Therefore, the study was conducted during lecture time, ensuring that participants see this as part of their work, and not as an additional form of work, with no relevance to what they were supposed to learn. According to Bertram (2004), purposive sampling means that the researcher makes specific choices about which people to include. Cohen, Manion & Morrison (2007) state that purposive sampling can be used to access those who have in-depth knowledge about a particular issue. The researcher hoped that pre-service teachers would provide rich information about the construction of knowledge in matrix algebra, since they were learning this concept for the first time. Gaining deep understanding of the mental constructions that students are able to make will help in understanding the misconceptions and other difficulties students have in the learning of these concepts. Purposive sampling is a sampling strategy for a case study (Pillay, 2008). As mentioned earlier, this study is qualitative by nature, and it is a case study.

4.5 Methodological framework

Dubinsky (1997) suggested that for students to develop conceptual understanding of a particular topic they needed to make constructions of mathematical concepts, as well as interact with each other and the instructor in the context of the problem situation that the course provides. He claimed that these interactions would bring out inconsistencies, contradictions and disagreements, the resolution of which leads to understanding on the part of the student. Furthermore, he proposes pedagogical strategies that begin with observation and analyses of the specific mental constructions that might be used to understand a certain concept. Then students were required to be presented with problem situation designed to foster their making of such constructions. Finally, the interaction among students, instructor and problem involves: (1) the students trying to obtain the solution to the problem; and (2) the instructor trying to get the students to revise their constructions so as to make them more effective in solving the problem and more consistent with the instructor's own constructions. This framework is elaborated in more detail in Asiala et al. (2004). Following this approach, this study commenced with four lessons of 90 minutes each on matrix algebra concepts. One lesson focused on matrices. Two lessons focused on determinants and matrix inverse, and a last lesson on solving the system of equations using Cramer's rule. This was aimed at helping students develop an understanding of these concepts. The instructions for the lessons were designed in relation to the proposed genetic decomposition. Then, activities that might help students make the necessary mental constructions of the matrix algebra concepts were selected and administered to whole group of students, but the analysis was only drawn from the 31 students selected to be part of the study. When students were solving the problems, the interaction among students was encouraged.

4.6 Research methods

The study was composed of two phases. Phase 1 was conducted in 2013. The main focus here was to test the research instruments and validate the activity sheets for the purpose of the main study. Phase 1 was conducted with 98 pre-service mathematics teachers. However, the analysis was based on the responses of 85 students who submitted their work. All 98 students had given consent to be part of the study. This allowed the researcher to conduct the session during lecture times. At the end of three lecture sessions, students were given an activity sheet, which they had to complete over two days, and then return. In this phase of the study, the analysis was only based on the students' responses due to time constraints. The results of this phase are included in the study as Appendix A1, and the full discussion is presented in Chapter Five.

In preparation for Phase 2, some changes were made to the activity sheet. The structure of the activity sheet was changed. Instead of one activity sheet with all the tasks, students were given separate activity sheets for different tasks. The skills and knowledge covered in phase one was still the same for Phase 2, with the addition of the solution of a system of equations. The skills and knowledge covered in the second phase consisted of determining the matrix transpose; the computation of matrices; evaluating determinants; and solving the system of equations using Cramer's rule. Students' understanding of these concepts was explored. In order to identify students' written work as it was collected, each student was allocated a numeric code together with pseudo-names. The pseudo-names are also used in the audiotape transcriptions of the interview process. When all the activities were completed and marked by the researcher, students' responses were categorised. The categories are shown in detail in the analysis chapter. For each category a sample of students was selected and interviewed. This was done in order to clarify some of their responses and to explore the individual mental constructions made in relation to the genetic decomposition.

The activity sheets comprised of four tasks, and each task has sub-questions. All in all there were 13 sub-questions that students had to solve. Task one focused on the computation of matrices and matrix transpose. Task two focused on the evaluation of determinants. Task three and task four were of higher order thinking, where students' conceptual understanding of the learnt concepts was explored. In task three, concerned with the concept of matrix inverse, an application of matrices was explored, and in task four, students' understanding of the solution of system of linear equations was explored. Once students' responses were analysed, the interview questions were designed. These were semi-structured allowing the researcher to probe further for more clarity where necessary. .

4.6.1 Data collection procedures

Qualitative research methods involve the systematic collection, organisation, and interpretation of textual material derived from talk or observation. They are used in the exploration of meanings of social phenomena as experienced by individuals themselves, in their natural context (Malterud, 2001). In qualitative research, the researcher is the primary instrument for data collection and analysis (Merrian, 2002). In this study, the researcher used her own experience and understanding of the concepts to design the genetic decomposition, which was used as the basis of designing the activity sheets as one of data collection methods. By its nature, qualitative studies use a variety of methods, such as interviews, observations, documents, etc., to gather data. In the data collection process, the decision as to which strategy to use is determined by the question of the study (Merrian, 2002). Merrian further argues that in qualitative research, there is often a primary method of collecting data. As a qualitative study, this study uses various methods like activity sheets, interviews, video and audio recordings as data collection methods, and each method used respond to a particular question of the study (see Table 1). This was done with the aim of enhancing validity of the findings. Mason (2002) articulates fives reasons for integration of methods in qualitative studies: (1) to explore different elements or parts of a phenomenon, ensuring the researcher knows how they interrelate; (2) to answer different research questions; (3) to answer the same research questions, but in different ways and from different perspectives; (4) to give greater or lesser depth and breadth to analysis; and (5) to triangulate and corroborate by seeking different data about the same phenomenon. Some of his suggestions conform to the methods for data collection used in this study.

Research Questions	Methods of Data collection	Participants under study	Time Frame
What do pre-service teachers' responses reveal about their mental constructions made when learning matrix algebra concepts and to what extent have they conceptualised the concepts in matrix algebra?	Activity sheets and interviews Video recorder	Pre- service teachers	February – March 2014
Why are pre service teachers able/not able to construct the necessary mental constructions?	Activity sheets And interviews Video recorder	Pre- service teachers	February- March 2014
To what extent, if any, are the students' mental constructions: action, process and object link with the preliminary genetic decompositions?	Interviews and activity sheets Audio recorder	Pre- service teachers	April- June 2014
What characteristics of the schema displayed by the pre service teachers can be adopted to create a modified genetic decomposition?	Activity sheets and interviews	Pre- service teachers	February- March; April- June 2014

Table 1: Data collection plan

In this study, data were captured using a multi-method approach. Phase 1 of the study was used to validate the research instruments. The findings are presented in the next chapter. Data analysis included: (1) results from phase one of the study; (2) analysis of students' responses in the tasks given in Phase 2 regarding their understanding of matrices, determinants, matrix inverse and system of equations; and (3) interviews conducted to analyse students responses to written tasks and class discussions. After the analysis of the tasks in Phase 1, the genetic decomposition was modified for phase 2. In Phase 2, while students were answering problems in activity sheets, they were video-recorded in order to capture their verbal and non-verbal behaviours, while solving problems and capturing the group discussions. An audio-recording was employed during the interview process. In the next subsections, the researcher discusses each data collection tool that was used in the study.

4.7 Stages of data collection

4.7.1 Stage 1: Structured activity sheets

According to Brijlall & Maharaj (2009a), structured activity sheets model the way in which meaningful mathematics teaching could be planned with the aim of simultaneously addressing the cognitive and affective domains when students solve problems. Matrix algebra problems require students to apply the algorithms and manipulation skills they have learnt in school algebra sections, but in a more critical way, showing their conceptual understanding of the learnt concept. The use of structured activity sheet can generate the required data that a researcher could use to understand the mental constructions that students employ in constructing the knowledge of matrix algebra and how these mental constructions link to the proposed genetic decomposition. Structured activity sheets are the best sources of data collection, since these give direction to learners on answering questions, thereby assisting them in constructing the required mental constructions (Ndlovu, 2012). While the students were working on the activity sheets, they were video recorded. This was mainly done so as to capture all the non-observant activities among students. Video recording documents non-verbal behaviour and communications and emotions (Marshall & Rossman, 2006). Flick (2006) concurs, as he points out that videos are a valuable tool in collecting data, as they can catch facts and processes that are too fast or too complex for the human eye. In this study, the use of video recording was aimed at capturing the essence of group discussions. The researcher believed that what might emerge in the group discussions might be informed by students' knowledge constructed of the concepts. For this reason, recording their interactions might also provide an insight about the mental constructions made.

4.7.1.1 Instruction

As it was stated, the data were collected during the tutorials. However, at first, the researcher delivered lectures on matrix algebra concepts, then prepared the tutorial tasks to enhance the student's understanding of the concepts. The teaching of the topic was guided by the preliminary genetic decomposition since it was going to be used to analyse data. Also to ensure that mental constructions that students are expected to make are actually taught to them. Individual activity sheets were prepared and one activity sheet per group was also prepared. These tasks were the main source of data generation. The tutorial sessions were mainly devoted to solving problems related to the concepts, and to discussions of the students' solutions. The students first solve the problems individually and then get into groups of six or seven where they discussed their solution. This was done so that in group discussion students will get a chance to explain their thinking strategies and think about their solution. The activity sheets were then collected after group discussions. In most of the problems, students were expected to justify their claims. This was done because students were expected to make their thinking strategies explicit. Aspects of the theoretical perspectives described in Chapter Three informed the instruction and design of the problems set. This was done so as to pay particular attention to the mental constructions that student might make. It was of importance to this study to know how students were thinking about these concepts and how they came to construct that thinking, because what they learnt during the lectures might not be what the researcher intended them to learn.

4.7.1.2 Key ideas targeted by the problems set

The problem sets were designed to provide experience with examples that could be used to motivate the learning of key ideas. The problems focused mainly on different aspects of learning of matrix algebra. For example, the students were asked to: find the transpose of a matrix; determine the order of a matrix; compute matrices; evaluate determinants; solve a system of equations; and show whether or not the matrix has an inverse (see Appendix A2). The activity sheet was divided into four sections, with each section covering certain aspects of these concepts. Initially, the researcher had planned to have four tutorial sessions, but after a meeting with the module coordinator, the researcher realised that matrix algebra was allocated only four lectures of 90 minutes each, and two tutorial sessions of 90 minutes each. In order for all the tasks to be covered, students were required to do two tasks in one session. This limited the time for whole class discussion. As a result of time constraints, the researcher realised that some students would get into group discussions without completing the tasks. This poses a challenge in understanding the mental constructions made, because when the question was not attempted, it was difficult to tell whether it was because no mental constructions had been made, or whether time proved to be a constraint. However, the researcher hoped the use of interviews would clarify these aspects.

4.7.2 Stage 2: Observation process

Observing is the process of studying classroom activities to determine teaching strategies and student responsiveness. It can be used to gain insight into planning, organisation, methods of presentation, behaviour management techniques, and individual student differences (Olsen, 2008). Observational data is used for the purpose of the description of settings, activities and people; and the meanings of what is observed from the perspective of the participants. Observation can lead to deeper understandings than interviews alone, because it provides knowledge of the context in which events occur, and may enable the researcher to see things that participants themselves are not aware of, or that they are unwilling to discuss (Patton, 1990). According to Cohen et al. (2011), observation offers an explorer the opportunity to gather live data from natural occurring social situations. Live data might give the researcher a clear perspective on how the participants constructed meaning, because in some cases, what people say might differ from what they do (Robson, 2002). Cohen et al. (2011) defines three types of observation: (1) highly structured observation, (2) semi structured observation and (3) unstructured observation. In highly structured observation, the researcher knows in advance what is sought and works out the categories before going into the field. In semi-structured observation, the researcher will draw an agenda of issues, but will gather data to illuminate these issues in a far less predetermined manner. In unstructured observation, the researcher is not clear on what is sought, and therefore first observes what is taking place before deciding on its significance for the research (Cohen et al., 2011).

Observation in the study was mainly used to gain insight into students' learning process of matrix algebra concepts. Moreover, it aimed at capturing their verbal and non-verbal behaviours during group discussions. Such behaviours can be later analysed and may provide more insight on the mental constructions that students may or may not have made. Such understanding helps to inform and refine the teaching instruction. Following Cohen et al. (2011), types of observation, this study used unstructured observation, and relied mostly on video recordings to capture important moments during the class interaction. However, to keep the students on the task, the researcher listened to group discussions and posed more questions to elicit higher thinking level from the students. The researcher observed that during group discussions, some students were not making their thought processes explicit, and in some cases, they remained silent. This posed a challenge in determining the specific mental constructions made, as students can adopt other students' thinking as theirs, and can complete their solution with the group thought processes that extend beyond their own. The researcher hoped that the interview might reveal the individual thought processes. Although the use of audio visual methods seem to be helpful in capturing what the human eye couldn't, Cohen et al. (2011) caution researchers against the problem of reactivity. To try and overcome this, the researcher used a third party person to video record classroom activity. Although this can also be considered to be intrusive, the researcher organised for third party to sit in a specific area, where he could get the view of everything that transpired, without disturbance to the group. In keeping with the ethics of best practice, all the students in this group gave consent. The researcher explained to them that no pictures showing their faces would be used. Instead, their responses and verbal texts would be used.

4.7.3 Stage 3: Using semi-structured interviews

Qualitative interviews may be used either as the primary strategy for data collection, or in conjunction with observation, document analysis, or other techniques (Bogdan & Biklen, 1998). Qualitative interviewing utilises open-ended questions that allow for individual variations. Patton (1990) writes about three types of qualitative interviewing: 1) informal, conversational interviews; 2) semi-structured interviews; and 3) standardised, open-ended interviews. Similarly, Cohen, Manion & Morrison (2011) group and discuss four main kinds of interviews, namely: the structured interview; the unstructured interview; the non-directive interview; and the focused interview. According to Denzin & Lincoln (2005), unstructured interviews provide greater breadth, with the main goal of understanding the phenomena. Using unstructured interviews allows the interviewer to probe where needed (Cohen et al., 2011).

This study is bound by an interpretive paradigm, and it sees humans as not just manipulable objects or data sources, but rather regards knowledge as generated between two humans through conversations (Cohen, Manion & Morrison, 2007). Therefore, interviews are suitable as data collection method, because this study aimed to understand students' experiences in constructing the knowledge to solve matrix algebra problems. According to Bertram (2004), interviews are a good data collection tool for finding out what a person knows. In this study, it was important to discover how students interpret tasks, which thus led to the way in which they constructed knowledge to solve the matrix algebra problems. Interviews enable participants to discuss their interpretation of the world, and to express how they regard the situations from their own point of view (Cohen, Manion & Morrison, 2007). This also is emphasised by Patton (2002), who notes that interviews allow us to enter into other person's perspective. Interviews are an important part of research, as they provide the opportunity for the researcher to probe and to delve deeper, to solve problems, and to gather data, which could not have been obtained in other ways (Cunningham, 1993). The use of interviews in this study provided the opportunity to gain a deeper understanding of students' mental constructions, which in some cases, were not explicit from their responses to the tasks.

After analysing activity sheets and reviewing student responses in Phase 2, the in-depth task-based semi-structured interviews were conducted so as to gain more clarity on students' thoughts about their solutions. These semi-structured interviews offered a versatile way of collecting data (Welman & Kruger, 2001) as they raised key questions and allowed the researcher to enjoy some natural conversation with the students. The rationale for the interviews followed the overall aims of the study to understand how students construct their conceptual understanding of matrix algebra and to explore the mental constructions made in the learning of matrix algebra. Therefore, the interviews were used as a means to gather feedback on how students constructed their knowledge, and to identify possible barriers to conceptualisation. Secondly, they were an opportunity to verify students' responses in relation to the mental constructions, which would have been made when analysing their solutions to the problems. As indicated earlier, students responses were put into certain categories based on their responses, and in each category, at least two students were selected for an interview. To cater for participants' withdrawal, sixteen participants were selected for an interview, but only ten students availed themselves for the interview. The six students who withdrew did originally give consent to be interviewed and be audio recorded, but later indicated their unwillingness to take part in the interviewing process, citing several reasons.

The interviews were conducted over four weeks in the months of April and May at the University. This was due to the fact that conducting these interviews depended on the availability of the participants. To elicit students' understanding, open-ended questions were used. This allowed the researcher to probe further for an in-depth understanding of how students came to make those mental constructions. It allowed the participants to express themselves freely, and to add or change whatever they wanted to. It also gave them another chance to relook at their responses and check if their understanding then is still the same, or if it has been improved. There were standard questions such as ‘can you clarify your response to this question’? Other questions were specific to a particular students’ response for example: “in Task 2, Question 1.1 you applied the rule for evaluating determinant but you did not even attempt Question 1.2. Why?” Before the commencement of the interview, participants were made aware that the interview would be audio-recorded and asked if they have any objection to this. Although they had given consent, it was important to remind them so that they would be made aware of how the process would unfold, and to develop a sense of trust. According to Denzin & Lincoln (2005), establishing rapport with the participants is of importance during the interview process.

The decision as to whether one relies on written notes or recording device appears to be largely a matter of personal preference. For instance, Patton (1990) says that a tape recorder is indispensable, but Guba and Lincoln (1994) do not recommend recording, except for unusual reasons. Guba and Lincoln base their recommendation on the intrusiveness of recording devices and the possibility of technical failure. Cohen et al. (2011) argue that using audio tape might constrain the respondent. However, Hoepfl (1997) argued that recordings have the advantage of capturing data more faithfully than hurriedly written notes might, and can make it easier for the researcher to focus on the interview. Patton (2002) concurs, as he pointed out that the audio recorder allows the interviewer to be more attentive and to increase the accuracy of data, catch what the interviewee said in his/her own words. According to Bernard (2000), tapes are a permanent record of primary information. Since interviews provide a rich and detailed explanation of how students construct knowledge, it is vital that each and every detail of the interview is captured. In this study, it was of importance to gain clarity on students’ explanations – therefore it was important to use an audio recorder to capture everything the participants were saying. This provided an in-depth understanding of how students used their experiences, and previous knowledge to construct new mathematical knowledge. After conducting the interviews, they were transcribed, and data were analysed inductively. The analysis of data is discussed in detailed in the chapters to follow. Mishler (as cited in Cohen et al., 2011) pointed out that audio tape is selective, where it filters out important contextual factors, neglecting the visual and nonverbal aspects of the interview. In this study, students were allowed to write down their explanation or to make sketches if they wished to do so, or to redo their solutions. This was done to supplement the verbal data recorded.

Although interviews are considered to be an important part of research, there are limitations, for example, interviews are lengthy and require more time. Bailey (2007) has highlighted the issue of bias on the side of the interviewer, by influencing the respondent responses. Using the semi-structured interview can produce data that were less systematic and comprehensive (Cohen et al., 2011). Since this study requires an in-depth understanding, it was important to spend some time with the participants. In order to ensure that the time spent was used to generate rich data, some of the questions were prepared ahead of time. Having those questions to begin with allowed for more probing during the interview. Since interviews are lengthy, there was the possibility of students deciding to withdraw. The researcher then decided to categorise students' responses according to their correctness, and to at least interview two participants in each category. This also gave more in-depth understanding as to the cognitive structures used by students to learn the concept. In certain cases, it appeared that although students were in the same category in terms of their responses to the activity sheet, they were actually at different levels cognitively, in term of knowledge constructed.

Although the issue of bias could not be totally eliminated, in this study the purpose was to understand the students' mental constructions, and so it was important that the researcher attentively listened to student's explanations, without interfering, so as to make sense of the mental constructions made. The questions were short and straight to the point, so as to avoid encumbering students' meaning. In formulating the questions, the researcher took great care in sequencing the questions moving from general or broad to specific or narrow. Students were allowed to explore as they liked, but I always referred back to the questions to check if the key areas had been explored and responses to it had been given. This helped with maintaining control of the interviews, without interfering with students' responses. The flexibility of the interviews also allowed students to provide more input, which probably was not said in their responses to activity sheets. This study was aimed at understanding individual mental constructions, therefore it did not strive for commonality among responses, where instead, the patterns are expected to emerge from the data themselves.

4.8 Authenticity and Trustworthiness of the study

This study is qualitative by nature, and therefore the issues of validity and reliability were considered. Validity determines whether a research instrument investigates what is intended to be investigated, while reliability refers to how consistent the results are. According to Bassey (2000), these concepts are problematic in case study research, since this manner of research is a study of singularity. This is not to say qualitative researchers should not concern themselves with reliability and validity, but that in qualitative research, validity might be addressed through honesty, depth, richness and the scope of data achieved (Cohen et al., 2007). Instead of using the terms validity and reliability in qualitative research, Guba & Lincoln (1994) have suggested the use of alternative terms. In this study, we adopt the two terms authenticity and trustworthiness. According to Cohen, Manion & Morrison (2007) "reliability in qualitative research can be regarded as between what researchers record as data and what actually occurs in the natural setting that is being researched" (p. 149). Creswell (2007) maintains that reliability can be addressed in several ways in qualitative research, such as obtaining detailed field notes, as well as employing good quality recording materials for ease of recording and transcribing. These are means to uncover participants' perspectives of the phenomena under study. The issue of authenticity of this study is discussed under triangulation.

4.8.1 Triangulation

When one observes something from different angles, it is possible to gain different perspectives, referred to as triangulation. Different scholars have given different definitions of triangulation. Cohen, Manion & Morrison (2007) define triangulation "as the use of two or more methods of data collection in the study of some aspect of human behaviour" (p. 143). Neuman (2006) has asserted that triangulation means it is better to look at something from several angles. According to O'Donoghue and Punch (2003), triangulation is a "method of cross-checking data from multiple sources to search for regularities in the research data" (p. 54). Denzin (2002) distinguished five types of triangulation: (1) data triangulation, which involves: time, space and persons; (2) investigator triangulation, which involves multiple researchers in an investigation; (3) combined levels of triangulation, which involves more than one level of analysis from three principal levels, namely: the individual level, the interactive level and the level of collectives; (4) theory triangulation, which involves using more than one theoretical scheme in the interpretation of the phenomenon; and (5) methodological triangulation, which involves more than one method to gather data, such as interviews, observations, questionnaires and documents.

In this study, data were triangulated over the use of different methods, activity sheets, interviews, and observations, use of digital technology. It also triangulated over space, time and persons. The data were collected over four lessons, and two tutorials, as well as over the prolonged time during interview process. The data procedure took place in the classroom with different students. The data in this study were collected at two different points, with three different instruments. First, it was through activity sheets, second through interviews, and third through viewing of video clips and the transcribing of interviews. Through analysing the activity sheets, one perspective regarding student's responses was evident. Through the interviews another perspectives, which in some cases supported what transpired in the activity sheets or differ were analysed. The video clips captured all the observant and non-observant activities among students that might have impacted on their construction of knowledge.

In qualitative methodologies, reliability includes fidelity to real life, context and situation, specificity, authenticity, comprehensiveness and meaningfulness to the respondents (Cohen et al., 2007). To ensure authenticity and dependability of data, several issues were taken into account during the data collection stage. During the first phase, data were collected over six months, and during the second phase, the data were collected throughout the year. While the data analysis was in progress, where some of the recordings were not clear, participants were called in to verify or to clarify what they were saying during the interview. To ensure that the participants clearly understood the purpose of this study, an informed consent letter was read, explained and given to all (122) students, so as to further read it by themselves and to sign it. The participants were assured that the data collected would not be divulged to anyone, except the university structures, and it will be kept safely by the university, where, if they would like to read the information before it is made public, they would be free to do so. It was also made clear that the data collected would solely be used for the purpose of the study, and not for continuous assessment. To ensure trustworthiness of the research instrument, the pilot study was conducted to evaluate whether the instruments would generate the required data, and to evaluate its suitability as an analytic tool. As much as students' responses were the primary source of data, for this study, it was not important that the answers were correct, but the focus was on thinking abilities, leading to their solutions. To validate data generated from activity sheets, selected students were invited for the interviews. Furthermore, the findings of the study had been presented to a panel of critical reviewers at this university where the study was conducted to ensure its trustworthiness and reliability. More over the researcher attended SAARMSTE workshop where the study was presented and feedback was incorporated for the authenticity of this study.

4.9 Ethical issues and limitations of the study

As this study involved human beings, ethical considerations were seen as crucial (Bertram, 2004). To ensure that all ethical issues were appropriately addressed, a letter outlining the nature, process and purpose of the study was given to the Dean, seeking permission to conduct the study (see Appendix B) . Letters of informed consent were given to all the participants to read and sign (see Appendix B). In the letter, it was clearly stated that participation was voluntary, and that participants could withdraw anytime they wanted to, only needing to inform the researcher if they wished to do so. Participants can become aware of their rights as participants when they read and sign the statement (Neuman, 2006). Before the commencement of the study, the researcher clearly explained and emphasised such issues to my participants. This was done to ensure that participants understand that they are under no obligations to take part in this study. Since activity sheets were one of data collection methods, it was possible that students may assume that this would impact on their assessment. At all times during the process of data collection, students were ensured that the data collected would only be used for the purpose of the study, and that the findings of the study will not benefit in terms of assessment but it hoped to help with their understanding of the taught concepts in matrix algebra. All the participants in the study were promised confidentiality and anonymity. The nature, process and purpose of the study were outlined to all the participants. To protect the identity of the participants, pseudo-names were used, and participants were ensured that all their details would be kept away from the public. Students were further assured that the information would be kept safely in the university, and would not be shared with anyone, except for the purposes of the study. If they wanted to read the information before it became public, they were informed that they would be free to do so. Students were also invited to ask questions to seek clarity on any issue or any uncertainty they were experiencing during the course of the study.

Before the researcher could commence with the study, it was necessary to seek ethical clearance from the university research office, which was granted, under ethical clearance number HSS/1470/013D. Also, the permission for conducting this study in the institution was granted by the Dean. This was granted after a summary of the proposal was presented to the institution's research committee.

As a case study, the sample use is quiet small, using a group of 31 pre-service teachers out of 122, and only in one university, therefore the findings cannot be generalised to other contexts. Even so, it is hoped that the findings would be informative enough to the mathematics community regarding what we can expect in the learning of matrix algebra in the South African context. The first set of data was collected during tutorials with a group of 31 students. To ensure that all students were engaging with the same task for their learning, the researcher delivered the lessons with the whole group of 122 students. Thereafter, the researcher held meetings with other tutors to ensure that during the tutorial, all students were doing the same tasks. Since the researcher was also a lecturer, power dynamics needed to be addressed. Although this did not mean that they would be able to be removed, certain measures were taken to ensure that they did not interfere with the study process. For the students to realise that this was not part of their assessment they were provided with an assessment plan at the beginning of the module, which served to assure them that the findings of the study were mainly formative, although not part of formal assessment. Throughout the whole process of data collection, the researcher attempted to diffuse tensions by allowing the participants to hold free discussions, without interference. To conduct interviews, a neutral venue was used, and students were allowed to speak in English or Isizulu since these are the common spoken languages at this university.

4.10 Conclusion

This chapter thus serves as an overview of how this study was conducted, with respect to methods and procedures. The chapter started with a list of the critical research questions and a discussion of the interpretive research paradigm used in this study. This discussion was followed by a discussion of the research design, methodology and methods adopted. As can be expected, the research methodology served as a guideline and point of reference for the study, with respect to data collection and procedures followed. The data collection process, together with the research instruments, were discussed at length in this chapter. Once all the data were collected and interviews transcribed, the data analysis process commenced. Issues of validity and reliability under the terms such as authenticity and trustworthiness of the study were discussed in length. In the next chapter, the researcher presents the findings of the first phase, and thereafter, the analysis of the data is discussed in detail in the chapters that follow.

CHAPTER FIVE

FINDINGS AND DISCUSSIONS OF PHASE 1 OF THE STUDY

5.1 Introduction

In the previous chapter, the research design, methodology and stages of data collection methods used in this study were discussed. A clear and a detailed description of the data collection process was provided. In this chapter, the discussion and findings of Phase 1 of this study are presented. To gather the required data for Phase 1 of the study, a qualitative method was used. Data was collected via an activity sheet. The activity sheet was designed to give insight into students' mental construction when learning matrices and determinants. This was guided by the belief that understanding the mental constructions the students made when learning mathematics concepts lead to improved instructional methods and curriculum development. The tasks chosen were those that the researcher identified as suitable for allowing students to make the necessary mental constructions. The activity sheets were administered to 98 pre-service teachers, who were registered for the module called Mathematics for Educators 210; however, the data presented here and analysed is from 85 students who completed the activity sheet. During Phase 1 of the study, students were asked to work individually in answering the activity, and were given two days to complete the tasks. Thereafter, data were analysed so as to assess their performance on their understanding of matrices and determinants. This was done with the aim of exploring the mental constructions made by students that were seen to concur with the genetic decomposition of matrix algebra concepts.

5.2 Analysis of items from activity sheet

The test instrument consisted of three questions. Each question had sub-questions, and all in all, there were eight questions that were analysed (see Appendix A1). The first question contained recall questions, which required students to show their understanding of the order of naming matrices. In terms of Bloom's taxonomy, these were mainly level one questions. The second question involved multi-step questions, where students were required to display their procedural fluency. The last question was of a higher order, because at this stage, students were expected to apply their problem solving skills, show an understanding of the relationship between concepts, and apply their knowledge and procedures in explaining the meaning of the concept. The analysis of student's responses for each of the questions and the extracts from students' responses are presented in the tables below. The marks were allocated as a means to group the responses, but the analysis was based on an individual's procedural and conceptual fluency. Each category in each particular question is discussed. The purpose of administering the activity sheet was explained to the students prior to the commencement of the data collection process.

5.3 Category A: Order of matrix

This category focused on exploring students' mental construction in identifying the order of a matrix. Question 1 is presented below as item 1. Item 1 involves determining the order of matrix which can be given by $m \times n$ where *m* refers to the rows and *n* refers to the number of columns or it can be given by $n \times n$, in square matrices, where the number of rows equals the number of columns.

Item 1

Identify the order of each matrix below:

a) $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$ C= $\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Answer: _______________ Explain:__________________________________

Item 1 was intended to provide insight into whether the students had developed an action conception of the concept of order of matrix.

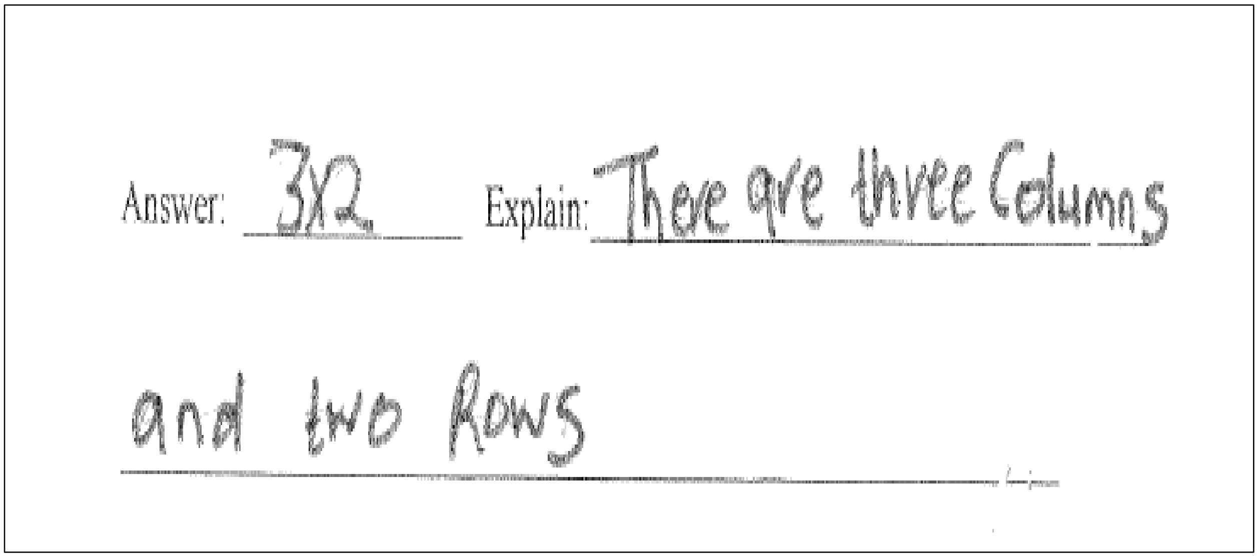
In Table 5.1 the allocation of scores for item 1 is displayed.

Category	1	2	3
Indicator	Not answered or incorrect response	Identify the order but no explanation given or incorrect explanation given	Provide the correct and complete explanation
No of responses	0	10	75

Table 5.1: Allocation of scores for Item 1

The responses of all the students in Category 2 indicated that they had not made the necessary mental constructions as indicated in Figure 3. The common errors that students made ranged from stating the correct matrix order but providing wrong explanation, to vice versa. In identifying the order of matrix A and B, six students in Category 2 provided the correct order. However, they reasoned that this would be because they were square matrices. This meant that they could not express themselves concisely, or they simply realised that the order of a matrix is defined for square matrices only. This explanation was insufficient for the researcher to conclude the APOS mental construction by means of which these students could be categorised. In Phase 2 of the study, the researcher will use the interviews with the aim to clarify such responses.

Four students in Category 2 correctly identified the order and provided explanation for square matrices. However, in matrix C, which was not a square matrix, students responses revealed some gaps in the knowledge constructed as they confused the rows and columns. One student out of the four provided the correct order for matrix C, stating that it is a 2 x 3 matrix, but in the explanation, the student simply wrote: ‘two columns and three rows’. This might be just an error, which Siyepu (2013) has referred to as a slip, or wrong answer, owing to processing. These are considered to be careless mistakes, made by both novices and experts, but can be easily corrected. A different case ensued with Aphile (see Extract 1).


Answer: 3X2 Explain: There are three Columns and two Rows

Extract 1: Aphile's written response for Item 1

Aphile is one of the four students in Category 2 providing an incorrect order and incorrect explanation, where she wrote 3 x 2 and provided an explanation for this that there are three columns and two rows. She could not correctly name the order of matrix, because her concept definition of a column and a row was in conflict with the concept definition accepted by mathematics community. She termed the first number in a_{ij} as a column, and the second as row, indicating the underlying difficulties with the understanding of the structure of a matrix and its entries. This seemed to be a barrier in understanding the concept and making the necessary mental constructions. Based on her response, it seemed that the action conception of naming the order of a matrix had developed.

The students' responses in Category 3 revealed that they had made the necessary mental constructions, as they provided correct and complete responses for all the matrices. However, four students, when providing their explanation, focused on the entries of the matrix, rather than just mentioning rows and columns. They provided the correct order, and when providing the explanation for their order, they stated the entries of the matrix (see Extract 2).

Identify the order of each matrix below

a) $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$

Answer: 2X2 Matrix Explain: Entry $a_{22}=3$

(In the row 2, column 2 entry is 3)

b) $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

Answer: 3X3 Matrix Explain: Entry $b_{22}=0$, Entry $b_{33}=4$

Extract 2*:* Amo's written response for Item 1

Although this was not the way in which students were taught to name the matrix, Amo, with the other three students, was able to show a relationship between the entries of matrix and the order. Instead of just counting the rows and columns, Amo looked at the last entry in the matrix and realised that it was $a_{22,}$ namely the last row and last column, therefore deducing it to be 2×2. Since this technique for determining the order of a square matrix was not discussed in class, the researcher presumed that Amo must had previously identified that the last entry in both the last row and the last column could be used. Amo must have done this at an action level. After doing this several times, he must have interiorised this into a process, since he did it all in the mind without expressing it verbally. The researcher thought that Item 1 was a reliable source by means of which to identify the mental construction as prescribed by the itemised genetic decomposition. Therefore, it will be used for the Phase 2 of the study.

5.4 Category B: Matrix transpose, computation of matrices and determinants

In the activity sheet, Question 2 comprised of five sub-questions (see Appendix A1). This analysis labels these sub-questions as items. These five sub-questions will be presented as Item 2 to Item 6. Item 2 was aimed at exploring pre-service teachers' knowledge of determining the transpose of a matrix.

Item 2

Let $A=\begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$	2.1.1 Determine A^T

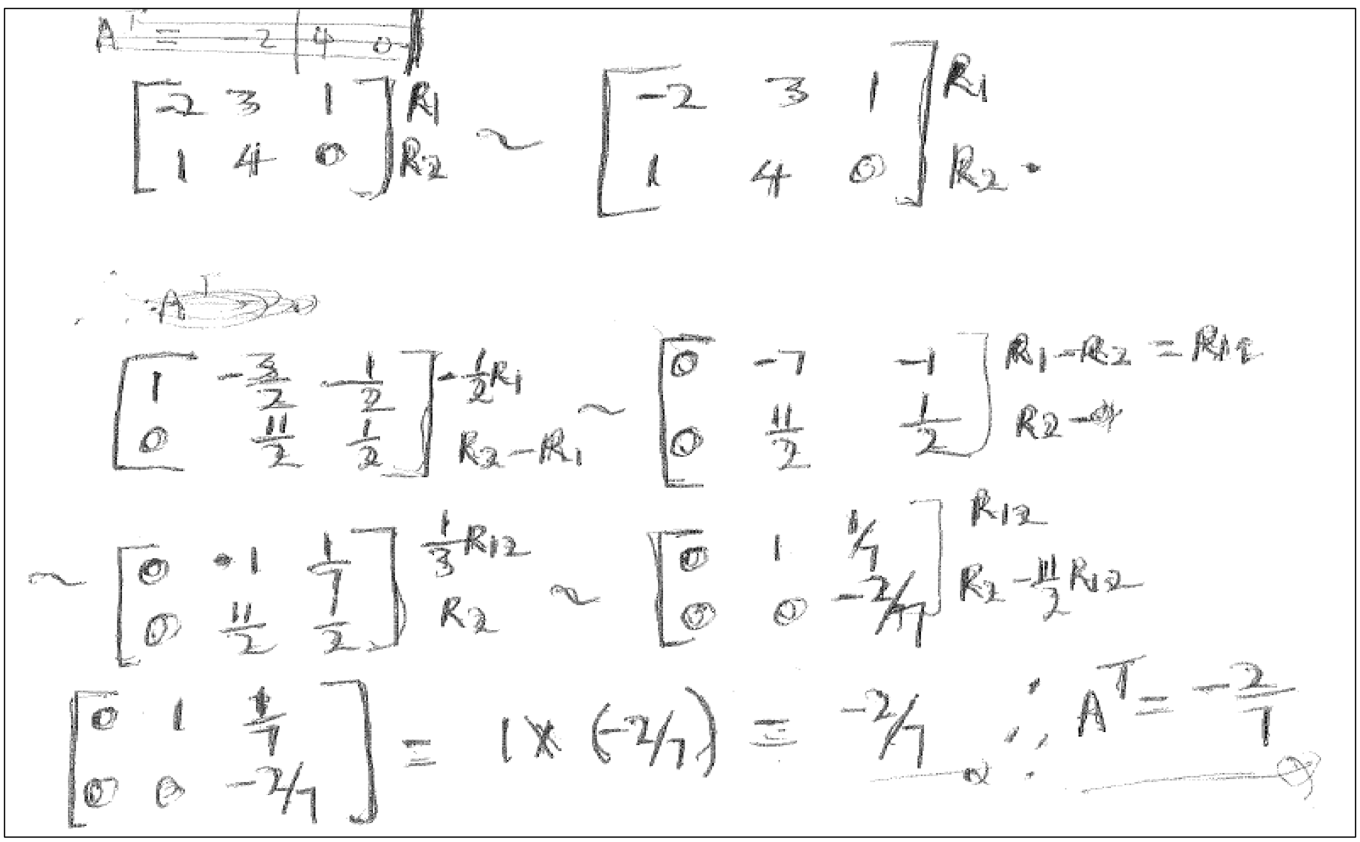
Item 2 was intended to provide insight into whether or not the students had developed an action conception of the concept of the matrix transpose.

In Table 5.2 the allocation of scores for Item 2 is displayed.

Category	1	2	3
Indicator	Wrong or not attempted	Partially correct	Correct
No of responses	2	1	82

Table 5.2: Allocation of scores for Item 2

In determining the transpose of a matrix, both students in Category 1 attempted the question, but totally missed the point. One of the two students, Thato, instead of determining A^T, tried to find the A^{-1} (see Extract 3). It seemed that Thato confused the notation of A^T with A^{-1}. He tried to determine A^{-1} using elementary row operations, but failed to apply the technique correctly. His response to Item 2 indicated that he does not understand the difference between A^T and A^{-1}. He might have memorised the algorithms for calculating matrix inverse, but did not understand how to apply it. He did not know the conditions for the existence of matrix inverse as matrix A was a non-square one. This meant he learnt the concepts as isolated facts, unable to see the interrelationship between concepts. This led to an over-generalisation of matrix inverse over matrix transpose. The use of elementary row operations indicated that in many instances, students had a tendency of applying rules even if they did not understand them, since this method of finding the inverse was never discussed during lectures. What was evident here was that he failed to interpret the nature of the problem, which was a result of poor conceptualisation of matrices. As a result, he applied the wrong procedures. According to Matz, (as cited in Siyepu, 2013), such errors persist due to surface level procedures, where an individual acquires knowledge by heart, without engaging with its meaning, which is what Thato appears to have done. His response also indicates that he does not understand the notation used in matrices, and its meaning. This caused a barrier in his attempt to construct a meaningful understanding of the learnt concept. Thatos' response revealed that he had not made the necessary mental construction of the concept.



Extract 3: Thatos' written response to item 2

The students in Category 2 made some procedural errors while solving this problem and therefore could not provide a complete response. They knew that it was necessary to interchange rows and column to determine the transpose but they also changed signs of some entries in the matrix. This could be considered as a structural error, since they knew the relevant rule, but failed to unpack the structure. This meant that the action conception of the matrix transpose has not fully developed. The student has not fully understand the rule. The 82 students in Category 3 made the necessary mental construction. These students demonstrated the understanding of the concepts, and applied the procedures of determining the transpose correctly.

Item 3 focused on exploring pre-service teachers' mental construction of matrix subtraction and scalar multiplication.

Item 3

Let $A=\begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ $B=\begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{bmatrix}$ $C=\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix}$ $D=\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$

2.1.2 Determine $3C - 2\,D$

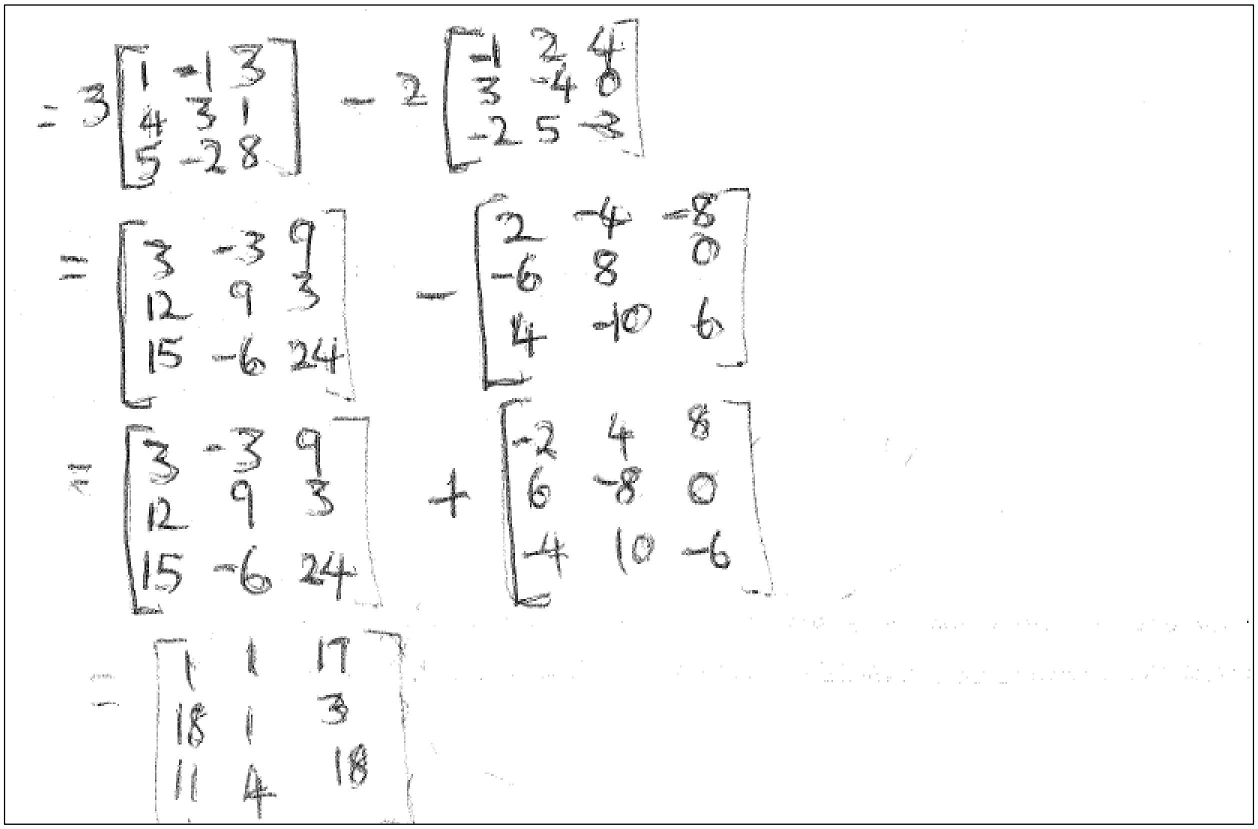
Item 3 was intended to provide insight into whether the students had developed an action conception of matrix addition and the process conception of the conditions defining matrix subtraction.

In Table 5.3, the allocation of scores for item 3 is displayed.

Category	1	2	3	4
Indicator	No attempt or totally incorrect.	Multiply matrices C and D by given scalars but made errors with multiplication.	Multiply correctly but the final answer is contains few computational errors.	Correctly determine the required matrix without actually showing all the calculation.
No of responses	0	10	7	68

Table 5.3: Allocation of scores for Item 3

In solving the problem in Item 3, students in Category 2 displayed mathematicaly inaccuracy. Inaccuracies in mathematics mostly arose when students failed to carry out manipulations or algorithms, though they understood the concept. Students made procedural errors indicating a lack of algorithm skills. What transpired here was that the ten students in Category 2 knew the procedure to use, but lacked the technique to carry out the procedures effectively (see Extract 4).



$$= 3\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix} - 2\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 9 \\ 12 & 9 & 3 \\ 15 & -6 & 24 \end{bmatrix} - \begin{bmatrix} 2 & -4 & -8 \\ -6 & 8 & 0 \\ 4 & -10 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 9 \\ 12 & 9 & 3 \\ 15 & -6 & 24 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 8 \\ 6 & -8 & 0 \\ -4 & 10 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 17 \\ 18 & 1 & 3 \\ 11 & 4 & 18 \end{bmatrix}$$

Extract 4: Aziwe's written response for Item 3

Looking at Aziwe's written response, certain mathematical errors were evident. In the first step she made computational errors as she multiplied by -2, but still kept the minus sign between the matrices, then multiplied by a negative again in the second step. She lacked the background knowledge of the effect of a negative sign in a mathematical problem. This is what Dubinsky (1997) alluded to when discussing students' difficulties with linear algebra concepts. Failure to carry out the procedures correctly meant that Aziwe had the action conception of matrix addition, but had not interiorised it into a process. Students in Category 3 also made some procedural errors, but very few. They displayed an understanding of operational rules of matrices and related this to arithmetic algebra. The students in Category 2 and 3 had the action conception of the concepts, but the process conception had not developed yet.

Students in Category 4 successfully made a link between algebraic manipulation and matrices, and therefore were able to perform the required operation accurately. They provided a complete and correct indication that they had constructed the suitable mental constructions necessary for developing a conceptual understanding of the concept (see Extract 5).

$$= 3\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix} - 2\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 9 \\ 12 & 9 & 3 \\ 15 & -6 & 24 \end{bmatrix} - \begin{bmatrix} -2 & 4 & 4 \\ 6 & -8 & 0 \\ -4 & 10 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -7 & 5 \\ 6 & 17 & 3 \\ 19 & -16 & 30 \end{bmatrix}$$

Extract 5: Aphile' written response for Item 3

Aphile's response revealed that she had made all the necessary mental constructions as she correctly applied the procedure for scalar multiplication, indicating that she has constructed the procedural knowledge of manipulating numbers. Her response in this item revealed that she could manipulate numbers and carry out the required procedure to solve the problem in Item 3 efficiently and accurately, meaning that she had constructed the process conception of the concept. This item will be used in Phase 2 of the study to explore the understanding of matrix subtraction and scalar multiplication.

Item 4 focused on pre-service teachers' understanding of and the ability to apply procedures accurately when finding the matrix product. Item 4 was intended to provide insight into whether the students had developed the process conception of the matrix product.

Item 4

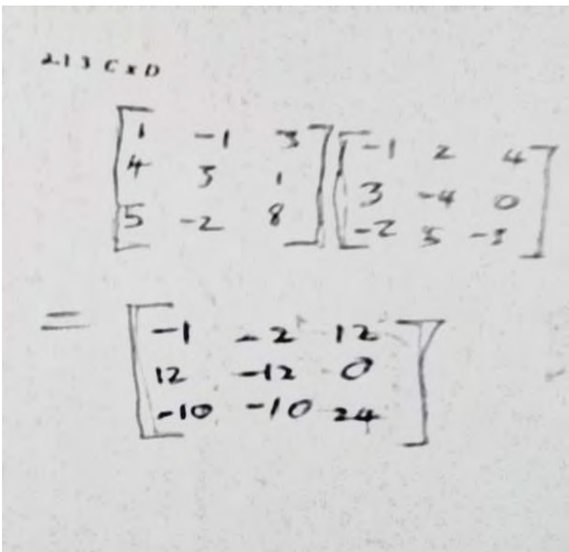
Let $A=\begin{bmatrix}-2 & 3 & 1\\ 1 & 4 & 0\end{bmatrix}$ $B=\begin{bmatrix}2 & 3\\ -3 & 1\\ 1 & -3\end{bmatrix}$ $C=\begin{bmatrix}1 & -1 & 3\\ 4 & 3 & 1\\ 5 & -2 & 8\end{bmatrix}$ $D=\begin{bmatrix}-1 & 2 & 4\\ 3 & -4 & 0\\ -2 & 5 & -3\end{bmatrix}$ Determine the following: 2.1.3 $C\ x\ D$ 2.1.4 $A\ x\ B$

In Table 5.4 the allocation of scores for item 4 is displayed.

