UNIVERSITY OF KWAZULU-NATAL

NEW REALISTIC SOLUTIONS FOR CHARGED MATTER WITH AN EQUATION OF STATE

Pedro Mafa Takisa

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Pedro Mafa Takisa

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Abstract

The purpose of this thesis is to provide new exact solutions to the Einstein-Maxwell system which are physically reasonable. We assume that the spacetime is static and spherically symmetric with a charged anisotropic matter distribution. The equation of state is linear. We show that the class of models found by Thirukkanesh and Maharaj (2008) has a singularity in the charge density at the centre of the sphere. Two new exact classes of solutions to the Einstein-Maxwell are found in terms of elementary functions. This contains models in which there is no singularity in charge density. From our general models, we can regain the Thirukkanesh and Maharaj (2008) models and other models as special cases. The physical analysis show that the solutions are relevant for the description of realistic compact relativistic stars. We demonstrate that the mass corresponds to a compact relativistic body with anisotropy in presence of charge. We show that the electromagnetic field appreciably affects the value of the mass. A detailed physical analysis of the matter variables and electromagnetic quantities is performed. The models generated are consistent with the quark stars containing strange matter.

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I, Pedro Mafa Takisa

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Contents

Chapter 1

Introduction

The theory of general relativity is clearly one of the greatest scientific achievements of all time, a theory derived from pure thought and physical intuition. It is capable of explaining and describing the behaviour of the gravitational field. The theoretical predictions of general relativity play a crucial role in observational astrophysics and cosmology. The reader is referred to Davies (1989) and Will (1981) for more detailed analyses of observations and experimental results related to general relativity. The key insight is that gravity is not a physical external force like the other forces of nature but rather a manifestation of the curvature of spacetime. The earlier Newtonian models are applied for weak fields and not in the case of strong gravitational fields, such as neutron stars. General relativistic models may be used for the analysis of strong gravitational fields and black holes where Newtonian models are not appropriate (Shapiro and Teukolsky 1983). In general relativity we use the Riemann tensor for describing the curvature of spacetime, and the energy-momentum tensor for matter distributions, which includes the contribution of the electromagnetic field tensor in presence of charge. The Einstein tensor is related to the total energy-momentum tensor by the Einstein field equations which satisfy the general covariance law, namely the Bianchi identity.

In relativistic astrophysics we include the study of highly dense objects such as neutron stars, and states of collapse leading to black holes, pulsating spheres referred to as pulsars, highly magnetic objects called magnestars, and extreme concentrations of energy such as gamma-ray bursts. Exact relativistic models are necessary to describe observed astrophysical processes. In this context spherically symmetric stellar models are important and they are extensively utilised in many applications. In astrophysics, the collapse of a star can be accurately modelled by a spherically symmetric gravitational field (Shapiro and Teukolsky 1983). In cosmology, spherically symmetric spacetimes have been utilised to model the behaviour and subsequent evolution of the early universe (Krasinski 1997). Under high pressures stars may possess a nonzero charge during the early stages of their evolution (Stephani 1990). This requires solving the Einstein-Maxwell system of partial differential equations.

The fundamental results (exact solutions) used in relativistic astrophysics can be listed as follows:

(a) Historically the Schwarzschild exterior solution is the first exact solution of the Einstein field equations. It is used to model the exterior gravitational field of a static spherically symmetric neutral star.

(b) The interior description of the gravitational field in a star is given by the Schwarzschild interior solution, with constant energy density. This is also a good approximation for small stars with low pressures. The interior and exterior solutions match smoothly at the boundary of star.

(c) The Reissner-Nordstrom solution is the exterior gravitational field valid for a static spherically symmetric charged star.

(d) The Kerr solution is characterised by mass and angular momentum and is related to the exterior of the rotating body. The Kerr-Neumann models are an extension to include the electromagnetic field.

In this thesis we are concerned with the stellar interior with a spherically symmetric charged matter distribution.