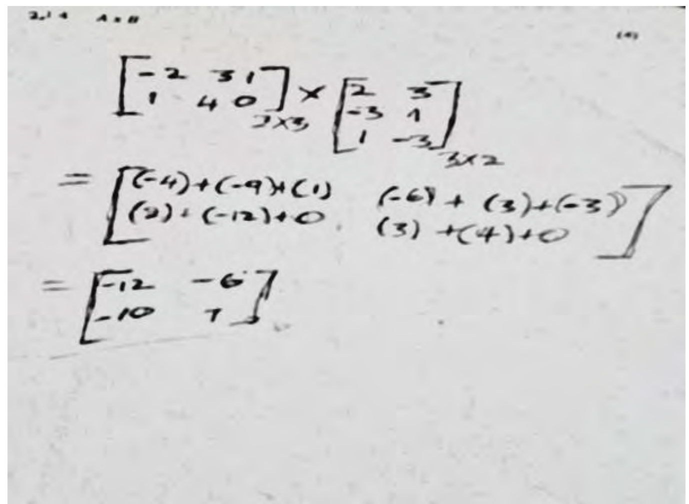
Category	1	2	3	4
Indicator	No attempt or totally incorrect.	Got one correct and the other totally incorrect or partially correct. Both are partially correct.	Techniques applied correctly but made errors when adding.	Provide a complete and correct solution.
No of responses	0	2	31	52

Table 5. 4: Allocation of scores for Item 4

In evaluating the product of CD and AB, Amanda correctly carried out the procedure and solved for AB, but could not apply the same procedure to solve CD (see Extract 6). This was surprising, since both the problems should be solved using the same technique. Her response was grouped under Category 2. In 2.1.3 she only wrote down the solution, which was incorrect, but in 2.1.4 showed all the steps leading to a correct solution. This made it difficult to determine whether she had made the necessary mental constructions or not. Her responses could not be explained in terms of APOS however, based on Extract 6b, the researcher was able to conclude that Amanda was at the intra-stage according to the Triad, since she displayed that she has the collection of rules, but could not recognise the relationship between them. According to Jojo (2011), students at intra-stage could solve some problems by simply applying memorised rules, and in some cases, could not remember correctly. Amanda might have memorised the rule of multiplying matrices, which then help her to be able to manipulate this to solve similar problems, but could not mentally collect all different cases. Although she could produce a correct answer in as seen in extract 6b, the knowledge constructed consists of a list of unconnected actions, processes and objects (Jojo, 2011).



Extract: 6a



Extract: 6b

Extract 6: Amanda's written response to Item 4

The responses of 31 students in Category 3 indicated that they held the action conception of the matrix product. They interpreted the question correctly, and applied the techniques correctly, but had difficulty with manipulating the signs, which indicated the lack of computation skills. This meant that process conception had not developed, as their responses contained procedural errors. In terms of APOS, they are operating at the action stage. They have constructed the collection of rules and assimilated them in their cognitive structures, but have not interiorised them into a process. Fifty-two students in Category 4 provided correct and complete responses. Their responses displayed procedural fluency and competence in determining the matrix product and manipulation of signs. This revealed that they had made all the necessary mental constructions of matrix multiplication, and that the process conception had fully developed. In terms of the Triad, they are operating at the trans-stage, since they displayed coherence of understanding of matrix multiplication. These students were able to use the constraints of matrix multiplication effectively (see Extract 7).

$$\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix} \times \begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -1-3-6 & 2+4+15 & 4+0(-9) \\ -4+9-2 & 8-12+5 & 16+0-3 \\ -5-6-16 & 10+8+40 & 20+0-24 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 21 & -5 \\ 3 & 1 & 13 \\ -27 & 58 & -4 \end{bmatrix}$$

Extract 7a

$$= \begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -4-9+1 & -6+3-3 \\ 2-12+0 & 3+4+0 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -6 \\ -10 & 7 \end{bmatrix}$$

Extract 7b

Extract 7: Aphile's written response for Item 4

This item will be kept for Phase 2 of this study, but 2.1.4 will be removed, because the researcher realised that one question will generate the required data. Also, 2.1.3 will be a suitable instrument to explore students' knowledge of identity elements, and the interrelationship between matrix product, identity elements, determinants and matrix inverse.

Item 5 aimed at exploring pre-service teachers' knowledge of evaluating and application of determinants. Item 5 was intended to provide insight into whether the students had developed the process conception of evaluating determinants and correct use of notations.

Item 5

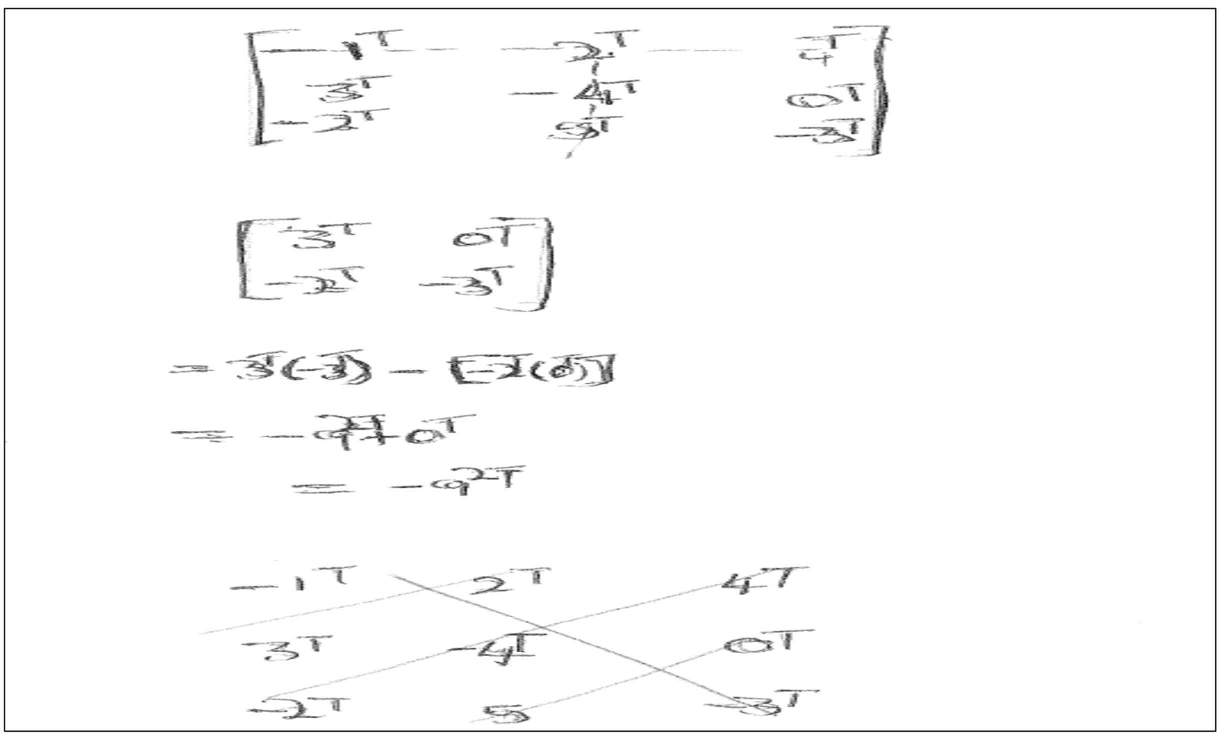
2.2 Find the determinant of C 2.3 Find the determinant of D^T

In Table 5.5 below the allocation of scores for item 5 is displayed.

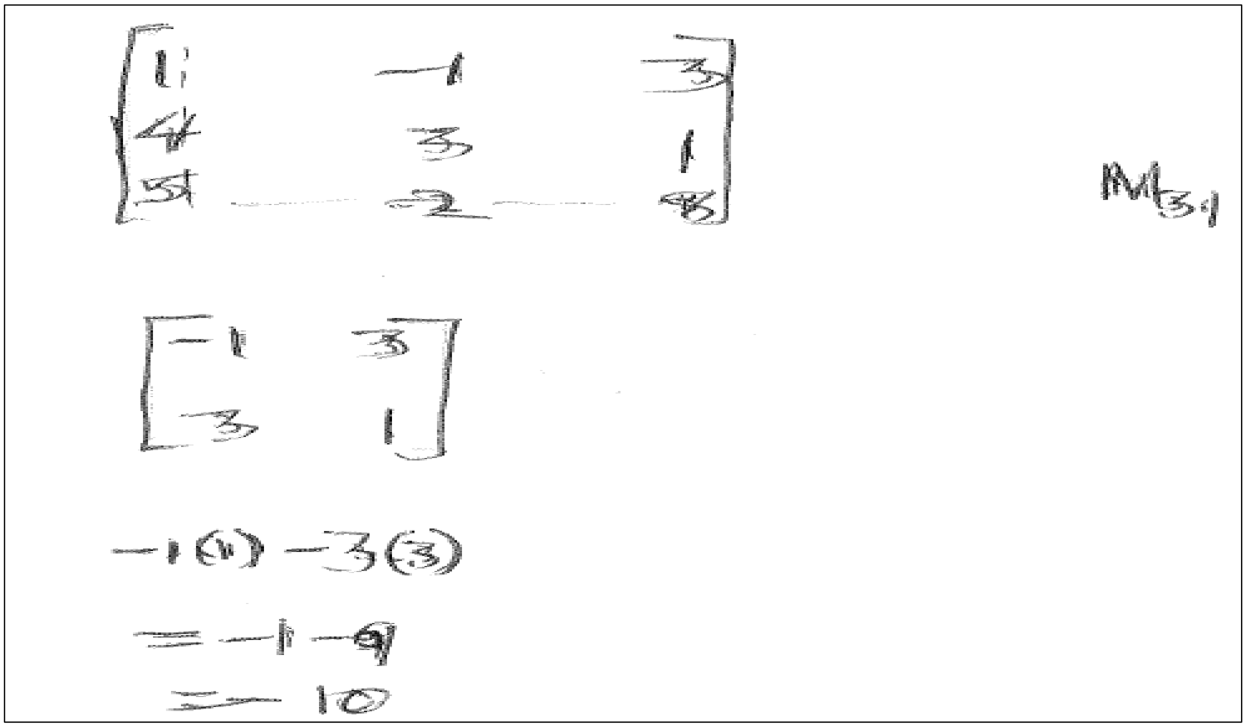
Category	1	2	3	4
Indicator	No attempt or totally incorrect.	Got one correct and the other totally incorrect or partially correct. Both are partially correct.	Made errors with minors, cofactors and use incorrect notation.	Provide a complete and correct solution.
No of responses	4	9	25	47

Table 5.5: Allocation of scores for Item 5

It was important to know whether students could associate procedures of evaluating the determinants of order > 2 to procedures of evaluating determinants of 2 x 2 matrices. Also for 2.3, it was important to know whether students would associate the determinant of matrix D to the determinant of its transpose. This would indicate that they are operating at the process stage in terms of APOS and have also developed the object conception of the determinant of a matrix and its determinant of its transpose. The response of students in Category 1 indicated they had made no mental constructions as discussed in APOS theory. In both the questions in Item 5, they provided incorrect responses, indicating an inability to apply the correct procedure. These students have no collection of rules, and demonstrated poor interpretation of the concepts, and as a result, they applied inappropriate procedures (see Bella's response in Extract 8). Bella's written response indicated that she had failed to grasp the concept and could not interpret the problem appropriately. Also, in trying to manipulate rules, she consistently made systematic errors which indicated the lack of both conceptual and procedural understanding of evaluating determinants. Since she had not made the mental constructions of the matrix transpose in Item 2, she lacked the mathematical skills and knowledge to solve other problems that involve the transpose. This was indicated by her response in extract 8b, as an error that was made in finding the A^T was repeated in D^T. In extract 8b, she found the minor by deleting row 1 column 2. Examining her response, it was evident that she lacked the skills to compute numbers (see the third, fourth and fifth steps).



Extract 8a



$$\begin{bmatrix} 1 & -1 & -3 \\ 4 & 3 & 1 \\ 5 & -2 & -8 \end{bmatrix} \qquad M_{31}$$

$$\begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}$$

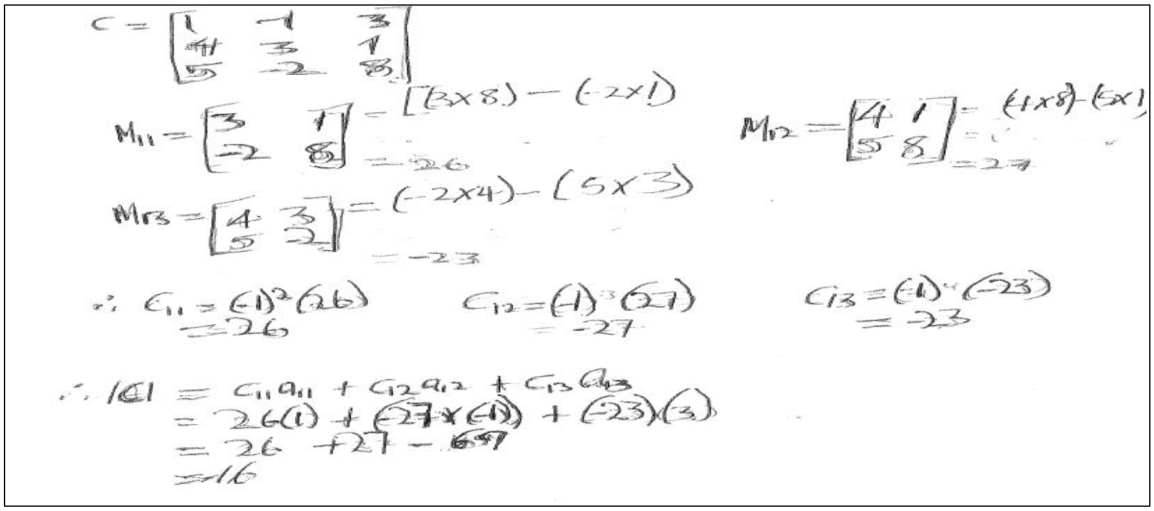
$$-1(1) - 3(3)$$
$$= -1 - 9$$
$$= -10$$

Extract 8b

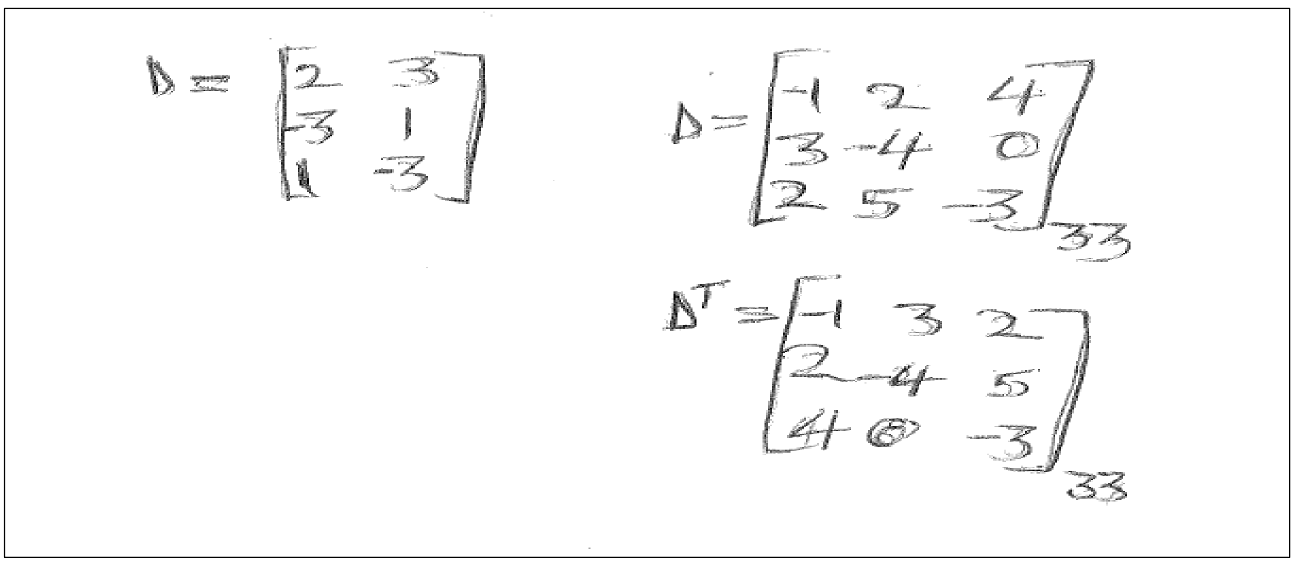
Extract 8: Bella’s written response for Item 5

It seemed that Bella could evaluate the determinant of a 2 x 2 matrix. This was observed since she found determinant of $\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$ to be -10. Although she knew the formula of finding the determinant of a 2 x 2 matrix, it seems that she could not apply it to solve determinants of order > 2, since in both cases, she calculated one minor, and consider it to be the determinant of matrix C or of D^T. In relation to APOS, Bella had not achieved an action conception of the determinant.

Two students in Category 2 obtained the correct determinant of matrix C, however, they could not even attempt the determinant of D^T. They did find D^T, but seem to experience difficulty in evaluating $|D^T|$ (see Extract 9). Based on their response, the researcher claimed that in terms of Triad, they are operating at the intra-stage, because they could evaluate problems similar to those they had encountered before, but could not apply the same procedures to other related problems. This indicated that the rules were memorised just to produce answers to related problems without developing any understanding of it.



Extract 9a



Extract 9b

Extract 9: Nhlanhla’s written response for Item 5

This could be attributed to what Matz (as cited in Siyepu, 2013) referred to as surface level procedure of evaluating a determinant. Based on his response in 9a, the researcher was able to conclude that Nhlanhla had memorised the rules, and that as a result, could solve problems of similar structure to the ones encountered previously, but could not apply this to other related problems. This assumption is based on the fact that during lectures problems similar to $|D^T|$, were not discussed. In terms of APOS, the action conception of the concept has not fully developed. In the genetic decomposition, the understanding of the structure of matrices was not accommodated, therefore the researcher adjusted the genetic decomposition for the Phase 2 of this study, so that such cases could be described in terms of the mental constructions made. In terms of APOS, students who displayed understanding of the different structures in matrices and reflect on this to solve related problems, are considered to be at the object stage.

Students in Category 3 could interpret the problem and applied the correct procedures to solve the problem. However, their workings contains computational errors such as: (1) omitting the sign when calculating cofactors; (2) deleting wrong entries when finding the sub matrix; (3) confusing minors and cofactor signs; (4) incorrect substitution to the formula; (5) using wrong notation. Anita provided the correct solution, but used incorrect notation in her solutions. In determining the determinant of matrix C, Anita wrote $|A|$ instead of writing $|C|$. This could be more than just a mistake. It could imply underlying difficulties that students are experiencing with constructing meaningful knowledge of the learnt concept. Students have the tendency to not interrogate what they write, focusing on producing the answer. Ndlovu (2012) highlighted that in many cases students tends to use notation loosely, where students tends memorised notation. Failure to construct a meaningful understanding of notation used in mathematical concepts means that the concept is not conceptually understood, and could be a barrier to meaningful learning.

All the students in Category 4 made the necessary mental constructions. Nancy's response was grouped under Category 4, since she provided a complete solution for both questions. Her responses indicated that she understood the relationship between concepts. Instead of finding $|D^T|$ she determined $|D|$, and used the solution for $|D|$ to determine $|D^T|$, indicating that she had encapsulated the process into an object. In extract 10, we observed that she correctly performed the necessary action on a process. She demonstrated a clear understanding of the concept, as she displayed a clear understanding of the relationship between determinant of the matrix and determinant of its transpose. In terms of APOS, Nancy has developed the object conception of the determinant. This item will be used for Phase 2 of this study, without any changes.

$$D \begin{vmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{vmatrix}$$

miners

$$M_{21} \begin{vmatrix} 2 & 4 \\ 5 & -3 \end{vmatrix} \quad -6-20 = \underline{-26} \therefore C = (-1)^{2} \cdot (-26) = -26$$

$$M_{22} \begin{vmatrix} -1 & 4 \\ -2 & -3 \end{vmatrix} \quad 3-(-8) = \underline{11} \therefore C = (-1)^{3} (11) = \underline{-11}$$

$$M_{32} \begin{vmatrix} -1 & 2 \\ -2 & 5 \end{vmatrix} \quad -5-(-4) = \underline{-1} \therefore C = (-1)^{5}(-1) = \underline{-1}$$

$$D' = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$= (3)(-26) + (-4)(-11) + (0)(-1)$$
$$= -78 + 44$$
$$= \underline{-34}$$

Extract 10: Nancy's written response for Item 5

Item 6 was aimed at exploring students understanding of the properties of the matrix product. It was intended to provide insight into whether the students had the conceptual understanding of the matrix product, and whether the students were able to explain their thinking beyond the application of rules.

Item 6

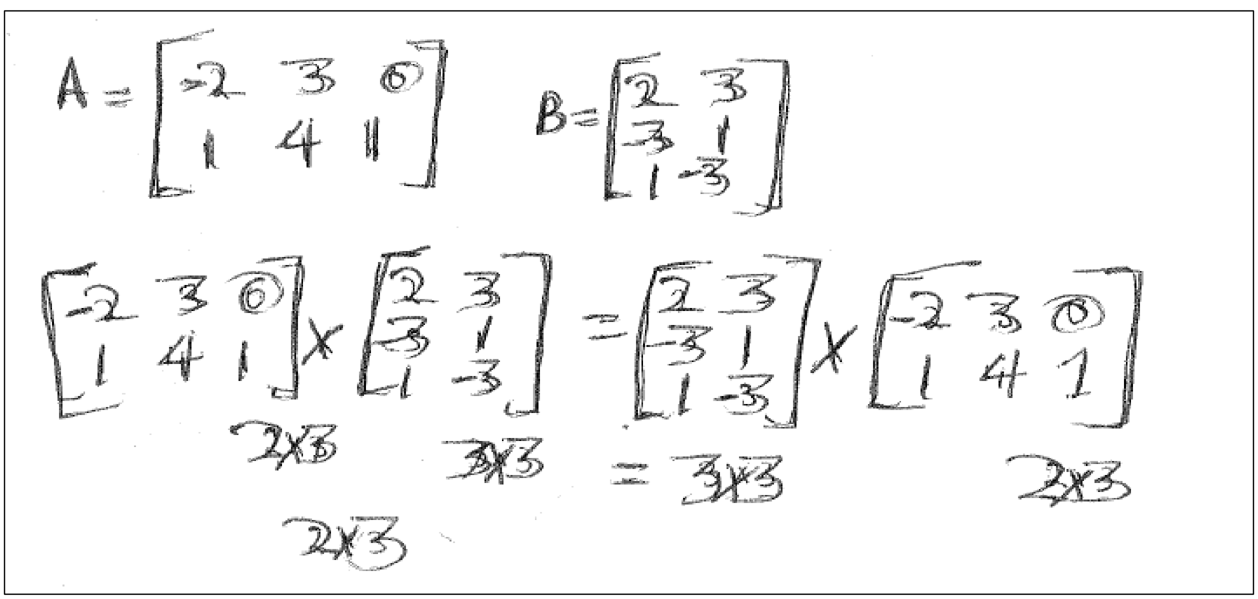
2.4 For any 3 x 3 matrices A & B, explain whether $A\ x\ B = B\ x\ A$

In Table 5.6 the allocation of scores for Item 6 is displayed.

Category	1	2	3
Indicator	No attempt or just state yes/no with no explanation.	State yes/no but the supporting arguments contradict the statement made.	Provide correct and clear explanation to support the statement made.
No of responses	39	8	38

Table 5.6: Allocation of scores for Item 6

In answering this item, students' responses in Category 1 revealed that they failed to interpret the question correctly, since 10 students could not provide any answer, and the rest just guessed 'yes' or 'no', without providing any explanation. This could be due to the fact that they were required to think of an arbitrary matrix. This meant that they had not constructed the mental constructions of the concept. Thirty-five of the students in Category 1 showed that they had the action conception of multiplication of matrices in Item 4. Their responses, or no response in item 6, indicated that they had not interiorised the action into a process. The eight students in Category 2 provided an explanation to support their answer, but two students tried to find a product of the two matrices without stating yes or no. This made it difficult to understand what they were trying to prove (see Extract 11). In this question, students were expected to use an arbitrary 3 x 3 matrix. Some students like Alex based their explanation on matrix A and B in Question 2. He identified that problem in Item 6 was about matrix product, and realised that the product of AB and BA in Item 2 was defined, and opted to use these matrices. Alex provided a correct and complete response to Item 1, which was about identifying the order of matrices indicating that he had the action conception. He could state the order of the matrix, but could not correctly apply that knowledge in Item 6, instead of reading the question with understanding, he picked up cues like the multiplication sign.



Extract 11: Alex's written response for Item 6

His responses indicated the following shortcomings: (1) misinterpretation of the question. It seemed that he wanted show that AB = BA. This was evident in his response in Extract 11, as he inserted the equal sign between the two products. At the same time, he was trying to show that multiplication of matrices is not commutative, by trying to show that the order of AB was 2×3 and for BA was 3×3. This was not clearly stated, indicating gaps in the knowledge constructed. The other shortcomings in his response were: (2) an over-generalisation of rules; (3) ignoring the constraints laid down in the question. This was evident in his response, as he used other matrices and not 3×3 as it was expected in the question. Instead, he was more focused in carrying out procedures of matrix multiplication. This indicated that the action conception of matrix product had not been interiorised into a process. The interpretation errors arise due to students' failure to interpret the question, owing to over-generalisation of certain mathematical rules involved in a mathematical problem (Siyepu, 2013). This item was of a higher order, where students were expected to show beyond just manipulation rules, the understanding of a matrix of products and its properties. Alex's responses reveal that he knew the rules, however, that he experienced difficulties in explaining his thought procedures, indicating inefficiency in the use of mathematical language.

All the students in Category 3 have interiorised the concept of matrix multiplication into a process, as they could, with or without calculation, show or explain that multiplication of matrices is not commutative (see Extract 12). Aviwe's response revealed that she had cognitively constructed the structure of a matrix and from that, could internally construct an arbitrary matrix. Her response indicated that she could carry out the procedures not just for the manipulation of numbers, but to yield understanding and meaning, which led to understanding of the concepts and can effectively think about and explain the properties of the matrix product. Based on her response, we can conclude that she is operating at the object stage in terms of APOS, since she can perform action on process and could reflect on the structure of the matrix and is able to link matrix multiplication with equality of matrices. She also displayed an understanding between the order of a matrix and equality of matrix. This we observed from her conclusions, as she stated that the 3 x 3 matrices are not equal. In terms of the Triad, she is operating at the trans-stage, as we could observe from her response that she displayed a coherent understanding of matrix multiplication as a rule, and as a process to think about.

$A \times B$

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \times B \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 15 & 12 \\ 19 & 17 & 8 \\ 16 & 12 & 12 \end{bmatrix}$

$B \times A$

$B \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{bmatrix} \times A \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 11 & 14 \\ 15 & 19 & 20 \\ 11 & 9 & 16 \end{bmatrix}$

$\therefore$ The 3x3 Matrices of A and B are not ~~the~~ Equal

$AB \neq BA$

Extract 12: Aviwe’s written response for Item 6

5.5 Category C: Matrix product and matrix inverse

The analysis of Question 3 presented as Item 7 and Item 8 will be discussed in the next sections. Item 7 aimed at exploring and describing the nature of pre-service teachers’ knowledge of the relationship of matrix product and square matrices. It was intended to provide insight into whether the students held a conceptual understanding of the matrix product in relation to other concepts.

Item 7

3.1.Suppose A and B are matrices with AB and BA defined. Explain whether AB and BA are square matrices

In Table 5.7 the allocation of scores for Item 7 is presented.

Category	1	2	3	4
Indicator	No attempt or just stated 'no'.	'Yes' with 'no' justification.	'Yes' with a partial attempt.	Provide a supporting argument that is clear and correct.
No of responses	52	30	11	2

Table 5.7: Allocation of scores for Item 7

In determining whether AB and BA are square matrices, many students experienced difficulties in interpreting this item. 52 students could not really provide a reasonable answer to the problem. Out of 52 students, five students did not attempt the problem. Thirty students stated 'no', but could not provide any reasons why. The other 17 students said 'no', and based their explanation on matrix A and matrix B in Question 2, since those two matrices were not square matrices. The students might have realised that the product of AB was defined, indicating their understanding of the constraints of determining matrix product, however, they could not identify that BA was also defined (see Extract 13). Students' failure to interpret the problem and reliance on rules seemed to be problematic, because before they could engage deeply with the problem, they immediately tried to do computation, hoping to get answers without first understanding the problem.

$$A = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} \times B = \begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -6 \\ 0 & 7 \end{bmatrix}$$ squre matrix

2×2

$$BA = \begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{bmatrix} \times \begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 18 & 2 \\ 5 & -2 & 1 \end{bmatrix}$$ BA is not a squre matrix

2×3

Extract 13: Andiswa's written response for Item 7

Andiswa based her response on matrix A and B in Question 2. It was evident that she had not achieved the action conception of the matrix product, as she could not carry out all the procedures correctly. This was based on her answer for BA. Her responses indicated that she did not understand the question, and as a result, opted for number grabbing. She could carry out procedures for the sake of providing some answer without interrogating how it related to the question asked. Her response revealed that she hadn't interiorised the actions of multiplying matrices into a process where she might internally think about and coordinate it with other processes to form new ones. The students in Category 2 said yes, however it was difficult to explore the nature of the knowledge constructed, since they did not provide any justification for their choices. The researcher hoped that in the Phase 2 of this study, using interviews might help in identifying the mental constructions made, and explore further responses. In terms of the proposed genetic decomposition, it was not clear as to whether they have made the mental constructions or not. Students in Category 3 said 'yes', however the reason provided showed gaps in the knowledge constructed. Five students stated that the matrices were defined, and that therefore, the product was a square matrix. This is the case only for square matrices, where the product is defined for AB and BA. This was an indication that these students had constructed limited knowledge for matrix multiplication to be defined, since there are other cases where both AB and BA are defined, such as matrix A and matrix B in Item 2. In terms of APOS, the process conception of the concept, we could conclude that it had not fully developed.

The two students in Category 4 had the process conception of the matrix multiplication as they provided correct and complete response (see Extract 14).

In order to find the product of 2 matrices the number of rows in the second matrix must equal to the number of columns in the first

If the matrices are reversed i.e AB and $\rightarrow$ BA

$\Rightarrow$ If the matrices are square, the product will be defined.

$\therefore$ AB and BA are square matrices.

Extract 14: Andries's written response to Item 7

Andries' responses revealed that the action of finding the matrix product had been interiorised into a process. In his explanation, he displayed the complete understanding of the meaning of the concept, as he deeply engaged with the procedure of finding the matrix product, to make meaning of the concept. In his response he provided an explanation for all the cases in which the product of AB and BA may be defined, and could abstractly think about arbitrary matrix, and construct its images. The second sentence is unclear, where he seems to indicate that both product will be defined if the order of the matrices can be reversed, e.g. 3 x 4 and 4 x 3. This assumption is based on the first sentence. The researcher will use this item in the main study, and through interviews more clarity to certain responses would share some light into the mental constructions made.

Item 8 explored pre-service teachers' knowledge of evaluating and understanding of matrix inverse and its relationship to other concepts such as determinants. Insight was sought into whether the students had a conceptual understanding of the matrix inverse.

Item 8

Does the matrix $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ have an inverse? If so what is the inverse? If not explain why?

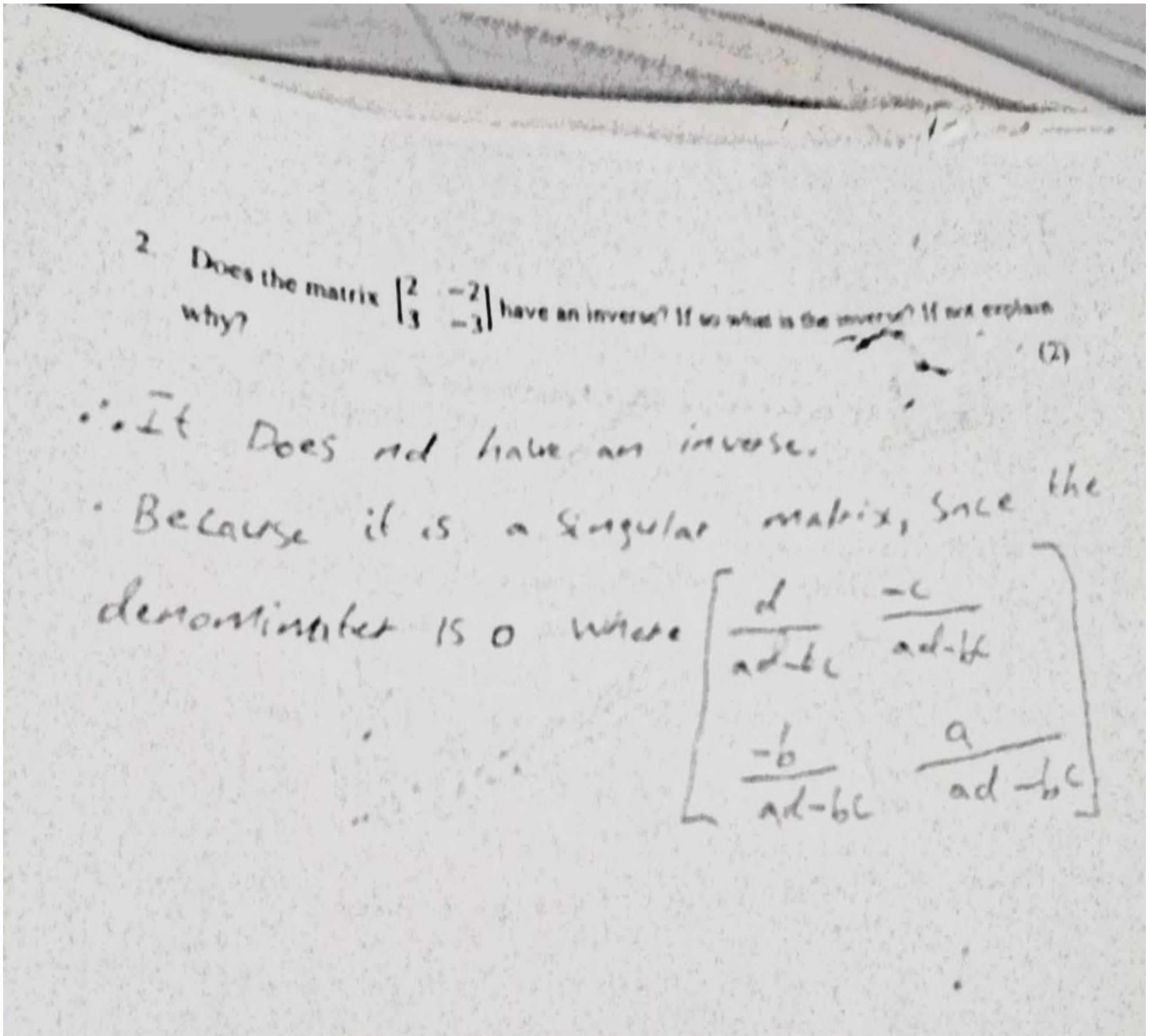
In Table 5.8 the allocation of scores for Item 8 is presented.

	1	2	3
Indicator	No attempt or stated yes.	Stated no. The explanation given does not show conceptual understanding of the concept.	Provide a clear and correct explanation.
	7	33	45

Table 5.8: allocation of scores for Item 8

In Category 1, three students did not attempt the question. The other four students attempted the problem and their responses were incorrect. They tried to evaluate the determinant, but could not carry out procedures correctly. Then they try to use elementary row operation, which likewise couldn't execute correctly. This leads the researcher to conclude that in terms of APOS the action conception has not fully developed. Based on the triad mechanism it seems that they are operating at the intra - stage as they show that they had the collection of rules of finding the inverse but have not constructed any meaningful understanding of the application of such rules.

Students in Category 2 stated that the matrix had no inverse. However, they couldn't provide a sound argument to justify their answer, in order to show that they understood the concept. In Category 3, 43 students had the process conception of matrix inverse, as they could apply the determinant to determine that the matrix was singular. Two students in Category 3 had encapsulated the process into an object. This was evident, as they clearly showed the understanding of the interrelationship between the determinants and matrix inverse. In Extract 15 we observed that Ela correctly performed the necessary actions on a process in order to show that the matrix is singular, and could use the appropriate mathematical language. He could make the link between school algebra and matrix algebra, and used the understanding of the relationship between the two to explain his thought processes. This was evident as he explained that the denominator is zero. This indicated that he knew that division by zero has an undefined meaning, where the determinant was zero. He further elaborated this by relating it to the determinant structure *ad*-*bc*. The determinant schema constructed allowed him to make sense of and construct new knowledge about the matrix inverse.


2. Does the matrix $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ have an inverse? If so what is the inverse? If not explain why? (2)

∴ It Does not have an inverse.

• Because it is a singular matrix, since the denominator is 0 where $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

Extract 15: Elano’s written response for Item 8.

5.6 Discussion

The researcher noticed that in some problems, students were assisting each other making it difficult to describe the nature of the mental constructions made by an individual. The researcher thought that the structure of the activity sheet caused students to think that the results would form part of their course mark, and as a result, tried to gather as much assistance as possible. This was more evident in Item 3 and Item 4, where in two questions of the same nature, the students provided a correct response for the one question, but a totally incomplete response for the other. For Phase 2 of this study, the activity sheet will be completed during the tutorials and the structure of the activity sheet will be changed. In order to describe in greater detail the nature of mental constructions made, the interviews will be used to explore why the mental constructions arise. In Phase 1, the data were gathered from the written responses only, however, the theoretical framework used, as well as the literature reviewed, provided useful insight into the nature of the mental constructions and the knowledge that the pre-service teachers had constructed of matrix algebra concepts. Also, it provided useful insight on some of errors that pre-service teachers made, which caused difficulty for their learning, and hindered their understanding of the concepts learnt.

The written responses revealed that most students were largely confident about applying the algorithms, but that they had difficulty in answering the questions where they were required to explain their reasoning. This indicated that students mainly know the rules and could carry out procedures, even if they had not made sense of them. This means that they mainly possessed procedural knowledge of the matrix algebra concepts. These findings seem to concur with what Naidoo (2007) eluded to in his findings that first year mathematics students rely on rules and algorithms. Also it seemed from their written responses that students had difficulty in interpreting the questions. The language of mathematics became a barrier for them to make sense of the questions, where, as a result, they weren't able to translate the statements into symbols. Errors made by students could be the result of their failure to conceptualise concepts, leading to poor interpretation of the nature of problems. With regard to the difficulties discussed in the literature review, this study supports the findings of Dubinsky (1997), that students' lack of background mathematical concepts generates more difficulties with the learning of linear algebra. It also supports the findings of Aygor & Ozdag (2012), that students' misconceptions in mathematics do cause learning difficulties for students. The written responses revealed that in term of APOS, many students were operating at the action and process stage.

The preliminary genetic decomposition that was used to analyse the students' mental construction will be used for Phase 2 of this study, but it will be modified, since the researcher realised that some responses could not be analysed using the designed instruments. In Item 4, Question 2.1.4 will be removed as repetitious, where no new knowledge was generated by its inclusion.

5.7 Conclusions and implications

This chapter provided the preliminary results of the Phase 1 of this study conducted to validate the research instrument. The research instruments proved to be a useful tool in analysing and revealing the nature of the mental constructions made by students. However, there were certain cases where the mental constructions made could not be explained through genetic decomposition, but only through Triad mechanism. The results could have been more enhanced with the use of interviews. This was evident in the Phase two of this study. The results revealed that students mainly possess procedural knowledge of the learnt concept, where they are operating at an action level in terms of the genetic decompositions. Most of the responses seemed to acknowledge the mental construction presented in the genetic decomposition. In the next chapter the analysis of data from Phase 2 of the study is discussed.

CHAPTER SIX
ANALYSIS OF WRITTEN RESPONSES AND INTERVIEWS FOR TASK ONE

6.1 Introduction

In this chapter, the analysis of students' responses to activity sheets and the transcription of students' interviews on selected tasks, based on their written responses to Task 1, are presented. The structure of the tasks was specifically designed to address the mental construction indicated in the genetic decomposition. Students' responses to activity sheets were categorised and some students were selected for the interview to provide clarity regarding their responses and to verify the mental constructions they seem to have made, based on their written responses. The selected participants were asked various questions, with the aim to extract how they constructed various mental constructions and to discover why these mental constructions arise in the context of matrix algebra. For this study, it was important to detect whether the knowledge constructed led to conceptual understanding of the concepts in matrix algebra and if the students could recognise and apply the required procedures appropriately in the given tasks. This chapter reports on the analysis of student responses (taken from both activity sheet and interviews) revealed about the mental construction (constructed or not) in relation to the preliminary genetic decomposition presented in Chapter Three. Based on the research framework, the students' responses to Task 1 were interpreted, and the data for each item and each participant who was interviewed is described using the APOS levels and the Triad mechanism.

6.2 The structure and analysis of the activity sheets

The activity sheet that consists of four tasks was administered to 31 students. For this study the questions will be referred to as items. Each task consists of number of items. Task 1 consists of six items, Task 2 consisted of two items, Task 3 consisted of two items, and Task 4 consisted of three items. These tasks address the following skills and knowledge: Task 1, Item1 focuses on students' understanding when identifying and explaining the order of the matrix given; Item 2 focuses on students' knowledge and skills in determining the matrix transpose; Items 3, 4 and 5 focused on exploring and describing students' understanding and application of procedures in solving problems related to matrix operations. Task 2 focused on exploring students' skills and knowledge when evaluating determinants and their understanding of the relationship between the determinant of a matrix and the determinant of its transpose. Task three, Item 1 focuses on exploring students' understanding of matrix multiplication and its relationship to square matrices and Item 2 focuses on exploring students' understanding of singular and non-singular matrices. Task 4 focuses on students' conceptual understanding of solution of the system of equations using Cramer's rule. Item 1 and Item 2 aim to explore students' level of knowledge constructed when solving problems related to the system of equations. All the thirteen items were coded (scored) using a 5 point rubric scale (see Table 6.1 adopted and modified from Jojo, 2012). The purpose of administering the activity sheet was explained to the students prior to each tutorial session and all the tutorial sessions were video recorded. Students were assured that their identity would not be revealed in any way.

Score	Description of mental construction	Behaviours
1	Showed no mental constructions.	No written response/incorrect response.
2	Mental constructions not properly developed.	No reasons to justify the answer/response show any understanding of the concept.
3	Mental constructions of the concept constructed but still display lot of gaps in the knowledge constructed.	Not totally complete in response to all aspects of the item and incomplete reasoning.
4	Mental construction constructed, limited gaps still prevail in the understanding of the concept.	A partially complete response with minor errors showing understanding of the main idea.
5	All the mental constructions proposed in the genetic decomposition of a particular concept are completely constructed.	Displaying complete understanding of the concepts.

Table 6.1: Scoring codes used

Once the scripts were analysed and the categories identified, one or two students were selected in each category for an interview. In some categories no written response fitted such as in Item 1, Category 1. The interviews were conducted in order to verify what has transpired in the students' written responses, as well as to clarify their responses where it was not clear how they found their solution.

6.3 The structure and analysis of the interview

The semi-structured interviews of one hour long were conducted by the researcher, with each of the eight participants selected from 31 first year and second year students. Based on what their responses from activity sheets revealed about mental constructions made or not made, an interview schedule was prepared by the researcher. The purpose of the interview was explained to each participant before the commencement of the interview. At all times, students were assured of their anonymity and pseudonyms were used. In ensuring that every aspect of the interview was captured, the interviews were audio recorded. In certain instances, during the interview process, students needed to write their answers on a piece of paper to explain their thinking, over and above expressing it verbally, and the researcher made provision for such cases. Although the interview questions were set before the interview commenced, probing questions were used to elicit more information about how participants constructed their knowledge and to ascertain their understanding of matrix algebra concepts. The probing questions were extensively used, because it was of importance in this study to clearly elicit the knowledge constructed from students so as to understand the mental constructions made and how those mental constructions were made. Students' difficulties and misconceptions that emerged from their responses in the activity sheet and during the interviews were analysed, with the aim of understanding the barriers that might have caused the students not to make the necessary mental constructions. The knowledge constructed by students when learning matrix algebra concepts and the levels at which they were operating in terms of APOS, were of significance to this study. Some of the questions used during the interviews aimed to find out whether students had just memorised the rules, or whether they actually understood the concepts at work.

6.4 Analysis and discussion of written responses and interviews

The objective of the activity sheets was (1) to understand the mental constructions that students make when learning matrix algebra concepts; (2) to understand the difficulties and the misconceptions that students display, which becomes a barrier in making the necessary mental constructions; (3) to explore the application of procedures in solving problems related to matrix algebra. The objective of the interviews was to: (1) get clarity on the written responses; (2) to verify the knowledge constructed as it transpired in the activity worksheets; (3) to classify the mental constructions made in relation to the preliminary genetic decomposition. During the interviews, students were requested to write down their explanations if they struggled to explain, in order to capture the substance of their mental constructions; and were allowed to use the language of their choice, in order to express their thinking strategies and knowledge construction procedures as clear as possible. Their explanations expressed in any vernacular language were then translated by the researcher into English. The students were asked to respond to the following issues: a) justifying their responses to particular questions in the activity sheet; b) looking at the strategies used in solving different questions; and c) examining their general understanding of matrix algebra concepts. Different questions based on the categories discovered on their responses to activity sheets were asked in order to elicit students' conceptual understanding of these concepts. Some common questions in understanding the strategies used in solving the problems were used, such as: (1) explanation of their strategy and why they use that strategy; (2) stating the order of matrices; (3) stating the constraints of addition and multiplication of matrices; (4) stating Cramer's rule and its constraints; (5) could you state the general formula of evaluating the determinants for 3 x 3 matrices. The analysis of students' responses to activity sheets, followed by the interview extract, is presented hereafter.

6.5 Order of matrices and matrix operation

Five items were analysed that explore students' knowledge constructions of order of a matrix and matrix operations. Item 1 concerns with determining the order of matrix, which can be given by $m \times n$ where m refers to the rows and n, refers to the number of columns. This can alternatively be given by $n \times n$, in square matrices, where the number of rows equals the number of columns. This addressed the first aspect of the action level in the genetic decomposition in Figure 3.3. It was designed to provide insight into whether the students had developed an action conception of the concept of the order of matrices.

Item 1

Identify the order of each matrix below

$A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$ C= $\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Answer:______________

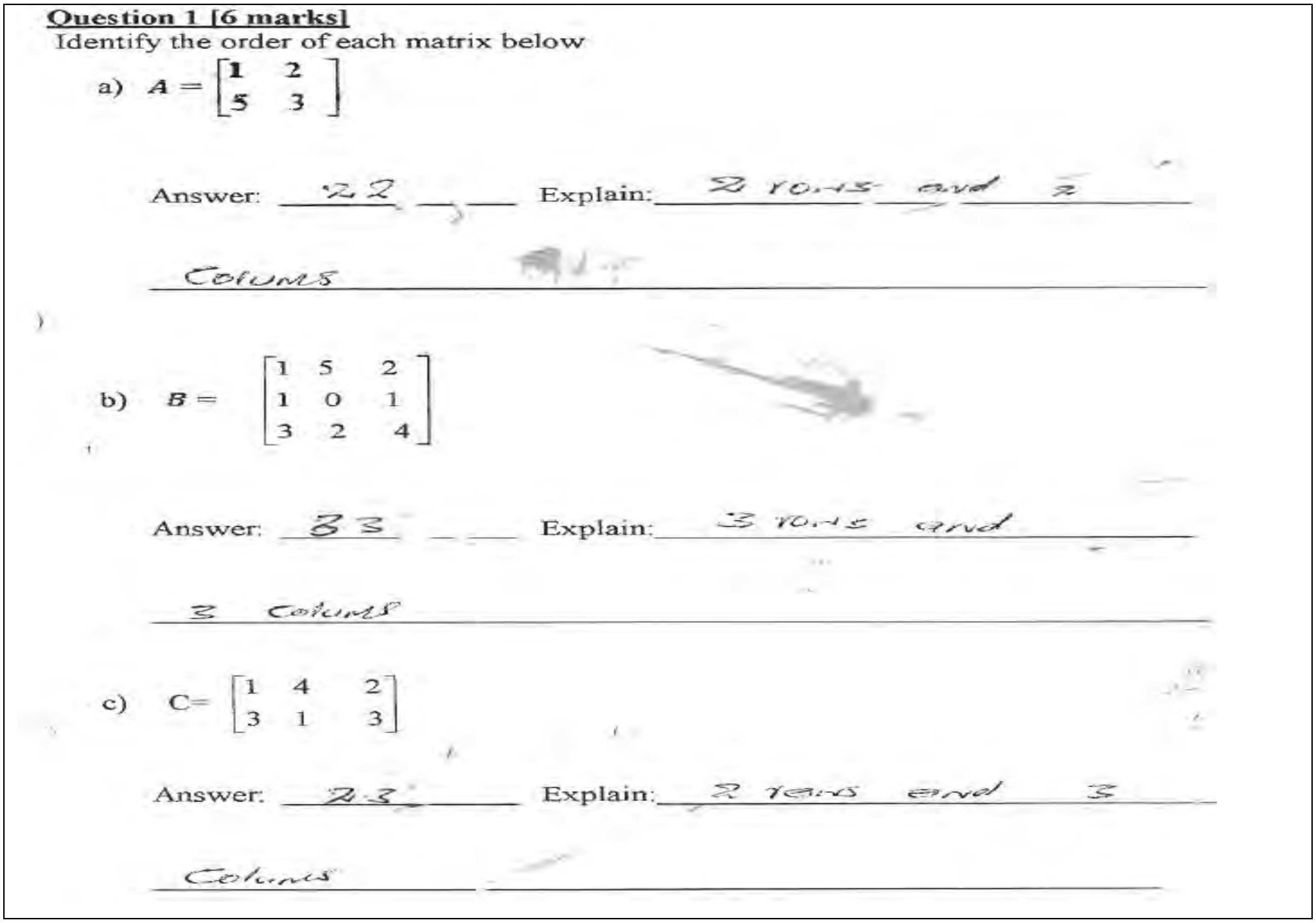
Explain:________________________________

In Table 6.2 the allocation of scores for item 1 is displayed.

Score	1	2	3	4	5
Indicator	No answer given.	Incorrect response with no explanation.	Order identified but provide a totally incorrect explanation in all the questions or vice versa.	Order correctly identified but explanation provided shows some gaps in the understanding of the concept in some of the questions or vice versa.	Order correctly identified with clear and correct explanation.
Number of students in each category	1	0	3	2	25

Table 6.2: Allocation of scores for Item 1

Eighty-one percent of the students gave correct responses to this item. This implied that they could clearly use the correct symbols to indicate the order and translate it into words. This may also mean they understand the structure of matrices in terms of rows and columns. These students have developed at least the action conception of the order of matrices. Students in Category 4 provided the correct explanation, but in some questions use incorrect symbols to represent the order. One of the two students in Category 4 wrote its a $n \times n$. It is possible that this student was trying to indicate that this was a square matrix, but misinterpreted the question such that in identifying the order of a matrix means stating whether or not the matrix is a square. Sydney in the same category wrote 2 2; 3 3; 2.3. Although this could be interpreted to be incorrect, the student knows the meaning of the order of matrix, but fails to represent it correctly mathematically (see Extract 1).



Question 1 [6 marks]

Identify the order of each matrix below

a) $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$

Answer: 2 2 Explain: 2 rows and 2 colums

b) $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

Answer: 3 3 Explain: 3 rows and 3 colums

c) $C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Answer: 2.3 Explain: 2 rows and 3 colums

Extract 1: Sydney's response to Item 1

It is possible that by using × to indicate the order during lectures and in textbooks might have been misinterpreted by some students to mean multiplication and that is why in 1 c) he wrote dots, indicating multiplication. Looking at his response for question 1 a) and b), one might assume that there is a dot (meaning times), but the writing is ultimately not clear. This would mean that notation has to be explained. The researcher had decided to incorporate the importance of notation in the modified genetic decomposition. This has huge implications for students' understanding of the structure of a matrix, and this might cause future learning barriers in the understanding of other related concepts if not given sufficient attention.

Two students in Category 3 stated the order correctly, but could not clearly translate the symbols into words so as to provide a clear explanation. Thabo, one of the three students in Category 3, stated the order correctly. When providing an explanation, he looked at the entries in the last row. By identifying the entry in the last row and last column such as in 1 a), he was able to tell how many rows and columns were there in that matrix, but in 1b and 1c, it is not clear how he decided on the number of columns (see Extract 2). In analysing his response in 1b, entry b_{22} would not help in identifying the order of the matrix. Although this was not how students were taught to name the matrix, Thabo was able to show a relationship between the entries of matrix and the order. He realised that the position of the last entry in a matrix could actually be used to determine the order, i.e a_{22} the last row and last column.

Identify the order of each matrix below

a) $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$

Answer: 2 × 2 Explain: a_{22} (the row 2, column 2 entry is 3)

b) $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

Answer: 3 × 3 Explain: b_{32} (row 3, column 2 is 2), b_{22} (row 2, column 2 is 0)

c) $C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Answer: 2 × 3 Explain: c_{23} (row 2, column 3 is 3) c_{12} (row 1, column 2 is 4)

Extract 2: Thabo's response to Item 1

Thabo must have done this at an action level. After doing this several times, he must have interiorised this into a process, since he did it all in the mind. However, the process conception is not fully developed, as he could not apply the same strategy correctly to all the questions asked. His response indicated he has constructed the necessary mental constructions regarding the order of the matrix as indicated in the preliminary genetic decomposition. The ability to use the elements of a matrix to determine the order of matrix shows that the student can abstractly think about the order of matrices. An interview with him indicated the following:

Researcher: In your explanation, you wrote, Row 2, Column 2, and Entry B_3. Can you clarify your response?

Thabo: *I did not understand how to go about explaining* [sic]. *I know it's 2 by 2, I know the columns and the rows. I did not know how to phrase the explanation, so I took it as row 2 column 2, it's a maximum of rows and maximum of columns - that is how I got 2 by 2, that gave a figure that is in that position that is 3. That is the maximum figure you can get. That is how I went about explaining.*

Researcher: Okay but in 1b you wrote: 'b_{22} is zero' - how does that relate to the explanation you just gave?

Thabo: [silent]. *Ya, here I made a mistake* [pauses] ... *I was trying to follow the same thing as the top but I mentioned the second one instead of mentioning the last one. I wasn't really looking at the question as a whole, because I was looking at the textbook – and one example that we did when asked for an explanation – that is how I saw they were giving the explanation. So I was looking at the book; so I used the first explanation as they did in the book and I tried the second one on my own but somewhere I went wrong.*

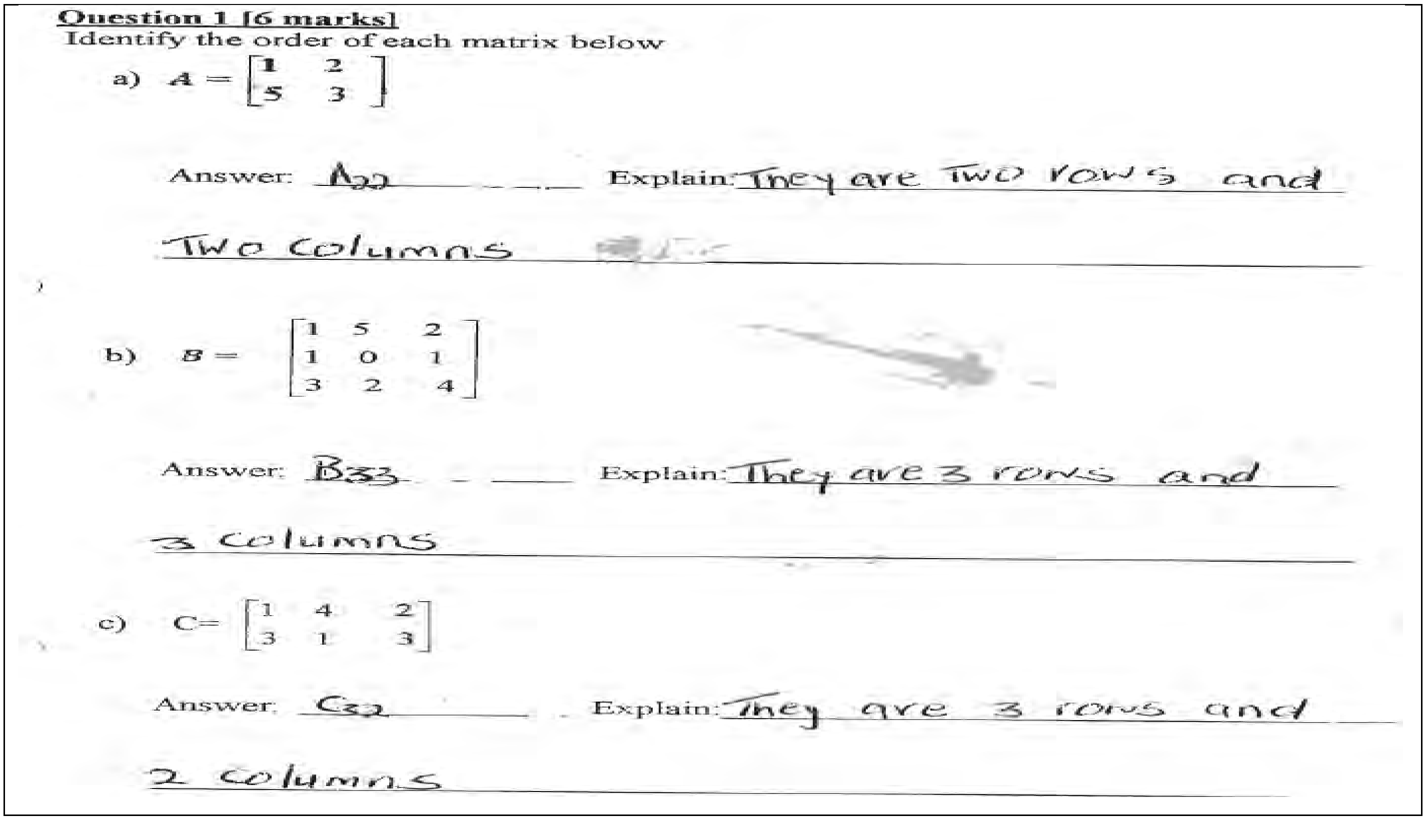
The above responses revealed that he knew how to identify the order, but did not know how to explain it and as a result tried to follow what is in the textbook, even if it did not make sense to him. This contradicts the assumption made above, based on his responses, that he must have done this several times and then interiorised this into a process, as he clearly stated he was just trying to follow the example in a textbook. In trying to understand his thought processes regarding the order of matrices, the interview continued as follows.

Researcher: According to your understanding is there any relationship between the entries of a matrix and the order of a matrix?

Thabo: *Yes, you look at how many entries there are, and if it's 4 entries, then it's a square matrix.* [...] *it's a '2 by 2', two in the first row and two in second row. By looking at the entries, you can decide whether it's 2 by 2, or 2 by 3 etc., but you need to understand what is a row and what is a column. The row comes first with an i and the column with a and j. If you know which is a row and which is a column, then you can decide whether it is a 2 by 3.*

In most cases, his explanation tended to focus on explaining the order of square matrices. He did not dwell too much on the order of non-square matrices. What is apparent though is that he understands the importance of differentiating between a row and a column, and that in stating the order of the matrix, the row is mentioned first. Although he seemed to know how to identify the order, he was having difficulty in articulating his thought processes clearly, as his explanation could not provide a clear understanding of the relationship between entries, and the order of matrices in a non-square matrix. His response, as mentioned earlier, contradicts the assumption made regarding his response to activity sheet, as it became clear here that his action conception of the order of the matrix had not been interiorised in the form of a process.

A different case ensued with Zinhle. Her response also falls under Category 3. Unlike the other two, she could not write the correct symbols to indicate the order of a matrix. She might have confused the symbols for stating entry of a matrix, i.e. $a_{22,}$ with the symbols for indicating the order of a matrix. By using capital letters, it seemed she knew that matrices were represented with capital letters in textbooks, as well as in lectures. As can been seen in her response, she was mixing elements of a matrix and the order of a matrix. Her explanation revealed that she could identify the order of square matrices, but not for non-square matrices, as is shown in 1c in extract 3. It seemed here she confused the rows and columns. It is possible that according to her, in using notation $c_{ij,}$ i represents the column and j represents the rows. This is, we observed, in her response to 1c. This could not be easily identified in 1a and 1b, since these are square matrices. Therefore, her response in 1c gives an indication that the action conception of the order of matrix is not fully developed, even though her responses in 1a and 1b were correct (see Extract 3).



Question 1 [6 marks]

Identify the order of each matrix below

a) $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$

Answer: A_{22} Explain: They are two rows and two columns

b) $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

Answer: B_{33} Explain: They are 3 rows and 3 columns

c) $C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Answer: C_{32} Explain: They are 3 rows and 2 columns

Extract 3: Zinhle's response to Item 1

Her response in an interview indicated that:

Researcher: Can you explain the meaning of these symbols you wrote here A_{22}; B_{33}; C_{32}?

Zinhle: *I was trying to clarify the what type of a matrix is. Like it's eh... it is like this one, it is like 2 by 2, this is a 3 by 3 and all of that* [sic].

Researcher: What does A_{22} stand for?

Zinhle: *It stands for* [the] *row and the column.*

Researcher: Do a_{22} and A_{22} mean one and the same thing, in the context of matrices?

Zinhle: *Hmm. I do not think so, because here (pointing at 1a) it says writing capital letter A and =, I was trying to name the matrix and write the rows and column.*

Her response in an interview does reveal that the action conception of the order of matrix is not fully developed. The knowledge seemed to be constructed from memorisation, as she could not clearly express her understanding of the different symbols used in matrices. Although she did mention that the first number represents the row and the second a column, she could not identify her mistake in 1c, indicating that she has not fully constructed the necessary mental constructions.

Furthermore, in her explanation, it became clear that she hasn't grasped the understanding and the correct meaning of notation used in matrices. A substantial contribution to APOS is the fitting that symbols and notation are vital for the development of an action conception in matrix algebra

Item 2 aimed to explore pre-service teachers' knowledge of determining the transpose of a matrix. It addressed the second aspect of the action level in the preliminary genetic decomposition in Figure 3.3. The aim was to gain insight into whether the students had developed an action conception of the concept of the matrix transpose.

Item 2

Let $A=\begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ Determine the following A^T

In the preliminary genetic decomposition, students' understanding of the transpose is considered to be at an action level, because it requires students to carry out the procedures and manipulate numbers externally. The notation in use ought to trigger a student's memory of the steps required for determining the transpose of a matrix. This could be done repeatedly at an action level, however, students also need to interiorise this into a process, so as to be able to solve other related problems such as determining the determinant of a matrix transpose. Understanding of the transpose would help students develop a conceptual understanding of why the determinant of matrix is the same as the determinant of its transpose. As Stewart (2008) pointed out, that action is an important step in developing an understanding of a concept.

In Table 6.3, the allocation of scores for Item 2 is displayed.

Score	1	2	3	4	5
Indicator	No response.	Attempts made but incorrect.	Partially correct with few errors.	Provide the correct answer but use incorrect notation or notation not used.	Both questions on transpose correctly answered.
Number of students in each category	1	3	0	0	27

Table 6.3: **Allocation of scores for Item 2**

In this item, 87% of the students gave the correct and complete response, indicating clear understanding of determining the transpose. The understanding of the relationship between transpose and its matrix will be better clarified through interviews, but their responses revealed that they had made the necessary mental construction about transpose of a matrix. Twenty-five out of the 27 students, who scored 5 here, also scored 5 in Item 1. The preliminary genetic decomposition suggests that both Item 1 and Item 2 are at the action stage. Therefore, students providing correct and complete responses in these tasks proved to have made the necessary mental construction needed for them. The interview with Sipho revealed the following.

Researcher: What is your understanding of the transpose of a matrix?
Sipho: *Transpose - I am going to reflect back to my mathematics and use reflections along the line. Y is going to be x and x is going to be y. Ok, perhaps we have for example, we have, hmmm... -1; 3; 4;1,*[where] *x becomes y and y becomes x, right? Ok we're going to follow the same method of the arm of y and arm of x. In the arm of y I got -2;1 and in the x I got -2;3;1, x becomes y; so it's going to swop* position [and] *be -2;3;1 and then* [...] *be -2;1* [sic].

Although he provided the correct response in item 2, his explanation was quite confusing. He knew that when transposing a matrix the columns becomes rows and vice versa. He then tried to relate that to transformation geometry, since he knew that reflecting along line means exchanging the coordinates. Therefore, to him, the columns represent *y* and the rows represents. The problem with his thinking is that these are not coordinates. Secondly, he tried to relate the structure of a matrix to a structure of the positive axis of symmetry. Although he had the correct answer the interview response seems to show that action conception has not fully developed. It is true, based on his response to the activity sheet, that he has an understanding of the action conception, however, it seemed he recalls the rules, but had not internalised them so as to construct processes to think about, meaning that he is operating at intra-stage in terms of the Triad mechanism. This we determined from the fact that his interview responses revealed some gaps in the knowledge constructed. Research has shown that students' previous knowledge plays a vital role in the construction of the new knowledge. However, if the previously learnt knowledge has not been conceptually formed, these could become a barrier in the students attempt to construct new knowledge. Tall (2008) emphasised that met before could positively or negatively impact the learning of the new concepts. This seemed to be the case with Sipho. His schema of transformation geometry seems not to have developed, where instead, he has developed a collection of rules, which he tried to use to solve problems he thinks might be of a similar nature.

To encourage him to think deeply about the transpose of matrix, more questions were asked.

Researcher: Given matrix A is a 2 x 1, what would be the order of the transpose?

Sipho: *The transpose would be 1 x 2.*

Researcher: If in matrix A, the element a_{11} was equal to -2, what would be the first entry in the transpose?

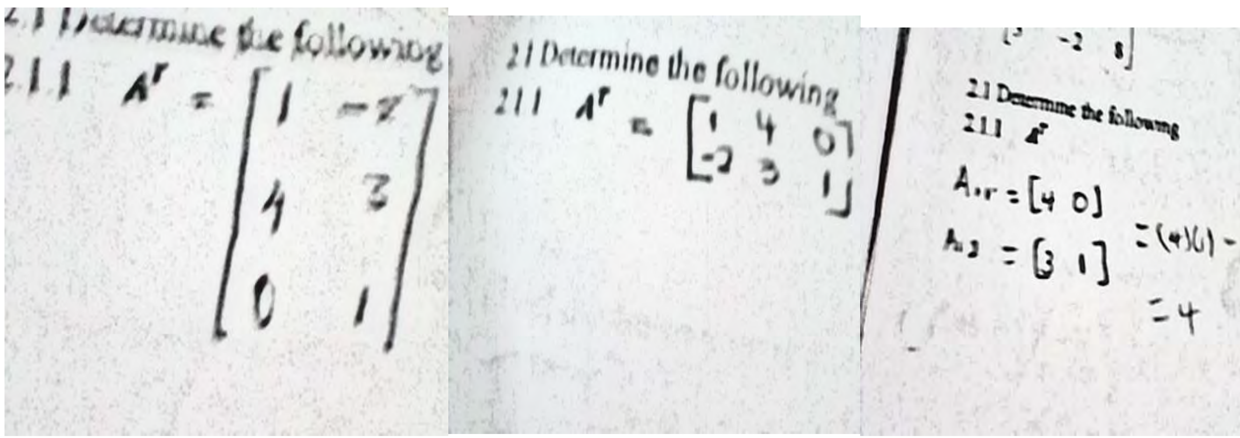
Sipho: *Can you make an example of this?*

Researcher: Okay matrix A = so what would be the first entry in A^T?

Sipho: *I may* [be using a] *law that is not required here, but the first number in the matrix even if it's transpose or not transpose, is not going to change, it will remain the same.*

This confirms that he had not internalised the action, as he could not construct an image internally of an arbitrary matrix. He needed to see the physical structure before he could make sense of the structure of a matrix transpose in question. Determining a transpose is considered to be an action level, as indicated in the preliminary genetic decomposition. However, the data analysis here shows that the ability to mentally think about the order of the transpose without actually doing the transpose or see the physical structure of the matrix, is necessary. This highlights that the process conception of a transpose is necessary. At a process stage, an individual uses the order of the matrix to create an image of the order of the transpose, and to construct an arbitrary matrix. The researcher has decided to incorporate this in the modified genetic decomposition.

Ten percent of the students provided an incorrect response. Sydney did change rows into columns, but then he also changed the entries, e.g., a_{11} in matrix A, which became a_{12} in the transpose. This revealed that the rule of transposing matrices was memorised without understanding that a_{11} is a pivot entry of both the matrix and its transpose. Nontando, in the same category, instead of interchanging the rows with columns, interchanged row 1 with row 2. John wrote down the entries in the second and third column and then calculated the determinants (see Extract 4). This revealed that in certain cases students just manipulate numbers without actually knowing what they were doing. It can be seem that often, instead of learning for understanding students mostly memorised the procedures because of their in efficiency in problems solving skills (Aziz, Mohmeerah & Tambychic, 2010). As a result, they struggle to make connections with concepts and just opt for number-grabbing when solving particular problems. The students' responses revealed that these students do not have the necessary mental constructions as expected in the preliminary genetic decomposition.



Extracts 4: Sydney, Nontando and John's responses respectively to Item 2

John's response revealed that he confuses concepts, transpose and determinants. The interview included the following exchange.

Researcher: In Item 1 can you clarify your response, your strategy, and what were you trying to solve?

John: *Njengalana nje* [here] *it says determine the transpose, nami angisaboni ukuthi ngenzenjani* [I also can't remember what I did here and why], *but okokuqala kwakumele ngishintshe amanumber of rows abe ama columns* [but I was supposed to interchange rows and columns] *sekwiyonake u A transpose* [that is a transpose of matrix A].

During the interview, John could not explain why he calculated the determinant. He indicated in the above statements that he also could not remember why he did what he did, but now he knows that to find a transpose he ought to have interchanged the rows with columns. This meant that the student had been learning these concepts again, and now seemed to understand it better after working on the activity sheet. This meant that this form of instructional method, as suggested by Dubinsky (1997), seems to be helpful to some students; as we observed the change in knowledge learnt by John.

Researcher: Do you understand now what a transpose is?

John: *Ja, njengoba ngishilo kwakufanele mina ngishintshe amacolumns abe ngamarows sekuyiyonake itranspose leyo* [Yes, as I have said I needed to interchange rows with column to get a transpose of a matrix].

Researcher: If matrix A was a 2 x 1 what will be the order of A^T?

John: [Tried to construct a matrix]. *Ukuthi angiyiboni ukuthi ngizoyibhala kanjani lento* [I cannot visualise how to write this matrix].

Researcher: It's a 2 x 1, how many rows are there?

John: *Two.*

Researcher: How many columns?

John: *One.*

Researcher: So, if A is 2 x 1, what would be the order of A^T?

John: *Oh!* [The] m*atrix is a 2 by 1, so you are asking the transpose, sizoyishintsha ibe ubani* [how we going to change it] *oh u-two columns* [frowning] *ukuthi angiyi understandi kahle* [I cannot imagine this].

Examining the response above, it is clear that when he said he understands the transpose, he was implying that he knows the rule. When asked to explain what a transpose is, he instead tells us how to find a transpose. The lack of understanding was also revealed when asked to state the order of the A^T, when A is 2 x 1. Firstly, he could not think about it, he needed to construct a structure which he also failed to do because he could not internally think and reflect on the structure of a given matrix. His interview response confirms that the process conception of a transpose has not developed. This was expected, since his written response revealed that the action conception has not developed. His response in an interview did however reveal that after working on the activity sheet, he is starting to develop the action conception of the concept. This we observed as he could describe the rule and say what he was supposed to do to produce the correct answer. Based on his interview response, it seemed as if that he is operating at the intra-stage in terms of the Triad Mechanism. Listening to him it became clear that he is aware of the collection of rules of determining the transpose, but is nevertheless unable to relate or use them appropriately. This argument is based on his failure to respond appropriately when asked what the order of A^T would be if matrix A was 2 x 1.

Item 3 addressed the third aspect of the action level and the first aspect of the process level in the genetic decomposition in Figure 3.3. It was intended to provide insight into whether the students had developed an action conception of matrix addition, and the process conception of the conditions defining matrix subtraction.

Item 3

Let $A=\begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ $B=\begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{bmatrix}$ $C=\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix}$ $D=\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$

2.1.2 Determine $3C - 2\,D$

The questions here were focused on exploring the knowledge constructed for scalar multiplication, and the addition and subtraction of matrices to explore the understanding that subtraction of matrices is defined in terms of the additive inverse.

In Table 6.4, the allocation of scores for item 3 is displayed.

Score	1	2	3	4	5
Indicator	No response or incorrect response.	Work contained many errors right through when multiplying by scalars.	Correctly multiplies by scalars, but fails to apply additive inverse. Final answer is totally incorrect.	Final answer contains few calculation errors.	Correct and accurate answer.
Number of students in each category	0	5	0	5	21

Table 6.4: **Allocation of scores for Item 3**

In this item, 68% of the students scored 5, and eleven of those also scored 5 in Item 1 as well as in Item 2. The results showed that most students had constructed the necessary mental constructions when it comes to addition and subtraction of matrices, as they were able to execute the answers without computational error. Also, their previous knowledge of arithmetic algebra seemed to be used effectively here, as these students showed the understanding that multiplying by a negative will change the sign to positive. Dubinsky (1997) has pointed out that the lack of previous knowledge has an impact on a students' understanding of linear algebra therefore that means a good grasp in the previous learnt concept will help students conceptualise concepts in linear algebra better. Based on their responses, those students with scores of fours and fives have constructed the necessary mental construction as indicated in the preliminary genetic decomposition. These students have therefore developed the action conception of addition of matrices. By continuously solving related problems, some student had interiorised this into a process where they were able to relate addition of matrices to subtraction of matrices. The schema developed of computation of integers e.g. 2 - 5 = 2 + (-5) assisted the students in constructing the necessary mental constructions. Tomita (2008) has argued that if the previously learnt knowledge is fully constructed, it can have positive impact in future learning. This was also echoed by Tall (2008).

During the interview with Sipho, the following exchange took place.

Researcher: Can you explain the strategy you used here in Item 3?

Sipho: *Because we are going to multiply and subtract, I am going to follow the Bodmas rule. I am going to multiply everything. Here we're going to have the matrices (pointing at matrix C) of C and multiply them by 3. So I am going to multiply all the entries in C (meaning matrix C) by 3 and* in [matrix] *D* [and] *I am going to multiply by -2. But because negatives sometimes confuse me, I am going to leave it like that. I am going to multiply by 2 inside, everything will be multiplied by 2; then continue, so now I know that I have multiplied what is left is to subtract. Subtraction is simple, I am going to subtract the matrices of C with matrices of D, which is why I didn't multiply by -2, I multiplied by 2, so that when I subtract, it's not going to confuse me. Then I subtract because if I multiplied by -2 here (pointing at his solution to item 3) I would have added the matrices.*

Researcher: If you multiplied by -2 not 2, would you have had a different answer?

Sipho: *Yes, you would have got the same answer, but* [I need to] *be sure of signs, because if I had multiplied by -2, the following step would have not been subtracting.*

His responses in the interview concur with what transpired in the activity sheet, where the necessary mental constructions have been made. Sipho seemed to have developed the understanding that matrix entries are real numbers, and that accordingly, most properties that apply to real numbers will also apply to matrix entries. Also, in many cases, when he responds to a question, he always tried to link what he is learning now to what he had learnt at school. Earlier in item 2, he tried to explain the understanding of matrix transpose through transformation, although that was inappropriate. For Item 3, matrix entries are real numbers which meant Bodmas was applicable. Furthermore when deciding on the method to use, he considers the method that would best solve problems effectively and efficiently. This is what Van de Walle (2007) referred to as one strategy of being mathematically proficient. When students need to be flexible before solving any particular problem they need to think about in how many ways they could solve the problem, and choose the one that will be easy for them to use. Based on his response to the activity sheet and interview, the researcher that Sipho had developed the action conception of matrix addition and interiorised it into a process to consider and to use the knowledge learnt to solve other problems.

Zinhle response was also grouped under Category 5. Her response to an interview revealed that she made the necessary constructions. She even went further to explain that since matrix C was a 3 by 3 and so is matrix D the product is a 3 by 3. When probed further the following discussion took place.

Researcher: If matrix C was a 3 x 2 and D was 3 x 3, would 3C-2D be defined?

Zinhle: *No, because mam you know u D has extra column okusho ukuthi* [that means] *it can't be added or subtracted with anything. You need C to have the same number of rows and columns as D for the solution to be defined.*

Her response indicated that she had constructed the required knowledge to understand the constraints of addition of matrices. Besides just mentioning the matrix order, she showed the knowledge that the order is made of entries, and that therefore, when talking about addition or subtraction of matrices, we talked about subtracting the entries in the corresponding order. This, we hear from her response, in the first line as she pointed out Matrix D has an extra column, which cannot be subtracted from C. This indicated that Zinhle might have done these operation at an action stage, but that she had since interiorised the action into a process that she could mentally think about, and was able to apply her reasoning to other related concepts. Her response revealed she has made the necessary mental constructions as indicated in the genetic decomposition, which means that in terms of APOS, she is operating at the process stage.

Sixteen percent correctly solved the given problem, but because of computational errors in the final answer, they could not provide a totally correct response. Although these students did not provide a complete response, their working out revealed that they are operating at an action stage. Like Thabo, all his workings were correct, but he made one computation error which could be considered a slip. The main problem was that Sipho, during the interview eluded to the fact that one needs to be careful in the computation of numbers. The responses of students in Category 4, including Thabo, revealed that they understood how to multiply with scalars, but lacked accuracy in their calculation, which made them unable to obtain the correct answer. Students seemed to be much careless in their manipulation of numbers, this shows incompetency in their calculations. In this item, students seemed to be making careless errors like: (1) ignoring the sign when multiplying or adding; and (2) adding or subtracting incorrectly. These errors are just computational errors; which are not rooted in their cognitive structures, and could be easily corrected. Olivier (as cited in Siyepu, 2013) differentiates the type of errors that students make when solving mathematical problems as follows:

> Slips are wrong answers owing to processing: they are not systematic, but are carelessly made by both experts and novices. They are easily detected and are quickly corrected. Errors are wrong answers owing to planning: they are systematic in that they are applied regularly in the same circumstances. Errors are the symptoms of the underlying conceptual structures that are the cause of errors and underlying beliefs and principles in the cognitive structure that are the cause of systematic conceptual errors are known as misconception (p. 578).

The errors made by students in this task are considered to be systematic. There can be underlying features of misconceptions about the fact that addition makes it bigger, or that subtraction makes it smaller, but what is displayed in this task was pure carelessness. Thabo's interview included the following exchange.

Researcher: Can you explain the strategy used to solve this problem?

Thabo: *In school, I was taught to always try your best to never work with negatives because when you work with negatives, you get confused, and end up writing the wrong thing.* [Since that is 99% true], *what I did was to multiply by negative all the entries, so I know if there is a positive and a negative, I just change the sign and multiply. I just change the sign and multiply the numbers, that is, what I did in the second step and in the third step, I think I made an addition error, 3+2 = 6 instead of 5.*

Like Sipho, Thabo also believes that working with negatives numbers is confusing and tried to make the problem simpler so that it will be easy for him to manipulate numbers. During the interview, he was able to pick out the mistake he made during computation, indicating that it was more of a slip than a lack of understanding or any form of misconception. What also transpired was that he had constructed many cues that helped him to work with numbers, such as his stated rule, where when finding positive and negative, one changes sign and multiplies. These cues might at times be a barrier to learning, if learnt without understanding. Also, he mentioned that working with negatives is confusing, but he multiplied with the negative. This gave the impression that negative numbers and negatives are two different things. This shows that in some cases, the correct knowledge could be constructed, but that gaps could cause future problems in the learning of related concepts. That said, with Thabo it seemed like it was just a matter of phrasing his sentences incorrectly as the interview revealed that:

Researcher: If you multiplied by 2, and not -2, do you think your answer would have been different?

Thabo: *I will get the same answer, because the signs in matrix will be opposite of what I have when I multiply by -2. If you have negative and positive, they* [will] *still give you a negative etc. You see, I would* [have gotten] *the same answer, but you see, people prefer to use different methods, which makes it easier for them.* [In] *maths, there is no one way to get the answer.*

Researcher: If you were given a 2 x 1 matrix and a 2 x 2 matrix can you add or subtract them?

Thabo: *Eh.....no.*

Researcher: Why?

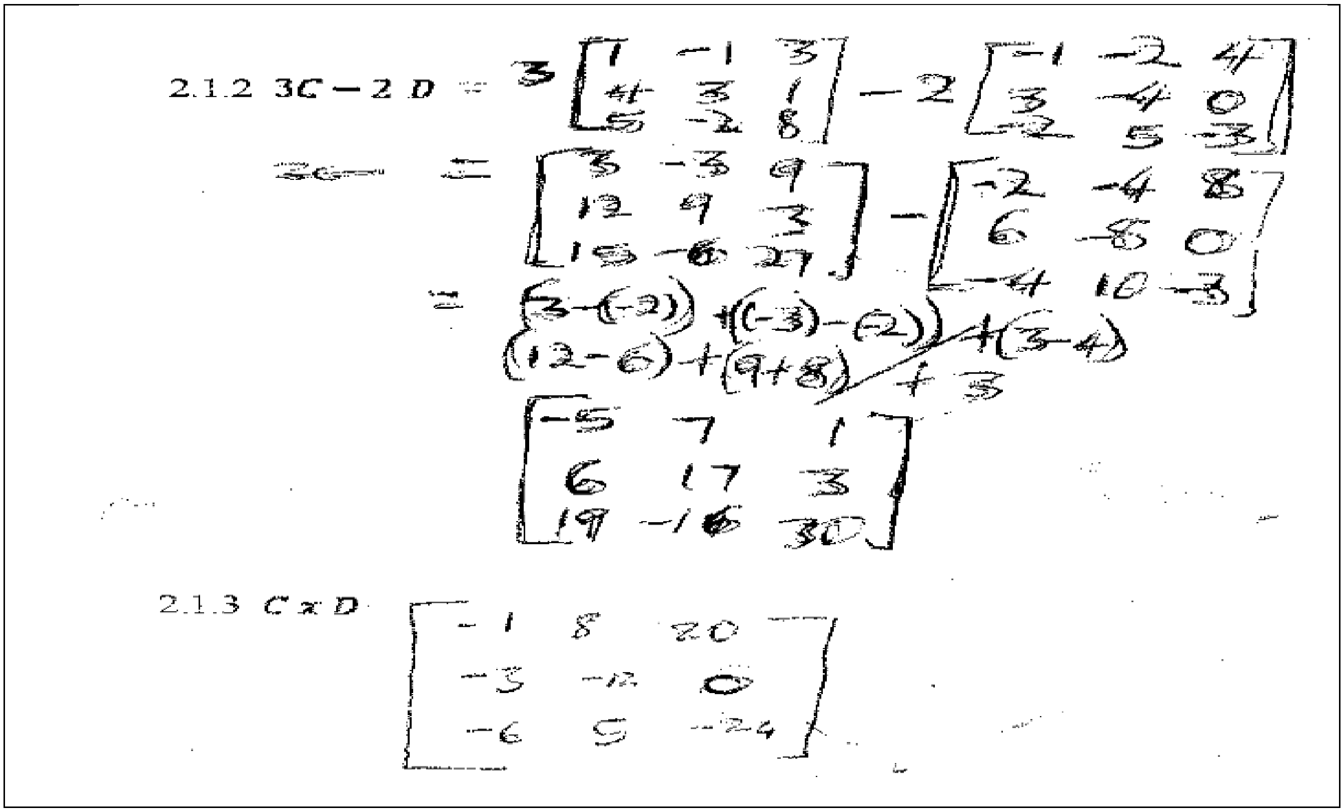
Thabo: *Because a row and a row, column and column: you need to have another column for this one* [pointing the 2 x 1]. *You can't just add any zeros there, because now you are changing the order of the matrix. The order you are making is 2* x *1 into a 2* x *2.*

Researcher: What would you say are the constraints of adding and subtracting matrices?

Thabo: *To add the order must be the same.*

Thabo's response from the interview revealed that he has made the necessary mental construction needed for this particular problem. Thabo seemed to have solved related problems at an action level but has now interiorised the physical action into a mental process. He could mentally construct the structure of matrix, and think about a solution, without actually doing it. This could be extracted from his response, as he could tell that we could not add a 2 x1 and 2 x 2 since the order is not the same. Also, he understands that although matrix entries are real numbers, not all conditions of real numbers apply to matrices. This, we observe, as he pointed out that inserting a row of zeros would change the order of a matrix. He knew that inserting a row of zeros or column of zeros to the given matrix is different to adding a zero matrix, which adheres to the rules of real numbers. What is interesting here is that Thabo chose a column of zero to add to support his argument. The property of zero from algebra he brought over to matrices. This meant he has developed the understanding of the relationship of the zero property from algebra with matrices. This we get from his response as he explained that when adding matrices, the order needs to be the same, and went on to indicate that the row and a row or a column and column meaning that row one in matrix A is added to row one in matrix B and so on.

Sixteen percent of students could not provide correct answers, because of several mistakes in their algebraic computation of matrix entries. Sydney copied the matrix D incorrectly, but correctly multiplied the given matrix with given scalars. He then committed another error when subtracting matrix 2D from 3C. As Sipho and Thabo stated, negative numbers can be confusing, and one needs to be sure of the signs when computing as Sydney appeared to struggle with when solving this sum. It is evident, as we could see his entries in the first row of the final answer, which contains many procedural errors [see extract 5].



Extract 5: Sydney's response to item 3

The written responses of all five students, including Sydney, revealed that the action conception of matrix addition and subtraction has not developed. The knowledge of working with integers seem to be missing, and as a result they could not apply it correctly in the computation of matrices. As Tall (2008) pointed out that students previous knowledge plays a major role in constructing new knowledge. In an interview with Sydney it appeared that:

Researcher: Can you explain how you solve this problem?

Sydney: *kuqala ebengikwazi ukuthi uthatha idata lesinayo, imatrix esinayo if ku 3C then umultiplaye wonke ama elements ngo 3, then nangalasimultiplaya ama elements a matrix D ngo 2.* (multiplied matrix C by 3 and matrix D by 2). *Thereafter mase siminus ama corresponding elements like 1 minus -1* (then subtract the entries like 1 minus -1) *but ngesikhathi ngisenza ngoba iyaconfuser uma isininingi kakhulu kwesinye isikhathi uzame ukuyisimplifier into uvele uthi nje "1-(-1)"bese uthola amalokhuzana awrong* (but when I do it because it can be confusing when there are too many entries. Sometimes you try to simplify it, then end up with wrong answers. *Sometimes uma ujomba ama steps yiwo kade ngiwaskipper* (Sometimes when you skipped some steps like I did you get wrong answers)

Here he notes that he tried to simplify by taking out entries in the matrix and wrote them aw 1-(-1), because it can be confusing for him, but that made him get wrong answers as well as trying to skip steps led to wrong answers.

Researcher: If you multiply matrix D by -2 not 2 as you did, do you think your answer would be different?

Sydney: *No, actually ibizofana* [the same] *but then ibizoshintsha ngoba uma sesimultiplaya ngo-2 besizoba no 1 bese kusal uplus la* [pointing between the two matrices as he explained that if he multiplied by negative 2, it will change to plus]. *cause uma ukhipha uminus uwuletha ngaphandle kuba icommon factor yakho konke lokho* (it was going to be the same but it will change because when we multiply by -2 there was going to be plus now between the matrices cause if I take negative inside and write it outside it will be a common factor)

Here he was saying the answer will be the same but the sign of the entries will changed since he is now multiplying by -2.

Researcher: If matrix C was a 3 x 2 and matrix D was a 3 x 3 would the 3C- 2D be defined?

Sydney: *No, it is not going to be defined, since the orders are not the same, so you* [are] *still going to have the extra order.*

Researcher: What do you mean by extra order?

Sydney: *Angisakhumbuli kahle iterm enikezwa lokho* [I do not remember the correct terminology] *kodwa kumele alingane* [but they have to be equal] *ama elements if the order it's a 3* x *3 then the other one should be a 3* x *3, because if the other one it's a 3* x *2 uzoba nelekhunja necolumn engeke ize icorresponde nalutho* [if the other one is a 3 x 3 and the other a 3 x 2, there will an extra column that does not correspond with anything].

After solving problems in the activity sheet, Sydney might have been engaging with these tasks, which led to him developing the action conception of the concept. This we observed as he is now able to explain and describe the conditions of matrix addition. When given an arbitrary matrix, without doing the whole computation, could explain if the sum is defined or not. Using his concept image of a matrix structure, he could reflect on the structure of matrix internally and visualise the addition procedures. From the interview we noticed that he did it all in real time in his head, while responding to an interviewer. This meant that in terms of APOS, he is operating at the process stage. Based on the Triad, we can conclude that he is operating at the inter-stage, as he had command of the collection rules, which could relate to any given matrix.

Dubinsky (1997) suggested that concepts that gives students' difficulties in linear algebra need to be analysed epistemologically. Then it will be possible to design instructional methods that foster the construction of the necessary mental constructions which the study intends to achieve. This form of instructional method seemed to of great help to the student participants, because now students do not just solve the problem and forget about it, but they seem to reflect on it and try to develop their understanding. This we observed as Sydney struggled to solve the problem in the activity sheet, and thereafter during the interview, seemed to have constructed the necessary mental construction. This implies that he reflected on his response and had developed the process conception of the concept. His response from the interview contradicts what appeared in the activity response. In item 3, the data analysis has shown that for these pre-service teachers to gain a proper understanding of matrix algebra, they needed to have at least a process conception of algebra involving real numbers. Therefore, when we modify the preliminary genetic decomposition, we will keep in mind a schema for algebraic procedures involving real numbers.

Item 4 focused on pre-service teachers' understanding of and the ability to apply procedures accurately when finding the matrix product. It addressed the first aspect of the process level in the preliminary genetic decomposition in Figure 3.3. It was intended to provide insight into whether or not the students had developed the process conception of matrix product.

Item 4

Let $C=\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix}$ $D=\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$

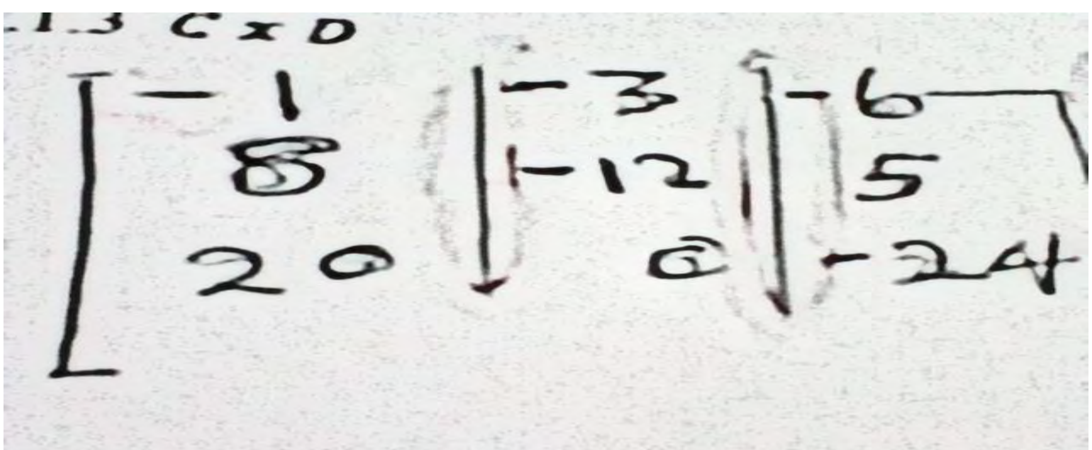
Determine the following: $C\ x\ D$

In Table 6.5 the allocation of scores for item 4 is displayed.

Score	1	2	3	4	5
Indicator	No response or incorrect response.	Numerous computational errors.	Working not shown. Shows understanding of matrix multiplication but some in-accuracy in the final response.	All the working shown with few computational errors.	Complete and correct response.
Number of students in each category.	13	0	5	4	9

Table 6.5: Allocation of scores for Item 4

This question explained students' understanding and application of rules of finding matrix product and their understanding of the constraints of matrix product. From the questions, students can show if they can execute the rule properly or not, but the interviews will explore the conceptual understanding of the concept. It was important to ascertain whether students could use the constraints of matrix multiplication to develop the understanding of matrix product. In this item, 42% of students were scored as 1, because of their failure to execute the rule accurately. 8 students out of 13 did not even attempt the problem. The other five did attempt it but provided totally erroneous answers; the student Sipho did not show any working, but looking at his response he multiplied row 1 of matrix C with column 1 of matrix D, row 2 with column 2, and row 3 with column 3 (see Extract 6).



Extract 6: Sipho's response to item 4

Sipho had memorised the rule incorrectly. He might have confused matrix multiplication with matrix addition. In terms of APOS, Sipho hadn't made any mental constructions as expected by the preliminary genetic decomposition. However, his response can be analysed in terms of the Triad mechanism. Looking at his response, we can conclude that he is operating at the intra- stage. Sipho, seemed to have wrongly memorised the rules of matrix computation, but could not recognise the differences between them as a result cognitively constructed the list of unrelated rules. When confronted with a situation to solve a mathematical problem and having the cluster of rules, he opted for what seemed to provide the answer in that particular context, without thinking about its meaning towards the understanding of whole concept. During the interview he stated the following.

Researcher: Can you clarify your steps in finding the product of matrix CD?

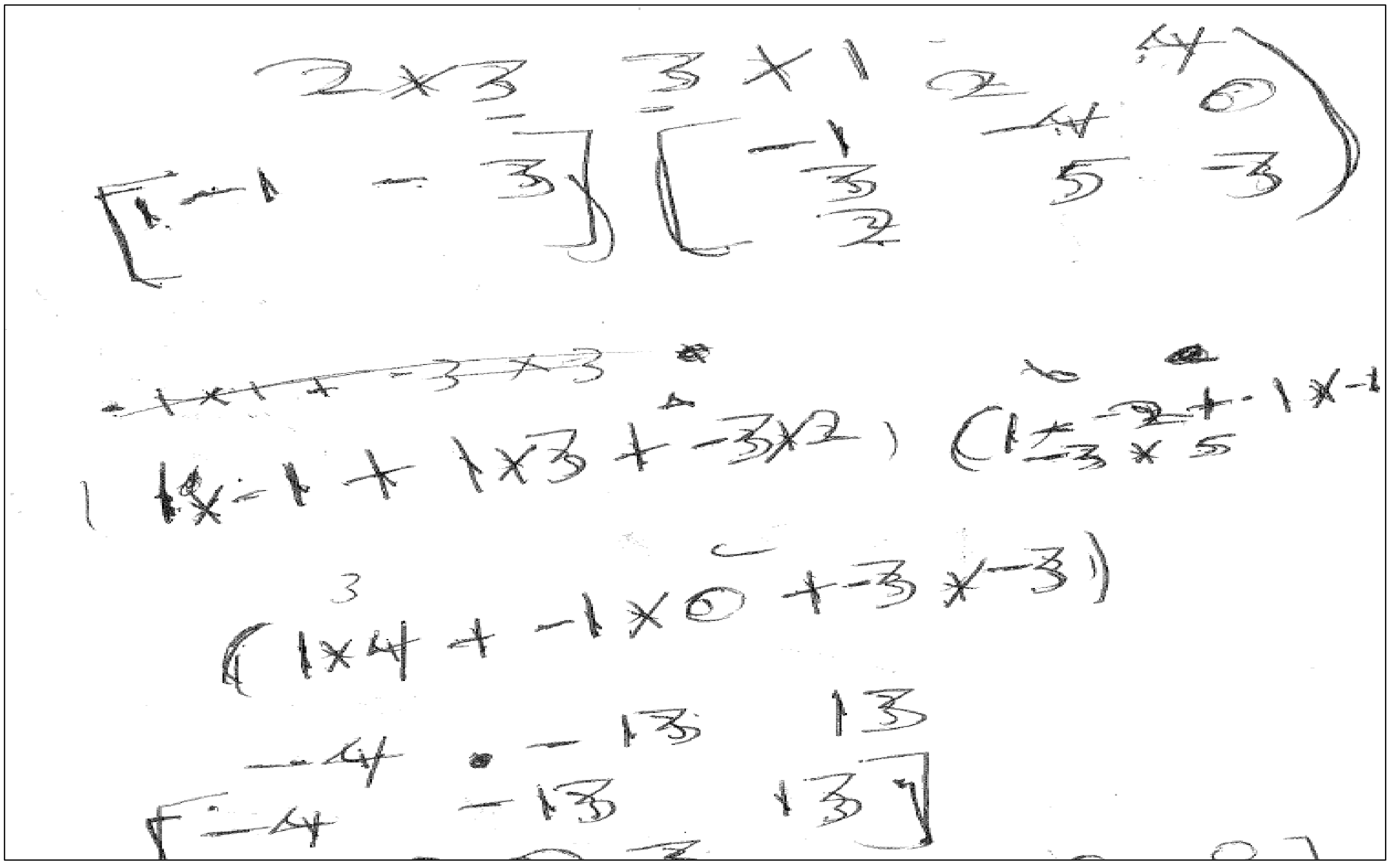
Sipho: *First in the matrices there is a law that says if we are multiplying* [...]*, columns must be equal to* [...]*columns must be equal to* [...] *in the first matrix columns must be equal to the rows in the second matrix. For example, matrix A is order of 2* x *3 so matrix of B must be equal to 3* x *2. So the columns in matrix A must be equal to the rows in matrix B.*

R: You said matrix A is a 2 x 3, does matrix B have to be a 3 x 2?

Sipho: *Yes, as long as the column in the first matrix equals the rows in the other matrix, it does not matter what the column in the second matrix might be.*

R: Did you apply the same procedure in your solution?

Sipho: *I think I made a mistake and that is not correct pointing to his solution, but I can do it now. I know the first row is multiplied with column 1, multiplied with column 2 and with column 3.* [He worked out the solution], *here is my answer for* [the] *first row.*



Extract 7: Sipho's solution during the interview

Although he articulated the rule well, his answer to the first row is incorrect. What transpired here is his failure to manipulate signs. Examining the entries of matrix C in row 1 are $[1 \quad -1 \quad 3]$. In the first line of his solution, one is able to notice the change of signs in the entries of matrix C. This led to him getting the incorrect answer, as indicated in his response in line 4. This concurs with the assumption made based on his written response that he wrongly memorised the algorithm. In his cognitive structures, he has constructed the list of the rules, which he could retrieve and use in certain context, but failed to use in other related contexts. His schema of basic arithmetic algebra is not developed and as a result, it affected his skills and knowledge of manipulation of matrix entries. Again, we observed that basic algebraic skills are necessary for matrix product. His response revealed that he is operating at the intra-stage in terms of the Triad mechanism, and in terms of APOS, he hasn't constructed the necessary mental construction as expected by the genetic decomposition.

Students in Category 3 wrote down solutions without showing any of their working. Their responses revealed that they knew the rule and constraints for multiplication, but made mistakes when manipulating numbers. Their solution contained computational errors such as ignoring the sign when multiplying, adding or subtracting incorrectly, and multiplying incorrectly. These are the careless mistakes students tend to make, because they do not interrogate the meaning of what they write. This is also emphasised by Maharaj (2014) that "students need to interrogate what they write and say in the context of a problem" (p. 66). One might also be reminded Polya's problem solving model, which includes a last stage of looking back. This is also linked with metacognition, that monitoring and evaluating whether the solution makes sense in relation to the problem is important. Shahbari, Daher & Rasslan (2014) emphasise that this step is important for successful problem solving and maximum student involvement.

Thirteen percent of students seemed to know the rule and its application to solve related problems. However, they could not provide a complete and correct response, and because of algebraic inaccuracy in their working, could not provide the complete response. They displayed procedural errors when computing, but looking at their solution, it was clear that most of the pieces of the knowledge of matrix product had been constructed (see Extract 7).

2.1.3 $C \times D$ (4)

$$= \begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix} \times \begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-1)+(-1)(3)+3(-2) & (1)(2)+(-1)(-4)+3(5) & (1)(4)+(-1)(0)+3(-3) \\ 4(-1)+(3)(3)+(1)(-2) & (4)(2)+(3)(-4)+(1)(5) & (4)(4)+(3)(0)+(1)(-3 \\ 5(-1)+(-2)(3)+(8)(-2) & (5)(2)+(3)(-4)+(8)(5) & (5)(4)+(-2)(0)+8(-3) \end{bmatrix} = \begin{bmatrix} -10 & 21 & -5 \\ 3 & 16 & 13 \\ -27 & 47 & 24 \end{bmatrix}$$

Extract 7: John written response to item 4

John's response indicated that he knew the rule to use, but struggled with manipulation of numbers. In the last row and second column of his working out, he multiplied (-3) by (-4). It is not clear where this resulted in (-3), because c_{32}= -2. This might again be a careless mistake, not indicating any misconception. This was a once off mistake, where all the other entries were multiplied correctly, indicating that this was an unsystematic error that could be corrected. Although John did not provide the complete response, his solution revealed that he has an action conception of the matrix product, and that he was operating at the inter-stage in terms of the Triad mechanism. This can be observed as he carried out procedures appropriately indicating that the collection of rules constructed form a systematic understanding cognitively.
John, in his interview, indicated as follows.

Researcher: Can you explain the rule you used when finding the product of matrix CD?
John: *Angiyazi irule but ngiyazi ukuthi ngenza kanjani. Indlela engayisebenzisa if ngifuna iproduct ngiye ngiqale ngitimes irow ngamacolumns. Hlambe njengoba sino C ngizothatha irow kaC ngiyitimes ngamacolumns ka D. ngizothatha irow yokuqala kaC ngiyitimes ngawowonke amacolumns* [I do not know the rule but I know what I did. The way I did it I multiply the rows with columns. For example in matrix C I will take one row and multiply it with all the columns in matrix D, like the first row in matrix C and multiply it with all the columns in matrix D].
R: In your working out, can you identify the mistakes you made?
John: [Worked it out again]. *The correct answer is 8, I do not know how I got 42. I really can't remember how I got this 42, because it does not make sense.*

John said he did not know the rule but could explain it properly. It is possible that he thought I was asking for a formula, or for him this is not a general rule it is his strategy as he did noted on self-reflection that he didn't know the rule' but 'knew what he did'. When asked if he could identify his mistake, he provided what was supposed to be correct answer, but in fact could not explain what he originally did. What was interesting though was that he decided to solve the problem again, instead of interrogating what he originally did. This shows that sometimes students do not read what they write, in order to judge whether or not it makes sense in the context of a problem. They just manipulate numbers.
R: Matrix C and matrix D were of the order of 3 x 3, so what would be the order of matrix CD?

John: *It will be a 3* x *3.*

Researcher: How can you tell?

John: *Noma umultiplier i 3* x *3 izokukhiphela I order ewu 3* x *3* [if you multiply 3 x 3 you will get 3 x 3].

Researcher: If matrix C was a 3 x 2 and matrix D a 2 x 4, what would be the order of matrix CD now?

John: *It will be a 2* x *2.*

Researcher: Without working out the whole product of any two matrices. How can you tell what will be the order of the product?

John: *I have to do it first, hlambe from there, ngizokwazi how to determine* [I have to do if first perhaps from there I will know how to determine it].

Researcher: Okay, use a 2 x 1 and 1 x 3?