Solutions of the Einstein-Maxwell system of equations for static spherically symmetric interior spacetime are important in describing charged compact objects in relativistic astrophysics where the gravitational field is strong, as in the case of neutron stars. The recent analyses of Ivanov (2002) and Sharma *et al* (2001) show that the presence of electromagnetic field affects the values of redships, luminosities and maximum mass of compact objects. The role of the electromagnetic field in describing the gravitational behaviour of stars composed of quark matter has been recently highlighted by Mak and Harko (2004) and Komathiraj and Maharaj (2007a, 2007b). In recent years, many researchers have attempted to introduce different approaches of finding solutions to the field equations. Hansraj and Maharaj (2006) found solutions to the Einstein-Maxwell system with a specified form of the electrical field with isotropic pressures. These solutions satisfy a barotropic equation of state and regain the Finch and Skea (1989) model. Thirukkanesh and Maharaj (2008) found new exact classes of solutions to the Einstein-Maxwell system. They considered anisotropic pressures in the presence of the electromagnetic field with the linear equation of state of strange stars with quark matter. Other recent investigations involving charged relativistic stars include the results of Karmakar *et al* (2007), Komathiraj and Maharaj (2007a, 2007b), Maharaj and Komathiraj (2007) and Maharaj and Thirukkanesh (2009). To get more flexibility in solving the Einstein-Maxwell system, Varela *et al* (2010) found a general approach of dealing with anisotropic charged matter with linear or nonlinear equations of state.

Our object in this thesis is to generate new physical solutions to the Einstein-Maxwell system which satisfy the physical criteria: the gravitational potentials, electric field intensity, charge distribution and matter distribution should be well-behaved and regular throughout the star. We find two new classes of solutions, which contain the Thirukkanesh and Maharaj (2008) model. In particular all matter variables and potentials are regular at the stellar origin.

In chapter 2, we rewrite the Einstein-Maxwell field equation for a static spherically symmetric line element as an equivalent set of differential equations using a transformation due to Durgapal and Bannerji (1983). By using this transformation with the linear equation of state, a new set of differential equations is obtained for the Einstein-Maxwell system. This system describes a gravitating charged fluid with anisotropic pressures in general. We also briefly review the physical properties required in interior solutions to the field equations.

In chapter 3, we provide a review of three classes of solutions found by Thirukkanesh and Maharaj (2008) for charged anisotropic matter with a linear equation of state. We observe that the class has a singularity in the charge density at the centre of the star.

In chapter 4, we present new exact solutions to the Einstein-Maxwell system. Plots are provided for the matter variables and the electromagnetic quantities. The singularity in the charge density is avoided in this class of solutions.

In chapter 5, other new solutions to the Einstein-Maxwell system are given. The results of this chapter contain special cases of previous models. The matter variables and the electromagnetic quantities are plotted. In addition we generate the tables of masses for charged and neutral matter. Our results are consistent with the conclusions of Dey *et al* (1998, 1999) for strange stars.

In chapter 6, we summarise the results obtained in this thesis.

Chapter 2

Field equations

2.1 Introduction

A realistic description of a spherically symmetric relativistic star is provided by the theory of general relativity. We give, in this chapter, the technical information necessary to generate a relativistic stellar model. The relevant differential geometry, structure of spherically symmetric spacetimes and physical criteria for a relativistic stellar model are reviewed. The reader may refer to the texts of Choquet-Bruhat *et al* (1982), de Felice and Clarke (1990), Gron and Hervik (2007), Misner *et al* (1973) and Straumann (2004) for more detailed treatments of these topics. In §2.2 we introduce basic spacetime geometry and curvature. Relativistic fluids, electromagnetics and Maxwell's equations are discussed in §2.3. The Einstein-Maxwell field equations are derived in §2.4 for a spherically symmetric spacetime. We evaluate the relevant equations for neutral and charged fluids, and express the field equations in a form that are easier to integrate, as demonstrated in later chapters. In §2.5 we consider exterior spacetimes matching to neutral and charged interiors, and list the physical conditions necessary for a realistic relativistic stellar model.

2.2 Spacetime geometry

In the theory of general relativity we take spacetime to be a four-dimensional differentiable manifold $\boldsymbol{M}$ containing the metric tensor field $\boldsymbol{g}$. The manifold of general relativity is pseudo-Riemannian since the metric tensor is indefinite. The tensor field $\boldsymbol{g}$ is symmetric and nonsingular with signature $(- + + +)$. The invariant element of distance between neighbouring points is given by

$$ds^2 = g_{ab} dx^a dx^b \tag{2.1}$$

Then using the fundamental theorem of Riemannian geometry we can generate the unique metric connection. We find that

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2}$$

which (2.2) are called the connection coefficients. The connection coefficients (2.2) are necessary for the definition of curvature on the manifold. The Riemann tensor or curvature tensor is defined by

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec} \tag{2.3}$$

On contraction of (2.3) we obtain the symmetric Ricci tensor

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac} \end{aligned} \tag{2.4}$$

On contracting the Ricci tensor (2.4) we obtain

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab} R_{ab} \end{aligned} \tag{2.5}$$

which is the Ricci (or curvature) scalar.