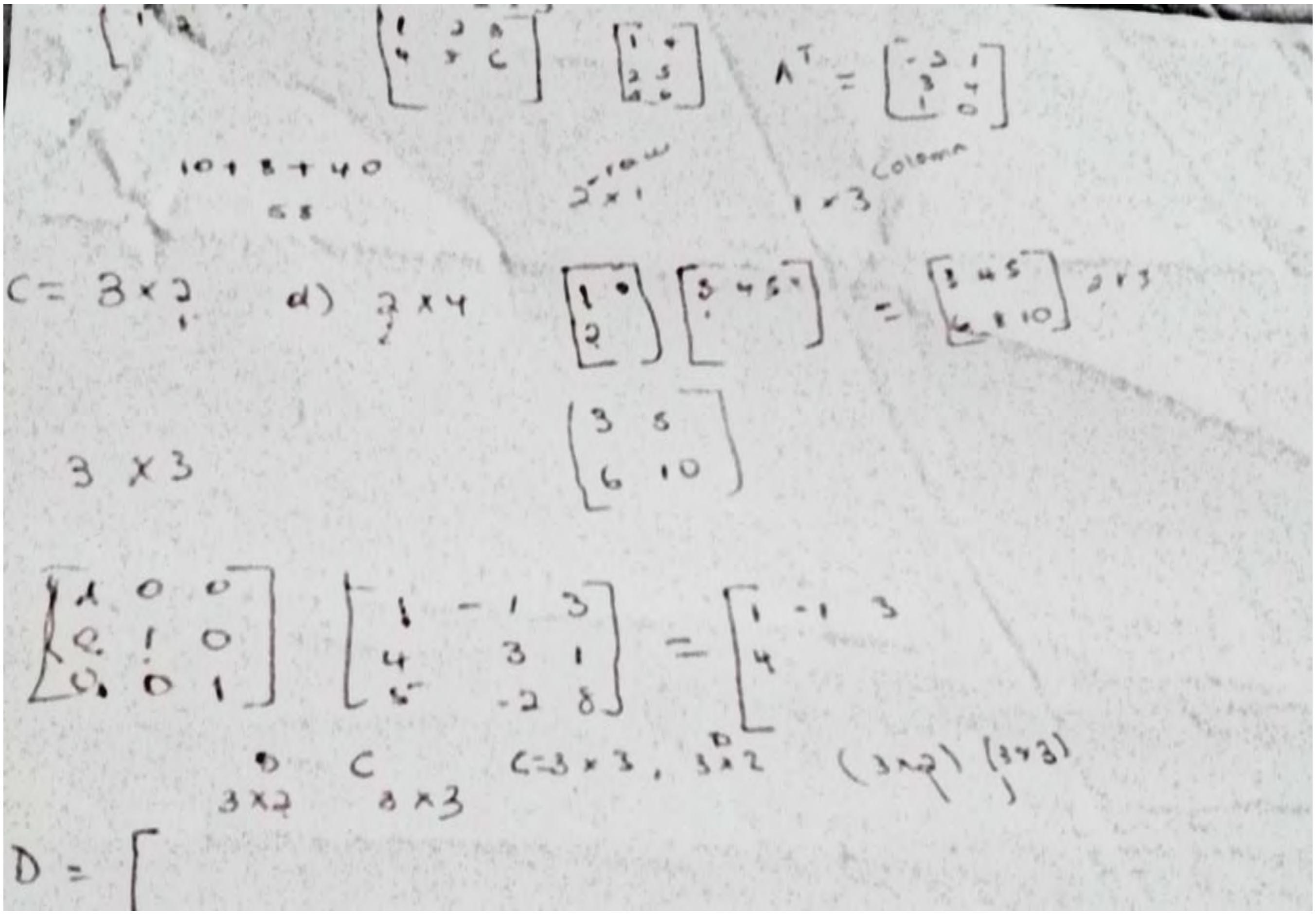
John: [tried to construct the structure matrices] [...] *let's do a 2 x 2 rather.*

Researcher: The original matrix that you solve was a square matrix, now I want us to talk about non-square matrices.

John: *Engikubonayo la ukuthi la... mmh, yini konje obyibulezile* [what I see here is that mmh what did you ask?]

Researcher: If these two matrices were matrix C and matrix D, what would be the order of matrix CD?

John: *Ukuthi I order uzoyithola kanjani sizothatha irow ka C then sizothola icolumn ka D ukuthi iwubani, then sizothatha icolumn yakhona* [to get order of matric CD, you take row in matrix C then take a column in matrix D].



Extract 8: John's attempt of solving the given matrices during the interview

Based on the above extract of an interview with John, it appeared that although he could manipulate the rule for multiplication of matrices, he has not constructed the meaningful understanding of the concept. He could carry out the action, but has not interiorised it into a process. This we observed, as he struggled to explain how the order of the product can be determined, where he could not conceptualise it without actually carrying out the whole calculation. As we see in the first step in his working out in Extract 8, he had to construct the structure of those matrices, and carry out the action of multiplication of matrices before he could tell what would be the order of matrix CD. Although he had solved matrix CD in the activity sheet, and performed the action again during the interview, he couldn't explain how he determined the order without first computing the product. Instead, he recited the rule again, such that the process conception of the matrix product had not developed. He is still operating at the action stage.

Researcher: Okay, let's use the above matrix 2 x 1 and 1 x 3. Is the product of these matrices defined?

John: *Yes.*

Researcher: How can you tell if the product is defined or not?

John: *Ubheka inamba yamacolumns ku C if ilingana neyama rows ku D then iproduct yethu izoba defined* (he is saying if the number of columns in C the same as the rows in D then the product is defined).

Researcher: Can you give a situation where CD is defined but DC is not defined? Think of specific matrix that will show such cases.

John: *Izoba kanje, ake sithi C is a 3 x 3 and D is 3 x 2. So u CD is defined but uma siqala ngalokhu (pointing at 3 x 2) siyabona ukuthi amarows awasalingani namacolums so DC is not defined*.

This response revealed that even though the multiplication of matrices was not yet fully interiorised into a process, nevertheless he was able to use the rule to understand the constraints of matrix multiplication. As he clearly explained, he multiplied the row with a column, which helped him to realise that row and the column needs to be the same. This meant that the inter-stage, in terms of the Triad mechanism, John was developing but has not yet constructed the coherence of all the interrelated systems of matrix multiplication.

Twenty-nine percent of the students provided a correct response, indicating a complete mathematical understanding of matrix product. Their written responses revealed that they knew the relevant rule of matrix multiplication. This is evident in the way in which they clearly showed their working. Although their responses indicate complete understanding of matrix product, during an interview, the following questions were posed to Zama, in order to gain an insight about her knowledge of the matrix product.

Researcher: Before we can multiply matrices, we need to check if their product is defined. How can we tell if the matrix product is defined or not?

Zama: [...] *when you can only multiply if the number of columns is equal to the number of rows*.

Researcher: Let us say I have matrix A and matrix B - how would I know that matrix AB is defined?

Zama: *Hmmm... . look for the number of rows and number of columns, if* [they] *are equal*.

Researcher: In which matrix must I look for rows and in which must I look for columns?

Zama: *In matrix A we look for rows and in matrix B we look for columns.*

Researcher: Okay, if matrix A is 3 x 1 and matrix B is a 3 x 3, is matrix AB defined?
Zama: *I think yes.*
Researcher: What would be the order of matrix AB then?
Zama: [shakes head].
Researcher: Since you say it's defined you mean you can multiply them?
Zama: *Yes.*
Researcher: Work it out then.
Zama: [tried to do it], *eish, I do not think so.*
Researcher: you said the rows in matrix A should be equal to columns in matrix B, and that these two matrices fit those conditions, but that you cannot multiply them. So, think carefully why the product here cannot be defined?
Zama: *Hayi I do not remember the condition for matrix multiplication.*

After some probing, it became clear that Zama had memorised the rule incorrectly, and so could not have interiorised it into a process where she could conceptualise it. Although she could manipulate numbers, she could not mentally unpack the structure of a matrix. What transpired here is that with square matrices, some students could write the order correct, and it would not be easy to identify whether they understand or not, unless one interrogates their thinking. Based on the interview, it seemed that Zama had not constructed the necessary mental constructions needed for matrix multiplication as indicated in the genetic decomposition. What *was* evident was that she had the collection of rules of finding matrix product in other situations, but could not apply the rule to non-square matrices. In terms of the Triad mechanisms she is operating at the intra-stage. This is based on the fact that she was able to apply the rule to square matrices, but unable to explain it and apply to other related contexts. She could manipulate the numbers at an action level for the sake of producing answers without constructing meaningful understanding of the concept.

We now present the analysis of Item 5. Item 5 was developed to ascertain the level of knowledge students had constructed of matrix product, beyond being able to manipulate numbers. It aimed to detect whether or not students had the process conception of the concept. Furthermore, it focused on exploring pre-service teachers' concept image of matrix multiplication. It addressed the first and third aspect, involving the process level in the preliminary genetic decomposition in Figure

3.3. It was intended to provide insight into whether the students had developed the process conception of matrix product or not.

Item 5

2.1.4 Let Matrix E be 3×2 and Matrix F be 3×3. Is the product of EF defined? Explain.

In Table 6.6 the allocation of scores for item 5 is displayed.

Score	1	2	3	4	5
Indicator	Yes with incorrect or no response.	Yes but shows some understanding of matrix product.	No without explanation.	No, with incorrect explanation.	No with correct explanation.
Number of students in each category.	9	0	0	9	13

Table 6.5: Allocation of scores for Item 5

This question was actually aimed at finding out whether students really understood the concept of matrix multiplication beyond knowing the algorithm. Twenty-nine percent said yes, where six out of the nine students did solve Item 4, even though not all of them provided the correct answer because of computational errors, but they applied the rule correctly. One student Zuko said, *"yes, because the number of columns in the first matrix equals the number of rows of the second matrix."* His response displayed gaps in his understanding of matrix multiplication and the naming of matrices, because the numbers of columns in E are not equal to the number rows of F. It is possible that he was referring to the rows of E and the rows of F. His response might have been triggered by E x F. By realising that it has to with matrix product the focus was placed on retrieving the rule of multiplication from his mental structures. This we observe as he showed difficulty in answering the question. His actions were not interiorised as he could not switch effortlessly from processes to do mathematics to concepts to think about (Tall, 2006). The knowledge constructed is in bits and pieces. This implies that the action conception has not been interiorised into a process.

Students grouped under Category 4 said no, but the explanation given by many in this category was not accurate, and it did not show that they understood the concept completely. What becomes evident here is that the constraints of matrix multiplication were not fully understood, as Jack noted thus: *"F has 1 extra column and E has 1 less column therefore, the product of EF cannot be* *defined*." Another problem that the students seemed to experience is the use of proper mathematical terminology. Some students could not express themselves correctly as they wrote number of numbers in rows and number of numbers in columns. In this case, it is possible they were referring to the entries in a row and entries in the column, but did not specify the matrix in which they should consider the row, and which they ought to consider the column. This revealed that some students could actually apply the algorithms correctly, even though they did not fully understand the concept because, except for (Jack and Sipho) in this category, all the seven students did solve item three and they did not score five due of computational errors. In terms of APOS it meant they are operating at an action stage. This we observe as they did apply the rule correctly in Item 4. However, in Item 5, they experienced difficulties in making thought procedures explicitly indicating that their concept image of matrix product was dominated by rules. They had not mentally make a collection of all cases related to matrix products.

Forty-two percent of the students provided a complete response, indicating the complete understanding of the concepts. They displayed the ability to switch from processes, to doing mathematics, to concepts they can think about. This meant that they were operating at the process stage, in terms of APOS, as it seemed that they made the necessary collection of elements of matrix product.

6.6 Conclusion

Most of the students' responses from the activity sheet and interview discussed in this chapter revealed that most students had not interrogated what they wrote. Their focus is on completing tasks they do not look back upon, whether or not their response has addressed the question. Therefore, it is important that teaching should instead focus more on getting students to reflect on how they construct their knowledge. This could help them reorganise the mental structures. We also found from the data analysis in this chapter that at least a process conception within algebra was necessary, as it became a barrier for students to interiorise their action mental construct into a process. In the next chapter, the researcher presents the analysis of students' written responses and interviews dealing with determinants.

CHAPTER SEVEN

ANALYSIS OF STUDENTS' RESPONSES FROM ACTIVITY SHEET AND INTERVIEWS TO TASK 2

7.1 Introduction

The analysis in this study is based on students' responses to activity sheets and transcripts from the interviews. Out of 31 participants, ten were selected for the interviews. In the previous chapter, six students' responses to activity sheet were analysed and discussed alongside these interviews. In this chapter, transcriptions of the students' interviews on task two, based on students' written responses, are discussed. Participants who took part in the interview process were asked various questions, with the aim of understanding how they constructed various mental structures. For this study, it was of importance to explore whether they could evaluate and apply determinants to solve other related problems. The responses to task two and interview transcripts of the ten students who took part in the interview are analysed and presented in this chapter. These were chosen on the basis that the researcher believed that they were the most suitable to providing insight on the mental constructions that students made or fail to make, as expected in the preliminary genetic decomposition. The genetic decomposition presented in figure 3.3 serves as an analytic tool as the mental construction developed are analysed in terms of the preliminary genetic decomposition. The Triad mechanism in conjunction with APOS describes the level at which an individual is operating.

7.2 Analysis of students' responses

In the activity sheet, Task 2 comprised three-questions (see Appendix A2). For this study, the researcher labels these questions as items. Question 1.1 and 1.2 will be referred to as item 1a and item 1b, and 1.3 will be referred to as Item 2. Item 1 aimed to explore pre-service teachers' knowledge of determining the determinant of a matrix. It addressed the second and the third aspect of the process level in the genetic decomposition in Figure 3.3. It was intended to provide insight into whether or not the students had developed the process conception of evaluating determinants, and a correct use of notations.

Item 1

Given: C= $\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix}$ D= $\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$

1.1Find the determinant of C

1.2. Find the determinant of D^T

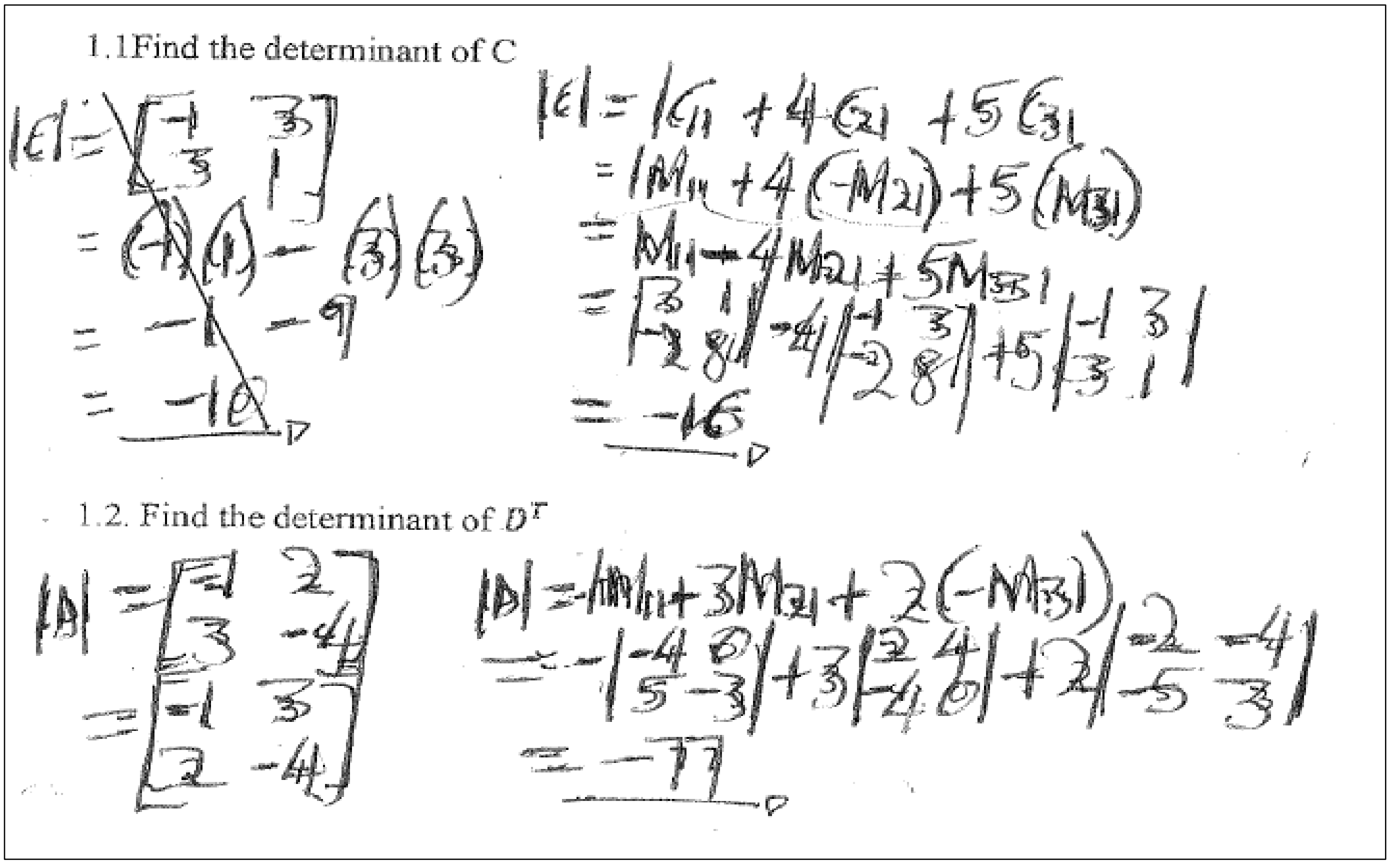
In Table 7.1 below the allocation of scores for item 1 is displayed.

Score	1	2	3	4	5
Indicator	No response or incorrect response in both questions.	Shows some knowledge of the formula but failed to carry out procedures correctly	One response is correct and one is totally incorrect or partially correct.	Correctly evaluated the determinants in both questions but use incorrect notation to express the answer	Complete and accurate response
Number of students in each category	6	5	18	0	2

Table 7.1: Allocation of scores for item 1

Only two students gave a complete response to item 1, indicating complete mathematical understanding of evaluating matrices. These students were scored 5, and in all the previous items in Task 1, they also provided complete responses, except for Nontando, who displayed the lack of understanding of matrix multiplication in Task 1, Item 5. Their responses indicated that they had constructed the necessary mental constructions as expected in the preliminary genetic decomposition. In terms of the triad mechanism, it seemed that they were operating at the inter-stage. This we observe as they show that they had the collection of all the procedures, and could apply them in all related context. Their responses showed that they had constructed the correct concept image of the concept. This was revealed where they proved to be clear on the relationship between the minors and cofactors, and on how they might correctly use this to evaluate the determinants of the matrices. Based on that, it seemed that the process conception of the concepts was fully developed. Fifty-eight percent of the students provided the correct response in only one of the two items. Sixteen out of the eighteen students provided a correct response to item 1a and a totally wrong response to item 1b (see Extract 1). The other two provided a correct response in item 1b, and a partially correct response in 1a. Their responses in item 1b was considered partially correct, because from their responses, it seemed that they knew the procedures to follow, but failed to manipulate numbers and therefore provided an incorrect response. There were a number of errors identified from the responses of sixteen students who provided a correct response in 1a and totally incorrect response in 1b. Some use the minors as cofactors or substituted incorrect cofactors. Some failed to manipulate the signs, ignoring the effect of multiplying by a negative number, and some did not even attempt to solve item 1b. Their responses or lack of response indicated that the schema of basic algebra has not developed, and since it is not developed properly, it impacted negatively on the new knowledge learnt. This confirms what Dubinsky (1997) pointed out, namely that students' difficulties with linear algebra emanate from the lack of background knowledge of school algebra and arithmetic algebra.

Zama's response in 1b indicates a number of shortcomings other than the one mentioned above for all sixteen students. She was unable to determine the transpose of D. Examining her solution on the left, she deleted and determined the transpose of that. Examining her response closely on the right hand side, she then attempted to evaluate $|D|$ not$|D^T|$. . One could assume that she knows the relationship between the determinant of a matrix and the determinant of its transpose, and therefore ignored the instructions and found the determinant of matrix D, which she could use to answer the question. However, she did not ultimately answer the question. If she knew the relationship between the determinant of a matrix and the determinant of its transpose, it is possible that she might have forgotten to answer the question at the end, or that she misread the question. There is also a possibility that she purposefully ignored the instruction, since she failed to determine the transpose of matrix D, and just solved for the determinant of D. This is what Siyepu (2013) referred to as conceptual errors, which arise due to "students' failure to grasp the concept involve or to appreciate the relationship involved in the problem" (p. 584).



Extract 1: Zama's response to Item 1

In finding the determinant, she expanded by column 1 and incorrectly determined the minors and cofactors (see the middle and last minors in Extract 1). She also failed to manipulate numbers correctly to determine the determinant. Her response indicated that she is operating at the action stage in terms of APOS. This we observe as she correctly determined the determinant of matrix C. Her overall response to item 1 could be explained in terms of the triad mechanism, which indicates that she is operating at the intra-stage. This we base on the fact that in her working out, she seems to use a number of rules, without relating them to each other, and as we observe she determinse an incorrect sub-matrix, determines its transpose, which she could not use anywhere in her solution, then write down the formula to evaluate the determinant of matrix D, but could not correctly apply or solve the related problem and fails to manipulate numbers. As Jojo (2011) stated, at this stage the student has the collection of rules, but had no recognition of the relationship between them, and this is what can observed in Zama's response to Item 1. During the interview the following discussion took place.

Researcher: In which type of matrices you can evaluate the determinant?

Zama: *In square matrices*.

Researcher: How do we evaluate the determinant of order?

Zama: *Silent.*

Researcher: Okay can you explain here how you evaluated the determinant of those matrices?

Zama: *I do not remember.*

Researcher: Let us look at your solution for matrix C, which row or column you expanded with?

Zama: *Column 1.*

Researcher: If you expanded by any other column or row would the answer be different?

Zama: *No, hmm, eish* [sic]*, I do not think so.*

Researcher: So you are saying the answer would have been the same?

Zama: *Yes.*

Researcher: So if you look at your answer for matrix C, can you explain how you did it as you correctly determined the difference?

Zama: *I first find the minors by deleting some rows and columns and then work out the determinant, because it was easy now since these matrices were 2* x *2 and there is a formula for that*.

Researcher: Did you do the same also in the second one?

Zama: *Eish* [sic]*, silent.*

Researcher: Let us look at the instruction, you were asked to determine but you worked out can you explain why you choose to evaluate determinant of D instead?

Zama: *Silent.*

Researcher: Okay like in matrix C, you expanded by column 1, but your answer here is incorrect. Can you identify the mistake you made here? You can work it out again and see.

Zama: *Couldn't solve it.*

After being assisted, she could see where she made a mistake, but she still could not clearly explain how to evaluate determinant of order. In every step she struggled to explain the step, indicating that even the action conception of evaluating determinant was not fully developed. She could apply the action to solve 2 x 2, but struggled when it came to 3 x 3. Also, her responses indicated that the rules were memorised, but not understood. This is based on her response when asked to explain how she evaluated the determinant of matrix C, and she indicated that she had forgotten. Her response in the interview confirms the assumption made based on her written response that she had not constructed the necessary mental construction as indicated in the genetic decomposition. It also confirms that she did not know the relationship between the determinant of a matrix and determinant of its transpose, which would indicate that she did not grasp the concept.

Thabo also provided the correct response in item 1a, but did not even attempt to evaluate the determinant of $|D^T|$. . He correctly determined D^Tbut not its determinant (see Extract 2). Due to notation, he thought he had answered the question. This implication is important. The clarity of notation and understanding of such leads to the possible correct solution to the mathematics problem.

1.1Find the determinant of C

R C

$M_{31} = -10$

$M_{32} = -11$

$M_{33} = 7$

$C_{31} = (-1)^4 \times -10 = -10$

$C_{32} = (-1)^5 \times -11 = +11$

$C_{33} = (-1)^6 \times 7 = 7$

$$
\begin{aligned}
|A| &= \boxed{a}_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\
&= (5)(-10) + (-2)(11) + (8)(7) \\
&= -50 - 22 + 56 \\
&= -16
\end{aligned}
$$

1.2. Find the determinant of D^T

$$D^T \begin{bmatrix} -1 & 3 & -2 \\ 2 & -4 & 5 \\ 4 & 0 & -3 \end{bmatrix}$$

Extract 2: Thabo’s response to item 1

In an interview with Thabo the following discussion took place.

Researcher: You correctly evaluated the determinant of matrix C, why did you not evaluate that for the transpose of matrix D?

Thabo: *I was slow by the time we had to do group discussion, I was still solving the determinant of matrix C, but I was not worried, because I knew what to do.*

Researcher: Okay, here, when evaluating the determinant of matrix C, you wrote $|A|$ instead why?
Thabo: *I am so used working with examples as the determinant of A, it was supposed to be ...*[sic].
Researcher: When evaluating the determinant of matrix C in your formula you wrote a_{11}. What is the meaning of this?
Thabo: *Was this also need to change. I did not know how to write c_{11}, since there was also C_{11} for a cofactor.*

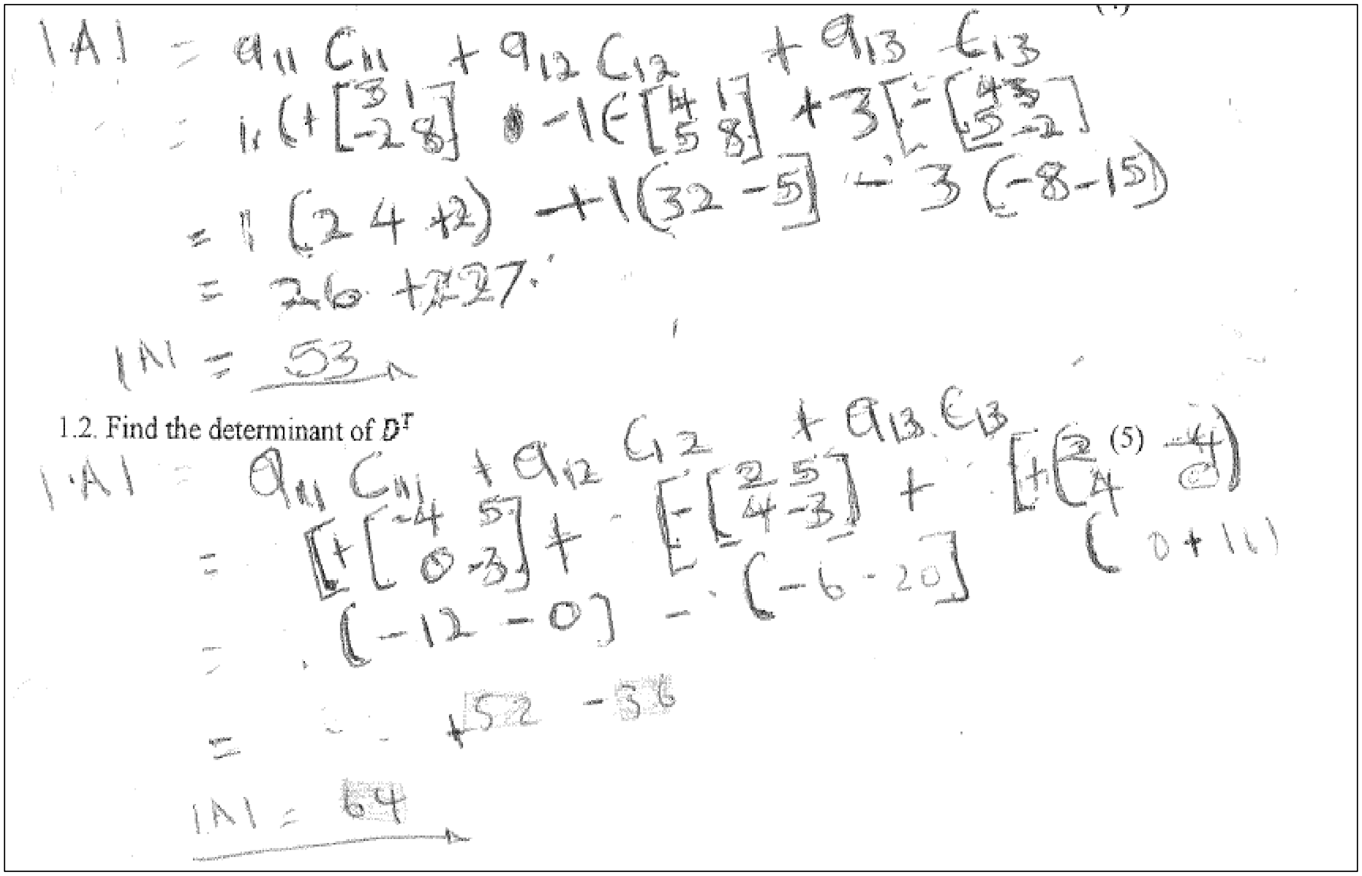
Based on his response to activity sheet and interview, it seemed that Thabo had the action conception of evaluating determinants. This we observe as he accurately carried out the procedure in Item 1, and from his response, as he stated that he knew what he was supposed to do in Item 1b, but due to limited time, he couldn't evaluate it. Also, even though he used the wrong notation, he could easily identify his mistakes, and highlight factors that caused him to write incorrect notation, as he pointed out that it was confusing for him to write $C_{11}C_{11}$, so he wanted to separate between the entry and the cofactors. To explore his conceptual understanding of the relationship of the determinant of matrix and the determinant of its transpose, the following conversation transpired.

Researcher: In evaluating the determinant of matrix C which row or column you expanded with?
Thabo: *Row 3.*
Researcher: If you expanded by any row or column would your answer have changed?
Thabo: *It will be the same; just different numbers in the equation.*
Researcher: Why?
Thabo: *All what we need is to find* [the] *sub-matrix of the original matrix, so it does not matter whether I find it in* [one particular] *row or column* [or another], *as long as I am consistent about the row or column that I used.*

His response reveals that the action conception of evaluating determinant has been interiorised into a process. This we observe as he clearly explains why using any row or column does not change the answer. This shows he is not just manipulating numbers or rules, he is actually thinking of process as a whole. Although he did not answer the question in an activity sheet, his response in an interview revealed that he correctly interpreted the question and could apply the relevant procedures. Thabo must have done this at an action level repeatedly, until he internalised it as he realised that answer remains the same regardless of the row or column used. Also, he showed that he understood that the matrix formed is a sub-matrix of the original matrix. His response revealed that he had made the necessary mental construction of evaluating determinants, as described in the genetic decomposition. His response could also be explained in terms of the triad mechanism, as it indicated that he is operating at the inter-stage. According to Jojo (2011) the inter-stage is characterised by the students' ability to begin to mentally collect all different cases and recognising their relationship. This we observe, as Thabo showed that he recognised why the relationship expanding by any column or row would give the same answer.

The responses of five students in Category 2 revealed that they experienced difficulties with the manipulation of numbers. They set up correct formulas, but substituted incorrect numbers, or failed to correctly solve for minors. Sipho's response showed that he had a poor understanding of notation, including the use of brackets. Both of his responses have $|A|$ instead of $|C|$ or $|D^T|$. The researcher believed that he does not understand the meaning of and use of notation. Research in the application of notation has shown that in many instances, students rote learn the notation and then tend to apply it in their solution without understanding its meaning (Jojo, 2011; Ndlovu, 2012; Siyepu, 2013). Also, Findell (2006) has pointed out that students tend to blur distinctions between closely-related signifiers and the ideas they intend to signify. Sipho's response also contains procedural errors and computational errors, emanating from his failure to manipulate numbers (see Extract 3).

According to Siyepu (2013), procedural errors arise when students fail to carry out manipulation or algorithm, although they understand the relevant concepts in the problem. Orton (cited in Maharaj, 2014) has referred to these as executive errors. Failure to manipulate signs seemed to be the biggest difficulty that Sipho is experiencing. This we observe since the same problem occurred in Task 1, Item 4. This seemes to be surprising, because in an interview regarding his response to Item 3 in Task 1, he emphasised the importance of being careful with signs when solving a problem, displaying that his schema of arithmetic algebra has developed. Examining his responses now, it seemed it has not fully developed. His responses reveales that he has not made the necessary mental constructions as expected in the preliminary genetic decomposition. Failure to carry out procedures indicates that the action level of evaluating determinants has not developed. His interview included the following exchange.



Extract 3: Sipho’s response to Item 1

Researcher: In item 1, the determinant of which matrix were you asked to evaluate?

Sipho: *Here,* [the] *determinant of matrix C.*

Researcher: In your solution you wrote 'Why'?

Sipho: *Oh ya, it’s a mistake, ayibo!* [sic] *Does that mean if I am finding a determinant of some matrix I must use those matrix letters in the formula? Eish!* [sic] *I really do not know.*

Researcher: Which row or column did you expand within Item 1 and Item 2?

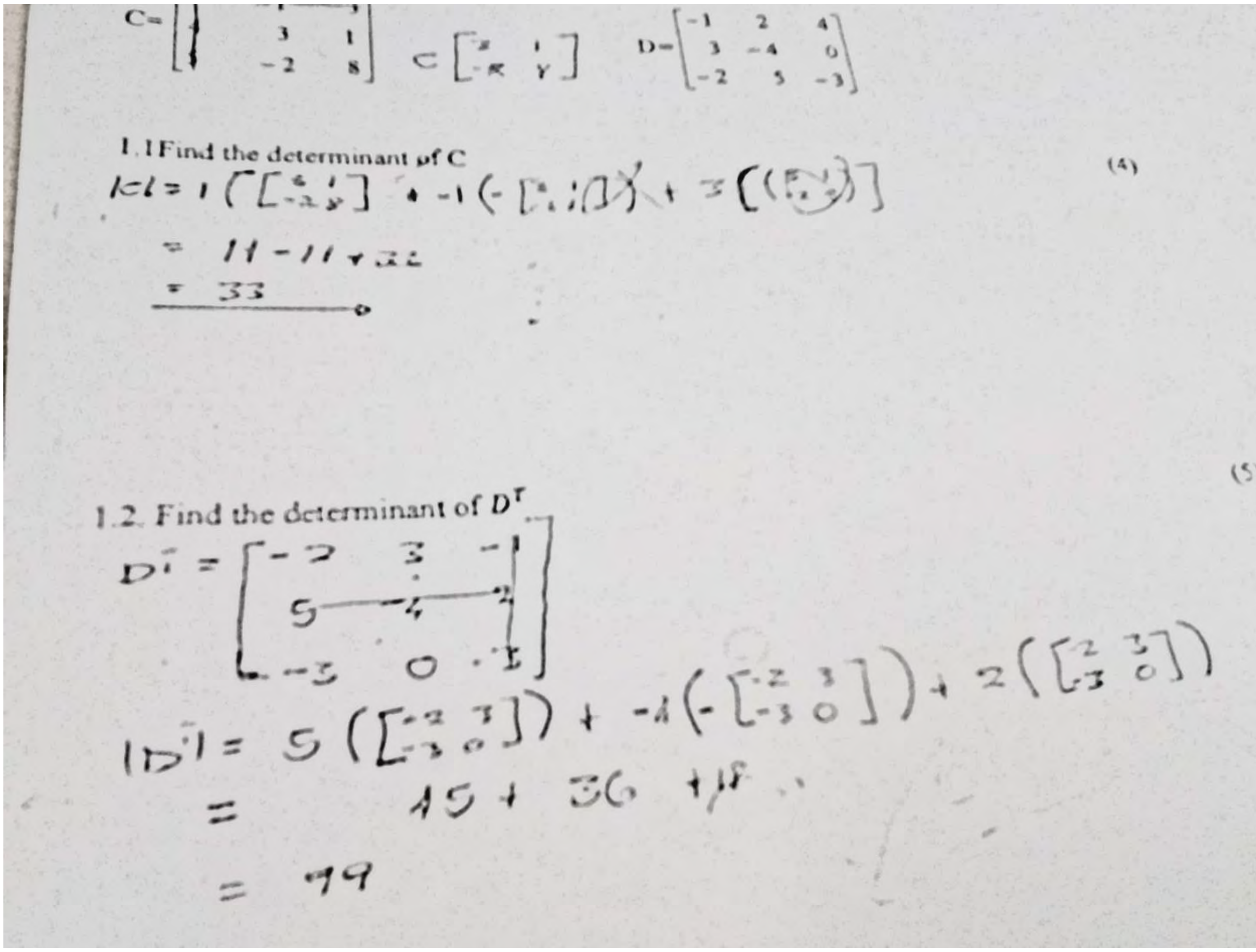
Sipho: *By row 1.*

Researcher: If you expanded by another row or column, will your answer be different?

Sipho: *No, because as long you follow your formula correctly, your answer will be correct.*

Sipho’s response confirms that he does not understand the meaning and correct use of notation and the ideas relating to it. In his thinking, it is just a letter with no meaning. This conclusion is drawn from his response in line three. Also, when asked to support his statement, he struggled to explain and then opted to talk about the formula. This revealed that he was not only experiencing difficulty with manipulation of numbers only, he was also experiencing difficulty with understanding the concept. His response showed that even though he could retrieve the formula from his cognitive structures, it was not fully constructed. Most of the things that he remembered and talked about seem to be retrieved from memorised knowledge, rather than conceptually constructed meaning. This indicated that he is operating at the intra-stage in terms of the Triad mechanism. By continuing to try to refer to formula when answering questions, which are also not properly constructed, confirms the assumption made earlier that the action conception is not fully developed.

Nineteen percent of the students did attempt to evaluate the determinants; however, their responses contain too many errors as a result they could not provide the correct response (see Extract 4). Sydney's response to activity sheet has the following shortcomings: (1) ignoring the cofactors when substituting in the first line in item two; (2) incorrectly determining the sub-matrix in step 1 in both item 1 and item 2. This revealed a complete misunderstanding of the methods used to evaluate 3 x 3 determinants; (3) incorrect manipulation of numbers in finding the determinants of sub-matrix in line two in both items. It was clear from his response that the mental construction necessary to understand the concept has not developed. Examining his response more carefully in item 1b, one realises that he deleted row 2 column 3 throughout, and ignored cofactors when substituting.



$$D = \begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$$

1.1 Find the determinant of C (4)

$= 11 - 11 + 22$

$= 33$

1.2. Find the determinant of D^T (5)

$= 45 + 36 + 18$

$= 99$

Extract 4: Sydney's response to Item 1.

The interview response included the following exchange.

Researcher: When solving these two problems which row did you expand with?

Sydney: *In matrix C, I used column one and in matrix D transpose I used row 2.*

Researcher: If you expanded by any row or column would you get different answer?

Sydney: *It would remain the same.*

Researcher: Why?

Sydney: *I do not know - you said that in a class.*

His response thus confirms that the necessary mental constructions, as discussed in the preliminary genetic decomposition, have not been developed. Even when probed further to clarify his understanding and knowledge of how expanding by any row or column does not change the value of determinant, he could not internalise this. For this item, he could not provide an accurate explanation, but instead, could only recall what he was taught. This revealed that even the action conception of the determinant has not fully developed and he also lacked what we call 'mathematical proficiency', as he failed to manipulate numbers and to identify his mistakes during the interview by himself. He seemed to realise what he did wrong after it has been explained to him by the researcher. This meant that the intra-stage started to develop in his cognitive structures.

Item 2

Item was designed to explore students' understanding of the determinant of a matrix and determinant of its transpose. It aimed to provide insight about students' conceptual knowledge in relation to matrix and transpose, and to address the process conception of determinants, as indicated in the preliminary genetic decomposition.

Without doing calculation what is the determinant of D? Explain your reasoning

In Table 7.2 below the allocation of scores for item 2 is displayed.

Score	1	2	3	4	5
Indicator	No response or incorrect response with no explanation	incorrect response with correct explanation	Correct response with no explanation	Correct response with incorrect explanation	Correct response with clear explanation
Number of students in each category	10	0	1	13	7

Table 7.2: Allocation of scores for item 2

Twenty three percent of students in this item provided a complete response. Fifty percent of these students scored 3 in Item 1, but their responses indicated they knew the relationship between the determinant of the matrix and determinant of its transpose, even though they committed computational errors in Item 1. In terms of the Triad mechanism, it seemed that they were operating at the inter-stage, as it appeared that they were able to recognise and describe the relationship between concepts. This indicated the development of a pre-schema of determinants in relation to solving related problems. Their explanation showed that these students could switch from actions, to doing mathematics, to concepts to think about meaning, that the process conception has developed. Their responses revealed that in terms of the preliminary genetic decomposition, they made most of the necessary mental construction of the concepts. Zama was unable to solve Item 1b, however, she was able to explain that the determinant of the transpose equals the determinant of its matrix.

Researcher: Your solution in Item 1b is for $|D|$ not $|D^T|$ so how did you conclude that $|D^T|$is 77

Zama: *From the knowledge that I already have, I learnt from the book was* $|D|$*would be the same as*$|D^T|$.

Researcher: Can you explain in why is that true?

Zama: Silent.

Researcher: What is a relationship between a matrix and its transpose?

Zama: Silent.

Researcher: If you are given a matrix and asked to find its transpose, how you would determine the transpose?

Zama: [...] *I think you interchange the numbers?*

Researcher: What do you mean by numbers?

Zama: *Eish…*[sic]

The above extract from the interview indicated that although Zama had the correct answer, she was experiencing difficulties in explaining the relationship between the matrix and its transpose. Furthermore, she also indicated that the knowledge she has on the relationship of determinants of matrix and its transpose came from the book, and that judging by her responses, she has not conceptualised that knowledge. This is based on her inability to explain the relationship between a matrix and its transpose. To further explore her knowledge construction the interview continued as follows.

Researcher: Can you determine the transpose of matrix D?

Zama: *She attempted it but instead tried to find the determinant.*

Researcher: Determine the transpose not the determinant of matrix D.

Zama: *Eh!* [sic]*… the transpose, I do not know how to find it in 3* x *3.*

Researcher: How did you determine it in Item 2 of Task 1?

Zama: *I interchange, no, I take the row and interchange it with columns*.

Researcher: Do you think it's done differently in 3 x 3?

Zama: *I am not sure isn't there some formula to use I think.*

Researcher: So in matrix C, you expanded by a column and that column is a row in a transpose of a matrix C, if you solve it the same way you did in Task 1, Item 2. Do you agree?

Zama: *Yes.*

Researcher: So what does that mean by the determinant of a matrix and the determinant of its transpose?

Zama: [...]

Researcher: ...what is the determinant of matrix C?

Zama: *-16.*

Researcher: what would be the determinant of its transpose?

Zama: *- 16, because they should be the same.*

Researcher: Good, that is true, the question is what makes that the case?

Zama: [...] *I just know that, I do not know how I can prove it to be true.*

The above extracts revealed that Zama has not fully constructed the process conception of determinants. She experienced difficulties when explaining the relationship between a matrix and its transpose, because it seemed that these concepts are learnt as isolated facts. In her mental structures, she has constructed the list of unrelated topics. This we observe as she struggled to determine the transpose of D, because it's a 3 x 3. It seemed as if she was confusing it with determinants, because she produced a correct answer in Task 1, Item 2, and also she seemed to think there is a different formula to use to find the transpose of 3 x 3. Her response also indicated gaps in the knowledge constructed when it comes to evaluating determinants. The fact that she is confusing the two concepts, namely the transpose and determinants, meant that she did not understand that determinants were only determined in square matrices. Although it seemed that she had developed the action conception of concepts, she has not interiorised the action into a process. This we observe as she evaluated the determinant of matrix C, and she knows that the determinant of a matrix equals the determinant of its transpose. However, she could not make the connection between the concepts as she struggles to explain the relationship between concepts. This is also verified by her interview response, as she failed to interiorise the action into a process, since, in determining the determinant, we expand by any row or column, and therefore, the column that she expanded within the matrix will be a row in its transpose, and therefore, yields the same answer. This shows that in terms of APOS, the process conception has not developed. Although she could articulate the rules, she hasn't fully constructed the necessary mental constructions as expected by the genetic decomposition.

Forty-two percent of the students were able to relate their answer for the determinant of matrix D to the determinant of its transpose, even if their answer was wrong, they knew that the two determinants are equal. Although students knew that they are equal, they could not explain the relationship properly. Jack, in his response, wrote *determinant is -44,* which is the answer he wrote in item 1b and his explanation was to say that: *because the same numbers remain in the matrix and only the order has changed.*

The lack of the use of correct language made his response incomprehensible. It seemed that he wanted to articulate that in the transpose, the entries are still the same as in the original matrix, where only the order changed, where a change in order does not affect the determinant. Based on his response, it seemed that he understood the relationship between the determinant of a matrix and the determinant of its transpose. Although he did not provide the correct answer in item 1b because he failed to manipulate numbers, it could be argued that the action conception had developed since he evaluated the determinant of matrix C and provided a partially correct answer in item 1b. Then, based on his response in item 2, it seemed that for him the process conception is still developing. For it to fully develop, he would have needed to conceptualise the importance of correct terminology in the learning of the concept.

Sipho wrote *'the determinant of D is 64*$|A|D = 64$ *because* $|A|D^T = 64$'. His response was really confusing especially the notation used. However based on his answer in item 1b, he incorrectly evaluated $|D^T|$ = *64.* Therefore we assume here he was indicating that the determinant of D equals the determinant of its transpose which is true. This we base on the fact that if you remove$|A|$, the statement read as determinant of D is 64 because D^T=64. However, mathematically this is wrong since $|D| \neq D^T$ but$|D| = |D^T|$. It is possible that it was just a slip that he wrote $|D| = D^T$instead of $|D| = |D^T|$ which means he might know the relationship between the determinant of a matrix and the determinant of its transpose. However, considering his solutions even in the previous items it seems that Sipho has not built necessary cognitive structures around notations to support its meaning. His use of notations seem to be too general, taking for granted the meaning attached to the notation used as a result could not make the necessary distinctions. In matrix algebra it is productive for students' understanding of the learnt concept to pay attention to differences in notation since they are significant. Based on his response it seems that the relationship between determinant of a matrix and the determinant of its transpose is instrumentally understood, meaning that he has not developed the process conception of the concept in terms of APOS

To gain more clarity about his response, he said the following in his interview.
Researcher: Why do you say the determinant of D is 64?
Sipho: *Because* [the] *determinant of a transpose is equal to that of its ordinary matrix.*
Researcher: Why do you think the determinant of a matrix equal the determinant of its transpose?
Sipho: *I think it's because when you find the determinant of a transpose, let us say, you have 26; now want to find the determinant of a matrix; like here, they are square matrices, so their determinant will be the same.*

His response in an interview reveales that as it was observed in Item 1, his action conception of evaluating matrices has not developed, therefore he could not interiorise the action into a process. He is still trying to make sense of the procedures used when evaluating matrices. As Stewart (2008) pointed out, the action is an important aspect in developing a basic understanding of mathematical topics. As has been observed here with Sipho, the lack of basic understanding of evaluating the determinant of matrix hinders his progress in understanding other related concepts.

Thirty-five percent of the students either did not attempt the question, or provided an incorrect response. Thabo did not attempt Item 1b and Item 2, and during an interview for Item1 he indicated that when they were required to get into groups, he was still solving 1a. The extract from an interview below revealed that although he was categorised as having not made any mental constructions of the concept, his responses during the interview showed that he has constructed the necessary mental construction.

Researcher: You did not answer Item 2, as you said you ran out of time. In your understanding what would have been the correct solution here?

Thabo: *To answer that, I will need to work it out, but I know if I did solve Item 1b, the answer here would have been the same, because the determinant of a matrix equals the determinant of its transpose.*

Researcher: What causes that to be true?

Thabo: *Not quite sure, but if we find the determinant, we expand by a row or a column and if we find a transpose we exchange the rows with the column. That means a column in a matrix is a row in the transpose. As you see here* [pointing at his solution of the determinant of matrix C] *I expanded by row 3, which will be column 3 in the transpose; so it will give the same answer there if I expand by it, because when we find determinants, we can expand by any row or column.*

His response clearly indicated that Thabo has done this at an action level repeatedly, and now has interiorised the action in to a process. He could mentally think about how the concept link. Over and above being able to carry out procedures, he is able to coordinate the process of evaluating determinants and determining transpose to form new processes, where he could switch from procedures to do mathematics to process to think about mathematics. He has cognitively constructed the collection of facts about the concept and could recognise the relationship between the necessary concepts. This meant that he is operating at the inter-stage in terms of the Triad mechanism. His response also revealed that he has constructed the necessary mental constructions as expected in the genetic decomposition.

Zinhle's written response was also placed under Category 1, because she provided an incorrect response to Item 2. She wrote that *where there is + there will be – in D.* Looking at her response, it seemed that the word transpose to her referred to changing the sign, because her determinant of the transpose was one hundred. This seems like this word triggered some misconception she may have been harbouring. As Aygur & Ozdag (2012) pointed out in their own study, students tended to confuse properties of matrices with that of determinants, and it seems that Zinhle did the same. Also, in school algebra when solving equations, students are told to transpose the number over the equal sign, and to change the sign. This might have created the misconception where the term transpose becomes associated with changing the sign. In matrices, she learnt that the transpose means interchanging the column, therefore she cognitively constructed the meaning that the word transpose is associated with, changing something, and in this case, linking it with changing the sign as she was taught to do at school. According to Aygor & Ozdag (2012), “these can be general features of students’ mathematical learning such as students’ tendency to over-generalise a correct conception” (p. 2899). When students’ over-generalise a correct conception without having conceptually understood it, they tend to formulate alternative frameworks of ideas, which are not appropriate (Zaslavsky & Shir, 2005). So, it seemed that Zinhle’s prior and current understanding of the word transpose impacted negatively on the construction of appropriate knowledge of the concept. This is what Tall (2008) referred to as met before and met after, which he says if not properly understood, may cause barriers to future learning. He further points out that the met-before and met-after are detrimental when one talks about psychology of mathematics, and the teaching of mathematics only revealed the positive effect of met-before, while the negative effects are also very damaging. During the interview with Zinhle the following conversation took place.

Researcher: Can you clarify your response here, what do you mean by saying where there is plus there will be minus?

Zinhle: *Actually, I did not have a full understanding of how the determinant of D would differ from the transpose. What I meant was that the cofactors, where the cofactor is positive D, will be negative.*

Researcher: How do you determine the transpose of a matrix?

Zinhle: *The transpose is a* [...] *is the opposite of [...] just exchange the rows with columns*.

Researcher: When evaluating determinant of matrix C, you expanded by column 1. If you were asked to find the transpose of matrix C, what was column 1 going to be in the transpose?

Zinhle: *Column 1 will be row 1.*

Researcher: If then you are were asked to find the determinant of the transpose of matrix C, and you expanded by row 1, what was going to be your determinant of the transpose?

Zinhle: *I do not know.*

Researcher: But you are expanding by row 1 in the transpose, which was column 1 in the matrix, so that means you are using the same entries. So what does that tell you about the determinant of the two?

Zinhle: *Oh man!* [sic] *It will be the same.*

After being probed several times she seems to understand that the two determinants are the same, however, her responses does not reveal whether or not she understood clearly what a transpose is. Her conception of the word transpose as meaning opposite seems to be causing difficulties, and it seems that this is where her misconception stems from. In terms of APOS theory, it seemed that the necessary mental constructions as indicated in the preliminary genetic decomposition have not been developed.

7.4 Conclusion

The analysis of students' response to task two revealed that the lack of basic skills of manipulating numbers seemed to be the cause of students' difficulties in producing correct answers, as well the students' tendency of memorising concepts, instead of understanding the meaning of the concepts. Most of the students' responses in Task 2 could be explained or described in terms of the preliminary genetic decomposition. In the next chapter, the analysis of students' responses to Items 3 and 4 is presented and discussed in terms of the preliminary genetic decomposition and the triad mechanism.

CHAPTER EIGHT

ANALYSIS OF STUDENTS' RESPONSES TO ACTIVITY SHEET AND INTERVIEWS FOR TASKS 3 AND 4

8.1 Introduction

In Chapter Seven, students' written responses to Task 2 and the interviews were analysed and discussed. In this chapter, transcription of the students' interviews for Tasks 3 and 4 and their written responses are presented. To examine the mental constructions made, as expected in the preliminary genetic decomposition in Figure 3.3, it was important to analyse the students' responses to activity sheets and to verify their responses, or to gain more clarity on how those mental constructions were made through the interviews. As Klasa (2010) suggested, lecturers need to proceed to a genetic decomposition of every mathematical concept in linear algebra before conceiving any pedagogical strategy aimed at bringing students from action to process, or to a more abstract level of object. During the interviews the participants were asked various questions and all the questions aimed at exploring certain mental constructions that students needed to make in order to develop a conceptual understanding of concepts related to matrix algebra.

In this chapter, the researcher explores whether students were able to interiorise the actions of computing matrices and evaluating determinants to a process stage, where they could think about and show an understanding of the relationship between concepts, as well as encapsulating processes of solving the system of equations to an object level. As in the previous chapters, the preliminary genetic decomposition presented in Figure 3.3 serves as an analytic tool by means of which to analyse the mental constructions made or not made by students. The APOS theory, in conjunction with Triad mechanism, describes the level at which an individual is operating.

8.2 Analysis of students' responses from the activity sheets and interviews

Task 3 consisted of two items and Task 4 consisted of three items (see Appendix A2). For coherence, reading, and understanding items in Task 3 and in Task 4 will be presented as Item 1 to Item 5. In the next section, the analysis of Item 1 of Task 3 is presented. Item 1 aimed at exploring and describing the nature of pre-service teachers' knowledge of the relationship of matrix product and square matrices. It addressed the first aspect of the process level in the preliminary genetic decomposition in Figure 3.3, and was intended to provide insight into whether the students had constructed the conceptual understanding of the matrix product.

Item 1

Suppose A and B are matrices with AB and BA defined. Explain whether AB and BA are square matrices.

In Table 8.1 below the allocation of scores for item 1 to task 3 is displayed

Score	1	2	3	4	5
Indicator	Wrong response with no explanation or did not attempt the problem	Yes, with no explanation	Stated yes with totally incorrect explanation	Yes, with partially correct	Yes, with correct explanation
Number of students in each category	20	0	8	3	0

Table 8.1: Allocation of scores for item 1

Ndlovu (2012) has pointed out that in many instances, students applied algorithms without actually conceptualising the concepts involved. The same was echoed by Siyepu (2013) in his study, where students applied the derivative rules without understanding the meaning of the notation used. To encourage students to think about what they write, Maharaj (2014) has suggested that the teaching should focus on the actions and processes that are necessary to interpret the structure of the relevant objects. Therefore, problems such as these presented in Tasks 3 and 4 force students to think about the procedures they apply and try and establish the relationship between concepts. Instead of simply determining the product, it was important to see whether students had interiorised the action into a process and could use the constraints of matrix multiplication to understand the concept of matrix multiplication in relation to square and non-square matrices. This item proved to be difficult for the students, since no-one provided a correct response. Sixty-five percent of the students totally missed the point. It was clear from their responses that they did not understand what the question required. Thirty-five percent of the 65% did not even attempt to answer the question. Those who attempted, gave responses that were totally wrong, ranging from stating the properties of matrix multiplication, like saying matrix multiplication is not commutative, doing some computation, such as finding the products of arbitrary matrices, or finding determinants and providing responses one could not unpack. The students whose response stated that matrix multiplication is not commutative, seemed to have conflated two terms “defined” and “commutative”. To them, 'not commutative' means it is not defined. The students might have solved problems where the product of matrix AB is defined and not of matrix BA, then concluded that since matrix multiplication is not commutative, this would mean it would it is not defined. Therefore, the statement declaring both matrix AB and matrix BA as defined, contradicted their constructed knowledge about matrix product being defined. In their response, they were trying to show that it was not possible, since matrix multiplication is not commutative. This indicated they had not constructed the meaning of terminology used when learning the concepts. This then led to students failing to interpret the nature of the problem.

The issue of using incorrect language in linear algebra has been discussed by (Findell, 2006; Stewart, 2009 and Parker, 2010). The students whose responses could not be understood was because they did not use appropriate verbal or mathematical syntax when referring to concepts or could not comprehend the language used implying that they had not constructed the necessary mental constructions to understand the concept. According to Findell (2006) students’ linguistic misbehaviours are interpretable as reflective of deficient understanding or of deficient expressive powers. Also, Parker (2010) has pointed out that students who had stronger language skills generally exhibit better understanding. This means that if students do not understand the concept, they will struggle to explain and express themselves mathematically. They may not be able to comprehend the problem and translate the word statement into symbols or make relationship between the words and symbols so as to construct proper understanding of the problem before attempting to solve it. However, those who have the ability to comprehend the language of mathematics are likely to display a better understanding of the concept learnt.

The pre-service teacher, John, also provided an incorrect response, indicating that he failed to interpret the nature of the problem. He did not unpack the whole statement before answering the question, but instead, he based his response on the last part of the question (see Extract 1). The evidence of this is included in his response, as he used two different orders to prove that matrix AB and matrix BA were non-square matrices. Again, his response indicated the knowledge constructed of the constraints of matrix multiplication to be insufficient. The evidence of this is seen in his response, as he struggled to show the relationship between the matrices and their products. Examining Johns' response in line 1, he created matrix A and matrix B, whose product was defined. Matrix A was a non-square matrix, while matrix B was a square matrix. It seemed he wanted to show that matrix AB was not a square matrix, but that it was defined. In the second line, he chose different order of matrices, where both were non-square matrices, and mentioned that they were not square, but were defined. In this way he was trying to show that matrix BA was not a square matrix. However, this was incorrect, since the product of matrix BA was not defined, indicating that he committed a structural error. Orton as (cited in Maharaj 2014), defines structural errors as those arising from some failure to appreciate the relationship involved in the problem or grasping a principle essential to solution. John wanted to support his statement, as he said no, and selected examples that would prove his statement to be true. Since he was more focused on proving his statement than on understanding the question, he did not recognise that the order of the matrix he constructed would not be defined, and therefore that matrix BA is not defined, (see the last line of his response in Extract 1). John had solved some problems related to matrix multiplication. However, his response revealed that he has not constructed the full understanding of the concept, indicating that he is still operating at the action stage.

No - because AandB is A3x2 B2x2 is not a square but you can
defined. If B is 4x3 and A is 2x4 are not a square but
is defined

Extract 1: John's written response to Item 1

To gain clarity on his response, the following discussion took place as part of his interview.

Researcher: What do you mean when we say matrix AB is 'defined'?

John: Sisho ukuthi singakwazi to [We mean that it is possible to] *multiply matrix A and matrix B and get the answer?*

Researcher: How can you tell if you can multiply matrix A and matrix B?

John: *Uma* [if] *columns of matrix A elingana nama* [is equal to the] *rows of matrix B.*

Researcher: Looking at your order of matrix B, and order of matrix A that you created, is matrix BA defined?

John: *Uyabo le eyesibili ayivumi. Nje yonke lento ayilandeleki, uyabo la bekumele ngisebenzise the same order to prove ukuthi matrix AB and matrix BA abasiwo amasquares* [the second one is not, this thing does not make sense. I was supposed to use the same order to show that AB and BA are not squares. I was trying to support my argument]. *Ngisho nawo umbuzo ngangingawuzwa nje kahle* [Actually, I did not understand the question].

During the interview John identified his mistakes. As was indicated earlier, he misinterpreted the question and mentioned that he was trying to prove that matrix AB and matrix BA were not square matrices. He also admitted that he did not really understand the question. After the researcher helped to unpack the question for him, John mentioned that if matrix A and matrix B were square matrices, then matrix AB and matrix BA would be square matrices. However, even when probed, he was unable to explain other cases, except square matrices, where both matrix AB and matrix BA would be defined. The interview responses revealed that John hadn't made the necessary mental constructions. His responses revealed that the collection he had made of the constraints of matrix multiplication was not related to other concepts. The collection made was conceived as a separate rule. This we observe, as he could state the algorithm for matrix multiplication, but seemed to be having difficulties in using it to explain cases where matrix product is defined in both ways. Based on his responses, it seemed that he lacked the metacognition and cognitive skills required, as he could not discern exactly what the problem required, and so failed to generate appropriate strategies to solve the problem. This meant that he had not interiorised the action into a process. He could perform some mathematics, but could not mentally interiorise it into a processes to think about.

A different case ensued with Thabo in Category 2. He did not answer the question, but instead constructed arbitrary square matrices of 2 x 2 and found the product of matrix AB and matrix BA. It seemed that he was proving that matrix AB and BA were square matrices. Like John, it seemed that he failed to unpack the whole question, and could not think about different scenarios where matrix AB and matrix BA would be defined. Although Thabo's response was not totally correct, it could be argued that while working at the action level, manipulating numbers to find the products of matrices, he might have recognised that in square matrices, both the product of matrix AB and matrix BA is defined, and that the resulting matrix is also a square matrix. This indicated that he was operating at the process stage as he could identify the relationship between the concepts. The evidence of this is based on his response, as he showed that in square matrices, both matrix AB and BA are defined, and are square matrices. To gain more clarity on the mental constructions made, the interview revealed the following.

Researcher: Why did you construct square matrices?

Thabo: *Because I wanted to show that in square matrices, the product would be defined, even in the reverse order, and that the resulting products would also be a square.*

Researcher: Is it that the question I was asking?

Thabo: *No, I was supposed to answer the question first. My answer was yes, then I couldn't explain* [further] *in words, so I wanted to show why I said yes.*

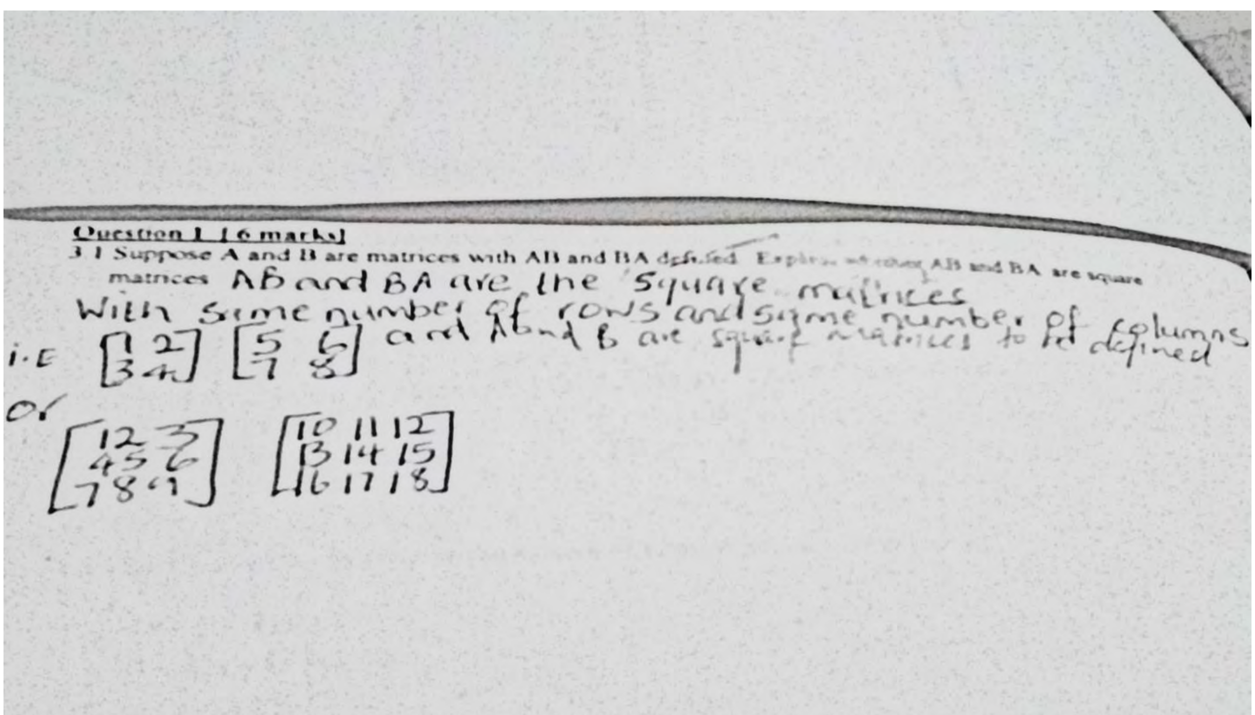
Researcher: Is it only in square matrices where both products will be defined?

Thabo: *Not really, squares is like 3 x 3 or 4 x 4 etc., but you can also have a situation where C is a 2 x 3 and D a 3 x 2 both it will be defined and the product is a square matrix. So, in all square matrices, and in some cases where the matrices can be written in the reverse order the product of AB is defined as well as matrix BA.*

Although in his response to an activity sheet, it was not clear if he had made the necessary mental constructions, Thabo's interview responses revealed that he had made the necessary mental constructions for square matrices as expected in the preliminary genetic decomposition. In his response to an activity sheet, he showed one scenario to prove that matrix AB and matrix BA were square matrices. After being probed further, he identified and explained other scenarios, indicating that he had the suitable schema required for matrix multiplication. His thinking strategies could be linked to what Sierpinska (2005) refers to as 'reason for thinking', where she argues that theoretical thinking focuses on developing an understanding of solving all problems of the certain type. This we observe, as Thabo identified and clearly explained all the scenarios relating to matrix product being defined. His interview responses further revealed that he had constructed a coherent understanding of the matrix product rules, and that the understanding of matrix product had been encapsulated into an object. Thabo had not just developed the understanding needed to calculate the matrix product, but he could de-encapsulate the product to its original matrices. Although he could not exactly determine the matrix entries of the original matrices, he could determine the order of the original matrices. This we observe as he pointed out certain scenarios in square and non-square matrices, which will yield a product which is a square matrix. This meant that when given a square product, he could figure out the order of the original matrices. This indicated that the schema of matrix product had moved to a trans-stage of the Triad mechanism. The trans-stage is noted as a schema development (Jojo, 2011). This we observe in his responses, as he displayed coherence in the understanding of a collection matrix product rules, and was able to reflect on the mental constructions already made to develop his schema of matrix product in relation to square and non-square matrices.

Twenty-six percent stated yes, meaning that matrix AB and matrix BA are square matrices, however, their explanations were completely wrong, where one of the explanations did not even relate to the problem, indicating that they must have just guessed correctly by saying yes. Their explanation revealed some misconceptions, such as confusing concepts, and applying the constraints of multiplication of matrices even when not applicable. Siyepu (2013) indicated that students tend to over-generalise rules and the author refers to such errors as linear extrapolation. Thula said: '*yes, because they are transpose of each other*.' In the previous tasks students solved problems related to transpose and Thula provided correct responses in Item 2 Task 1 and determine the transpose of matrix D in Item 2, Task 2. His response in this task indicated that he had not fully understood the concept. He could manipulate the numbers, but has not constructed the meaning of the concept. Failure to conceptualise the concept caused him to fail to recognise the context where the concept is applicable. As pointed out in the analysis of previous items, students do not interrogate the meaning of their solution. When they encounter difficulties, they just write for the sake of giving an answer, without actually thinking about the meaning of their solution to the solved problem. The same appeared in the responses of the other students in the same category.

They said 'yes', noting their reason to be *because the rows and columns are the same*. They recognised that the problem had to do with matrix product, and without comprehending the problem, they stated whatever they could remember. It is clear that their responses were supporting square matrices, or that the product of the matrices were defined. The researcher thinks that such misconceptions were necessary to incorporate in the revised genetic decomposition, because students' failure to interrogate the meaning of what they wrote prevents them from constructing the necessary mental constructions to conceptualise the concept. Nine percent of the students stated 'yes', and their responses revealed that they had made certain mental constructions but had not fully understood the concept. Their responses revealed that even though they did not provide a complete and correct response, they nonetheless attempted to comprehend the whole question. This, we conclude, as they attempted to visualise the order of matrix A and matrix B, and based their responses on what they had visualised matrix A and matrix B to be. This meant that they did not pick some cues in the question that made sense to them, however, that they tried to show the understanding of the whole question, as well as to link the product of the matrices to its original matrices, matrix A and matrix B. To try and clarify her response, Zinhle further constructed arbitrary square matrices of 2 x 2 and 3 x 3 (see Extract 2). Her response had the following shortcomings.



Extract 2: Zinhle's written response to Item 1

Zinhle said 'yes', but then constructed two matrices of order 2 and two matrices of order 3, without showing how they relate to the problem to be solved, and there is no link in her explanation, i.e, it is not clear how the first sentence linked with the last sentence. Examining her responses, it seemed that she wanted to show that for product of matrix AB and matrix BA to be defined, matrix A and matrix B ought to be square matrices. This assumption we base on her last sentence, and the matrices shown there are square matrices. It seemed that Zinhle knows some cases where the product of matrices is defined, even in the reverse order. She seemed to have recognised the relationship between square matrices and the matrix product, and without computing the matrices she could mentally construct the order of the product of the square matrices. Even though this was not explicitly expressed in her response, by constructing the different square matrices, she might have wanted to show that when multiplying square matrices, the product is a square matrix, and similarly for the other case. Her response revealed that she had not fully constructed the necessary mental constructions as expected in the genetic decomposition, since she struggled to clearly articulate the relationship between a matrices and its product. Also, she only looked at one case of matrices, namely the square matrices where both products are defined.

Researcher: Here you said yes and you provided an explanation. Can you clarify your explanation? Which rows and columns are you referring to when you said AB and BA have the same number of rows and column?

Zinhle: *Since I said yes, I was trying to explain what I mean by saying* [that] *they are square matrices.*

Researcher: Okay [sic], here you have two matrices, one is 2 x 2 and the other is a 3 x 3. Which one represents AB and which one represents BA?

Zinhle: *None, all what I was showing here was that is how square matrices look, they have the same number of rows and the same number of columns.*

Researcher: Let’s get back to the question. Here you have two matrices: A and B, and their product is defined when you multiply A and when it is B$\times$ A. If AB and BA are defined, what is the order of these of these two matrices A and B?

Zinhle: *I think they are the same. Ya* [yes]*, the order should be the same.*

Researcher: Okay, let’s say matrix A is 2 x 1 and matrix B is 2 x 1. Are the orders the same?

Zinhle: *Yes.*

Researcher: Is the product of AB defined?

Zinhle: *Ha!* [sic] *It's not defined; that means matrix A and B should be a square, like I* ['ve] *shown it here. Ya* [yes] that *is what I was trying to show here, that if A is a square matrix and B is a square matrix, their product is also a square, and if we go back, the same will happen.*
Researcher: Is it only in square matrices where both products would be defined?
Zinhle: *Uyazi ke, angikaze ngiyicabange enye* [you know I had never thought of any other]. *I think there could be just that we need to have rows, I mean columns equals to the rows. If we have something like this 2* x *3 and 3* x *2, it could be defined, because if we look at the columns here, it is equal to the rows there, and if you take it to the other side, you see its 2 and 2.*
Researcher: You said earlier when multiplying two square matrices, that the product is a square matrix- what would be the order of your products here?
Zinhle: *AB will be a 2* x *2 and BA, let's see, yes,* [sic] *it will be 3* x *3. It will be a square matrix.*
Researcher: What can you conclude based on your observation?
Zinhle: [I am] not *sure how to put it, but when multiplying square matrices, the product will be a square, and you get the answer in both ways, and also if the matrices you are multiplying are the opposite of each other, the same will happen.*

Her response to the first two questions confirmed that she did not comprehend the question, as she said she was defining what square matrices were. Unlike the other students in the category one and two, after being probed, she began to understand what the question required, and she recognised other cases where both the product of matrix AB and matrix BA would be defined. She also internalised her thoughts. This we observe as she started to think about the rules of multiplication of matrices and use them to develop better understanding of matrix product. Although her response to activity sheet indicated that she was still operating at the action stage, during the interview she started to interiorise the action into a process for square matrices. This we observe as after she had discovered all the cases and made connections between them, she was able to draw conclusion indicating that she has made the necessary connections between the concepts, and that she is able to coordinate processes to develop a pre-schema of matrix product.

Item 2 was aimed at exploring pre-service teachers' knowledge of evaluating and understanding of matrix inverse and its relationship to other concepts, such as determinants. It addresses the first aspect and the second aspect of the object level in the genetic decomposition in Figure 3.3. It was intended to provide insight into whether the students had the conceptual understanding of the matrix inverse.

> Does the matrix $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ have an inverse? If so what is the inverse?
>
> If not explain why?

In constructing the necessary mental construction of the matrix inverse, students who just applied the formula to determine the determinant of the matrix saying there is no inverse, without providing any explanations, were considered to be operating at the action stage. Students who evaluated the determinant and tried to do use it to solve other related problems were considered to be at a process stage. When a student is able to perform other actions on the process, this could reflect on operations applied in determining the matrix inverse, where the student becomes aware of the process as a totality in which symbols or notations such as $\frac{1}{A}$ or A^{-1} evoked the understanding of new actions that could be performed, and shows coherent understanding of all the rules involved in determining inverses, and that the student has encapsulated the process into an object. In this item, the process and object conception of this concept will be more substantially explored through the interviews. Students written responses will be interrogated further, in order to explore the mental constructions made when learning this concept.

Table 8.2 displays the allocation of scores for item 2

Score	1	2	3	4	5
Indicator	Yes with no explanation or Left it blank or did some incorrect calculations	Yes, but shows some understanding of application of determinants	No, with no justification of the answer or just did calculations without relating it to the problem	No, with incorrect justification	Complete correct answer
Number of students in each category	3	0	0	0	28

Table 8.2: Allocation of scores for item 2

Ten percent of the students did not attempt the problem, indicating that they had not made the necessary mental construction required to understand this concept, as expected in the preliminary genetic decomposition. This indicated that the action conception of the concept had not developed. This item proved to be manageable for almost all the students. Ninety percent of students gave a correct answer, indicating that they made the necessary mental constructions. To determine their solution, they applied the determinant formula, and used the determinant to explain why the matrix had no inverse. This revealed they recognised the relationship between the matrix inverse and the determinant, and used that relationship to construct new knowledge (see Extract 3).

> NO, BECAUSE THE DETERMINENT OF THE METRIX IS 0 THUS
> THERE IS NO INVERSE FOR THIS MATRIX

Extract 3: Jabu's written response to Item 2

Although Jabu did not show his workings, the explanation given shows that he has made the coherence understanding of the relationship between the determinants and matrix inverse, indicating that he has made the necessary mental constructions. This we base on his response in the first sentence, as he stated the determinant to be zero. Since he had constructed the understanding of the application of determinants to other concepts, he could relate it to the matrix inverse. An interview with him revealed the following.

R: In your explanation you said there is no inverse because the determinant is zero. Why do you think the matrix will have no inverse when the det = 0?

Jabu: *In the book, it says* [that a] *matrix with a determinant of zero does not have an inverse. We know that to find an inverse of a matrix, you multiply the matrix by a reciprocal of a determinant, then if the determinant is zero, then it will be undefined, as we cannot divide by zero, and so this thing (pointing at the matrix) is undefined, so there is no inverse.*

Researcher: If so, what would that mean?

Jabu: *It would mean the matrix has an inverse.*

Researcher: How would you tell if the two matrices are inverse of one another?

Jabu: *Hayi Ma'am, angeke ngikwazi ngokuwabuka nje* [no Ma'am. I can't by just looking]; *but I know that if I multiply the number with its reciprocal, and the answer is one, then that number is the inverse of the other one. Nalana ke* [even here] *if I multiply the two matrices, the answer should be one.*

Researcher: True. What is the correct terminology that we used to refer to that product?

Jabu: Hmm.... *ayi angisakhumbuli but engikwaziyo ina same* [I do not remember; what I know is, it has the same] *order as the matrices you multiplied and all the entries are zero except abo 1 kumadiagonals.*

His responses revealed that Jabu made some necessary mental constructions, as expected by the preliminary genetic decomposition. This we observed as he clearly made a link between matrix inverse and school algebra. In his explanation he displayed a coherent understanding of how the rules linked, as he explained that since division by zero is undefined, this makes the whole matrix undefined, as he will be multiplying by the reciprocal. For him, the notation evoked other concepts, such as the reciprocal, or quotients, where he displayed coherent understanding of how they relate to matrix inverse and he could mentally perform those actions and construct new knowledge that the inverse of such a matrix is undefined. When probed further, Jabu displayed coherent understanding of the relationship of matrix inverse with the inverse of real numbers. He could relate his previous knowledge of finding an inverse of real numbers to finding an inverse of matrices. Arnawa, Sumarno, Kartasasmita & Baskoro (2007) pointed out that in abstract algebra, students might extract common features from many mathematical systems that they have used in the previous mathematics courses such as school algebra. Students began to link a common idea behind the inverse of a number and the inverse of a matrix. This is what Jabu displayed as he deeply engaged with the procedures of determining the matrix inverse, displaying a well-organised collection of mathematical procedures. According to Hasebank (2006), the goal of mathematics instruction is to help students develop a well-organised collection of versatile mathematical procedures that they can call upon to solve problems in a variety of situations. Jabu might had done this at an action level; and through interacting with several problems he was able to construct the necessary mental constructions that helped him to recognise and understand the application of determinants to solve other related problems, indicating that the schema of determinants and matrix inverse was starting to take place. This means that his object conception of the concepts has developed.

Item3

Consider the system of equations below and answer the questions that follow. Can the system of equations be solved using Cramer's rule? If yes solve it, if not explain why. $2x+y-z=3;$ $x+y+z=1$ $x-2y-3z=4$ $3x-y-z=2$

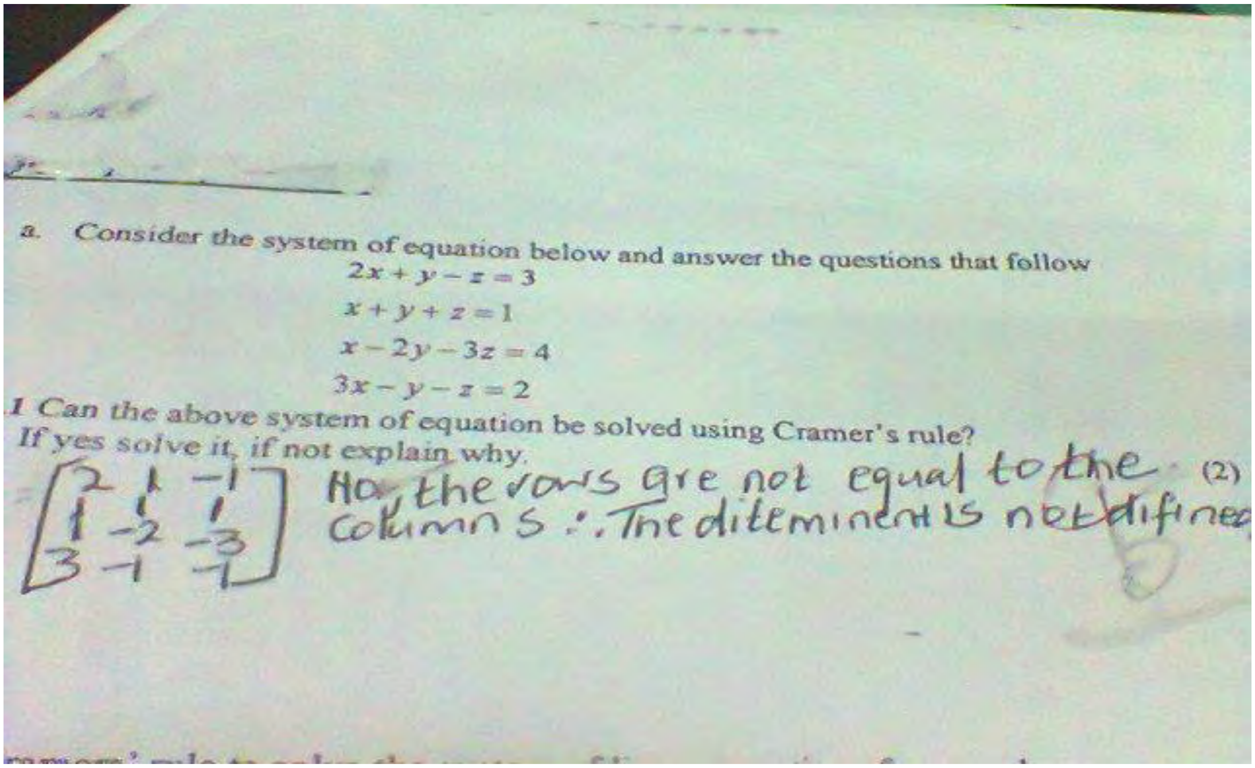
Item 3 was aimed at exploring students' conceptual understanding of constructing a solution of a system of equations and its multi representation. It addressed the action and process level of the concept as expected by the preliminary genetic decomposition in Figure 3.3.

Table 8.3 below allocation of scores of Item 3

Score	1	2	3	4	5
Indicator	Left blank or incorrect response	No, without explaining	No, with incorrect explanation	No, explanation is partially correct	Complete and Correct answer
Number of students in each category	3	2	16	1	9

Table 8.3 Allocation of scores for item 3

In this item, only 29% of the students gave a correct response indicating that they have made the necessary mental constructions. Their responses revealed that these students have interiorised the action into a process, as they could internally make the necessary connections regarding the structure of matrix that could be solved using Cramer's rule. This we observe as they clearly explained that since this system is a non-square system, it could not be solved using Cramer's rule. Also, their explanation revealed that they understood the cases in which the rule could be applied, as well as the cases where it could not be applied, and why. Zinhle also scored 5, because her response was accurate and correct. Moreover, she first showed the relationship between matrices and system of equations (see Extract 4).



Extract 4: *Zinhle's response to Item 3*

She first changed the structure of the system of equation to that of a matrix form, after which she realised that it was a non-square matrix, and therefore, that Cramer's rule could not be applied. Her response revealed that she could represent the system of equations using different structures, and she has made the connections about the relationship between these structures. Also, the constraints related to the application of Cramer's rule. Based on her response we can conclude that Zinhle was operating at the process stage in terms of APOS when it came to solving system of equations using Cramer's rule. These findings contradict what Hong, Thomas & Kwon (2000) said that at the beginning of a course, students do not fully understand the multi-representation of the solution of the system of linear equations. This we observe as she was able to pass from one representation to another. The interview included the following exchange.

Researcher: Why did you change the structure of the system of equation to that of matrices?

Zinhle: *It was not changing* [it]*, it just another way of writing it, because the rows refers to the number of equations and the columns are the variables: x; y ; z.*

Researcher: What did you mean by saying the determinant is not defined?

Zinhle: *When solving the system of equations using Cramer's rule, Cramer's rule has something to do with determining the determinants; so here, the rows are not equal to the columns, so we cannot work the determinant out, because the rows are not equal to the columns. It is not a square matrix.*

The above extract revealed that Zinhle had developed algebraic reasoning, since she displayed the knowledge of relating the structure of a matrix to a solution of a system of equations, as well as making the connections between variables and the columns of a matrix. According to Panasuk (2010) algebraic reasoning is a way of reasoning involving variables, generalisations, different modes of representation, and abstracting from computation. Her responses revealed that Zinhle had performed the actions of using Cramer's rule and had described it verbally. She interiorised the action into a process. This we observe, as she could both point out and explain the possible restrictions of Cramer's rule, without first trying to perform the calculations, and as she explained the relation between number of equations and the number of unknowns. By solving several problems of system of equations using Cramer's rule, she had come to realise that it could only be applied in square matrices. Her response revealed that she had made the collection of all the rules required or necessary, and that without actually performing the procedures, she has constructed an understanding of all the procedures. Hasenbank (2006) suggested that students whose procedural knowledge becomes deeper and more connected are less likely to make mistakes.

Forty-eight percent of the students knew that Cramer's rule could not be applied to solve the given system, however, they failed to provide a meaningful explanation to support their answer. There is a possibility that they simply guessed the answer, indicating that they had not made the necessary connections of the application of Cramer's rule. Their responses revealed that they might have the collection of rules, but had no recognition of the relationship amongst concepts. That meant they are operating at the intra-stage in terms of the Triad mechanism. This we observe, as their responses showed gaps in the knowledge constructed, and they stated that Cramer's rule could only be used in three equations with three unknowns. As it was noted for the previous items, students do not always interrogate what they write, and it seemed to be the case here. This we observe as we examined some of the responses. Thamsaqa stated: *No, because Cramer's rule works in a three linear equations in three variables* and Thula said *No, this is a 4* x *4 matrix, so the system of equation*[s] *cannot be solved using Cramer's rule, it is not a square solution.* Thula's response revealed that the understanding of a definition of a square matrix had not fully developed. Evidence of this can be found in his response, as he said it is "*4* x *4 and it is not a square solution*". This indicated that he lacked the basic concepts of the order of a matrix. The knowledge of such concepts forms the basis of understanding and solving for other concepts in matrix algebra. Thus, a lack of basic knowledge leads to difficulties in developing better understanding of learnt concepts. Thamsaqa's responses also revealed the gap in the knowledge constructed. Like Thula it seemed that he could not recognise the interrelationship between concepts. He constructed knowledge and kept it in different compartments, the nature of which prevented him from developing the link between them. This meant that he had constructed the list of unrelated actions and processes. These responses revealed that students have not made the necessary mental constructions, and that based on this, it seemed that they are still operating at the action stage in terms of APOS.

In an interview with Thamsaqa, the following exchange took place.

Researcher: In your explanation, are you saying Cramer's rule can only be applied in three equations with three unknowns?

Thamsanqa: *Yes, I think so.*

Researcher: So are you saying it cannot be applied when there are two equations with two unknowns?

Thamsanqa: *Ya* [yes], *you can apply it.*

Researcher: If there are four equations, with four unknowns, can you apply it?

Thamsanqa: *Yes ingasebenze* [it can work].

Researcher: So what should be the order of the system where Cramer's rule can be used?

Thamsanqa: *Kumele ibe nama equations, mhmm, ayi kodwa angisakhumbuli kahle madam kodwa ngiyazi iyasebenza la kuna three equations and nama three variables* [it should be, mmh, I do not remember but it does work where there are three equations with three unknowns].

Researcher: When using Cramer's rule to solve the equations, what do we need to calculate first?

Thamsanqa: *Determinants.*

Researcher: What are the order of matrices where we can evaluate the determinants?

Thamsanqa: *Square matrices.*

Researcher: So, what should be the order of the system of equations that we can solve using Cramer's rule?

Thamsanqa: *Square system or matrix.*

Thamsanqa seemed quite confused about which type of system of equation could be solved using Cramer’s rule. When asked if it could work in other square matrices like 2 x 2 or 4 x 4, he said 'yes'. At the same time, he could not draw conclusions about which type of system of equations Cramer’s rule could be applied to. Although he could state the rule, his responses revealed he had no recognition of the relationship between concepts. The evidence of this is that he failed to make connections between the applications of a determinant, and Cramer’s rule, in order to explain the types of system where Cramer’s rule could be applied. Furthermore, he had not made the connection between the structures of a system of equations with a matrix structure. He could not think of the system of equations in terms of matrices. This is observable where he could not provide the answer when asked about the order of the system of equations, but when asked about the order of the matrices, he did provide an answer, indicating that he considered these to be two different concepts. His responses in the interview, even after being probed, confirmed that the process conception has not fully developed. This we observed, as he could not internalise the procedures of computing using Cramer’s rule. According De Vries & Arnon (2007), students who are still operating at the action stage could not think of an action as whole, or predict its outcomes, or sometimes struggle to describe it verbally. Six percent of the students just stated no, and did not explain why they said so. Nine percent did not even attempt the question. The students in category two might have just guessed the answer, and as a result, could not explain, indicating that they hadn’t made the necessary mental constructions as expected by the preliminary genetic decomposition. It is possible that they knew that the system given was not a square matrix, but lacked the language to express their thoughts. This meant that they might know the rule, but lacked the flexibility of reasoning or expression.

Item 4 aimed at exploring students’ mental construction when solving the system of equations to see if they understood the meaning of their solution in relation to the problem. It aimed to address the process conception of the concepts, as well as the first aspect of the object conception of the concepts, as indicated in the preliminary genetic decomposition.

Item 4

Use Cramer’s rule to solve the system of linear equations for x and y $kx+(1-k)y=1$ $(1-k)x+ky=3$

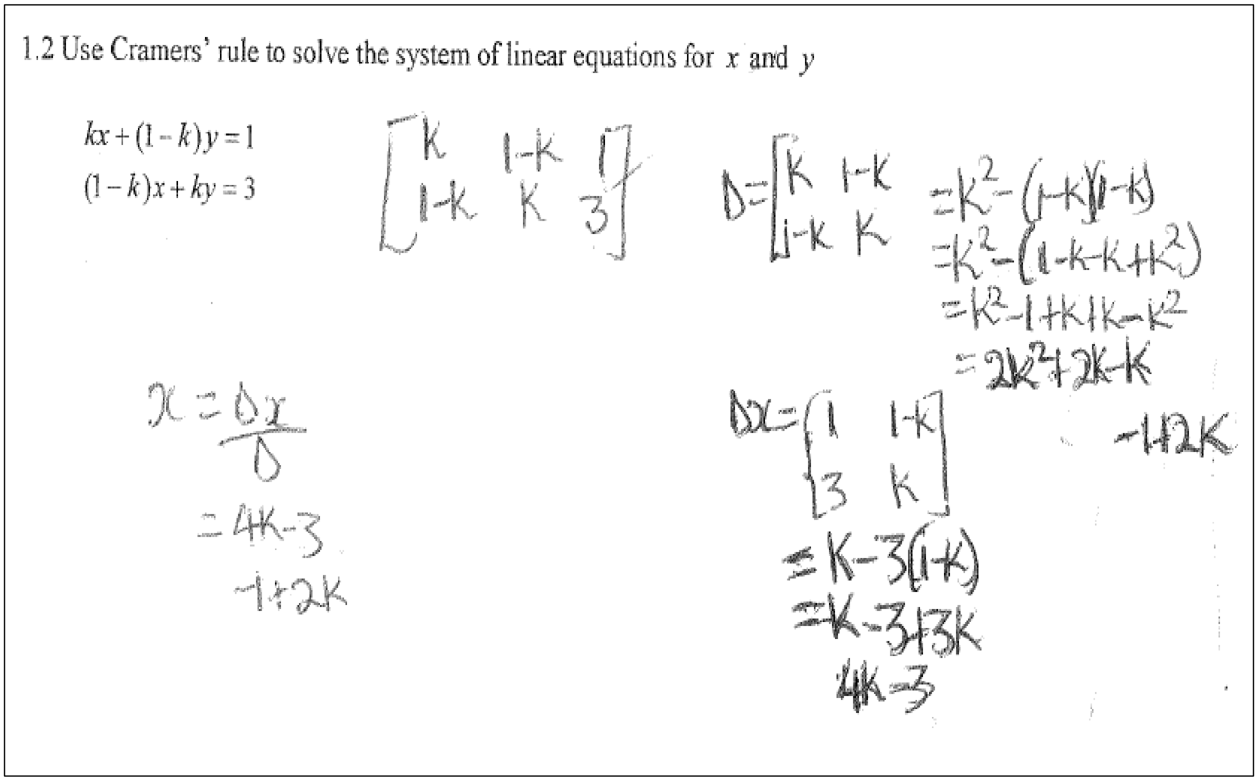
Table 8.4 below shows the allocation of scores for Item 4.

Score	1	2	3	4	5
Indicator	Left blank or incorrect response	Represent the system of equations in matrix form but failed to solve the problem.	Not all determinants correctly determined but shows understanding of the application of Cramer' rule.	Shows understanding of the method used, solutions contains mathematical errors	Complete, correct and accurate answer
Number of students in each category	8	6	5	3	9

Table 8.4: Allocation of scores for item 4

Among the eight students in Category 1, three students did not even attempt to solve the problem. The other five students' responses were totally incorrect. Their solution had the following shortcomings: (1) structural errors; (2) failure to appreciate the relationship between matrices and system of equations; (3) executive errors, where some students were able to write the system of equations as matrices, but failed to evaluate the determinants of the variables; (4) procedural errors, where some students failed to apply Cramer's rule correctly; and (5) incorrect use of parentheses. These five students could not represent the system of equations as matrices. They lacked what Berman, Koichu, Shvartsman (n.d.) have called 'representational understanding' since they could not represent the system of equation as a matrix structure. Cutz (2005) pointed out that some students' experience difficulties when required to pass from one representation to the other. This could be attributed to poor application of mathematical concepts and to a failure to appreciate a multi-representation of the system of equations. When solving mathematical problems, students mostly believed that there is only one way to obtain the answer and as a result, they could not think of presenting their solution in different format using different heuristics. In this case, students' misconceiving of notation also created difficulties for them. Maznichenko (2002) argued that misconceiving a concept leads to inadequate perception of reality. Based on their responses, it seemed that the action conception had not developed, indicating that they had not made the necessary mental constructions.

The solution for students in category two had the following shortcomings: (1) incorrect use of parentheses; (2) procedural errors; and (3) replacing variables with a concrete number (see Extract 5). Similar findings were discovered by Burhanzade & Aygor (2013) in their study about problems that students face when solving equations. They discovered that calculation errors and inability to form connection between conceptual and operational information were the main challenges. The authors argued that these were a result of memorising without understanding. To get more understanding of the students' solutions some responses were further examined. Examining Thula's response, he used any form of brackets without taking into consideration the context of the problem and the meaning of the brackets used. This might indicate some underlying misconceptions in relation to the use of parentheses. In some cases, teachers and lecturers do not take enough time to explain the mathematical meaning of an application of the brackets when using them. They interchange the use of brackets without making students aware of the meaning of the brackets used, with the result that some students end up constructing incorrect concept image of parentheses. Goris & Dyrenfurth (2010) have pointed out that misconceptions originate from prior learning. During lectures when the researcher was teaching matrix algebra, the types of brackets to be used and the meaning associated with each bracket used was clearly explained. Furthermore, during tutorials, this was emphasised. However, Thula did not use the appropriate examples in his solution. Examining his response closely, he set up the system of equation as a matrix. When determining the determinant of the matrix, he used half open brackets (see the first line in extract 5). He then committed procedural errors as he failed to manipulate variables correctly, see the fourth step of his calculation. One is not sure how he ended up with the correct answer written on the right hand side of his working determiningD_x. He correctly solves for x but not for y, which was confusing, since the procedure is the same. Based on his response, it seemed it could be argued that the process conception has not fully developed. This we base on his response, as he could not determine all the determinants and failed to provide an accurate solution for the system.



1.2 Use Cramers' rule to solve the system of linear equations for x and y

$kx+(1-k)y=1$

$(1-k)x+ky=3$

$\begin{bmatrix} k & 1-k & 1 \\ 1-k & k & 3 \end{bmatrix}$

$D=\begin{bmatrix} k & 1-k \\ 1-k & k \end{bmatrix}$
$=k^2-(1-k)(1-k)$
$=k^2-(1-k-k+k^2)$
$=k^2-1+k+k-k^2$
$=2k^2+2k-k$
$-1+2k$

$Dx=\begin{bmatrix} 1 & 1-k \\ 3 & k \end{bmatrix}$
$=k-3(1-k)$
$=k-3+3k$
$4k-3$

$x=\frac{Dx}{D}$
$=\frac{4k-3}{-1+2k}$

Extract 5: Thula's response to Item 4

The interview included the following exchange.

Researcher: Can you clarify how you determine the determinant of the system of equations?

Thula: *I took these co-efficients,* [the] *co-efficient of x and y, and this became my elements in the matrices. Firstly I needed the determinant; then I removed the constant column; then left with co-efficient of x and y; then I worked out the determinant. Oh, I made a mistake here; but this is the correct answer* [pointed at the fifth step just below the fourth one, where he incorrectly added k^2 with k^2 instead of subtracting].

Researcher: Can you explain how you determine

Thula: *I will exchange the co-efficient of x with the constant column and multiply to get the determinant.*

Researcher: Why you did not solve for *y*?

Thula: *Oh! It's time Ma'am, I did not finish.*

Researcher: Okay, if the determinant of the co-efficient was equal to zero, would you be able to find the solution?

Thula: *No.*

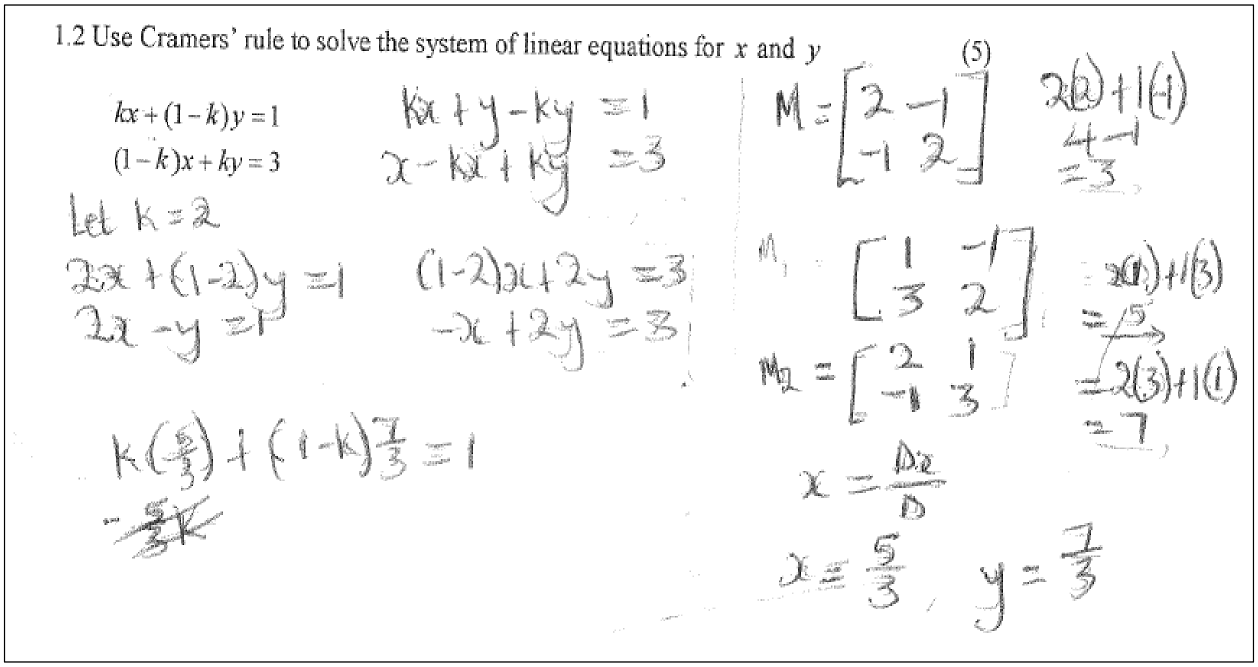
Researcher: Why?

Thula: *Any number divided by zero is undefined, the solutions will be undefined.*

Researcher: How would you be able to tell if the system will have a solution or not?

Thula: The d*eterminant of the co-efficient must not be equal to zero.*

The above extract revealed that Thula knows how to apply Cramer's rule in solving the system of the equation. When asked to explain his solution to the problem, he was able to identify where he made mistakes and to correct his computational errors. This we observe as he identified that his fourth step was incorrect and provided the correct answer. The problem seemed to be to clearly explain step by step the procedures he followed. This was observe as he was explaining how he determined the determinant of x, he only notes that he multiplied, he did not mention subtraction of the product of the diagonals, which he did do when solving the problem. This indicated that the solution of a system of equation is still at the action level, since he could only explain his action one step at a time, and not as a whole. After being probed to think deeper about his solution, he began to think and verbalised the relationship between solutions in the real number system. This we observe as he explained that if the determinant is zero, this means that the solution is undefined, since division by zero is undefined. He was now able to apply zero property of division of real numbers to matrices, indicating that he has now identified that some properties of real numbers are applicable to matrices. This revealed that by allowing students to talk about their responses, they are assisted in developing the necessary mental constructions needed to develop conceptual understanding of the concept. This contention is supported by Hamdam (2005), as he pointed out that allowing students to express their thoughts about concepts in words can be a powerful tool in developing conceptual understanding. In category 2, Jack also substituted K for a number (see Extract 6).



1.2 Use Cramers' rule to solve the system of linear equations for x and y (5)

$kx+(1-k)y=1$

$(1-k)x+ky=3$

$kx+y-ky=1$

$x-kx+ky=3$

Let $k=2$

$2x+(1-2)y=1$

$2x-y=1$

$(1-2)x+2y=3$

$-x+2y=3$

$k\left(\frac{5}{3}\right)+(1-k)\frac{7}{3}=1$

$\cancel{-\frac{5}{3}k}$

$M=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ $2(2)+1(-1)$ $4-1$ $=3$

$M_1\ \begin{bmatrix}1 & -1\\ 3 & 2\end{bmatrix}$ $=2(1)+1(3)$ $=5$

$M_2=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}$ $=2(3)+1(1)$ $=7$

$x=\frac{D_x}{D}$

$x=\frac{5}{3}$, $y=\frac{7}{3}$

Extract 6: Jack's response to item 4

His response to activity sheet displayed the following shortcomings: (1) incorrect notation used; (2) oversimplification of rules; and (3) changing the structure of the problem. In the first line, he expanded the equation and substituted k with 2. When evaluating the determinants on the right hand side, he used M, M_1 for D_x and M_2 for D_y. It was possible that by 'M_1 etc.' he was referring to minors, which were not needed here, since this was a 2 x 2 matrix. Sometimes students memorised rules and used them inappropriately due to a lack of understanding of the rule. As with his solution on the right hand side, he oversimplified the rule for finding determinant of 3 x 3, and tried to fit in when solving 2 x 2. As a result, he ended up using incorrect notation when referring to the determinant. Once he had the solution of x and y, he substituted it back in to the first equation, (see the last step on the left in Extract 6). His solutions became co-efficients of k. This changed the structure of the original problem. He did not substitute the value of x and y in the second equation. He might have got confused about what he was doing and decided not to continue. This revealed that in many instances, students rushed to apply algorithms before actually understanding the problem they want to solve. This would indicate that they may be more focused on providing the answer, regardless of whether the answer is meaningful or not in terms of the problem. The same applies to Jack, because if he understood the meaning of the solution to a system of equations, he probably would have realised that his solution did not make the statement true, and therefore, it could not be a solution of this system of equations. Tall (2008) pointed out that previous knowledge learnt could either positively or negatively impact on the new knowledge learnt. It seemed that the knowledge Jack constructed about the solution of system of equations has a huge impact in his construction of new knowledge. Replacing k with a numerical value might have been triggered by previous experience in working with variables, where one variable stood for another. Therefore he decided to replace k with any number. Also, it might be the case that his experience of dealing with equations is based on solving equations with a numerical co-efficient. Therefore, he was trying to make an unfamiliar situation familiar.