It is now possible to construct the Einstein tensor $\boldsymbol{G}$, in terms of the Ricci tensor (2.4) and the Ricci scalar (2.5), as follows

$$G^{ab} = R^{ab} - \frac{1}{2} R g^{ab} \tag{2.6}$$

which is necessarily symmetric. A defining characteristic of the Einstein tensor is that it has zero divergence

$$G^{ab}{}_{;b} = 0 \tag{2.7}$$

which follows from the definition of the Einstein tensor (2.6). This property is sometimes called the Bianchi identity and generates the conservation of energy-momentum via the Einstein field equations.

2.3 Fluids and electromagnetics

In our model the matter distribution is described by a relativistic fluid. The energy-momentum tensor for uncharged matter is described by the symmetric tensor $\boldsymbol{M}$, where

$$M^{ab} = (\rho + p)u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab} \tag{2.8}$$

In the above ρ is the energy density, p is the isotropic (kinetic) pressure, q^a is the heat flux vector ($q^a u_a = 0$) and π^{ab} is the anisotropic pressure (stress) tensor ($\pi^{ab} u_a = 0 = \pi^a{}_a$). All these quantities are measured relative to a comoving fluid four-velocity $\boldsymbol{u}$ which is unit and timelike ($u^a u_a = -1$). It is convenient to express the stress tensor in the form

$$\pi_{ab} = (p_r - p_t)\left(n_a n_b - \frac{1}{3}h_{ab}\right) \tag{2.9}$$

where p_r is the radial pressure, p_t is the tangential pressure, $\boldsymbol{n}$ is a unit radial vector orthogonal to $\boldsymbol{u}$, and $h^{ab} = g^{ab} + u^a u^b$ is the projection tensor. In perfect fluids there are no heat conduction and stress terms ($q^a = 0, \pi^{ab} = 0$). For a perfect fluid energy-momentum tensor, (2.8) becomes

$$M^{ab} = (\rho + p)u^a u^b + pg^{ab} \tag{2.10}$$

The perfect fluid form (2.10) is applicable in many situations in relativistic astrophysics and cosmology (Stephani *et al* 2003). For many physical applications it is necessary that the matter distribution satisfies the barotropic equation of state

$$p = p(\rho)$$

and the pressure depends only on the energy density ρ.

We define the electromagnetic field tensor $\boldsymbol{F}$ in terms of the four-potential $\boldsymbol{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b}$$

which is skew-symmetric. The electromagnetic field tensor may be expressed solely by the electric field, $\boldsymbol{E} = (E^1, E^2, E^3)$, and the magnetic field, $\boldsymbol{B} = (B^1, B^2, B^3)$. We obtain the following matrix representation

$$F^{ab} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

The electromagnetic contribution $\boldsymbol{E}$ to the total energy-momentum tensor has the form

$$E_{ab} = F_{ac}F^c{}_b - \frac{1}{4}g_{ab}F_{cd}F^{cd} \tag{2.11}$$

In order to analyse the effect of $\boldsymbol{E}$ on the gravitational field we require a covariant formulation of Maxwell's laws. The governing equations are given by

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.12a}$$
$$F^{ab}{}_{;b} = J^a \tag{2.12b}$$

In the above system $\boldsymbol{J}$ is the four-current defined by

$$J^a = \sigma u^a$$

and σ is the proper charge density. For further information on Maxwell's field equations (2.12) see Misner *et al* (1973) and Narlikar (2002). The Maxwell equations (2.12) are the basic equations for the electromagnetic field in a manifold $\boldsymbol{M}$ with a metric tensor field $\boldsymbol{g}$.