Wilensky (1991) pointed out that in many instances, students try to concretise the concepts they learn, so as to come as close as possible to what they know. It seemed that Jack was trying to generalise the school algebra of solving the system of equation, and trying to reduce the abstraction of the problem by substituting the value of k with a numerical number. Soylu and Sahin (2011) have pointed out that students have a tendency to reduce a variable to constant, and struggle to find a connection between verbal expressions and variables. This revealed that the conceptual understanding of the concept had not yet fully developed, as he could not make connections so as to solve the equation in terms of k. He could not generalise the learnt knowledge to an unfamiliar context. His response also revealed some misconception about the concept of a solution of a system of equations. It seemed that Jack had internalised the solution of a system as a numerical value. Panasuk (2010) highlighted that internal representations are usually associated with mental images that individuals create in their minds. This meant that when a student had constructed a particular concept image, it would always impact the way in which the individual conceives any knowledge related to that particular concept. It seemed that Jack's concept image of the solution of the system of equation prevented him from developing the conceptual understanding of the concept. This might be a result of his previously learnt knowledge in school algebra, where the solution of system of equation is always presented as a numerical value. Thompson & Logue (2006) have stated that students become fixated on a concept even if the context of the problem doesn't allow its use, and are reluctant to modify the identified misconception. His responses revealed that Jack knows how to use Cramer's rule to solve the system of equation but that he has not made connection to understand the meaning of the solution, indicating that he had not made the necessary mental constructions. The process conception of the concept had not developed.

The interview excerpt

Researcher: here you said let $k = 2$, why did you propose $k = 2$?

Jack: *You see Ma'am, when you have too* [many] *variables to work with, you tend to confuse numbers. In order to make my life a bit simpler I used "k" as a number, just an integer.*

Researcher: Ok, so if someone else said let k = 3, would that still be fine?

Jack: *K can be any number.*

Researcher: Does your values of x and y make the system of equation true?

Jack: *I do not follow, can you repeat the question.*

Researcher: If you substitute your values of x and y here, will the system be true?

Jack: *Hmm!* [sic] *I never checked that. Honestly I do not know. I can only check it now.*

Researcher: Is this system of equation consistent or inconsistent?

Jack: *Hmm, eish,* [sic] *I do not know.*

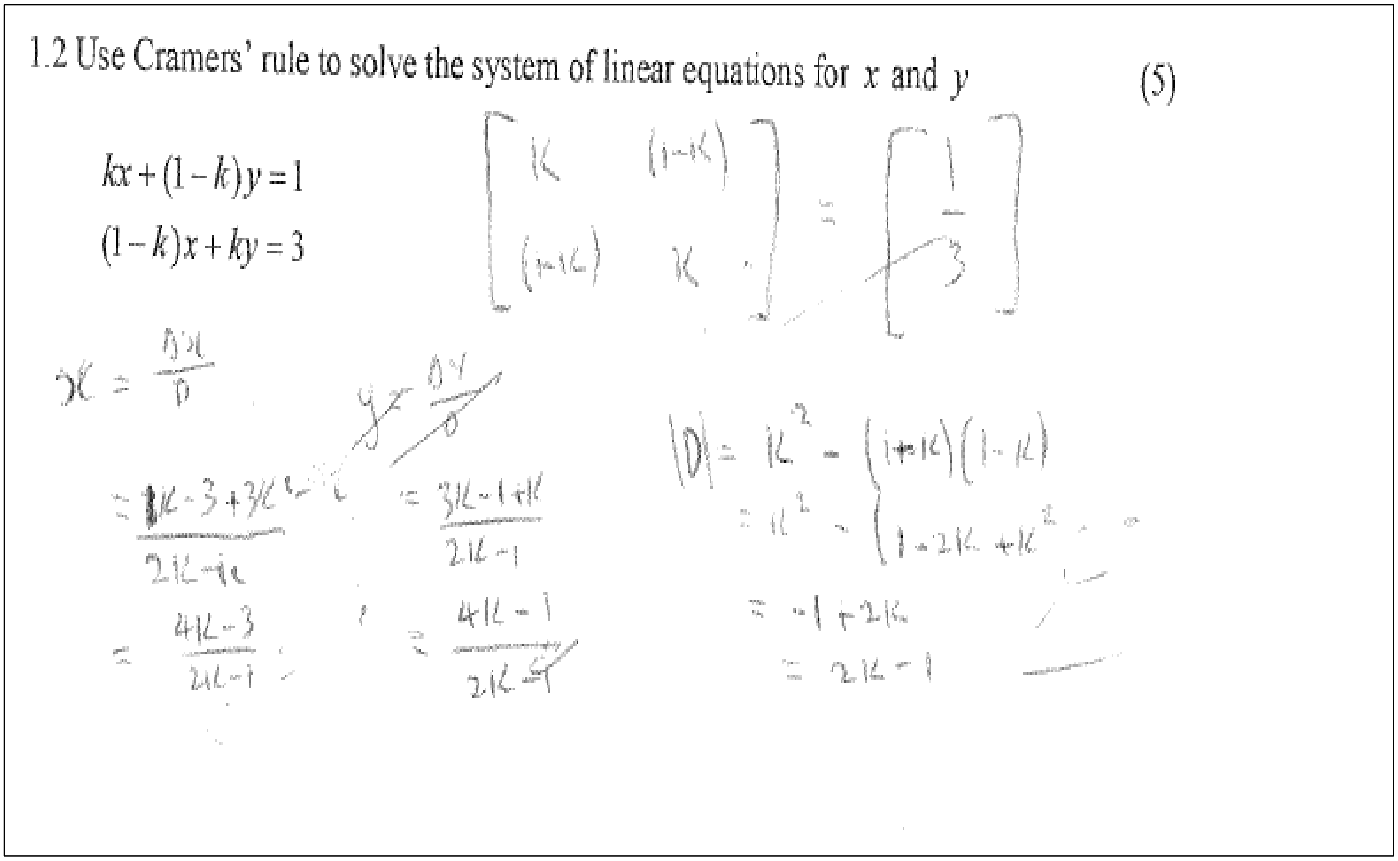
Researcher: Using Cramer's rule, how would you decide if the system is consistent or inconsistent?

Jack: *I do not know.*

The above extracts reveal Jack confirming that he had not developed conceptual understanding of the concept. Students find abstract algebra quite difficult, opting to replace the abstract part of the problem with a concrete part, in order to provide a solution. This is what Jack implied in his first part of the response, where he was trying to make it more concrete for himself to solve, regardless of whether it was mathematically correct. Jack knew how to use Cramer's rule to find the solution of a system of an equation, however, he had not yet interiorised the action into a process. This we observe as he pointed out that he had never thought about the meaning of the solution to the given system. Furthermore, he was unable to explain how using Cramer's rule helped him decide if the system is consistent or not. What transpired during the interview, and from his response to the activity sheet, was that Jack has not constructed the necessary mental construction to develop the process conception of the concept as he could not differentiate between a consistent system and an inconsistent one.

The responses of students grouped under Category 3 and Category 4 showed some knowledge of the application of Cramer's rule to solve the system of equations. The difference in their responses was that those in Category 3 did not evaluate all the determinants. Some determined the determinants of the co-efficient and one of the variables or determined them all, but failed to express the solution in terms of k and used incorrect notation to represent the determinants. The students in category four evaluated all the determinants, however, their solutions contained few computational mistakes. This indicated that they had made most of the mental constructions required to develop the conceptual understanding of the concept. Students in Category 4 showed evidence of having collected some of the rules necessary to solve the system of equations. They showed evidence of being able to identify the relationship between Cramer's rule and the determinants. Although they have identified the relationship, it seemed that they have not constructed the underlying structure of the relationship. This revealed that they were developing the process stage in terms of the APOS.

Only 29 % of the students made all the necessary mental constructions as expected by the genetic decomposition. They correctly solved for x and y, indicating that they had constructed a coherent understanding of the collection of rules and could apply them accordingly. These students could solve the system of equations in its abstract form and presented appropriate solutions. Thabo's written response indicated that his schema of this concept had developed (see Extract 7). Without performing all the steps, he could present the solution to the system. He could generalise his knowledge of solving the system of equations to unfamiliar contexts, without trying to reduce the abstract nature of the problem, and indicating a certain degree of coping with the level of abstraction of mathematical problems.



1.2 Use Cramers' rule to solve the system of linear equations for x and y (5)

$kx + (1-k)y = 1$

$(1-k)x + ky = 3$

$$\begin{bmatrix} k & (1-k) \\ (1-k) & k \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$x = \frac{Dx}{D}$ $\quad$ $y = \frac{Dy}{D}$

$= \frac{k-3+3k}{2k-1}$ $\quad$ $= \frac{3k-1+k}{2k-1}$

$= \frac{4k-3}{2k-1}$ $\quad$ $= \frac{4k-1}{2k-1}$

$|D| = k^2 - (1+k)(1-k)$

$= k^2 - (1-2k+k^2)$

$= -1+2k$

$= 2k-1$

Extract 7: Thabo's response to item 4

The interview included the following exchange.

Researcher: You determined the solution for *x* and *y*. Is your solution true for the above system of equations?

Thabo: *Yes it is true.*

Researcher: How do you know?

Thabo: *I did take my values and substitute them back to see if they were correct. That is what I normally do when I solve equations.*

Researcher: How would you represent your solution graphically?

Thabo: *With K's I cannot do it exactly, but let me think. These are two straight lines with one solution, so there will be two straight lines intersecting at one point looking like this* [makes a rough sketch]. *You see Ma'am, both these equations have a negative slope and positive y-intercepts, they will go like this and meet somewhere, I think.*

The above extracts from an interview revealed that Thabo's schema of solving the system of equation has developed. He was able to present his solution algebraically and to cognitively construct its concept image, and therefore could also present it graphically without actually having the concrete numbers and plotting the points. He has gone beyond just determining the solution, as he could reflect on it and construct an explicit structure of the system of equations. As Jojo (2011) pointed out, students at the trans-stage are capable of operating on the mental constructions which make up his/her collection. Thabo identified the structure of solution of the system and reflected on it, performing other actions on processes so as to understand the meaning of his solution. He internally operated on the mental constructions constructed, which then made him recognise the relationship between the system of equations and the solution. He could think about its structure graphically. This indicated that the object conception of the concept is fully developed. This we observe as he could perform actions on a previous process.

Item 5 addressed the last part of the preliminary genetic decomposition. It aimed at identifying whether students have constructed the object conception of the solution of system of equations.

Item 5

For what value(s) of k will the system be inconsistent?

Table 8.5 below shows the allocation of scores for item 5

Score	1	2	3	4	5
Indicator	No response	Provide an incorrect.	Response provided shows understanding of the concept but use wrong methods to solve for k	Use trial and error method to solve for	Correct and completely solve for k
Number of students in each category	10	9	0	3	9

Table 8.5: Allocation of scores for item 5

Examining closely how the students' responses vary, it seemed that many students had not constructed the object conception of the concept. This item required student understanding of the relationship between concepts, their knowledge of application of Cramer's rule and their understanding of the meaning of the solution of a system. In this item, it was imperative for students to have constructed the knowledge of the relationship between the determinants and the solution of system of equations, the relationship of proportionality with the solution of a system, as well as their knowledge of analytic geometry. Examining the responses, only 29% of the students answered the question correctly, indicating that they had made the necessary mental constructions. Their responses revealed that they had constructed the knowledge of the relationship between the determinants and the solution of system of equation. The evidence is that all of them had correctly identified the value of *k* by equating the determinant of the co-efficient to zero. This means that they have made the connection between the determinants, Cramer's rule and the system of equations, as they used the determinant to determine whether the solution existed or not. This revealed that the schema for solution of system of equations had developed. This is also based on their responses from Item 3 and Item 4.

The interview with Zinhle included the following exchange.

Researcher: Why did you use 2*k* -1 to solve for *k,* and not the other determinants?

Zinhle: *Since 2k - 1 is a denominator, obviously from lower grades we know that if you take a number and divide by zero, you are going to get an undefined answer. So, I took 2k-1 to find the value of k that will make the denominator zero, and if the denominator is zero, the answer will be undefined.*

Researcher: What do we mean by saying the solution of a system is inconsistent?

Zinhle: *I think it's where there is no solution.*

Researcher: Given the system of equations, how can you identify whether or not the system is inconsistent?

Zinhle: When you *try to find the solution, you divide by the determinant in Cramer's rule, so there will be no solution if the determinant is zero.*

Researcher: So if *k* in this system of equation will have no solution. If you were asked to represent these two equations graphically, what would they look like? You can draw rough sketch.

Zinhle: *Hmm, if k = both these equations; if you look here* [writing the equation in the form y= mx +c]*, will have the same gradient. I know that from school that straight lines with the same gradient are parallel, so I think these too will be like that. Oh yes, bazoba nje* [there will be like that] *since we say they are inconsistent*.

Her responses above clearly indicates that she has the object conception of the concept. She acted upon her process conception of Cramer's rule. This we observed as she reflected on the knowledge already constructed to develop new knowledge, where she was able to project the knowledge constructed to a new level. She did that as she identified and realised what an 'inconsistent solution' meant. She linked this to analytic geometry, to application of determinants; as well as in explaining the term 'undefined' in mathematical sense, and showing its applicability in this context. Also, the evidence of that we observe as she was able to describe the form of the solution of this system. This meant that Zinhle could generally coordinate actions internally, and identify and explain interrelationship between concepts, which is what reflective abstraction is about. As Von Glaserfeld (1995) indicated, in reflective abstraction, the subject is able to project the structures created by his/her activities on to a new level, and reorganise them, but is also consciously aware of what has been abstracted. Based on her responses from the activity sheet and interviews, it seemed that the schema for the solution of system of equations had developed. Moreover, she displayed coherent understanding of the collection of rules and application of such rules in constructing meaningful understanding of the concepts, indicating that she is operating at the trans-stage in terms of the Triad mechanism.

The responses of the students in Category 4 revealed that they had constructed most of the mental constructions. In determining the value of k they used trial and error, indicating that they had not fully internalised the process. Ten students in category one did not attempt to solve the problem or provide any explanation. Seven out of the ten were scored one in Item 4, indicating that they had not constructed the necessary mental constructions needed in order to develop the understanding and application of Cramer's rule to solve the system of equations. Failure to make the necessary mental constructions in Item 4 led to these students experiencing difficulties in constructing the meaning of the solution of a system. This resulted in their failure to make connections between concepts. Three students were scored with 4 in Item 4, but did not attempt Item 5. This makes it difficult to make certain assumptions as to why these students were unable to attempt this problem. However, it is possible that even though their responses in Item 4 indicated that they constructed the necessary mental constructions, it seemed they have not made a link between the solution to a system of equations and Cramer's rule. Two of these students withdrew from taking part in the interviews.

The interview with Sydney included the following exchange.

Researcher: In Item 4, you first determine the determinant of the co-efficient? Why?

Sydney: *I found the determinant because uma sibheka amasolutions ukuthi ah… like la sasifuna ukuthola if the equation has a solution* [If we check the solutions... like if we wanted to find out if the equation has a solution].

Researcher: Using Cramer's, how you can tell if the system is inconsistent or consistent?

Sydney: *I forget.*

Researcher: If det = 0, what does it mean about the solution of the system of equation?

Sydney: *It means when solving the system of equation, I will have the exact value for x and y which make the system valid.*

Researcher: What do you mean by valid?

Sydney: *It has a solution.*

Researcher: What do we mean when we say the system is inconsistent?

Sydney: *It has no solution for values of x and y.*

Researcher: How would an inconsistent system be shown graphically?

Sydney: Silent. *Hayi Ma'am, this one* [sic] *I do not think I can* [do] *since I do not know the value of k. Kodwa uma unganginika enye enezinamba hlambe ngingayisolver then ngidwebe* [but if you can give me another equation with numbers I might solve it and sketch it].

The above extract revealed the gaps in the knowledge constructed. He knew that the determinant tells us about the type of solution of the system. He did not know what the value of the determinant would be in order to decide whether the solution existed or not; indicating that the concept had not fully developed. Also, he knew the meaning of the word inconsistent but failed to link it to the context of the problem. This meant that he had developed instrumental understanding of the solution of the system. Here, his responses in the interview revealed that the concept had not fully developed. In terms of APOS, it seemed that the process conception had not fully developed. This we observe as he struggled to predict how an inconsistent solution could look graphically, and needed concrete numerical values to plot.

Unlike the students in category one, 29% of students attempted the problem and provided incorrect responses, indicating that they had not made the necessary mental constructions as expected by the preliminary genetic decomposition. What mostly transpired in their responses was: (1) students did not understand the meaning of the term inconsistent; (2) they did not link the term consistent with x value of the solution of a system; and (3) could not link the problem in Item 5 to the problem in Item 4. In his response, Jack wrote $k \neq 3,5$.

The interview excerpt:

Researcher: What do we mean when we say the system is inconsistent?

Jack: We mean that it *doesn't give the pattern that you expect, or something* [sic]. *Zero may be an inconsistent zero.*

Researcher: If the system is inconsistent does it mean it has a solution, or not?

Jack: *It does not have a solution.*

Researcher: How can you tell if the system is inconsistent?

Jack: *What do you mean?*

Researcher: Given a solution using Cramer's rule, how can you tell if the system is inconsistent or consistent?

Jack: *You know* [sic], *determine the zeros, if the values of x and y are zeros that will be inconsistent.*

Researcher: In your answer you said, can you clarify this, how did you identify this value for k?

Jack: *Not sure, hey* [sic]. *I was thinking it will be inconsistent when k has the same value as x, honestly I do not know what I was doing here.*

At first his response revealed that he does not understand the mathematical meaning of the term inconsistent. In his definition, he tried to link it to the conventional English usage, as he said "something that doesn't give you a pattern you expect". It seemed his misunderstanding of the term created a barrier for him in constructing the necessary knowledge of the concept, leading to a conceptual understanding. Furthermore, when he tried to explain it in a mathematical context, he seemed to be confused. When probed, he said it meant there was no solution. For him, no solution meant that values of the variables are zero. This meant that he hasn't constructed the correct knowledge of the meaning of the solution of a system. Arnawa et al. (2007) pointed out that students need to learn about importance of precise language in mathematics, and about the role of supporting such precision. This meant that failure to appreciate the importance of language in mathematics would cause barriers in developing conceptual understanding of the concept. His response revealed some misconception about zero as a solution, that when variables equal zero, this meant that there is no solution. This misconception originates from properties of zero in multiplication, where multiplying by zero, the answer becomes zero, and zero equal nothing. Jack seemed to have incorrectly generalised that to mean that the solution does not exist if one of the variables is zero. This revealed a lack of understanding of basic algebra, especially the zero property. Dubinsky (1997) pointed out that the lack of basic knowledge of other related concepts is the main source of students' difficulties with conceptualising linear algebra concepts. Another misconception revealed by Jack's responses, was his association of the solution of the system with the value of x. This we base on his last response, when he said it will be inconsistent if k is the same as x. This misconception can be traced back to school teaching, where, when solving equations, there is so much emphasis place on the variable x even in the textbook, to the point where some students, when they think of a solution of a system of equations, think of variable x.

Also, the understanding, especially when dealing with functions that x is an independent variable and y depends on x. This indicated that when concepts are memorised, students like Jack tend to over-generalised knowledge learnt in other contexts to an unrelated context. As mentioned earlier, his responses revealed the he hasn't made the necessary mental constructions. For Jack, it seemed that the process conception of the concept had not fully developed.

8.3 Conclusion

In this chapter, the data analysis of students' responses to Tasks 3 and 4 were presented and analysed. Students were prompted through guiding questions to explain the knowledge constructed of the related concepts. Through guiding questions, they were prompted to collect constructed sets of knowledge of matrices and system of equations in order to examine the mental structures formed, the mental representation of concepts, the knowledge of structures and concept image formed of related concepts. The interviews conducted with various participants aimed to clarify some responses, understanding the mental constructions formed and helping the participants learn to interrogate what they write, which in turn helped them in constructing and reconstructing the mental objects formed. This chapter provides an explanation of the mental constructs students were able to make, and the difficulties students experienced when trying to make those mental constructs of matrices, matrix inverses and system of equations. Analysis presented from students' responses and interviews served to explore the conceptual understanding of these concepts through APOS and the Triad mechanism. The next chapter concludes the study by discussing these findings in response to the main questions, recommendations and limitations of the study

CHAPTER NINE
FINDINGS AND DISCUSSIONS

9.1 Introduction

This study is a contribution to research in undergraduate mathematics education, focusing on matrix algebra. It sought to describe students' understanding of matrix algebra in the context of an undergraduate course at one of the South African Universities. Despite the centrality of the subject in the undergraduate curriculum for students in mathematics, research on learning and understanding of matrix algebra concepts is rather slim in South Africa. This study is an attempt to fill this gap and it is guided by the belief that better understanding of student's difficulties leads to improved instructional methods.

In Chapter eight the themes that were uncovered when analysing task three and task four were discussed. These themes focused on specific mental constructions made/ not made by pre-service teachers when solving problems related to matrix product, matrix inverse and the use of Cramer's rule in solving a system of equations. In this chapter a summary of the study, synthesising the themes that emerged from chapter six to chapter eight are presented. This chapter begins by describing the mental constructions that students made, followed by the difficulties that pre-service teachers experienced when solving problems related to matrix algebra which caused them not to make the necessary mental constructions, then by exploring the link between students' mental constructions and the ones indicated in the preliminary genetic decomposition. Lastly the modified genetic decomposition for matrix algebra concepts which the researcher modified based on what transpired from pre-service teachers' responses is presented.

Using APOS theory, which is a theory of learning advocated by Dubinsky (1991), the study identified prominent characteristics and components of students' mental constructions for central concepts in matrices, determinants, matrix inverse, up to and including the concept of solution of a system of linear equations. The analysis and the ensuing results are based largely on students' responses to activity sheet and from transcribed interviews conducted with ten participants from the class of 31 pre- service FET mathematics teachers. Video recordings helped the researcher together with students' responses to the activity sheets to make certain inferences about the mental constructions which students were able to make or not make in the first phase of the study. The written responses were verified or clarified through the interviews. Detailed results for each of these analysis are organised according to the relevant mathematical concepts, are found in chapters 6-8. This chapter presents a synthesis of the findings that transpired in pre-service teachers' responses. The method of analysis emerged through the analysis itself, and the research questions evolved as part of the process. The preliminary genetic decomposition served as analytic tool through which students' mental constructions were analysed.

The main question that this study aimed to answer was:

How do pre-service teachers' **mental constructions** of concept in matrix algebra concur with a preliminary genetic decomposition?

The main question will be answered through the following sub questions.

- *What levels of conceptualisation of action, process and objects are reflected by pre-service teachers' mental constructions of matrix algebra?*
- *What difficulties do the students experience in their effort to construct the necessary mental constructions in matrix algebra?*
- *To what extent do the students' mental constructions of action, process and object link with the preliminary genetic decompositions?*
- *What characteristics of the schema displayed by the pre service teachers are adaptable to a genetic decomposition of matrix algebra?*

Below the researcher presents the main findings of the study addressing each of the above research questions and the modified genetic decomposition of matrix algebra concepts. The main aim of the study was to explore and explain the pre-service teachers' mental constructions of matrix algebra concepts. Under matrix algebra for the purpose of this study the following concepts were taught, (1) matrices, and (2) solution of the system of equations using Cramer's rule. Under matrices, pre-service teachers' knowledge of naming and stating the order of matrices, computation of matrices, evaluating determinants and determining and identifying matrix inverse were covered. For the purpose of this study four tasks were administered with each task covering certain concepts as described in chapter 6 -8. To describe the nature of mental constructions made by pre-service teachers and the difficulties they experienced, it was important to analyse their responses to each task. This revealed their level of understanding of concepts covered in those particular tasks. Once their responses were analysed the emerging patterns that explained why certain mental constructions were made or not made were identified. Analysing emerging patterns helped in identifying the main themes. These themes were then synthesised with the aim of answering the research questions posed in this study. In the next sections the researcher will present the findings as they relate to each of the four sub-questions presented above.

9.2 Students mental constructions of matrix algebra concepts

One of the goals of this study was to answer the following research sub-question:

- *What levels of conceptualisation of action, process and objects are reflected by pre-service teachers' mental constructions of matrix algebra?*

This study is not about giving statistical comparisons of students' responses but aims to reveal the nature of mental constructions that students were able to make when learning matrix algebra concepts. The theoretical framework that underpinned the study provided one way of revealing the nature of mental constructions made. Evidence from chapter six revealed that for many students' the understanding of matrices was at the action stage. The responses revealed that many students provided correct answers in items where they needed to perform step by step computations or needed to recall a rule triggered by external stimuli. This was mainly observed in item 1 and item 2 in chapter six. In these two items, for students to solve them efficiently, they needed to have developed the action conception of those concepts. For example the results of the 31 students who responded to item 1 (for identifying the order of matrices) revealed that 81% were able to perform the action of naming and explain the formation of the order of matrices. Also, in item 2 (determining the transpose of a matrix), 87% of students showed that they had made the necessary mental construction showing they could determine the matrix transpose. As it was explained in chapter one the action stage is about physical action, where an individual performs step by step calculation. At some point this action could be triggered by external stimuli. In item 2, seeing the notation A^T would have cognitively triggered the rules to be performed. The results from these two items suggested that the pre-service teachers have constructed the necessary mental constructions, meaning that they could identify and name the order of matrices and determine the transpose of a matrix. When examining the video clips it was evident that the majority of students understood how to solve these items since there was not much discussion around them. What also transpired from the video clips was that the pre-service teachers were not at all concerned with the method used; rather their focus was checking on the others answers. If all the answers were similar, they assume they are correct and move on to the next item.

Although many pre-service teachers carried out related procedures, it seemed that when dealing with problems involving multi-steps which required the internalisation of procedures, pre-service teachers were experiencing difficulties. Evidence from item 3 in chapter six revealed that only 68% of the students represented their understanding in the manner described as process conception of addition of matrices. These pre-service teachers were able to recognise the relationship between real numbers and matrices and therefore the rules applied in addition and subtraction of real numbers were accurately applied in the addition and subtraction of matrices. Furthermore, based on the data from interviews, there is evidence that many pre-service teachers could verbalise the conditions of matrix addition and subtraction. Thabo said " *to add or subtract matrices the order must be the same because entries in row of matrix A must correspond with entries in row of matrix B (*making an example). Also Sipho said "*because we gonna multiply and subtract I am going to follow bodmas rule"* This meant that entries in the matrix are recognised as real numbers and as it seemed their schema of real numbers has developed so many have interiorised the action into a process. Also, before multiplying by scalar k, they thought about it's effect in whole matrix and develop a short cut or an effective way to get the answer as Sipho said " *I am going to multiply by 2 inside instead of -2 of matrix D since it will be easy now to subtract the matrix C with matrix D rather that if I multiplied by -2".* The results showed that many pre-service teachers had developed the process view of scalar multiplication as well addition and subtraction of matrices. Although there is no intention to make strong claims (since this is a small scale study) it could be argued that matrix C and matrix D could be considered concrete when viewed by some of the pre-service teachers who provided the correct answer. Thabo and Sipho could imagine the array of numbers as a single object which they could manipulate. When solving item 3 they looked at the effect of multiplying by these scalars to the entire matrices and its effect in their final answer. This is drawn from their discussion during group interaction. When analysing the video recordings the researcher found that Thabo tried to explain to his group the effect of multiplying by negative and why he chose to multiply by 2 instead.

As mentioned in the above paragraph some pre-service teachers experienced difficulties when performing multi steps computation. It seemed that as the problem required them to carry more procedures and explain their solution they struggled to solve such problems. Evidence in item 4 and item 5 showed that many of them could not switch from actions to do mathematics to process to think about mathematics. In item 4 (matrix multiplication) only 29% of the students seemed to have interiorised the procedures of multiplying matrices and only 42% recognised the relationship between the rules of matrix multiplication and constraints associated with matrix multiplication. It was noted that in item 5, thirteen pre-service teachers gave a correct response compared to nine in item 4. The nine pre-service teachers showed that they knew the procedures of multiplying matrices but made few computational errors e.g $3 \times 2 = 5$, that means their action conception of matrix multiplication had developed. Stewart (2008) pointed out that action is the basis of understanding of mathematical concepts. This meant that when students have correctly and cognitively constructed the meaning of the rules applied, they could verbalise it and internally think about its application to other concepts. The findings in item 3, 4 and item 5 revealed that for computation of matrices pre-service teachers could easily perform the relatively easy procedures but struggle when required to mentally think about application of such rules and link it to other related context. It was interesting to note that students who displayed an effective schema of basic algebra were the ones who displayed their thinking about computation of matrices at the process stage, thus indicating that they were developing the conceptual understanding of computation of matrices. Moreover it was noted that the majority of these pre-service teachers struggled with manipulation of numbers indicating the lack of a well-constructed basic algebra schema. This was also supported by what transpired in video clips, unlike item 1 and item 2, there was much of discussion around the solution to item 4 and 5 indicating that they had different answers. At some point students would argue for the correctness of their answers. It was observed in one group that students did not agree about the answer. Since they had to submit what they did as group, they came to consensus that the group leader solution would be submitted as the group response. What was interesting to note was even when students realised during group discussion that their answer was wrong they did not change their answer in the activity sheet, they wrote the new answer in their exercise book. This then confirms that the mental constructions were actually made by a specific individual not influenced by another.

Regarding the computation of matrices the findings are consistent with findings in the literature that previous knowledge can either have a positive or negative effect in the conceptalisation of the new knowledge (Tall, 2008, Tomita, 2008, Ndlovu, 2012). Also, students develop conceptual understanding by making mental constructions of mathematical objects and processes (Dubinsky, 1997). The literature suggests concepts such as computation of matrices are less abstract and therefore students should do well in these types of problems (Carlson et al, 1993). However the findings of this study proved otherwise as many students struggle with manipulation of numbers when computing matrices.

Evaluating determinants of order > 2 was considered to be at the process stage in terms of the presented preliminary genetic decomposition. To carry out the required necessary procedure and construct necessary mental constructions an individual needed to know the formula of evaluating 2 x 2 determinants which is considered to be at the action stage. Evidence from chapter seven item 1 (evaluating determinants) revealed that only 6% of students had interiorised the action of evaluating determinant into a process. This indicated that the rules were mainly memorised since they could not solve problems similar to those done during lectures. The findings showed that two pre-service teachers understood the procedures of evaluating determinants of order > 2 to a point that they could explain the connection made between their general statement of evaluating the determinants and its applicability. It was interesting to note that 58% provided a correct response for either item 1a or item 1b. Among these eighteen students, sixteen evaluated $|C|$ only. The other two evaluated $|C|$ correctly but their solution for $|D^T|$ contained computational errors. As a result they provided an incorrect answer for item 1b. This was surprising since both items needed to be solved using the same procedure. Some students did not even attempt item 1b. Among those who did not attempt 1b, the interview indicated that time was a factor. Suggesting that some of them might have known how to solve item 1b. In this regard Thabo stated that "*I was slow by the time we had to do group discussion I was still solving the determinant of matrix C, but I was not worried because I knew what to do*". This seemed true because when he was asked to explain what he was supposed to do to solve item 1b, he was able to provide mathematically correct explanations on how he was going to solve the problem. The findings of this study suggest that the majority of the students had the action conception of evaluating the determinants. The carrying out of procedures correctly of evaluating $|C|$ indicate that the difficulty was not with performing the action but with unpacking the structure of $|D^T|$. This meant that they have not made the connection between the determinant of the matrix and determinant of its transpose. Also, indicating that the rules were mainly assimilated as a list of unconnected actions. Hence we found that most of the pre-service teachers could not identify the use of the same rules in other different contexts unfamiliar to them.

In item 2 of chapter six (recognising the relationship between the determinant of a matrix and determinant of its transpose) only 23% could clearly explain the relationship between the $|D^T|$ and$|D|$. The 23% comprised the two pre-service teachers who provided the correct answer in item 1 and the five students in category two in item 1. The students in category two failed to manipulate numbers and provided incorrect responses to both items 1a and 1b. Although the five pre-service teachers had an incorrect answer for item 1b, their responses together with the other two showed that they have made a correct distinction between the determinant of a matrix and determinant of its transpose. This meant that they had developed the process stage of the concepts as they could clearly explain the relationship between the determinant of a matrix and determinant of its transpose. What was noted though was that although they seemed to have the process view of the concepts, it was not fully developed for some pre-service teachers since they showed misunderstanding of notations used and they haven't developed the strong language skills to help them communicate their thinking strategies clearly.

Regarding the concept of matrix inverse presented in chapter eight, the evidence showed that many students have a well-developed process conception of the concept. Students accurately carried out procedures of evaluating the determinants and used the determinants to determine whether the matrix had an inverse or not. This indicated that these students have recognised the relationship between these concepts. Jabu without carrying out the procedures of evaluating the determinants did all the calculation in the mind and explained why he said the matrix was singular. In an interview when asked to elaborate on his written responses of why he said the matrix had no inverse he said: "*We know that to find an inverse you multiply the matrix by a reciprocal of a determinant. Then if the determinant is zero it will be undefined as we cannot divide by zero. I know that if I multiply the number by its reciprocal and the answer is one, then the number is the inverse of the other one, nalana ke (even here) if I multiply two matrices that are inverse of the other the answer should be 1*". He used his background knowledge of inverse of real numbers to explain his understanding of the matrix inverse. He displayed coherent understanding of the relationship between quotients, reciprocal and zero property and its link to matrix inverse. Based on the responses in the activity it was concluded that 90% of the students had the process view of the concept.

The concept of a system of linear equations forms part of school curriculum. This means the pre-service teachers have some background knowledge of solving system of equations. At this point they are expected to generalise their knowledge of solving a 2 x 2 system to solving n x n system of equations. Since this concept is not fairly new to students it was assumed that they would have conceptualised the concepts. However, the evidence from the findings showed that only few students have at least the process conception of the concept. Only 29% of the students were able to verbalise their thinking and were able to relate the structure of a system of equations to the matrix structure. By identifying and making connection of the relationship between the number of equations and the number of the variables they could tell that the given system could not be solved using Cramer's rule. This meant that they could think about the action without performing it. According to De Vries & Arnon (2004) when a student can envisage the action without actually performing it meant that their conception is at the process stage. This statement concurs with the findings that some of these students were representing their thinking at the process level. The other students experienced number of difficulties in expressing their thinking but these will be discussed later under difficulties.

In items 4 and 5, only 29% provided the correct answer. These items aimed to explore the object conception of a system of equations as students were expected to generalise their knowledge of solving concrete system of equation to solving an abstract system of equations. Some students who had the correct answer in item 3 could not solve this system because of its abstract nature. This meant they could not reflect on the knowledge constructed and apply it to unfamiliar contexts. The nine students were able to generalise their knowledge to unfamiliar situations. This was observed as they could solve the problem in its abstract nature. This was also supported by the evidence from an interview. Thabo was asked if he could represent his solution graphically and he stated said "*These are two straight lines with one solution so there will be two straight lines intersecting at one point. Since both have negative slope and positive y-intercept they will be like this" (drawing the image).* Thabo reflected on his solution and constructed the meaning of the solution in relation to the given system and represented his solution geometrically. Without trying to find values to substitute he imagined and explained what the solution looked like graphically. Zinhle also, was able to use the given system with a solution to show how it would look like if it had no solution, algebraically as well as graphically. It meant she transformed the consistent system to an inconsistent one. She reflected on her solution of the item and internally performed manipulations on it. She saw the determinant of the matrix (D) as an object and performs actions on it to form new knowledge as she described the value of k, which would make the system inconsistent and how she determined it. Also she reflected on her solution and was able to link the new knowledge to school algebra as she explained that equations with the same gradient are parallel meaning that the system of equation had no solution. Based on the evidence from their responses in the activity sheet and from the interviews conducted these pre-service teachers have conceptualised the meaning of a solution to a system. This indicates that some of the pre-service teachers had gone beyond thinking about a solution to an equation to thinking about the solution that satisfy the system of equations.

The action/process/object distinction is portrayed as developmental and hierarchical, with a process view considered to be more sophisticated than action and object to be the most sophisticated one (Brown, Dubinsky, McDonald, Stenger & Weller, 2004). The findings in this study seem to concur with such contention. Students who were unable to manipulate numbers to carry out procedures mostly struggle to verbalise their thinking. For item four the students who could not manipulate numbers and carry out procedures failed to identify the relationship between the rule applied and the constraints of matrix multiplication. In chapter seven the students who struggled to evaluate determinants and determining transpose had difficulties in explaining the relationship between the determinant of a matrix and the determinant of its transpose. Similarly in chapter eight students whose process conception of the system of equation had not fully developed could not solve item 4 in its abstract nature. Instead they tried to make it concrete by substituting k with a constant number. In summary few students were able to make the necessary mental constructions indicated in the preliminary genetic decomposition. What was interesting though was that student's responses revealed other important concepts that need to be considered in order to develop conceptual understanding of matrix algebra concepts such as notation, terminology and basic algebra. These will be discussed in more detail when we look at modifying the preliminary genetic decomposition.

9.3 Difficulties in making the necessary mental constructions

From the results presented in chapters 6 - 8 it was evident that in most of the items students were experiencing difficulties in making the necessary mental constructions. Literature has shown some of these difficulties relating to concepts such as linear transformation, vectors, and other concepts.

The literature has been silent when it comes to students' difficulties relating to computation of matrices. There are some limited discussions around students' misconception of matrix properties and determinant properties as well as system of equations. These studies have been done internationally and they look at these concepts separately. This study examined students' mental constructions of these concepts since at this university these concepts are taught as one module and matrix operation is the prerequisite for all the others. While literature emphasise the importance of identifying students' mental constructions that students are able to make of a particular concepts (Dubinsky, 1997; Klasa, 2010) to improve instructional methods, it also vital to understand the reasons that led to students' inabilities to construct the necessary mental constructions. To address the issue relating to students' abilities or inabilities to make the necessary mental constructions in matrix algebra the following research question was posed:

What difficulties do the students experience in their effort to construct the necessary mental constructions in matrix algebra?

In the first section the constructed knowledge that impacted positively in their learning has been discussed. In this section, students constructed knowledge that seem to impact negatively in their attempt to construct the necessary mental constructions is discussed. These will be discussed under four distinct sections, (1) incorrect use of notation (2) difficulties in comprehending and using the language appropriately (3) lack of background knowledge and (4) misconceptions of matrix algebra concepts and in mathematics in general.

9.3.1 Incorrect use of notation

In the preliminary genetic decomposition, pre-service teachers' use of notation and its role in in developing conceptual understanding of matrix algebra concepts was not explicitly stated. While students were solving problems and when examining their responses, it was noted that the students' use of notation had the combination of both formal and informal aspects of mathematics. For example in chapter six, item 1 and item 2 for students to fully construct the action conceptions of those concepts they needed to have complete understanding of the notation to use. If a pre-service teacher misinterpreted the notation it was difficult to construct the correct meaning of the concepts. In item 1, to represent the order of a matrix such as 2×2 or 2×3, Sydney used a dot 2.2 to indicate the order, meaning that he considered $\times$ to mean multiplication. Such misunderstanding of notation could have detrimental effect in the understanding of the naming of order of matrices.

If students see this as multiplication they will construct an incorrect understanding of the order of matrices. This meant that it is very important to explain the meaning of the notation in its context before it can be used. As Tall (2008) pointed out that met befores can either have a positive or negative effect in students' attempt to learn the new knowledge. Another student, Zinhle in item 1, used capital letters to indicate the entries of the matrix A_{22} and confused the rows and column, c_{ij}. This could not be easily identified in the square matrix. This was evident in a non- square matrix as she wrote C_{32} instead of c_{23}. Learning the notation requires building some cognitive structures around the notation to support its meaning and its use (Findell, 2006). This suggests that notation should not be memorised but understood if it had to support learning. The findings of this study seem to show that pre-service teachers who memorised notation failed to use it in its proper context and to construct the associated meaning of the concept. It can be argued that Zinhle knows that $_{ij}$ refers to rows and column. However to her $_i$ columns and $_j$ rows, as she wrote in item 1c as C_{32} there are three rows and two columns while if we look at the structure of matrix C in item 1 it was the other way round.

Similarly, the same problem of notation emerged in chapter seven. Students used $|A|$ to represent the determinant of any matrix and used a_{11} to represent entries of any matrix. This was more evident to those pre-service teachers who wrote down the formula first when evaluating $|C|$. It seemed that many students were thinking of $|A|$ as the standard notation for determinants. When the researcher asked other students why they used $|A|$ to represent$|C|$, Thabo said " *I am so used in using A"*. For using a_{11} instead of c_{11}, "*was this also need to change, I do not know how to write it since there was also c_{11} for indicating cofactors*". Sipho said "*it was a mistake*". When probed further he changed and said "*does it mean if I am finding the determinant of some matrix I must use those letters?"* In this case it seemed that the notational distinctions that students were making of matrix algebra concepts were not necessarily the ones considered to be standard and focusing on the meaning of the concepts. These results coincide with the findings in the literature that learning new notation may be better seen as accommodation rather than assimilation (Findell, 2006). Also students in general do not interrogate what they write (Maharaj, 2014). Contrary to the literature in many cases when I intervened during interviews, trying to get students to develop a good sense of the meaning of standard notation, students were able to identify their mistakes and seem to make the conceptual distinction. This meant that it is vital not only in getting students to provide solutions to mathematical problems, but there is dire need to get students to talk about their responses, about the meaning of their solution to the problem. Also, this has implication for the teaching and learning of matrix algebra concepts. The clarity of notation and understanding of such leads to possible correct solutions to a mathematical problem. This has substantial contribution to APOS, that symbols and notation are vital in the development of an action or process conception in mathematics. Therefore lecturers need to make sure that use and meaning of each notation is clearly understood by students.

9.3.2 Students' had difficulty comprehending the language and terminology used

Learning advanced mathematics involves learning concepts, language, notation and relation among them (Findell, 2006). This means comprehending the meaning of a concept requires one to comprehend the language used and the meaning of the terms used in the context of a problem. Therefore understanding the language and terminology will help the students construct the meaning of the concepts. In this study it was noted that students had several difficulties related to the use and understanding of language. The results revealed three kinds of difficulties that students had relating to language.

The first difficulty students had was the inability to explain their thinking using correct mathematical language. For example in chapter six item 5, 29% of students said yes but failed to clearly support their statement. This was mainly due to inappropriate language or terminology used. One response was "*number of numbers in rows is not equal number of numbers in columns*". There are two issues here that might lead to such errors (1) not interrogating what they write or (2) ideas themselves are somewhat muddled resulting in failure to form a coherent sentence to support the statement. In chapter seven, item 2 students were asked to explain why they say $|D| = |D^T|$. Zama said it's what she learnt from the book. When probed further to explain the relationship between the matrix and its transpose, she said "*you interchange numbers*". Sipho said "*when you find determinant of a transpose let's say you have 26 now you want to find the determinant of matrix they are square matrices so their determinant will be the same*". It is difficult to make sense his explanation. Only when the researcher had to interrogate term by term that he used and its meaning he started to make sense of the concept and was asked to express his thoughts in any language. This shows that the concept definition students construct if it's incorrect can impede their understanding of the concepts. Also, if the language used to construct that definition is not appropriate for the concepts it can cause barriers to learning.

Secondly, difficulties were caused by the name itself or confusing the terms. In chapter eight, some students failed to explain the meaning of consistent system or an inconsistent system of equation. Some responses about inconsistent system were that inconsistent because it does not give you a pattern that you want, or it gives zero. Some pre-service teachers regarded consistent to mean "it is valid". This showed that not having the correct mathematical meaning of the term could impact negatively when constructing appropriate knowledge of the concepts. It is vital for students to understand the use of the term in a mathematical context. As Stewart (2008) pointed out that some terms have colloquial meaning. It has different meaning when used in an English context to when its use in a mathematics context. The findings show that some students had the same problem with the term inconsistent as they said it does not give you a pattern. Another difficulty with terminology was evident in chapter eight item1 when some pre-service teachers formulated a causation relationship between the term defined and commutative property regarding multiplication of matrices. They concluded that since multiplication of matrices is not commutative that means it's not defined. They were unable to make distinctions between the terms. The distinction that students make are not necessarily the same as those made by mathematics teachers (Findell, 2006). He further pointed out that learning mathematical vocabulary and appropriate syntax is sometimes a complicated process with much potential for a misstep. Students can use the terminology without actually constructing its proper meaning if they have not made the necessary constructions to understand them as processes or objects (Trigueros, Oktac & Manzanero, 2007). This will result in students applying a term in wrong context or unable to use it when needed. Therefore if the meaning of the terms in relation to the learnt concepts had not been understood, students would struggle to formulate appropriate mathematical sentences to make meaning of the learnt concepts.

Thirdly, the difficulties are caused by students' failure to comprehend the question. This was more evident in chapter eight, item 1. Students were unable to provide the answer because they misinterpreted the question. Thirty nine percent of the students tried to prove that matrix AB and matrix BA are square matrices. Sixty five percent could not solve the problem or some did not even attempt the question. When John was asked to explain his response, he indicated that he wanted to prove that matrix AB and matrix BA are not square matrices. That is how he interpreted the question. This showed that when students failed to interpret the question, they look for cues and try to respond to a certain part of the question. The results are consistent with observation by Parker (2010) as he said that students with stronger language abilities tend to do well when solving linear algebra concepts. This study demonstrated that when boundaries between concepts are not clearly drawn they can be a barrier to learning. The results of this study showed that for pre-service teachers to have at least the process conception of the matrix algebra concepts the use of correct language and terminology should be emphasised.

9.3.3 Lack of background knowledge and unable to connect concepts

It was hypothesised that when dealing with matrix algebra concepts, students should be able to generalise their knowledge of arithmetic and school algebra to formulate new knowledge. The results of this study showed the many difficulties that students experienced, like failing to manipulate numbers, emanate from the lack of basic algebra. For example in chapter six, Sipho tried to explain his understanding of transpose by linking it to reflection in transformation geometry. Transpose of a matrix and reflection are not related concepts, since his schema of reflections has not developed as a result he is linking it to any concepts he thinks is of similar nature. This was also evident in item 3 and item 4 in chapter six. Many students could not provide the correct answers because of computational errors such as (1) unable to carry out computation involving negative numbers, indicating the lack of basic algebra schema. In item 1 chapter seven it was evident that those pre-service teachers who recognised matrix entries as real numbers were able to carry out computation effectively. When Sipho was asked to explain the strategy used to solve item 3 he said "*since these are real numbers I know from school that we should apply bodmas rule*". This suggested that he had made the connection between real numbers and matrices and understands that the same computation rules would apply.

Furthermore when Thabo was asked the constraints of addition and subtraction of matrices he indicated that the order needed to be the same since you add a column from matrix A with another column in matrix B. He said "*if the order are not the same you can't add because adding the column of zero will change the order*". This could mean that he had the understanding that although the matrix entries are real numbers, not all properties of real numbers were applicable to matrices. This indicated that his schema of real numbers has developed. The findings showed that when students had developed the schema of basic concepts they are more likely to make the necessary mental constructions. Similarly in chapter eight some students were unable to generalise their school knowledge of solving system of equation and their knowledge currently learnt of using Cramer's rule to solve the system/or incorrectly tried to generalise it. When Sydney was asked how an inconsistent system would be shown graphically, he said "*I do not think I can since I do not know the value of k, but if you can give me one with numbers I can"*. This showed finding the solution of a system was rather instrumentally understood. As Biggs & Tang (2007) pointed out that students who mainly have the surface understanding of mathematical concepts and that would cause barriers in conceptualising the learnt concepts. The results revealed that for pre-service teachers to gain a proper understanding of matrix algebra concepts they needed to have at least a process conception of algebra involving real numbers.

It was further noted that students are struggling with connecting their understanding of one concept to other related concepts. Some pre-service teachers solved a problem using a particular rule, but failed to recognise other related context where the same rule could be applied. This indicated that the rules were incorrectly memorised. In chapter six item 2, 87% determined A^T, then in chapter seven students were asked to evaluate $|D^T|$ only 6% provided a correct response. Fifty two percent evaluated the determinant of C not $|D^T|$. One student evaluated $|D|$. These results showed that the challenge was to determine $|D^T|$ not evaluating the determinant. When Zama was asked why she evaluated $|D|$ not $|D^T|$, she couldn't answer. It was noted from the video clips when students were not sure of how to solve a particular problem, they checked through their notes to see if they had done a similar problem. If they could not find any similar problem, they become very frustrated. One student was spotted stroking his head when solving problems in task 4. This shows that if the mental construction of a particular concept has not been made it becomes very difficult to identify problems of the same nature. Instead of making the necessary connection, they opted for repetition looking for similar examples where they can just copy.

In chapter eight only few students were able to describe the connection between Cramer's rule and determinant and between the structure of a matrix and the structure of the system of equations. When Thula was asked what would be the order of a system of equation in which Cramer's rule can be applied to? He said "I *do not remember*". However, when asked about order of matrices in which Cramer's rule could be applied, he was able to give the correct answer. Only when he was probed did he realise that the system of equations can be represented using matrices. Duval (2006) pointed out that students need to understand the different representation of matrices. These findings suggest that to develop the process conception of the concepts pre-service teachers should be able to recognise the related concepts and be able to describe that kind of relationship. Moreover it is imperative that students experience the links and the sub - concepts of one to another, independent and dependent variable in different representation (Hong, Thomas & Kwon, 2000). The findings revealed that in most cases where students provided an incorrect answer, when given a chance to interrogate their responses and explain their thinking strategy they start to display some kind of understanding of the concepts. This means that students should not only write their solutions but need to be given a chance to talk and explain their solution in order for them to conceptualise what they write. This was also noted from video clips that during group discussions when they had different answers, they will try and explain what they did and some were able to realise their mistakes. From the video clips we gathered why some students provided a wrong solution in the activity sheet and during the interview they clearly explained it correctly.

9.3.4 Misconceptions of matrix algebra concept

In search of the reasons to why students could not make the necessary mental constructions, the results indicated students' misconception of matrix algebra concepts was one of the reasons. Also, the misconceptions the pre-service teachers had of other related concepts impacted in their understanding of matrix algebra concepts. In elaborating and synthesising the results of this study, the researcher identified certain misconceptions that seemed to cause students not to make the necessary mental constructions. The evidence from the results revealed that students tend to over-generalise rules. In item 4, chapter six, Sipho confused the rule of adding matrices with that of multiplying matrices. He multiplied row 1 of matrix C with row 1 of matrix D, etc. He treated multiplication of matrices as addition of matrices. Sipho, always tried to link computation of matrices to real numbers, since multiplication of numbers is the same as repeated addition, he might have tried to generalise it to the multiplication of matrices. There is no data to support this claim but it is based on the trends that the researcher had observed. In an interview he stated the rule correctly and he realised where he made a mistake. When he was asked to do it again, he applied the rule correctly but struggled with the manipulation of numbers and ended up with a wrong answer. When observing a video clip the researcher noticed that Sipho and Thabo were more dominating in their group discussion and tried to defend their answer even if it was wrong. Another misconception was that students become fixated on a rule rather than understanding a concept. In several cases in this study when students were asked to explain the concept they stated a rule. John was asked to explain how he could determine the order of the matrix without solving the matrices, he said "*I have to do it first perhaps I will know*". When he was asked to state the rule of finding the matrix product he clearly stated it. This indicated that knowing the rule does not necessary mean one understands the concepts.

Furthermore in chapter seven some students associated the word transpose with opposite or change of sign. Zinhle incorrectly determine$|D| = 100$, when asked to determine $|D^T|$, her answer was – 100. This misconception might have originated from school algebra. In most cases students are told to transpose the number over the equal sign and it has to change sign. Again this misconception might have been perpetuated when determining the transpose of a matrix that one needs to interchange the rows with column. Therefore Zinhle associated the word with change. She decided to change the sign to -100. This meant she confused the properties of matrices with the properties of determinants. These findings are consistent with the findings of Aygor & Ozdag (2012). In their study students confused the properties of matrices and that of determinant. When Zinhle was asked to clarify her response, she said "*what I meant was that the cofactors, where the cofactor is positive matrix D will be negative"*. The results showed that when the ideas are not conceptually understood they could be easily muddled up and become a barrier to learning.

In chapter eight, item 4, some students' responses showed misconceiving of a variable. The previous conception that a variable stands for "any number" might have triggered some of the students ideas that they needed to substitute any number for k. In item 5 students' misconceptions about the inconsistent system and the zero solution were revealed. When Jack was asked how he could determine if the system was inconsistent he said if the solution of x and y are zero that means the system is inconsistent. This revealed that he had not constructed the meaning of the solution of the system. Again this showed the over- generalisation of zero property and the properties of multiplication of real numbers to an understanding of an inconsistent system. The knowledge that division by zero is undefined might have been incorrectly associated with inconsistent system of equation. Also, he might have confused the concept of an inconsistent system with matrix inverse.

The idea that when the determinant = 0, it means there is no inverse. Therefore he tried to link it to the meaning of an inconsistent system. This implied that he confused the matrix inverse properties with the inconsistent system properties. Again he might have confused the multiplication of real numbers with inconsistent system. Multiplying by zero equal zero which is considered to be nothing and therefore if the solution is zero it means there is nothing and the system is inconsistent. This shows that the lack of schema arithmetic algebra impacted negatively in the understanding of the meaning of solutions of system of equations. These findings confirm what Trigueros, Oktac & Manzanero (2007) pointed out in their study that good understanding of elementary algebra is really important for students to learn concepts related to systems of equations. Another misconception was that Jack associated the value of x with solution of the system. He suggested that the system was inconsistent when k had the same value as x. This might be as a result of huge emphasise being placed by mathematics teachers when solving equations to the value of x. These findings indicated that when a concept is not properly understood and its meaning is not appropriately constructed cognitively it could become detrimental to future learning. These findings are consistent with other studies as it could be argued from the results of the study that these misconceptions were mainly caused by lack of background knowledge (Dubinsky, 1997) as well as misunderstanding of the previous concepts which are related to matrix algebra (Tomita, 2008; Tall, 2008). Tall (2008) emphasised the previous knowledge learnt can either have positive or negative effects to the constructions of knowledge of the new concepts.

9.4 The preliminary genetic decomposition as an analytic tool

To explain the extent to which the students' mental constructions link with preliminary genetic decomposition, the following research question was considered:

To what extent do the students' mental constructions of action, process and object link with the preliminary genetic decompositions?

For this study a framework based on the framework for research and curriculum development and APOS was conducted. Under theoretical analysis, the preliminary genetic decomposition describing the necessary mental constructions for matrix algebra concepts was presented. From the researcher's point of view this preliminary genetic decomposition provided an excellent starting point for making sure that the concepts were constructed carefully and presented from many angles in the lectures and preparing the tasks that students were to engage with. Furthermore this preliminary genetic decomposition proved to be a valuable tool in analysing pre-service teachers' thinking by providing evidence of their level of thinking based on the specific mental constructions as explained in terms of APOS. By using the preliminary genetic decomposition in analysing pre-service teachers' solutions we could observe the level at which the pre-service teacher was operating. Also, we were able to trace and explain the reasons which caused these pre-service students not to make the necessary mental constructions.

The findings of this study revealed that the majority of the pre-service teachers were operating at the action/process stage in terms of APOS after completing the teaching of the topic. What was most prevalent was that in all the items where the action conception was required, students seem to have constructed the required knowledge. In task 1, item 1, 81% of the responses showed that the action conception of the concept has developed, meaning that students can state the order of matrix and explain it. This concept was considered to be at the action level. However, some students' responses proved that their thinking went beyond the action conception. In their explanation of the order instead of looking at the columns and rows, they looked at the entries of the matrix indicating that they described the relationship by making connection between the entries of a matrix and the number of rows or column of the given matrix. The mental constructions of the majority of pre-service teachers in item 1 of chapter six did link with the mental constructions indicated in the preliminary genetic decomposition. In item 2 of chapter six, the focus was on performing an action. The results showed 87% of students provided the correct solution meaning that their mental constructions linked with those presented in the preliminary genetic decomposition. Although most students had the action conception of the concepts the results showed that this was not enough for them to make connection between transpose and other related concepts. For students to make such a connection they needed to at least have the schema of the matrix order, positions of the entries hence process conception was required. By position I mean since in a transpose the columns becomes rows vice versa, that also applies to the entries of a transpose, i.e., a_{12} becomes a_{21} in the transpose.

For item 3 in chapter six more than half of the students were considered to be at the process stage. That means they were at the level expected as it was indicated in the preliminary genetic decomposition. Although one might consider manipulation of numbers to be at the action stage, the researcher considered that students who provided a complete solution, indicating that those that had an effective schema of basic algebra, were at the process stage. The preliminary genetic decomposition for this item did not clearly articulate specific aspects of the concepts that students need to conceptualise in order to make the necessary mental constructions such as computation of integers and computation of real numbers. Therefore the researcher decided to include the basic algebra schema in the modified genetic decomposition. This will clearly help to differentiate between students who have the action/process conception of the concepts. In the preliminary genetic decomposition the object conception was not considered in this item. However, few student responses display the object stage. This was observed as Thabo and Sipho transform the structure of the given matrices and present it as addition but still manages to show that subtraction of matrices is determined through understanding of additive inverse indicating the relationship between real numbers and matrix entries.

In item 4 of chapter six, the formulation of process stage went beyond just carrying out procedures. It also requires the ability to identify and explain connection between the rule and the constraints of matrix multiplication. In this item only 29% of the students constructed the necessary mental constructions indicating that they have not conceptualised the concept. Since most students were at the action stage, their responses did not link with the mental constructions indicated in the preliminary genetic decomposition. This was due to the lack of basic algebra skills and misunderstanding of matrix algebra terminology and notation. The same applied to item 5. What was mostly noted though was that when the pre-service teachers were probed to think about their response they were able to identify their mistakes and display some understanding of the concepts. This means that it needs to be emphasised to students to make meaning of their solution in relation to the solved problem. Since the preliminary genetic decomposition assisted us to identify the gaps in the knowledge constructed it means the pre-service mental constructions for this item did link with ones indicated at in the preliminary genetic decomposition. However, the results showed that many have not yet developed the process conception of the concept

On examining chapter seven we noted that many pre-service teachers evaluated determinants of the similar structure, but have not constructed the understanding of the relationship between $|A|$ and $|A^T|$. For this item pre-service teachers were expected to at least show the process conception of the concept. However the results showed that many have the action conception of the concept.

For the students to develop the process conception they needed to understand the relationship between $|A|$ and A^T, have the formal understanding of the notation and be mathematically proficient in the manipulation of numbers. All these aspects, together with the understanding of the relationship between the determinant of a matrix and the determinant of its transpose, will be included in the modified genetic decomposition. Looking at chapter eight, the analysis of students' responses indicated that many pre-service teachers had the process conception of matrix inverse. However, when engaging with some pre-service teachers it was revealed that they had the object conception of the concept as it has been discussed in the first section of this chapter. Based on the mental constructions indicated in the preliminary genetic decomposition we conclude that the majority of the pre-service teachers' mental constructions display a satisfactory developed schema for matrix inverse. The results revealed that many students had the action conception of the solution of the system of equation. However, their mental constructions did not link with the preliminary genetic decompositions since most of them could not even predict the type of system in which Cramer's rule could be applied. Only few students reached the object stage of the solution of a system. The object stage of a solution of a system was mainly explored during the interview which the researcher considered to be the limitation of the preliminary genetic decomposition because not all students were interviewed. However, since only 29% of students provided the correct response in the items related to the solution of a system of equations it could be argued that these were the students who developed the object conception of the concepts.

In summary the results showed that in relation to matrix algebra concepts pre-service teachers' mental constructions were mainly at the action and process stage. The object stage is considered to be more sophisticated because it is at this level where one is believed to have fully developed the conceptual understanding of the learnt concepts. Therefore this means that for many of the pre-service teachers their conceptual understanding of the concepts is still at the developmental stage. It could be said that they have constructed procedural understanding of the concepts but as indicated that is not enough for them to understand relationship between concepts. This is an important aspect of mathematics learning. The results of this study showed that in the learning of matrix algebra concepts the framework used as indicated by literature proved to be true. Brown et al (2004) indicated that the Action-Process-Object stages are hierarchical, meaning that for a student to develop the process stage, he/she must have developed the action conception of the concepts. This was evident since students who had not constructed the meaning of the rules used had difficulty in predicting the outcomes or verbalising their actions. What the study showed was that having the correct answer was not necessarily an indication that one had developed the conceptual understanding of the concepts. Students can dwell in the action/process stage and pass the course while they might have not reached the object conception of the concept which means they have not conceptualised the concepts. According to Stewart (2008) this is actually a common trend. Tall (2004a, p.30) stated that “There are many occasions when individuals do not encapsulate a given process into thinkable object and instead carry out procedures in a routinised way based on repetition and interiorisation of learned operation. This happens not only with students who fail, it can happen in a very successful way, in which familiar procedures are performed on symbols that do not have natural conceptual embodiment for the individual concerned”

9.5 Changes to the preliminary genetic decomposition

The following research question was mainly concerned with the modification of the preliminary genetic decomposition.

What characteristics of the schema displayed by the pre service teachers are adaptable to a genetic decomposition of matrix algebra?

The preliminary genetic decomposition proved to be a useful tool to evaluate students’ thinking and analysing the mental construction made. However the findings showed that some of the responses could not be explained in terms of the presented genetic decomposition. Also some of the characteristics of the students’ mental constructions were not considered in the preliminary genetic decomposition. Therefore the researcher decided to modify the preliminary genetic decomposition based on what transpired in the data findings of this study. The findings of this study suggested a number of changes. These will be considered per item and at the end presented as a combined modified genetic decomposition for matrix algebra concepts. In the modified itemised genetic decompositions the changes made will be in bold. As indicated at the beginning, this study is a contribution to undergraduate mathematics teaching that is why the researcher decided to do an itemised genetic decomposition for each item. It has been observed in the literature that in some institutions these concepts are not taught in a single module. Therefore the instructors could use the itemised genetic decomposition specific to the concepts they teach or use the combined itemised genetic presented as Figure 9. 6. Item 1 and item 2 of task 1 (see Appendix

A) were considered to be at the action stage. However students' responses of item 1revealed that identifying the order using the entries is actually at process stage. As well as in item 2 more than just interchanging the rows and column students needed to have the schema of the matrix order in order to internally think about the transpose without having the physical structure and show understanding that the entries will change position in the same order. This is included because during the interviews the researcher realised that even though many students determined the transpose, they could not apply it to solve related problems, let alone explain it clearly. Also some pre-service teachers did not understand that $(A^T)^T=A$. The modified itemised genetic decomposition for order of matrices is presented below in Figure 9.5.1

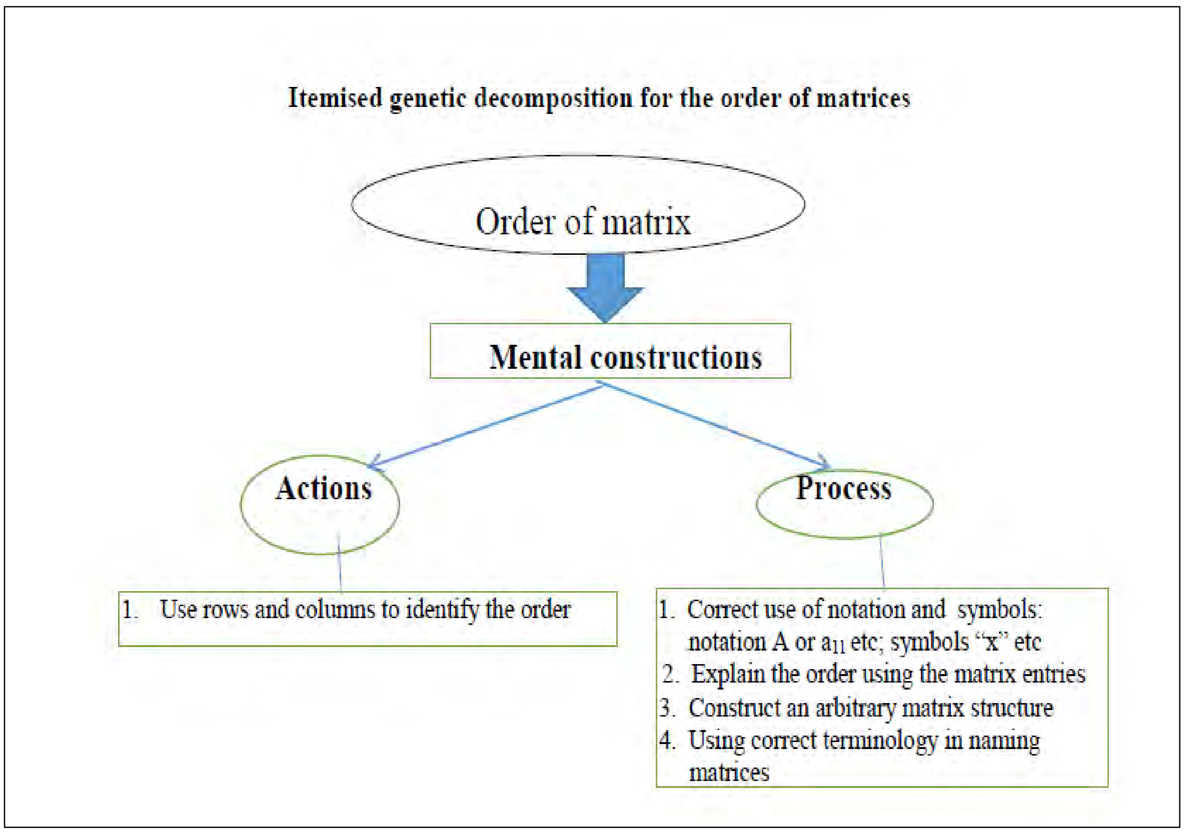


Figure 9.5.1: A modified itemised genetic decomposition for matrix order

In figure 9.5.2 below the modified itemized genetic decomposition for matrix transpose is presented.

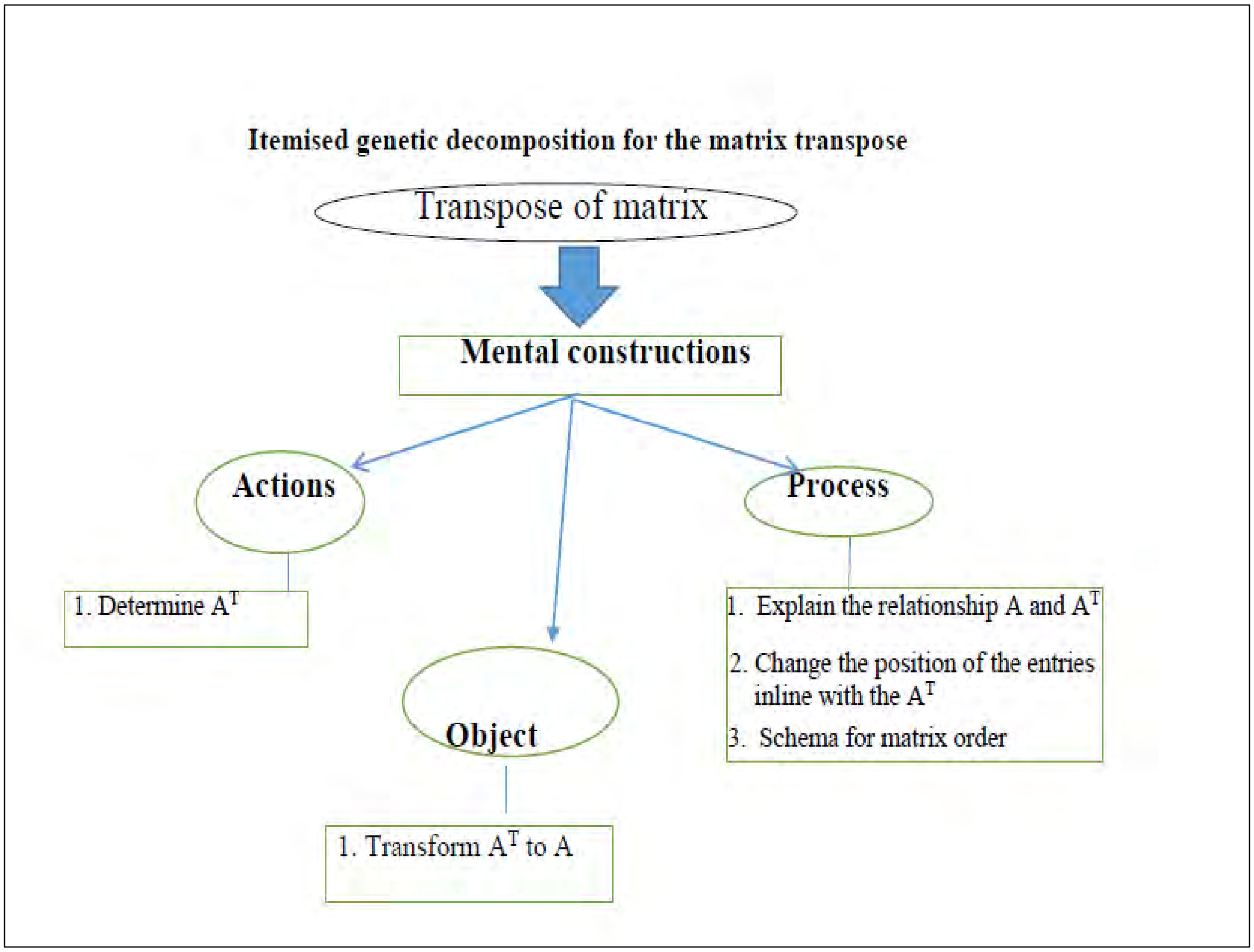


Figure 9.5 2: A modified itemised genetic decomposition for matrix transpose

Items 3, 4 and 5 of task one focused on different computations of matrices. Item 3 focused on scalar multiplication, addition and subtraction of matrices. Items 4 and 5 focused on determining matrix product as well as explaining the constraints of matrix multiplication. Item 1 in task 3 also focused on matrix product. It aimed to extend students' knowledge beyond application of rules and understanding and explaining the constraints of matrix multiplication. The main aim of this item was to help students identify the relationship of matrix product to other related concepts and be able to unpack the structure of a matrix internally. In item 3 of task 1 students at the action stage multiply each element of the matrix with a scalar without thinking about the effect of scalar for the whole matrix as without seeing the solution as an object. Students who thought about effect of the scalar and invented a short cut to get to the answer were considered to be at the process stage. Those thought about the effect of the scalar for whole solution and transform the structure of the matrix were the object stage. These were some of the characteristics of the schema displayed by some of the pre-service teachers. For item 4 and 5, it was noted that students needed to have well-grounded basic algebra schema, real number schema and strong language skills to develop the process conception of the matrix product. For item 1 in task 3, to develop the object stage needed to identify the case where the product is defined e.g both matrix AB and matrix BA is defined. The modified itemised genetic decomposition for the computation of matrices is presented below (see Figure 9.5.3).

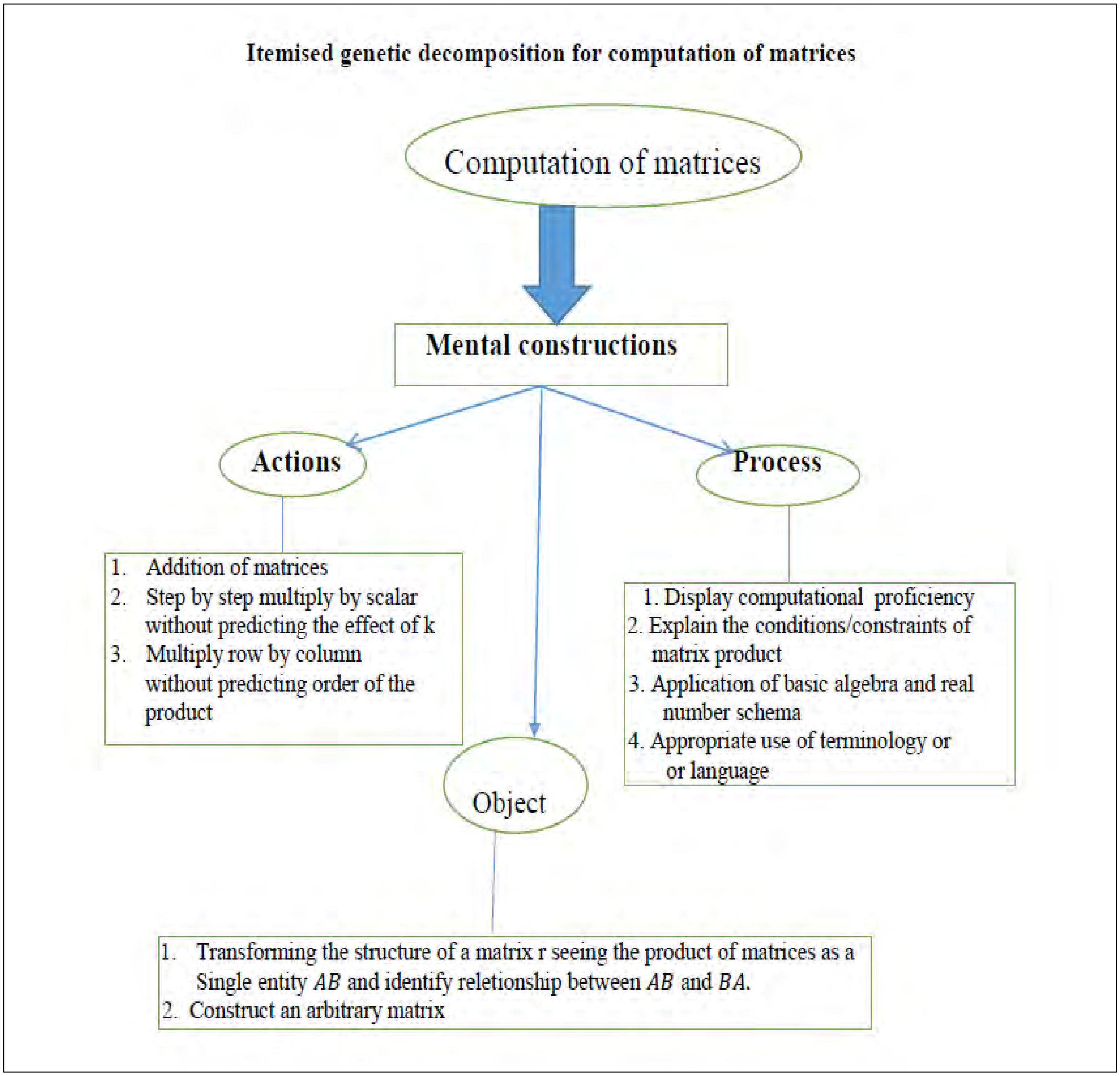


Figure 9.5.3: A modified itemised genetic decomposition for matrix operation

Students' difficulties with notation and lack of basic arithmetic skills made it difficult to focus on their ability to evaluate determinants and the application of determinants to other related concepts. Item 1 and 2 of task two in chapter seven originally intended to focus a great deal on the evaluation of determinants and identifying the relationship between the determinant of a matrix and the determinant of its transpose. Instead it ended up being a long discussion about problems with notation and lack of background knowledge as well as misconceptions students' have regarding the transpose. Although this discussion was not originally the focus of the question it help in understanding the difficulties that hindered the construction of the necessary mental constructions. Among the five students interviewed, four showed that they could not make connections and explain beyond the knowledge of rules the relationship between the determinant of a matrix and determinant of its transpose. This suggested that the action or a process concept of determinant and matrix transpose is not enough for students to develop conceptual understanding of these concepts. Students must have an object conception to be able unpack the structure of a matrix and construct transformations of these relationships. These aspects were considered when modifying our preliminary ideas and appear in Figure 9.5.4. The misconception students make could be explained as problems with terminology (Findell, 2006), confusing matrix properties with determinant properties (Aygur & Ozdag, 2012), met befores (Tall, 2008; Nogueira de Lima & Tall, 2008). Since some students were unable to make the necessary mental constructions shows that pre-service teachers knowledge of matrices and determinants consists of misconceptions that could be detrimental to their conceptual understanding of these concepts and other related concepts. Therefore the modified itemised genetic decomposition for evaluation of determinant is presented.

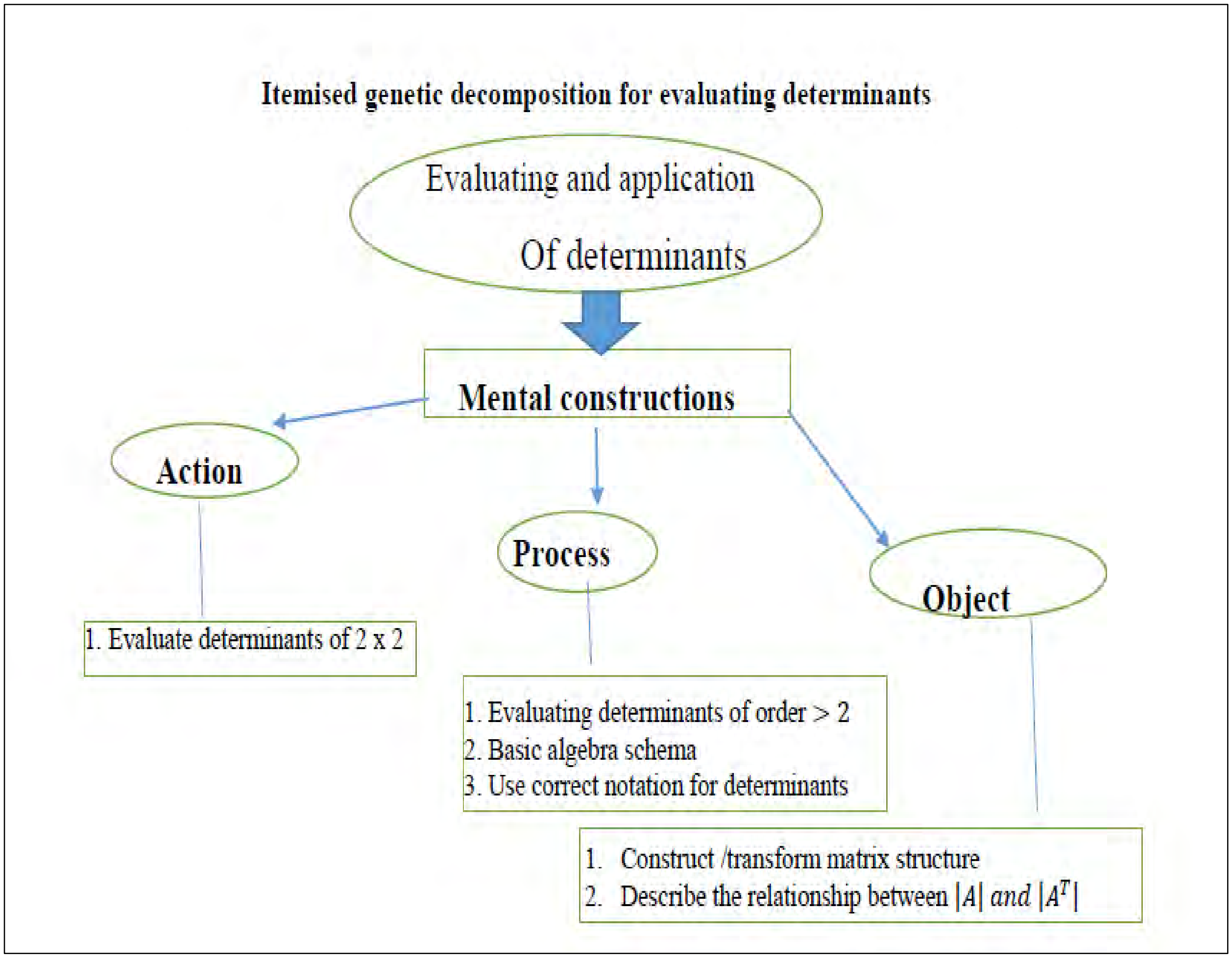


Figure 9.5 4: A modified itemised genetic decomposition for evaluation of determinants

For matrix inverse some students had the object conception of the concepts. The results suggested certain characteristics of the schema need to be included in order for students to show if they have conceptualised the concepts. The schema an individual must bring to the study of matrix inverse are inverse of real numbers, zero property of division, determinant. This was based on the mental constructions that students were able to make and those that were not able to make when engaging in an interview with the researcher. These characteristics were used to modify the preliminary genetic decomposition. These were discussed in detail in chapter eight as well as in the above sections. The modified itemised genetic decomposition for matrix inverse is presented in Figure 9.5.5.

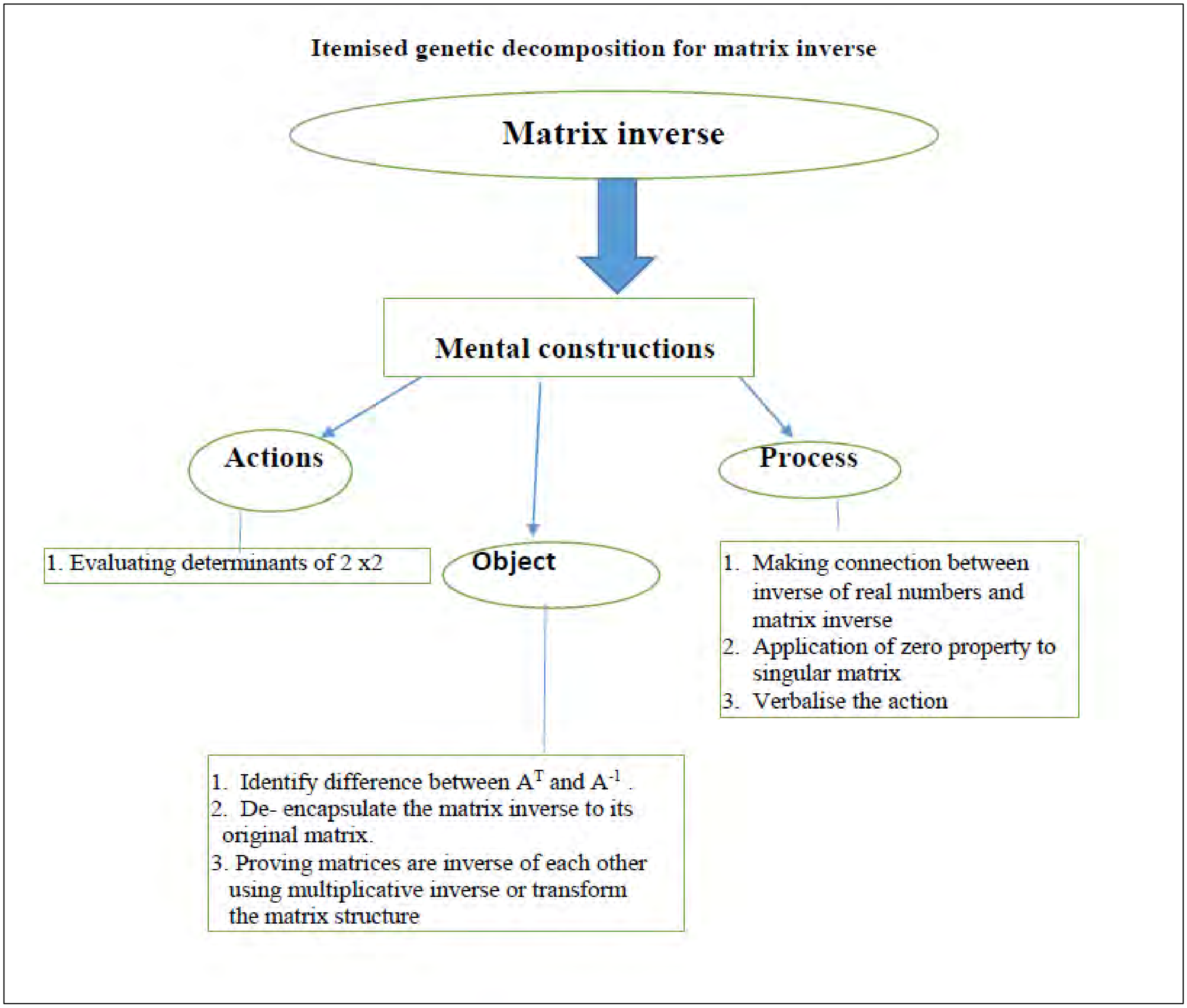


Figure 9.5. 5: A modified itemised genetic decomposition for matrix inverse

For pre-service teachers to produce a correct answer when determining the matrix inverse they should have the process conception of the Cramer's rule. In the preliminary genetic decomposition the mental constructs relating to this item were formulated together with those identified as needed for system of equation. This was done because these two concepts are interrelated. After analysing students' responses the researcher decided it would be better for future purposes to have mental constructs specifics to application of Cramer's rule. The mental constructs indicated in this modified genetic decomposition were those that students had when answering this item. Pre-service teachers who failed to answer this question were assumed to be at the action stage, assuming that they could use Cramer's rule to determine the solution. The modified itemised genetic decomposition for application of Cramer's is presented in Figure 9.5.6.

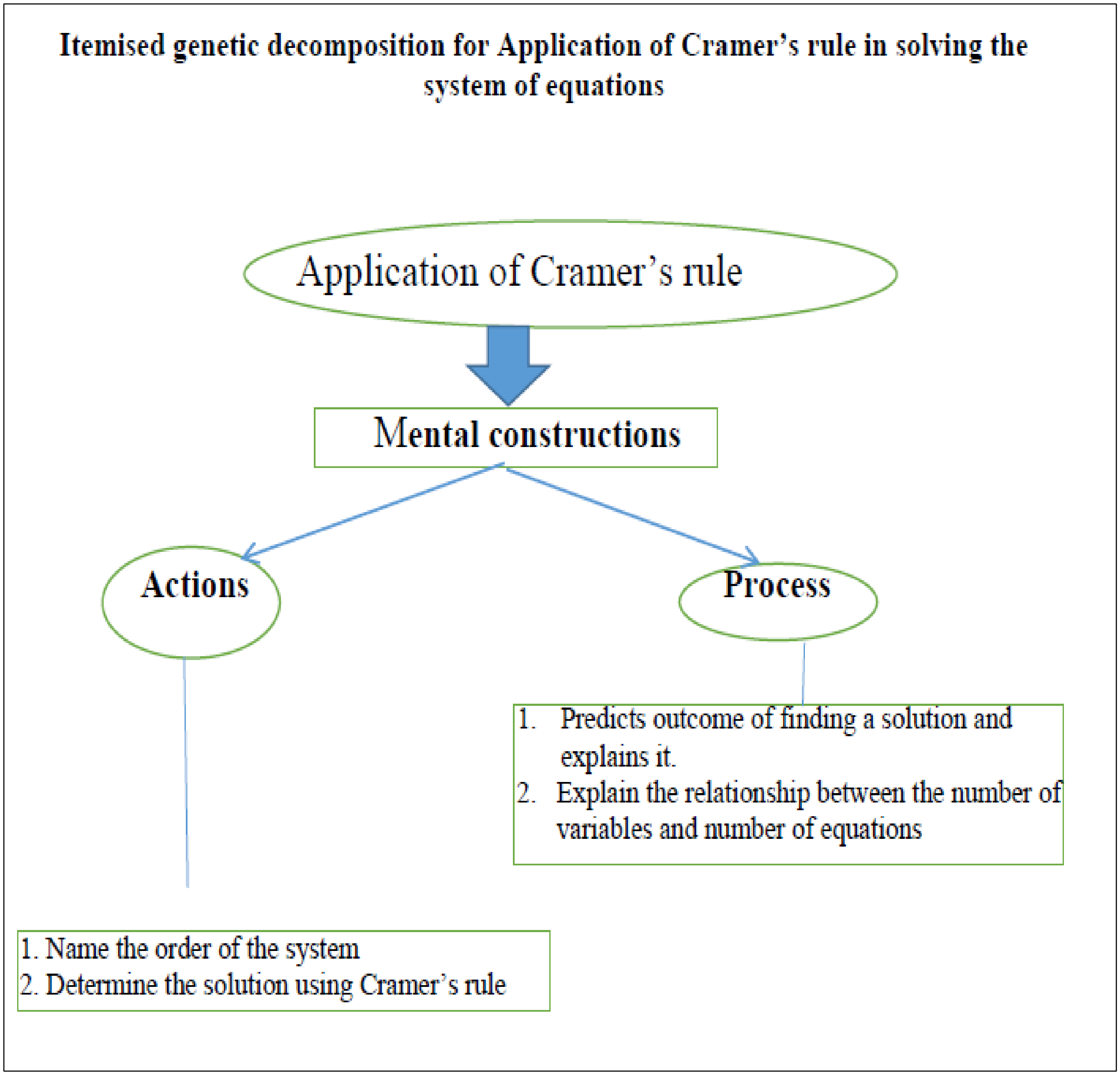


Figure 9.5.6: A modified itemised genetic decomposition for Cramer's rule

The original design of these two items assumed that students were familiar with the solution of system of equations and therefore they could generalise their knowledge to solve problems in its abstract nature. This assumption was not entirely correct. It is true that students were familiar with solution of system of equations. However, since most of them had not yet conceptualised the meaning of the solution to the given system, they could not make connections between the different representations of the solution of a system, could not explain the meaning of variable and the relationship between the number of variables and the number of equations so they were unable to generalise their knowledge of dealing with concrete system of equations to solve the system of equation in its abstract nature. By this time it was assumed that students would have the object conception of the solution of the system of equations. However, the results showed that many students were still operating at action level. What transpired in the students' mental constructions was that there were some gaps in their knowledge about the solution of the system of equations. Therefore for students to develop the conceptual understanding of this concept these need to be addressed. In the object stage indicated below "D" refers to determinant of the system since the formula, for example, to determine the x coordinate is given by $x = \frac{D_x}{D}$. These aspects are incorporated in Figure 9.5.6 and Figure 9.5.7. The modified itemised genetic decomposition for the system of equations is presented in Figure 9.5.7.

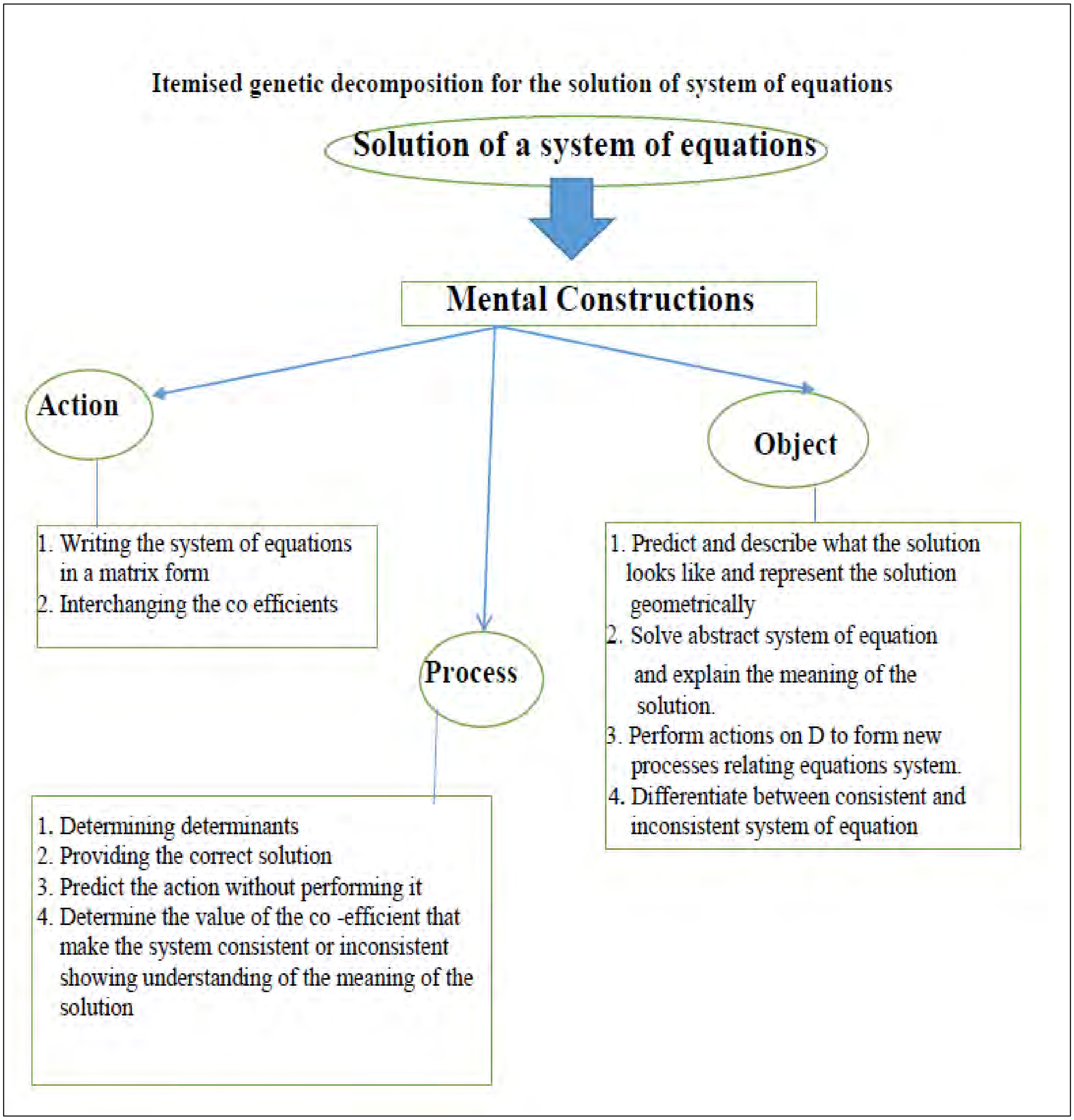


Figure 9.5 7: A modified itemised genetic decomposition for system of linear equation

9.6 A modified itemised genetic decomposition for matrix algebra concept

After considering the itemized genetic decompositions (that is the genetic decompositions for each item) we now provide an overall modified genetic decomposition in Figure 9.6.

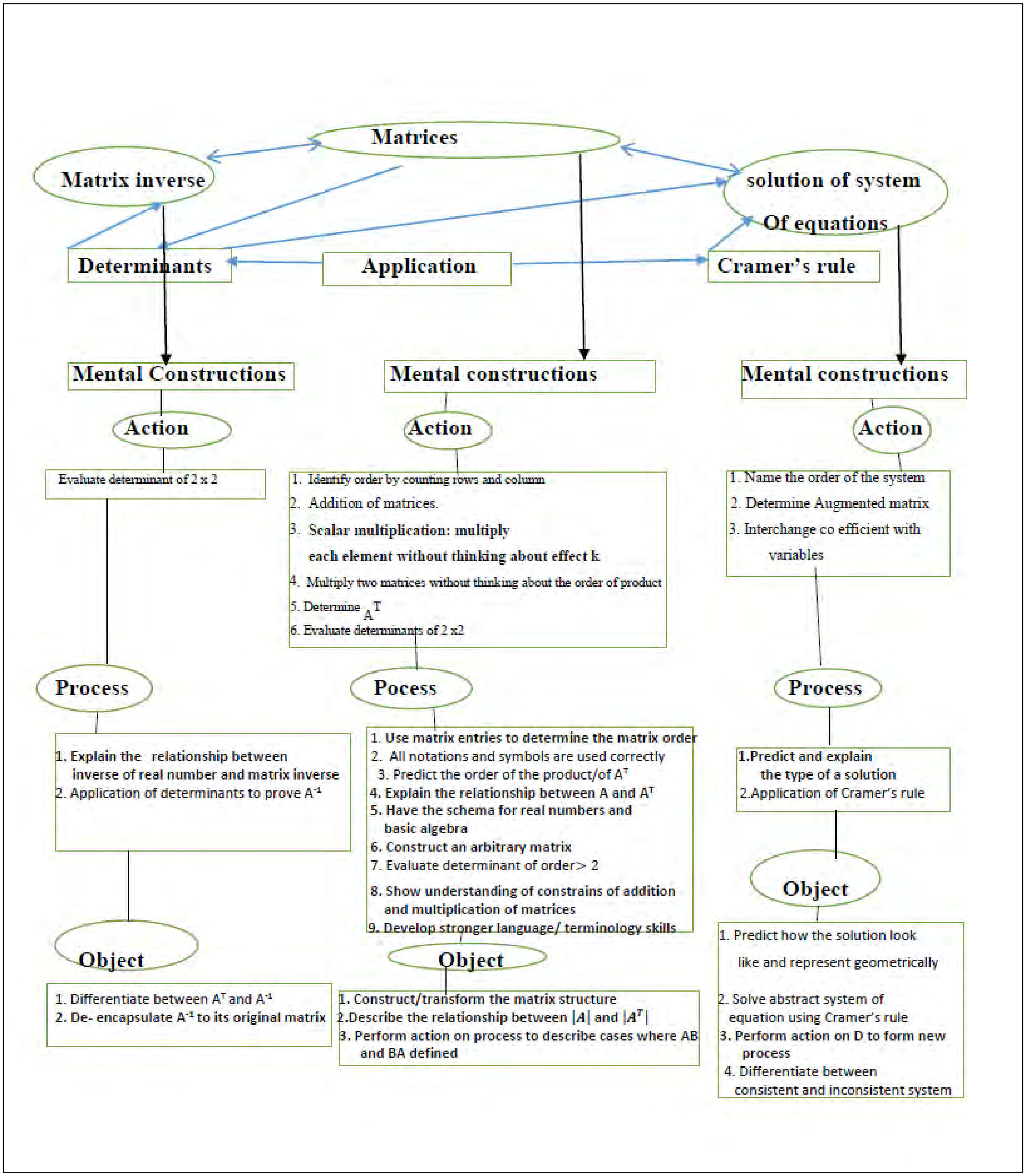


Figure 9.6: A modified itemised genetic decomposition for matrix algebra concepts

9.7 Conclusion

This chapter commenced with the researcher revisiting the aims of this study. A summary of the research study followed. Within this summary key aspects directly related to each critical research question was discussed. This study focused on exploring pre-service teachers mental constructions of matrix algebra concepts. Based on data collected in this study, the preliminary genetic decomposition used to analyse pre-service teachers specific mental construction of matrix algebra proved to be useful. Also the inclusion of the triad mechanism proved useful in describing the level of thinking of pre-service teachers especially those not catered for in the preliminary genetic decomposition. Through the use of preliminary genetic decomposition as an analytic tool, pre-service teachers' difficulties with matrix algebra concepts were identified. These helped in explaining why students' could not make the necessary mental constructions where they did not. It also helped to identify the concepts where necessary mental constructions were made. This chapter concludes by providing the modified genetic decomposition of matrix algebra based on the characteristics of the schema displayed by pre-service teachers.

CHAPTER TEN

CONCLUSION AND RECOMMENDATIONS

10.1 Introduction

> "If we want our students to be enthused by mathematics, to approach it eagerly and positively, and if we want them to appreciate them to appreciate what mathematics is like as a discipline rather simply a body of definitions, theorems, proofs and techniques, then it behoves us to be mathematical with and in front of our students. This does not mean that is effective to walk in and solve lot of problem, formulate definitions and prove theorems in front of them, mindless of their presence. On the other hand, neither is it effective to give a truncated and stylised presentation which supports impression that mathematics is completely cut, dried and salted, that it is something that one can either pick up easily or not at all"(Mason, 2002, p.4).

The main aim of this study was to explore the specific mental constructions of pre-service FET mathematics teachers when learning matrix algebra concepts. Matrix algebra is one of the topics in the module called mathematics for educators 210. All the students planning to teaching mathematics have to do it since it is one of their major modules at this university where the study was conducted. The researcher believes that students need to have conceptual understanding of the concepts taught in their first major module since these concepts form the basis for other modules they will learn as future mathematics teachers. Also, as future mathematics teachers it is imperative to develop conceptual understanding of mathematical concepts so that they will teach mathematics with confidence and to have the requisite knowledge enough to help learners at school. The data generated revealed issues related to pre-service teachers' conceptualisation of matrix algebra concepts. These were discussed in chapter six to eight and the general findings were discussed in chapter 9. In this chapter, the researcher aims to present recommendation for future teaching of matrix algebra, limitations of the study, and conclusions made.

10.2 Conclusion

This chapter commenced with the researcher revisiting the aim of this study, followed with researcher recommendations and limitations. It concludes by making suggestion for further research in matrix algebra related concepts in the context of South Africa. In this study the APOS framework was used to construct a genetic decomposition of the schemas needed to understand the conceptual stage of matrix algebra concepts. Some students responded well to the activity sheet in terms of completing the problems. However the majority were unable to provide correct and complete response to all the task especially those items whose level of thinking were supposed to be at the object stage in terms of APOS. The use of APOS theory to study the conceptual steps of matrix algebra proved to be useful since in this study the focus was on helping pre-service teacher's construct schema for specific matrix algebra concepts. APOS theory has proved to be useful in these cases as a web of concepts can be constructed. By linking APOS and triad mechanism helped in making sense of all specific mental constructions that pre-service made or not made. What was most prevalent was that APOS theory does enhance the development of conceptual understanding of some mathematical topics especially in abstract algebra. Therefore it could lead to the design of more effective instructional methods in the teaching of advanced mathematics in the undergraduate level.

10.3 Recommendations

The recommendations made are structured under the following headings: (1) pedagogical instructions, (2) use of APOS in exploring pre-service teachers' mental constructions, (3) re-examining the content of matrix algebra.

10. 3.1 Pedagogical instructions

Part of the rationale of this study was to bring new knowledge in the teaching of matrix algebra concepts. As Dubinsky (1997) suggested, before pedagogical strategies are considered, the concepts that give students difficulties in linear algebra need to be analysed epistemologically. The researcher had observed that pre-service teachers mostly had difficulties with conceptualising matrix algebra concepts. According to the APOS theory, students need to perform mathematical tasks, discuss their results and listen to fellow students and lecturer. On the other side, the lecturer needs to provide a theoretical analysis modelling the epistemology of the concepts in which the specific mental constructions that a learner might make in order to develop his/ her understanding are described (Tziritas, 2011). Then design and implement instruction that will foster the making of those mental constructions. This will provide an opportunity to observe and assess the type of mental constructions made. This provides an opportunity for an inter-play between teaching and learning since both lecturers and students are constantly evaluating the knowledge learnt and knowledge provided. Teaching for meaning goes beyond solving of routinised problems, it requires students to be part and parcel of the learning activities which will instill in them the skills and knowledge to explore meaning and reasoning. Mason (2002) asserts that:

> "By promoting the actions of specialising and generalising in concert, lecturers can promote students use of their own powers to make sense of mathematics and to make mathematical sense of situations. If concepts are always introduced through examples, or always introduced through definitions followed by examples students are more likely to become dependent on the expert for the specialising and generalising that they could be doing themselves" (p.38).

The above statements recommend some mathematical reform in the teaching of mathematical concepts. As part of pedagogical considerations this study provides a genetic decomposition for matrix algebra concepts. This is hoped to result in instructional treatment that would guide students to make necessary mental constructions relevant to matrix algebra and leads to improvement of their understanding of relevant concepts. Moreover by focusing on specific mental constructions, the preliminary genetic decomposition provided a deeper analysis of students' understanding of matrix algebra concepts. On the other hand it proved to be an effective data collection and data analysis tool which worked effectively on the methodological aspect. Through the use of APOS, the researcher makes two suggestions about teaching of matrix algebra concepts. These are based on what transpired in the results of the study.