The total energy-momentum tensor $\boldsymbol{T}$ is the sum of $\boldsymbol{M}$ and $\boldsymbol{E}$ for a charged gravitating fluid. This is given by

$$T^{ab} = M^{ab} + E^{ab}$$

The gravitational interactions on matter and electromagnetic fields are governed by a relevant set of field equations. These interactions are contained in the Einstein-Maxwell system of equations

$$\begin{aligned} G^{ab} &= T^{ab} \\ &= M^{ab} + E^{ab} && (2.13a) \\ F_{ab;c} + F_{bc;a} + F_{ca;b} &= 0 && (2.13b) \\ F^{ab}{}_{;b} &= J^a && (2.13c) \end{aligned}$$

The system (2.13) is a highly nonlinear system of coupled, partial differential equations governing the behaviour of gravitating systems in the presence of an electromagnetic field. In (2.13a) we use units in which the coupling constant in the Einstein equations is unity. It is necessary to solve the system (2.13) to generate an exact solution. We need to specify forms for the matter distribution and electromagnetic field on physical grounds, integrate the system of partial differential equations, and find explicit forms for the metric tensor field $\boldsymbol{g}$. For uncharged matter the only equations that have to be satisfied are the Einstein field equations

$$R^{ab} - \frac{1}{2} R g^{ab} = M^{ab}$$

which are given by equation (2.3) with $\boldsymbol{E} = 0$. Also observe from (2.7) and (2.13a) we have

$$T^{ab}{}_{;b} = 0 \tag{2.14}$$

which is conservation of total energy-momentum.

2.4 The basic equations

The field equations (2.13) are highly nonlinear; to generate exact solutions we need to make simplifying assumptions. Since our intention is to model a relativistic astrophysical compact star it is reasonable to assume that spacetime is static and spherically symmetric. This assumption is normally made in most treatments to describe highly dense relativistic objects when the effects of rotation may be neglected. Our approach is similar to Shapiro and Teukolsky (1983) and Straumann (2004) in the modelling of physical processes for compact objects.

In standard coordinates $(x^a) = (t, r, \theta, \phi)$, the line element (2.1) has the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.15}$$

The functions $\nu(r)$ and $\lambda(r)$ are related to the gravitational potentials. The nonvanishing connection coefficients (2.2), for the line element (2.15), are given by

$$\Gamma^0{}_{01} = \nu' \qquad \Gamma^2{}_{12} = \frac{1}{r}$$

$$\Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)} \qquad \Gamma^2{}_{33} = -\sin\theta\,\cos\theta$$

$$\Gamma^1{}_{11} = \lambda' \qquad \Gamma^3{}_{13} = \frac{1}{r}$$

$$\Gamma^1{}_{22} = -re^{-2\lambda} \qquad \Gamma^3{}_{23} = \cot\theta$$

$$\Gamma^1{}_{33} = -re^{-2\lambda}\sin^2\theta$$

On substituting the above connection coefficients in the Ricci tensor (2.4), we generate the following nonvanishing components:

$$R_{00} = \left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r}\right] e^{2(\nu-\lambda)} \tag{2.16a}$$

$$R_{11} = -\left[\nu'' + \nu'^2 - \nu'\lambda' - \frac{2\lambda'}{r}\right] \tag{2.16b}$$

$$R_{22} = 1 - \left[1 + r(\nu' - \lambda')\right] e^{-2\lambda} \tag{2.16c}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.16d}$$

From (2.16) and (2.5) we obtain the Ricci scalar

$$R = 2\left[\frac{1}{r^2} - \left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{1}{r^2}\right) e^{-2\lambda}\right] \tag{2.17}$$

The Ricci tensor components (2.16) and the Ricci scalar (2.17) generate the corresponding nonvanishing components of Einstein tensor (2.6):

$$G^{00} = \frac{1}{r^2} e^{-2\nu} \left[r(1 - e^{2\lambda})\right]' \tag{2.18a}$$

$$G^{11} = e^{-2\lambda}\left[-\frac{1}{r^2}(1 - e^{2\lambda}) + \frac{2\nu'}{r} e^{-2\lambda}\right] \tag{2.18b}$$

$$G^{22} = \frac{1}{r^2} e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) \tag{2.18c}$$

$$G^{33} = \frac{1}{\sin^2\theta} G^{22} \tag{2.18d}$$

for the line element (2.15).

Now we find the form of the energy-momentum tensor $\boldsymbol{T}$ with the comoving four-velocity vector $u^a = e^{-\nu}\delta^a_0$ for the line element (2.15). For uncharged fluids ($E^{ab} = 0$) we obtain the nonvanishing components

$$T^{00} = e^{-2\nu}\rho \tag{2.19a}$$

$$T^{11} = e^{-2\lambda} p_r \tag{2.19b}$$

$$T^{22} = \frac{1}{r^2} p_\perp \tag{2.19c}$$

$$T^{33} = \frac{1}{\sin^2\theta} T^{22} \tag{2.19d}$$

For a charged fluid we make the choice

$$A_a = (\phi(r), 0, 0, 0) \tag{2.20}$$

which is a simple form that allows for a nonzero electromagnetic field. The form (2.20) permits one nonzero component of the electromagnetic field tensor $\boldsymbol{F}$ given by

$$F_{01} = -\phi'(r)$$

and the corresponding skew-symmetric component F_{10}. The associated contravariant component of $\boldsymbol{F}$ has the form

$$F^{01} = e^{-(\nu+\lambda)}E(r)$$

where we have defined $E(r) = e^{-(\nu+\lambda)}\phi'(r)$. The quantity $\boldsymbol{E}$ may be interpreted as the electric field intensity. The proper charge density takes the form

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)' \tag{2.21}$$

The components of $\boldsymbol{E}$ may be evaluated using (2.11), and then the total energy-momentum tensor has the nonzero components

$$T^{00} = e^{-2\nu}\left(\rho + \frac{1}{2}E^2\right) \tag{2.22a}$$

$$T^{11} = e^{-2\lambda}\left(p_r - \frac{1}{2}E^2\right) \tag{2.22b}$$

$$T^{22} = \frac{1}{r^2}\left(p_\perp + \frac{1}{2}E^2\right) \tag{2.22c}$$

$$T^{33} = \frac{1}{\sin^2\theta}T^{22} \tag{2.22d}$$

for spherically symmetric spacetimes (2.15). This system reduces to (2.19) for uncharged matter.

2.4.1 Neutral fluids

From equations (2.18) and (2.19) we obtain

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho \tag{2.23a}$$

$$-\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p_r \tag{2.23b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p_\perp \tag{2.23c}$$

which are the Einstein field equations for uncharged anisotropic matter distributions. From the conservation law (2.14) we have

$$\frac{dp_r}{dr} = -\frac{1}{r}\left[2(p_r - p_\perp) + r(\rho + p_r)\frac{d\nu}{dr}\right] \tag{2.24}$$

When $p_r = p_\perp (= p)$ the anisotropy vanishes and we obtain the relationship

$$\frac{dp}{dr} = -(\rho + p)\frac{d\nu}{dr}$$

for a neutral perfect fluid. The result (2.24) is a direct consequence of the field equations and it may be used to replace one of equations in (2.23).

The field equations (2.23) may be expressed in a variety of equivalent forms to simplify the integration process. A second form of the field equations is obtained if we introduce the equivalent transformation

$$x = Cr^2 \tag{2.25a}$$

$$Z(x) = e^{-2\lambda(r)} \tag{2.25b}$$

$$A^2 y^2(x) = e^{2\nu(r)} \tag{2.25c}$$

which was first used by Durgapal and Bannerji (1983). The quantities A and C are arbitrary constants. With the help of the transformation (2.25), the Einstein field equations (2.23) can be written as

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{2.26a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p_r}{C} \tag{2.26b}$$

$$4xZ\frac{\ddot{y}}{y} + (4Z + 2x\dot{Z})\frac{\dot{y}}{y} + \dot{Z} = \frac{p_\perp}{C} \tag{2.26c}$$

where dots represent differentiation with respect to x. Note that the new metric functions y and Z now depend on the new variable x. From (2.26b) and (2.26c) we generate the differential equation

$$4x^2 Z\ddot{y} + 2x^2\dot{Z}\dot{y} + \left(x\dot{Z} - Z + 1\right) y = \frac{x}{C}\left(p_\perp - p_r\right) y \tag{2.27}$$

Equation (2.27) is called the condition of pressure isotropy. The advantage of (2.27) is that it is linear in the metric functions y and Z (and is homogeneous when $p_r = p_\perp$ for isotropic pressures).