First, it is important that pre-service teachers have a sufficient view of the matrix algebra concepts not only as concrete concepts but also the abstract nature of it. Therefore the teaching of matrix algebra should involve problems that encourage students to explain their thinking strategies. Teaching should not only focus on solving problems but students should be provided opportunities to talk about their solution. The findings discussed in chapter nine showed that when students were allowed to talk about their solution, they began to show some characteristics of the necessary mental constructions as Jack said " *You see now I understand what do we mean by inconsistent, by that time I said* $k = 3{,}5$ *I wasn't sure what I was writing*. This came after we were discussing his response to item 5 in chapter eight. That means students should be given opportunities to interrogate what they write. As Maharaj (2014) suggested that this will help them to reorganise and refine their mental structures and schemata. Moreover, allowing students to talk about their solution forces them to use the language of mathematics thereby improving their understanding and correct use of mathematical terminology. The results of this study showed that it could help them change their beliefs about the body of mathematics. The response to some of the general questions asked, in your group did everyone explain their strategies and did that help you in any way? John said "yes *we did share and try and explain how we find our answers but some were not sharing. For mina kwangisiza ngoba ngesikhathi ngikhuluma ngangibona lengenze khona amaphutha. Nala ngingakwazi khona kahle ukuchaza, sengibona ukuthi angazi kahle. Then ngibuyela emuva ngawubhekisisa, yingakho ubona manje sengikwazi ukukuchazela kahle kodwa ebe ama answer emi ewrong. Uyazi imaths iyabhora if uzolokhu ukhuluma ngamarules kodwa kungcono uma usuzama ukwahlangainisa nento oyibhalile. (It helped because by explaining I was able to identify my mistakes. Where I couldn't explain I realised I do not understand, then go over that work again, that is why now I can explain some of the things that were wrong as my answers. Maths can be boring if you always talk about rules but it becomes better when you start to relate those rules to your solution).*Yong-Loveridge, Sharma, Taylor, Hawea, (2006) emphasized that in most cases student beliefs about mathematics are a result of their experiences in mathematical classes.

Second, it is insufficient to only examine the mental constructions that students make. It is also important to analyse those mental constructions that students could not make and the possible reasons that cause them to fail to make those mental constructions. Therefore the teaching would then focus on addressing those challenges. It is therefore recommended that lecturers in the mathematics discipline try to design teaching material that target the development of conceptual understanding of the concepts by helping students make the necessary mental constructions of the learnt concept. For example in the teaching of solution of a system of equations lecturers should emphasise on the multirepresentation of the solution since from the findings of this study it appeared that students had difficulties with that aspect of the concept. This will help in building rich relational schemas among students that contain internal representations of the external ones (Hong, Thomas & Kwon, 2000). Also, to increase the development of advanced mathematical thinking among students there is a need to get students to solve tasks that are more abstract. This will help student's gain formal understanding of the concepts. As Findell (2006) pointed out that through abstract algebra students learn the importance of precise language in mathematics and the role of definition in supporting that precision.

Harel (2000) suggested three principles for the teaching and learning of linear algebra as discussed in chapter two, the researcher recommends that necessity principle and generalisability principles are more applicable in the teaching and learning of these concepts in matrix algebra. Research into the teaching and learning of algebra suggested that using technology would support students' in constructing meaningful meaning of the learnt concepts by focusing more on the content then doing tedious computations. The researcher concurs with this contention but adds that this study suggests that doing hand calculation is beneficiary especially in the matrix algebra concepts. As it was revealed from the results that proficiency with procedures helps in linking previously learned ideas with new ones and turn vague ideas into process and possibly into the development of object conception.

10.3.2 Using APOS in exploring pre-service teachers' mental constructions

While APOS proved to be useful in exploring pre-service teachers conceptual understanding of matrix algebra in this study resulting in a genetic decomposition of the concepts, it is further recommended for researchers to look into development of action/process/object. The results of the study revealed that some of the results could not be really categorised as action/ process/ object since they were not fully developed. Vinner (1991) introduces the notion of pseudo-conceptual understanding and characterises it into two types. One is when students do not understand the topic but want to appear as if they do. In this case they use mathematical terminology and do as much as they can remember to make it look like they understand. For example Sipho when describing the transpose of matrix, tried to link it to many unrelated concepts. The other issue we found to be careful about is when students think they understand but in reality they do not. In this case they may produce a partially correct answer. Many students evaluated $|C|$ but struggled to evaluate $|D^T|$. One might argue that most of the students' responses could be classified as pseudo-process or pseudo- object unless the concepts have been conceived in their totality. This would then suggest that the concepts are still developing. For example some students were considered to have an object view of the solution of system of equations as they could represent their solution on a plane or surface. Based on the findings of this study it could be argued that not all interiorised processes or encapsulated objects are the same. This means that the focus should rather be on characterisation of different kinds of action/process/ object conceptions for the specific concepts. The results of this study agree that the action/process/object is developmental and it allows for the distinction of mental constructions made.

10.3.3 Re-examining the content of matrix algebra

One could argue that the matrix algebra should be a module on its own so as to cover all the important topics such as linear (in) dependence, eigenvalues, eigenvectors at this university where the study was conducted. Knowing how to evaluate determinants is not enough but students need to see its application to other concepts outside of matrix algebra such as volumes and areas as well as the application of matrices to real life. Usiskin (2012) pointed out for students to understand mathematics they need to see its application to real life. It is the researcher's view, based on the findings of this study that students need to start to engage with abstract algebra as early as in their first year of study. Therefore concepts like linear (in) dependence, linear transformations, eigenvalues, vectors etc would be suitable for students. Abstract algebra could be a setting in which pre-service teachers develop a deep sense of the nature and role of definitions and proofs in mathematics (Findell, 2006). Therefore if we hope for our secondary school learners to develop the sense of mathematical reasoning, then at the outset the same idea needs to be instilled in the teachers. The better place to start is with undergraduate students, especially in their first year. It is insufficient for pre-service teachers to only have a concrete view of concepts such as inverse, identity and solution of system of equations. The secondary mathematics curriculum includes concepts such as inverse functions, solutions of system of equations and geometry. Therefore pre-service teachers need to develop sufficient sense of dealing with more abstract concepts in order to do justice in the teaching of these concepts at school level.

10.4 Limitations of the study and suggestion for further exploration

This study has some limitations. First, this was a small scale study with one group of first years and second years so the results could not be generalised to the other groups. We are aware that variables differ from one setting to the other and from one discipline to the other. However the researcher hoped that the genetic decomposition presented is rich enough to be adopted by other lecturers teaching the same concept to the other groups. Second, the issue of lecturer/ researcher might have impacted in the way students presented their responses in the activity sheet and also in the interviews. Students might have tried to get as much information as they could from the textbook in order to produce a correct answer. Also students might have not spoken freely during the interview since they might have felt they must present their answer in a particular way. However the researcher did address some aspects of biasness by interviewing selected participants who volunteered to take part in the interview. Also, by allowing students to speak in any language they are comfortable with other that English the researcher thought that these pre-service teachers could feel free. Moreover students were encouraged to ask the researcher anything they wanted to know related to the study.

Third, because of time constraints due to students' protest not enough time was allocated to the teaching of matrix algebra instead of four 90 minutes tutorial session, the study ended up being conducted over two 90 minutes tutorial sessions. Only the main tasks that addressed specific mental constructions were administered. Furthermore the venues for the tutorials were not set up properly, at the start of each session, students needed to get furniture from other venues which put pressure on them rushing to complete within a short space of time and did not allow for whole class discussions after they completed the tasks. It would have been wise to have also a revolving camera, which would have been able to capture all the students' discussions and their expressions when argued for correctness of their responses.

This study explored pre-service teachers' mental constructions of several concepts in matrix algebra using the preliminary genetic decomposition. It would be interesting to explore pre-service teachers mental constructions of the same concepts using other framework such as Tall's three world (2002) or Sfard (1991) theory of reification since these frameworks also focused on cognitive growth of mathematical concepts and compare the analysis of results. As suggested that other concepts such as eigenvalues/ eigenvectors should be incorporated in the first year course, therefore it is suggested that pre-service teachers' mental constructions of these concepts be explored. This will result in the genetic decomposition of many mathematical concepts, which hopeful will improve instructional methods and develop deeper understanding of those concepts among pre-service teachers. Further study could evaluate the effectiveness of the genetic decompositions in developing deeper understanding of mathematical concepts of various concepts in a South African context.

REFERENCES

1. Ahmet, I., Konyahoglu, A.C ., & Sabri, I. (2003). On the teaching linear algebra at the University level: The role of visualisation. *Journal of the Korea Society of Mathematical Education Series D, 7*(1), 59-67.
2. Alves, D., & Artique, M. (1995). *Articulation of Problems between different Systems of Symbolic Representation in Linear Algebra.* Paper presented at the Proceedings of the 19th Conference of Psychology of Mathematics Education, Universidade Federal de Pernambuco, Recife, Brazil.
3. Angen. (2000). Evaluating Interpretive Inquiry: Reveiwing the validity debate and opening a dialogue. *Qualitative Health Reserach, 10*(3), 378-395.
4. Appova, A., & Berezovski, T. (n.d). *Commonly Identified Student's Misconceptions about Vectors and Vector operations*.
5. Arnawa, I. M., Sumarno, U., Kartasasmita, B.,& Baskoro, E.T. (2007). Applying the APOS theory to improve students ability to prove in elementary algebra. *Journal for Indones Mathematics Association, 13*(1), 133-148.
6. Arnon, I., Cottril., Dubinsky,E.,Fuentes S.R.,Trigueros M.,& Weller, K. (2014). *APOS Theory: A Framework for Research and Curriculum Develoment in Mathematics Education*. New York: Springer.
7. Artique, M. (1999). The teaching and learning of mathematics at the university level. *Notices of the AMS*, 1377-1385.
8. Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (2004). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. *Research in Collegiate Mathematics Education, 6*, 1-21.
9. Asiala, M., Brown,A., DeVries,D., Dubinsky,E., Mathews,D., & Thomas K. (1997). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. *Research in Collegiate Mathematics Education 6*, 1-21.
10. Attorps, L. (2006). *Mathematics teachers' conception about equations.* Doctor of Philosophy, University of Helsinki, Finland.
11. Aydin, S. (2009). The factors affecting teaching linear algebra. *Procedia Social and Behavioural Sciences, 1*, 1549-1553.

12. Aygor, N., & Ozdag, H. (2012). Misconceptions in linear algebra: the case of undergraduate students. *Procedia Social and Behavioural Sciences, 46*, 2989-2994.
13. Aziz, Z., Meerah, T.S.M.,& Tambychick, T. (2010). Mathematics skills Difficulties: A mixture of intricacies. *Procedia Social and Behavioural Sciences, 7*, 171-180.
14. Bassey. (2000). *Case Study Research in Educational Settings*. Philadephia: Open University Press.
15. Berman, A., Koichu, B., & Shvartsman, L. (n.d). Understanding Understanding Matrices. Retrieved Retrieved December 15, 2013 from Google Scholar database
16. Bernard, R. (2000). *Social Research Methods*. California: Sage.
17. Berry, J. S., Lapp, A.D, & Nyman, A.M. (2010). Students connections of linear algebra concepts: an analysis of concepts map *International Journal of Mathematics Education in Science and Technology, 41*(1), 1-18.
18. Berth, E. W., Piaget, J. (1966). *Mathematical Epistemology and Psychology*. Dodrecht: Reidel.
19. Beswick, K. (2006). The importance of mathematics beliefs. *AMT, 62*(4), 17-22.
20. Biggs, J. (2003). Aligning Teaching and Assessing to course objectives. *Teaching and Learning in Higher Education, 2*, 13-17.
21. Biggs, J., & Tang, C (2007). *Teaching For Quality Learning at University*. McGraw Hill: Open University Press.
22. Bogomolny, M. (2006). *The role of example-generation tasks in students' understanding of linear algebra.* A thesis submitted in partial fulfilment for the degree of Doctor of Philosophy, Simon Fraser University, Canada.
23. Bogomolny, M. (2007). *Raising students' understanding of linear algebra.* Paper presented at the Proceedings of the 37th Conference of the international group for the psychology of mathematics Education.
24. Brijlall, D., & Bansilal, S. (2010). *A Genetic decomposition of the Riemann Sum by student teachers*. Paper presented at the Proceedings of the Eighteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education, University of KwaZulu Natal, Durban.
25. Brijlall, D., & Maharaj, A. (2009a). *An APOS analysis of students' onstruction of the concept of continuity of a singled - valued function*. Paper presented at the In Proceeding

of the 7th Southern Hemisphere Conference on Mathematics and Statistics Teaching and Learning, Stellenbosch.

26. Brijlall, D., & Maharaj, A. (2009b). Using an inductive approach for definition making: monotonicity and boundedness of sequences. *Pythagoras, 70*, 68-79.
27. Brijlall, D., & Maharaj, A. (2010). *Exploring a theoretical framework to understand student learning of concepts in differential calculus*. Paper presented at the Proccedings of the16th Annual Ameas National Congress, University of KwaZulu Natal, Durban.
28. Brijlall, D., & Maharaj, A & Jojo, Z. (2013). From Human Activity to Conceptual Understanding of the chain rule. *A Journal of research in Mathematics Education, 2*(1), 77-99.
29. Brijlall, D., & Ndlovu, A.Z,. (2013). High school learners' mental construction during solving optimisation problems in calculus: a South African case study. *South African Journal of Education, 33*(2), 180-197.
30. Brijlall, D., Jojo, Z., & Maharaj, A. (2013). Schema development for the chain rule. *South African Journal of Education, 27*(3), 645-661.
31. Britton, S., & Henderson, J. (2009). Linear algebra revisted: an attempt to understand students' conceptualm difficulties. *International Journal of Mathematical Education in Science and Technology, 40*(7), 963-974.
32. Brown, A., Dubinsky, E., McDonald, M., Stenger., & Weller, K. (2004). Intimations of Infinity. *Notices of AMS, 51*(7), 741-750.
33. Burhanzade, H., & Aygor, N. (2013). Problems that the students face while solving 1st degree equations with two unknowns. *Global Journal of socilogy 3*(1), 9-16.
34. Carlson, D. (1993). Teaching Linear Algebra: Must the fog always roll in? *College Mathematical Journal, 12*(1), 29-40.
35. Carlson, D., Johnson, C.R., Lay, C.D.,& Porter, D. (1993). The linear algebra curriculum group recommendations for the first Course in Linear algebra. *The College Mathematics Journal, 24*(1), 41-46.
36. Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. F. T. A. Romberg (Ed.), *Mathemetic classroom that promotes understanding* (pp. 19-32). Mahwan: Erlbaum Associates.

37. Carrizales, R. (2011). *Cognitive constructs in linear algebra: Metaphors, Metonymies, Modes.* Master of Arts in Teaching Theses presented to the Facultybof the Graduate School of the University of Texas at El Paso, University of Texas at El Paso, Texas. (1518191)
38. Celluci, C. (2013). Philosophy of mathematics:Making a fresh start. *Studies in History and Philosophy of Science, 44*, 32-42.
39. Clark, J. F., Cordero, J., Cottrill, B., Czarnocha, D.J., DeVries, D., St. John, G.T., & Vidakovic, D. (1997). Constructing s Schema: The Case of a Chain Rule. *Journal of Mathematical behaviour, 16*(4), 345-364.
40. Cobern, W. W. (1995). Constructivism. *A Journal of Educational and Psychological Consultation, 4*(1), 105- 112.
41. Cohen, L., Manion, L., & Morrison, K. (2007). *Research Methods in Education* (6th ed.). London: Sage.
42. Cohen, L., Manion, L., & Morrison, K. (2011). *Research Methods in Education* (7th ed.). USA: Routledge.
43. Cooley, L., Dexter,S., Loch,S.,Martin,M. & Vidakovic,D. (2010). Integrating learning theories and application-based modules in teaching linear algebra. *Linear algebra and its application, 432*, 2089-2099.
44. Cooley, L., Loch, S., Martin, B., Meagher, M.,& Vidakovuc, D. (2006). *The learning of linear algebra from a APOS perspective.* Paper presented at the Proceedingd of the 28th annual meeting of the North American Chapter of the international Group for the psychology of Mathematics Education, Merida: Mexico.
45. Corbin, J., & Strauss, A. (2005). *Basics of Qualitative Research* (3rd ed.). Thousands Oaks: Sage.
46. Cresswell, J. W. (2009). Mapping the field of mixed methods research. *Journal of mixed method research, 3*(2), 95-108.
47. Creswell, J. W. (2007). *Qualitative Inquiry and Research Design* (2nd ed.). London: Sage.
48. Cutz, B. (2005). *Un estudio acerca de las concepciones de estudiantes de licenciatura sobre los sistemas de ecuaaciones y su solucion.* Master's degree, University of Mexico, Cinvestav.
49. Day, J. M., & Kalman, D. (1999). Teaching Linear algebra: Issues and Resources. *The College Mathematics Journal, 32*(3), 162-168.

50. De Vries, D., Arnon, I. (2004). *Solution- What Does It Mean? Helping Linear Algebra Students Develop the Concept while Improving Research Tools.* Paper presented at the Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway.
51. Denzin, N., & Lincoln, Y. (2005). *Qualititative Reserach* (3rd ed.). London: Sage.
52. Denzin, N. K. (2002). The Interpretive Process. In A. M. H. B. M. Miles (Ed.), *The Qualititative Research Companion* (pp. 349-366). London: Sage.
53. Devatak, I., Glazer,S.A.,&Vogrinc,J. (2010). The role of qualitative research in science education. *Eurasia Journalof Mathematics, Science and Technology, 6*(1), 77-84.
54. DeVillies, M. (2007). *Mathematical Application: Modelling and Technology* Paper presented at the Proceedings of the 13th Annual National Congress of AMESA, Uplands College, Mpumalanga, SA.
55. Dikovic, L. (2007). Interactive learning and the teaching of linear algebra by Web technologies. *The teaching of Mathematics, 10*(2), 109-116.
56. Dorier, J. L. (2000). On a research programme concerning the teaching and learning of linear algebra in the first -year of a French science university. *International Journal of Mathematical Education in Science and Technology, 31*(1), 27-35.
57. Dorier, J. L. (2002). Teaching linear algebra at University. *ICM, 3*, 1-3.
58. Dorier, J. L., Robert, A., & Rogalski,M. (2000). On Research Programme about Teaching and Learning of Linear Algebra in First Year of French Science University. *International Educational Journal of Mathematics Education in Sciences and Technology, 31*(1), 27-35.
59. Dossey, J. (2007). Modelling and Application in Mathematics. In I. W. B. M. Niss (Ed.), *Mathematical Modelling on the Catwalk* (pp. 88-95). Netherlands: Springer link.
60. Dreyfus, T., Hillel, J., & Sierpinska, A. (2000). Capri Based Linea Algebra: Transformations. *European Research In Mathematics Education, 2*, 209-221.
61. Dubinsky, E. D. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Kluwer: Dordrecht.
62. Dubinsky, E. D. (1997). *Some Thoughts on a First Course in Linear Algebra at the College Level*. Purdue University.
63. Dubinsky, E. D. (2001). Using a Theory of Learning in College Mathematics courses. *MSOR Connections, 1*(2), 10-15.

64. Dubinsky, E. D., & McDonald M.A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education at university level. *An ICM Study*, 273-280.
65. Dunlap-Dogan, H. (2006). Lack of set theory relevant prerequisite knowledge. *International Journal of Mathematics Education in Science and Technology, 37*(4), 401-410.
66. Dunlap-Dogan, H. (2010). Linear algebra students' models of reasoning: Geometric representations. *Linear algebra and its Application, 423*, 2141-2159.
67. Dunlosky, J., & Metcalfe, T. (2009). *Metacognition*. Los Angelos: Sage Publication.
68. Durkay, M., Senel, O., Ocal, O., Kaplan, A.,Aksu,Z., & Konyahoglu, A.C. (2011). Pre-service mathematics teachers' multiple representation comptencies about determinant concept. *Procedia Social and Behavioural Sciences, 15*, 2554-2558.
69. Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in A Learning of Mathematics. *Education studies in Mathematics, 61*, 103-131.
70. Egodawatte, G. (2011). Is Algebra Really Difficult for All Students? *Acta Didacta Napocensia 2*(4), 101-106.
71. Ernest. (1991). *The Philosophy of Mathematics Education*. London: The Falmer Press.
72. Ferriri-Mundy, J., & Graham, K.G (1991). Issues for Learning, Teaching and Curriculum development. *The American Monthly 98*(7), 627-635.
73. Findell, B. R. (2006). *Learning and Undertsanding in Abstract Algebra.* Doctor of Philosophy, University of New Hampshire.
74. Flavell, J. H. (1976). Metacognitive Aspects of Problem Solving. In I. L. B. Resnick (Ed.), *The Nature of Intelligence* (pp. 231-236). Hillsdale: Erlbaum.
75. Flick, U. (2006). *An introduction to qualitative Research* (3rd ed.). London: Sage.
76. Frorer, P., Hazzan, O., & Manes, M. (1997). Revealing the faces of abstraction. *International Educational Journal of Computers For Mathematical Learning, 2*(3), 217-228.
77. Goodson, T. E. (2014). Reflective Abstraction as an Individual and Collective Learning Mechanism. *Forty Years of Radical Constructivism in Educational Research 9*(3), 1-3.
78. Goris, T., & Dyrenfurth, M. (2010). *Students' Misconception in Science, Technolgy and Engineering*. Paper presented at the ASSEE/Indiana Section Conference, Purdue University.

79. Grace, S., Higgs, J., & Horsfall, D. (2009). *Writing Qualitative Research in Practice*. Australia: Sense.

80. Gray, E., Pinto,M.,Pitta, D., & Tall, D. (1999a). Knowledge construction and divergent thinking in Elementary and Advanced Mathematics. *Education studies in Mathematics, 38*, 111-133.

81. Gray, E., Pinto,M.,Pitta, D., & Tall, D. (1999b). Knowledge Construction and Diverging Thinking in Elementary & Advanced Mathematics. *Educational Studies in Mathematics, 38*(1), 111-133.

82. Guba, E. G., & Lincoln, Y. (1994). Competing Paradigms. In *I. N. D. Y. Lincoln* (Ed.), *Handbook for Qualitative Research* (1st ed., pp. 105-117). Thousands Oaks: Sage.

83. Guthrie, G. (2010). *Basic Research Methods*. London: Sage Publication.

84. Habgood, K., & Arel, I. (2012). A condensation based application of Crammr's rule for solving large linear system. *Journal of Discreet Algorithms, 10*, 98-109.

85. Hamdan, M. (2005). Nonlinear learning of linear algebra: active learning through journal writing. *International Journal of Mathematics Education in Science and Technology, 36*(6), 607-615.

86. Hapaaalo, T. (2004). *On Instrumental Genesis with Procedural and Conceptual Thinking*. Finland: Univesity of Joensuu.

87. Harel, G. (1997). The Linear Algebra Curriculum Study Group Recommendations: Moving beyond concept defeintion. In C. J. In D. Carlson, D. Lay, D. Porter, A. Watkins, & W. Watkins (Ed.), *Resources for Teaching Linear Algebra* (pp. 107-126). MAA Notes, Washington: Mathematical Association of America.

88. Harel, G. (2000). Principles of Learning and Teaching Mathematics, With Particular Reference to the Learning and Teaching of Linear Algebra. In I. J. L. Dorier (Ed.), *On the Teaching of Linear Algebra* (pp. 177-189). Dordrecht: Kluwer Academic Publishers.

89. Hasenbank. (2006). *The effects of a frameowrk for procedural understanding on college algebra students' procedural skill and understanding.* A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, Montana State University.

90. Hazzan, O. (1999). Reducing abstract in abstract algebra. *Education studies in Mathematics, 40*, 71-90.

91. Hierbert, T., & Carpenter, T. (1992). Learning and Teaching with Understanding. In i. D. A. Grouws (Ed.), *Hand book of Research On Mathematics Teaching and Learning* (pp. 65-97). New York: McMillan.

92. Hierbert, T. J., & Lefevre, P. (1996). Conceptual Knowledge in mathematics: The case of mathematics In T. Hierbert (Ed.), *Conceptual and Procedural Knowledge* (pp. 1-27). Hillsdale: Erlbaum.

93. Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In I. J. L. Dorier (Ed.), *The Teaching of Linear Algebra in Question* (pp. 191-207). Dodrecht, Netherlands: Kluwer Academic Publishers.

94. Hillel, J. (2002). Computer algebra Systems in the teaching and learning of linear algebra: Some examples. *Teaching and Learning in Higher Education, 7*, 371-380.

95. Hobden, P. (2006). *Mathematical Profieciency - a way of describing succesful mathematics learning*. Unpublished. Mathematics, Science and Technology. University of KwaZulu Natal. Durban.

96. Hoepfl, P. (1997). Research Methods. *Journal of Technology Education, 9*(1).

97. Hong, Y. Y., Thomas, M., & Nam Kwo, O. (2000). *Understanding linear algebraic equations via super -calculator representations.* Paper presented at the Proceedings of the 19 th mathematics education research group of Australian Conference, Melbourne.

98. Jojo, Z. (2010). *Students' understanding of the Chain rule.* Masters Degree a thesis submitted in partial fullfillment for the academic requirement for the Masters degree in mathematics Education, University of KwaZulu Natal, Durban.

99. Jojo, Z. (2011). *An Exploration of Conceptual Understanding of the Chain rule in Calculus by First year Engineering Students.* Doctor of Mathematics Education, University Of KwaZulu Natal, South Africa.

100. Jojo, Z. (2014). Mental Constructions formed in the Conceptual undertsanding of the Chain rule. *Mediterranean Journal of Social Sciences, 5*(1), 171-179.

101. Kardash, C. M., & Howell, K.L. (2000). Effects of Epistemology Beliefs and Topic-Specific Beliefs on undergraduates' Cognitive and strategic processes of Dual-Positional text. *Journal of Educational Psychology, 92*(3), 524-535.

102. Klasa, J. (2010). A few pedagogical designs in linear algebra with Cabri and Maple. *Linear algebra and its Application, 432*, 2100-2111.

103.Kuzle, A. (2013). Patterens of Metacognitive Behaviour During Mathematics Problem Solving in a Dynamic Geometry Environment. *International Journal of Mathematics Education, 8*(1), 20-39.

104.Leron, U., & Dubinsky, E. (1995). An abstract Algebra Story. *AM. Math Monthly, 102*(3), 227-242.

105.Liu, P. (2010). Are beliefs believable? An investigation of college students' epistemological beliefs and behaviour in mathematics. *The Journal of Mathematical behaviour, 29*, 86-98.

106.Magiera, M. T., & Zawojeski, J. (2011). Characteristics of Social -based and Self-based Contexts associate with students awareness, evaluation and regulations of their thinking during small group mathematical modelling. *Journal for Research in Mathematics Education, 42*(5), 486-520.

107Maharaj, A. (2014). An Apos Analysis of Natural Science Students' Understanding of integration. *Journal of Research in Mathematics Education, 3*(1), 54-73.

108.Maoto, S., & Wallace, J. (2006). What does it mean to teach mathematics for understanding:When to tell and when to listen. *African Journal of Research in SMT Education, 10*(1), 59-70.

109.Marshall, C., & Rossman, G.B. (2006). *Designing Qualitative Research*. London: Sage.

110.Mason, J. (2002). *Mathematics Teaching Practice: A guide for university and college lecturers*. Chichester: Albion/Horwood Publishing House.

111.Masson, J. (2002). *Qualitativr researching* (2nd ed.). London: Sage.

112.Maznichenko, M. A. (2002). *Overcoming of Professional and Pedagogical errors as a stage in Professional Development of the teacher.* Doctor of Philosophy, Sochi State Pedagogical University, Russia.

113.Mendick, H. (2005). A beautiful Myth? The Gendering of Being/Doing Good at Maths. *Gender Education, 17*(2), 203-219.

114.Merrian, B. (2002). *Qualitative Research in Practice* (1st ed.). San Francisco: Jossey-Bass Publishers.

115.Michener, E. R. (1978). Understanding Understanding Mathematics. *Cognitive Science, 2*(361-383).

116.Naidoo, J. (2011). *Exploring Master Teachers' Use of Visuals as Tools In Mathematics Classrooms.* Doctor of Mathematics Education, University of KwaZulu Natal, South Africa.

117.Naidoo, K. (2007). First Year Students' Understanding of Elementary Concepts in Differential Calculus in a Computer Laboratory Teaching Environment. *Journal of College Teaching and Learning 4*(9), 55-66.

118.Ndlovu, A. Z. (2012). *Grade 12 Learners' Construction of knowledge of Optimisation Problems.* A Masters Degree thesis submitted in partial fullfillment of the academic requirement for the degree of Masters in Mathematics Education, University of KwaZulu Natal, Durban.

119.Nehme, Z. (2011). Blogging and the learning of linear algebra concepts through contextual mathematics. *Mathematics teaching*(225), 43-48.

120.Neuman. (2006). *Social Research Methods* (6th ed.). USA: Pearson Education Inc.

121.Noguera de Lima, R., & Tall, D. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics, 67*, 3-18.

122.Noss, R., & Hoyles, C. (1996). *Windows on Mathematical Meanings- Learning Cultures and Computers*. Dordrecht: Kluwer Academic Publishers.

123.O'Donoque, T., & Punch, K. (2003). *Qualitative Educational Research in Action: Doing and Reflecting*. London: Routledge.

124.Ollerton, M. (2006). *Getting the Burgers to Add Up* (2nd ed.): Bloomsbury Publishing.

125.Ozdag, H., & Aygor, N. (2012). Undergraduates Students' Performances in Linear Algebra: factorization of a determinant of n * n dimensional Matrices. *Procedia Social and Behavioural Sciences, 46*, 125-129.

126.Panasuk, M. R. (2010). Three phase ranking framework for assessing conceptual understanding in algebra using multiple representations. *Educational research Journal, 131*(2), 236-255.

127.Parameswaran, R. (2010). Expert Mathematician approach to Understanding Definitions. *The Mathematics Educator, 20*(1), 43-51.

128.Parker, F. (2010). How intution and langauge use relate to students' understanding of span and linear independence. *Langauge and Education, 19*(2), 1-5.

129.Parraguez, M., & Oktac, A. (2009). Construction of the vector space concept from view point of APOS theory. *Linear Algebra and it's Application, 432*(2010), 2112-2124.

130.Patton, M. Q. (1990). *Qualitative research and evaluation methods*. London: Sage.

131.Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). London: Sage.

132.Piaget, J. (1978). *Success and Understanding*. London and Henley: Routledge and Kegan Paul.

133.Piaget, J. (1985). *The Equilibrium of Cognitive Structures*. Cambrigde, Massachutte USA: Harvard University Press.

134.Piaget, J., Garcia, R. (1989). *Psychogeneses and History of Science*. Columbia: University Press.

135.Pillay, E. (2008). *Grade 12 learners' understanding of the concept of derivative.* A thesis submitted in partial fullfillment for the academic requirement for the degree of Masters of Mathematics Education a theses submitted in partia fullfillment of the academic requirement for the degree of Masters of Mathematics Education, University of KwaZulu Natal, Durban.

136.Pinto, M., & Tall, D. (2001). *Building Formal Mathematics on Visuary Imagery: A case study and theory.* Paper presented at the Proceeding of the 24th International Conference for Psychology of Mathematics Education, Haifa, Israel.

137.Presmeg, N. (2002). Beliefs about Nature of Mathematics in the bridging of everyday and school mathematical practices. In E. P. In G. Leder, & G. Tomer (Ed.), *Beliefs: A hidden varaible in mathematics education* (pp. 293-312). Dordrecht: Kluwer.

138.Rabin., M. J., Sweeney, G.F., & Wawro, M. (2011). Subspace in linear algebra: Investigating students' concept images and interactions with formal defintion. *Educational Studies in Mathematics, 78*, 1-19.

139.Rittle-Johnson, B., Sigler, R.S., & Alibali, M. (2001). Developing Conceptual Understanding and Procedural Skill in Mathematic: An Iterative Process. *Journal of Educational Psychology, 93*(2), 346-392.

140.Robert, A., Robinet,J. & Rogalski, M. (2000). On a reseach programme concerning the teaching and learning of linear algebra in the first-year of a French Science University. *International Journal of Mathematics Education in Science and Technology, 31*(1), 27-35.

141.Rolka, K., & Rosken, B. (2007). Integrating Intution: The Role of Concept Image and Concept Definition for students Learning integral. *The Montana Mathematics Enthuisiast, 3*, 181-204.

142.Rossman, B. G., & Marshall, C. (2011). *Designing Qualitative Research*. London: Sage Publication.

143.Sahin, O., & Soylu, Y. (2011). Mistakes and Misconceptions of Elementary School Students about Concept of a Variable. *Procedia Social and Behavioural Sciences, 15*(2011), 3322-3327.

144.Samson, D. (2007). Patterns of Visualisation. *Learning and Teaching Mathematics, 5*, 4-9.

145.Scherr, R. E. (2007). Modelling Students' thinking: An example from Special Relativity. *American Journal of Physics, 75*(3), 272-280.

146.Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook for Research on mathematics teaching and learning* (pp. 334-370). New york: McMillan.

147.Schultz, A. (1962). On Multiple Realities. *Collect Papers, 1*, 207-259.

148.Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and objects as different sides of the same coin. *Education studies in Mathematics, 22*, 1-36.

149.Sfard, A., & Linchevski, L. (1994). The gains and Pitfalls of Reification- Case of Algebra. *Education Studies in Mathematics*(26), 191-228.

150.Shahbari, T. A., Daher, W.,&Rasslan, S. (2014). Mathematical Knowledge and the Cognitive and Metacognition Processes Emerged in Models. *Journal on New Trends in Education and Their Implications, 5*(2), 209-219.

151.Sierpinska, A. (1994). *Understanding in Mathematics*. Washington D.C: The Falmer Press.

152Sierpinska, A. (2000). On Some Aspects of Students' thinking in Linear Algebra. In I. J. L. Dorier (Ed.), *On The Teaching of Linear Algebra*. Dordrecht, Netherlands: Kluwer Academic Publishers.

153.Sierpinska, A. (2005). Discoursing Mathematics Away. *Meaning in Mathematics Education, 37*, 205-230.

154.Siyepu, S. W. (2013). An exploration of students' errors in derivatives in a university of technology. *The Journal of Mathematical behaviour, 32*, 577-592.

155.Skemp, R. R. (1976). *The Psychology of Learning Mathematics* (2nd ed.). Harmondsworth, England: Penguin Books LTD.

156.Stewart, S. (2008). *Understanding Linear Algebra Concepts Through the Embodied, Symbolic and Formal Worlds of Mathematical Thinking.* Doctor of Philosophy, University of Auckland.

157.Stewart, S., & Thomas, M.O.J. (2007). *Embodied , Symbolic and Formal aspects of basic linear algebra*. Paper presented at the Proceedings of the 31st conference of international group for the psychology of mathematics education.

158.Stewart, S., & Thomas, M.O.J. (2008). *Linear Algebra Thinking: Embodied, Symbolic and Formal Aspects of Linear Independence*. Paper presented at the The 11th Conference of Special Interest Group of Mathematical Association Of America on Reserach in Undergraduate Mathematics Education, San Diego, California, USA.

159.Stewart, S., & Thomas, M.O.J. (2009). *Linear algebra snapshots through APOS and Embodied, and Formal Worlds of mathematical thinking.* Paper presented at the Proceedings of the 32nd Annual Conference of the MathematicsEducation Group of Australasia, Palmerston North.

160.Swan, M. (2001). Dealing with Misconceptions. In I. P.Gates (Ed.), *Issues in Mathematics Teaching* (pp. 147-165). London: Routledge.

161.Tall, D. (1988). Concept Image and Concept Definition. In I. J. D. L. M. Doorman (Ed.), *Senior Secondary Mathematics Education* (pp. 37-41). Utrecht: University of Warwick Press.

162.Tall, D. (1990). Inconsistences in the Learning of Calculus. *Focus, 3*(4), 49-63.

163.Tall, D. (1991). Psychology of Advance Mathematical Thinking. *Advanced Mathematical Thinking, 11*, 3-21.

164.Tall, D. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity and Proof. In I. D. A. Gouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495-511). New York: MacMillan.

165.Tall, D. (1997). Functions and Calculus. In I. A. J. Bishops (Ed.), *International Handbook of Mathematics Education* (pp. 289-325). Dordrecht: Kluwer.

166.Tall, D. (1999). *Reflections on APOS theory in elementary and advanced mathematical thinking.* Paper presented at the Proceedings of the 23rd Conference of PME, Haifa.

167.Tall, D. (2002). Using Technology to support an Embodied Approach to Learning Concepts in Mathematics. In L. C. G. in L.M Calvalno. (Ed.), *The developing theory of three modes of thinking Embodied, Proceptual and Formal* (pp. 1-28). Rio de Janeiro: Brasil Press.

168.Tall, D. (2004a). Building theories: The three worlds of mathematics. *For the learning of mathematics, 24*(1), 29-32.

169.Tall, D. (2004d). *Thinking Through Three Worlds of Mathematics.* Paper presented at the Proceedings of the 28th Conference of the International Group For the Psychology of Mathematics Education, Bergen, Norway.

170.Tall, D. (2006). A theory of Mathematical growth through Embodiement, Symbolism and Proof. *Annels de Didactiquet de Sciences Cognitives, 11*, 195-215.

171.Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5-24.

172Tall, D., & Vinner,S. (1981). Concept Image and Concept Definition in Mathematics. *Education studies in Mathematics, 12*, 151-169.

173Thompson, F., & Logue, S. (2006). An Exploration of Common Students' Misconceptions in Science. *International Educational Journal, 7*(4), 553-559.

174.Tiberghien, A. (2007). *Construction of Students' knowledge in relation to teaching sequence* France: University de Lyon.

175.Tomita, M. K. (2008). *Examining The Influence of Formative Assessment on Conceptual Accumulation and Conceptual Change* Doctor of Philosophy, Stanford University.

176.Tossavainen, T., Attorps, I., & Vaisanen, P. (2012). Some South African Mathematics Teachers' Concept Images of the equation concept. *African Journal of Research in Mathematics Science and Technology Education, 16*(376-387).

177.Trigueros, M., Oktac, A., & Ku,D. (2010). *Spanning set: an analysis of mental constructions of undergraduate students*. Paper presented at the Proceedings of the 6th Congress of the European Society for research in mathematics education, CERME.

178.Trigueros, M., Oktac, A., & Manzanero, L. (2007). Understanding of Systems of Equations in Linear Alegbra. *Cerme, 5*, 2359-2367.

179.Trigueros, M., Oktac. A., & Manzanero, L. (2007). *Understanding the system of equations in linear algebra.* Paper presented at the Proceedings of the 8th Congress of the European Society for Reserach in Mathematics Education.

180.Tucker, A. (1993). The growing importance of Linear algebra in Undergraduate Mathematics. *The College Mathematics Journal, 24*(1), 3-9.

181.Tziritas. (2011). *APOS Theory as a Framework to study the Conceptual Stages of Related Rates Problems* Master' Degree In Mathematics Education, Concordia University, Montreal, Quebec, Canada.

182.Unlu, M., & Aktas, G.S. (2013). Beliefs' of Pre-service Elementary Mathematics Teachers' and Mathematics Department Students' about Mathematics. *Social and Behavioural Sciences, 93*, 742-746.

183.Usiskin, Z. (2012). *What does it mean to understand some mathematics*. Paper presented at the 12th International Congress on Mathematical Education, South Korea.

184.Van De Walle, A. J. (2007). *Elementary and Middle School Mathematics: Teaching Developmentally* (6th ed.). USA: Pearson Education Inc.

185.Vinner, S. (1991). The role of definition in the teaching and learning of mathematics. In I. D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht: Kluwer.

186.Von Glaserfeld, E. (1995). *Radical Constrauctivism: A Way of Knowing and Learning*. London, Washington: Falmer Press.

187.Vosniadou, S. (2004). Extending the Conceptual Change Approach to Mathematics Learning and Teaching. *Learning and Instructions, 14*, 445-451.

188.Wawro, M. (2011). *Individual and Collective Analyses of the genesis of students Reasoning Regarding the invertible matrix theorem in linear algebra.* a thesis submitted in partial fullfillment of the academic requirement for the degreein Doctor of Philosophy University of California, San Diego.

189.Weber, K. (2004). Traditional Instructions in Advanced Mathematics courses: A case study of one Professor's Lectures and Proofs in an introduction of real analysis course. *The Journal of Mathematical behaviour, 23*(2), 115-133.

190.Weyer, R. S. (2010). APOS theory as a conceptualisationfor understanding mathematics learning. *Summation*, 9-15.

191.Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematics Education. In I. I. H. S. Papert (Ed.), *Constructivism* (pp. 193-203). Norwood, NJ: Ablex.

192.Wilson, J., & Clarke, D. (2004). Towards the Modelling of Mathematical Metacognition. *Mathematics Education Research Journal, 16*(2), 25-48.

193.Young-Loveridge, J., Taylor, M., Sharma, S., & Hawera, N. (2006). *Students' Perspectives on the Nature of Mathematics.* Paper presented at the Proceedings of the 29th Annual Conference of the Mathematics Education Research Group of Australia, Adelaide.

194.Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education, 36*(4), 317-346.

APPENDIX A1

ACTIVITY SHEETS FOR PHASE ONE

Question 1

Identify the order of each matrix below

b) $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$

Answer: _______________ Explain:__________________________________

____

c) $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

Answer: _______________ Explain:__________________________________

____

d) $C= \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Answer: _______________ Explain:__________________________________ ___

____

Question 2 [28 marks]

Let A= $\begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ B= $\begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{bmatrix}$

C= $\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix}$ D= $\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$

2.1 Determine the following

2.1.1 A^T

2.1.2 $3C - 2\,D$

2.1.3 $C\ x\ D$

2.1.4 $A\ x\ B$

2.2 Find the determinant of C

2.3 Find the determinant of D^T

2.4 For any 3 x 3 matrices A & B, explain whether $A\ x\ B = B\ x\ A$

Question 3

3.1 Suppose A and B are matrices with AB and BA defined. Explain whether AB and BA are square matrices

3.2 Does the matrix $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ have an inverse? If so what is the inverse? If not explain why?

APPENDIX A 2

ACTIVITY SHEETS FOR PHASE TWO

Task 1

Question 1 -Item 1

Identify the order of each matrix below

e) $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$

Answer: ______________ Explain:________________________________

____

f) $B = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

Answer: ______________ Explain:________________________________

____

$C= \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Answer: ______________ Explain:________________________________ ___

____

Question 2 Item 2 to Item 5

Let $A=\begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ $B=\begin{bmatrix} 2 & 3 \\ -3 & 1 \\ 1 & -3 \end{bmatrix}$

$C=\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix}$ $D=\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$

2.4 Determine the following

2.4.1 A^T (3)

2.1.2 $3C - 2\,D$ (5)

2.1.3 $C\ x\ D$ (4)

2.1.4 Let Matrix E be 3x2 and Matrix F be 3 x 3

2.1.4 a) Is the product of EF defined? Explain (3)

Task 2

Question 1 - Item 1 and Item 2

$C=\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix}$ $D=\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$

1.1Find the determinant of C (4)

1.2. Find the determinant of D^T (5)

1.3. Without doing calculation what is the Determinant of D? Explain your reasoning. (3)

Task 3

3.1 Suppose A and B are matrices with AB and BA defined. Explain whether AB and BA are square matrices

3.2 Does the matrix $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ have an inverse? If so what is the inverse? If not explain why?

Task 4

<u>Question 1[9 marks]</u>

4.1 Consider the system of equation below and answer the questions that follow

$$\begin{aligned} 2x + y - z &= 3 \\ x + y + z &= 1 \\ x - 2y - 3z &= 4 \\ 3x - y - z &= 2 \end{aligned}$$

4.1.1 Can the above system of equation be solved using Cramer's rule?

If yes solve it, if not explain why. (2)

4.2 Use Cramers' rule to solve the system of linear equations for x and y (5)

$$\begin{aligned} kx + (1-k)y &= 1 \\ (1-k)x + ky &= 3 \end{aligned}$$

4.2.1 For what value(s) of k will the system be inconsistent? (2)

APPENDIX B

CONSENT FOMS/LETTERS

Letter of Consent

Humanities & Social Sciences Research Ethics Committee

Dr. Shenuka Singh (Chair)

Westville Campus

Govan Mbeki Building

Ximbap@ukzn.ac.za, snymanm@ukzn.ac.za, mohunp@ukzn.ac.za,

031 260 3587/ 4557/ 8350

To: Participant(s)

Research Project**: Exploring pre-service teachers' mental construction of concepts in matrix algebra**

Year: 2013

I **Annatoria Zanele Ndlovu** (First year Ph.D Student) am doing a study through the **School of Education, Mathematics Education at the University of KwaZulu-Natal** with **Professor Deonarain Brijlall.** His contact numbers are 031-373 2126 (work). We want to research pre-service teachers' mental construction of concepts in matrix algebra at one of the University in KwaZulu-Natal: South Africa.

Students are asked to help by taking part in this research project as it would be of benefit to lecturers and interested educationists and/or mathematics teachers. However, participation is completely voluntary and has no impact or bearing on evaluation or assessment of the student in any studies or course while at the University. Participants may be asked to take part in the interviews after the questionnaires have been completed. These interviews will be tape-recorded in the future. All participants will be noted on transcripts and data collections by a *pseudonym* (i.e. fictitious name). The identities of the interviewees will be kept strictly confidential. All data will be stored in a secured password and not been used for any other purpose except for the research.

Participants may leave the study at any time but need to inform the researcher of their wish to do so. Participants may review and comment on any parts of the researchers' written reports.

__________________________ ____________________

(*Researcher's Signature*) (*Date*)

DECLARATION

A

I,_________________________________ (*Participant's NAME)*_______________(*Signature)*

___________________ (*Date*)

☐ Agree.

N.B. Tick ONE

☐ Disagree.

To participate in the research being conducted by **A. Zanele Ndlovu** concerning *pre-service teachers' mental construction of concepts in matrix algebra.*

I,_________________________________ (*Participant's NAME)*_______________(*Signature)*

___________________ (*Date*)

B

☐ Agree.

N.B. Tick ONE

☐ Disagree.

To be video recorded and audio recorded in the research being conducted by **A. Zanele Ndlovu** concerning *pre-service teachers' mental construction of concepts in matrix algebra.*

C

N.B. Underline One

I hereby consent/ do not consent to have this interview recorded.

C

N.B. Tick the box

I acknowledge that I have the right to withdraw at any stage should I wish to do so. ☐

To: Communications/Research officer: University of KwaZulu Natal

Dean of School of Education: Professor G. Kamwendo

Year: 2013

Research Project:

I **Annatoria Zanele Ndlovu** (First year PhD student) am conducting a study through the **School of Education, Mathematics Education at the University of KwaZulu-Natal** under the supervision of **Professor Deonarain Brijlall**. Proposed research looks towards "Exploring pre-service teachers' mental construction of concepts in matrix algebra at the University of KwaZulu Natal which aims at providing better understanding of "learning and teaching" of matrix algebra concepts necessary for Cramers' rule". In particular, this inquiry looks at how pre-service teachers learn and construct knowledge necessary for matrix algebra and the relationship between their mental constructions with the preliminary genetic decomposition of matrix algebra.

Students are requested to assist through participating in this research project as it would be of benefit to their understanding of this concept of matrix algebra with specific reference to Cramers' rule and interested educationalists/researchers and/or mathematics teachers. However, participation is *completely voluntary* and has no impact or bearing on evaluation or assessment of the student in any studies or course while at University. Participants will be video recorded as they engage with structured questionnaires (*only for analysis purposes)*. Also participants will be asked to take part in the post-course open-ended interviews after the questionnaires have been completed. These interviews will be recorded (*only for analysis purposes*) as the time progresses, but initially no recording will take place. All participants will be noted on transcripts and data collections by a *pseudonym* (i.e. fictitious name). The identities of the interviewees will be kept strictly confidential, even the name of the University will not be disclosed. All data will be stored in a secure password protected server where authentication will be required to access such data.

Participants may revoke from the study at any time by advising the researcher of this intention. Participants may review and comment on any parts of the dissertation that represents this research before publication.

_________________________ _______________

(*Researcher's Signature*) (*Date*)

DECLARATION

I, ____________________________ __________________________(**NAME** and **SIGNATURE**)

Dean on this day of ______ month_______ 2013, hereby grant permission to the researcher to go ahead with the research in the above-mentioned University following the terms of reference noted in this request letter

To: Communications/Research officer: University of KwaZulu Natal

Dean of School of Education: Professor G. Kamwendo

Approval granted 20/9/13

Year: 2013

Research Project:

I **Annatoria Zanele Ndlovu** (First year PhD student) am conducting a study through the **School of Education, Mathematics Education at the University of KwaZulu-Natal** under the supervision of **Professor Deonarain Brijlall**. Proposed research looks towards "Exploring pre-service teachers' mental construction of concepts in matrix algebra at the University of KwaZulu Natal which aims at providing better understanding of "learning and teaching" of matrix algebra concepts necessary for Cramers' rule". In particular, this inquiry looks at how pre-service teachers learn and construct knowledge necessary for matrix algebra and the relationship between their mental constructions with the preliminary genetic decomposition of matrix algebra.

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Participants may revoke from the study at any time by advising the researcher of this intention. Participants may review and comment on any parts of the dissertation that represents this research before publication.

________________ 20 SEPTEMBER 2013

(*Researcher's Signature*) (*Date*)

DECLARATION

I, PROF GH KAMWENDO ________________ **(NAME** and **SIGNATURE)**

Dean on this day of 20 month SEPT 2013, hereby grant permission to the researcher to go ahead with the research in the above-mentioned University following the terms of reference noted in this request letter

APPENDIX C

ETHICAL CLEARANCE CERTIFICATE

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

17 January 2014

Ms Annatoria Z Ndlovu (994246135)
School of Education
Edgewood Campus

Protocol reference number: HSS/1470/013D
Project title: Exploring pre-service teachers' mental construction of concepts in matrix algebra: A South African case study

Dear Ms Ndlovu,

Full Approval – Expedited

In response to your application dated 03 October 2013, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and the protocol have been granted **FULL APPROVAL.**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number.

Please note: Research data should be securely stored in the discipline/department for a period of 5 years.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

……………………………………
Dr Shenuka Singh (Chair)

/ms

Cc Supervisor: Professor D Brijlal
cc Academic Leader Research: Dr MN Davids
cc School Administrator: Mr Thoba Mthembu

Humanities & Social Sciences Research Ethics Committee
Dr Shenuka Singh (Chair)
Westville Campus, Govan Mbeki Building
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Telephone: +27 (0) 31 260 3587/8350/4557 **Facsimile:** +27 (0) 31 260 4609 **Email:** ximbap@ukzn.ac.za / snymanm@ukzn.ac.za / mohunp@ukzn.ac.za
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APPENDIX D

EDITORS CERTIFICATE

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CERTIFICATE OF EDITING

This is to certify that the thesis entitled:

Exploring pre-service teachers' mental construction of concepts in matrix algebra: A South African case study

by

Zanele Ndlovu

has been language edited on the author's behalf for submission.

December, 2014

Genevieve Wood

PhD candidate in the Humanities
Wits University

APPENDIX E

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