2.4.2 Charged fluids

By equating (2.18) and (2.22), and incorporating (2.21), we generate the system of equations

$$\begin{aligned}
\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' &= \rho+\frac{1}{2}E^2 && (2.28a)\\
-\frac{1}{r^2}(1-e^{-2\lambda})+\frac{2\nu'}{r}e^{-2\lambda} &= p_r-\frac{1}{2}E^2 && (2.28b)\\
e^{-2\lambda}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right) &= p_\perp+\frac{1}{2}E^2 && (2.28c)\\
\sigma &= \frac{1}{r^2}e^{-\lambda}(r^2E)' && (2.28d)
\end{aligned}$$

The above system of equations (2.28) governs the behaviour of the gravitational field for a charged anisotropic matter distribution. We note that when $E=0$, equations (2.28) reduce to (2.23) for a neutral fluid. As a consequence of (2.14) we have

$$\frac{dp_r}{dr}=-\frac{1}{r}\left[2(p_r-p_\perp)+r(\rho+p_r)\frac{d\nu}{dr}\right]+\frac{E}{r^2}\frac{d}{dr}(r^2E) \qquad (2.29)$$

Sometimes it is convenient to use equation (2.29) as a starting point to integrate the system (2.28). When $p_r=p_\perp(=p)$ the anisotropy vanishes and we obtain

$$\frac{dp}{dr}=-(\rho+p)\frac{d\nu}{dr}+\frac{E}{r^2}\frac{d}{dr}(r^2E)$$

for a charged perfect fluid.

If we utilise the transformation (2.25) the Einstein-Maxwell system has the equivalent form

$$\begin{aligned}
\frac{1-Z}{x}-2\dot{Z} &= \frac{\rho}{C}+\frac{E^2}{2C} && (2.30a)\\
4Z\frac{\dot{y}}{y}+\frac{Z-1}{x} &= \frac{p_r}{C}-\frac{E^2}{2C} && (2.30b)\\
4xZ\frac{\ddot{y}}{y}+(4Z+2x\dot{Z})\frac{\dot{y}}{y}+\dot{Z} &= \frac{p_\perp}{C}+\frac{E^2}{2C} && (2.30c)\\
\frac{\sigma^2}{C} &= \frac{4Z}{x}\left(x\dot{E}+E\right)^2 && (2.30d)
\end{aligned}$$

where dots denote differentiation with respect to the variable x. From (2.30b) and (2.30c) we obtain the differential equation

$$4x^2Z\ddot{y}+2x^2\dot{Z}\dot{y}+\left(x\dot{Z}-Z+1-\frac{xE^2}{C}\right)y=\frac{x}{C}\left(p_\perp-p_r\right)y \qquad (2.31)$$

Equation (2.31) is called the condition of pressure isotropy generalised to include the electromagnetic field.

The mass contained within a radius x of the sphere is given by the expression

$$m(x) = \frac{1}{4C^{3/2}} \int_0^x \sqrt{\omega}\rho(\omega)\mathrm{d}\omega \tag{2.32}$$

For a physically realistic relativistic star we expect that the matter distribution should obey a barotropic equation of state $p_r = p_r(\rho)$. For our investigations, we assume the linear equation of state

$$p_r = \alpha\rho - \beta \tag{2.33}$$

where α and β are constants. Then we can write the system (2.30) in the simpler form

$$\frac{\rho}{C} = \frac{1-Z}{x} - 2\dot{Z} + \frac{E^2}{2C} \tag{2.34a}$$

$$p_r = \alpha\rho - \beta \tag{2.34b}$$

$$p_t = p_r + \Delta \tag{2.34c}$$

$$\Delta = 4CxZ\frac{\ddot{y}}{y} + 2C\left[x\dot{Z} + \frac{4Z}{(1+\alpha)}\right]\frac{\dot{y}}{y} + \frac{(1+5\alpha)}{(1+\alpha)}C\dot{Z} - \frac{C(1-Z)}{x} + \frac{2\beta}{(1+\alpha)} \tag{2.34d}$$

$$\frac{E^2}{2C} = \frac{1-Z}{x} - \frac{1}{(1+\alpha)}\left[2\alpha\dot{Z} + 4Z\frac{\dot{y}}{y} + \frac{\beta}{C}\right] \tag{2.34e}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2 \tag{2.34f}$$

where $\Delta = p_t - p_r$ is defined as the measure of anisotropy. The system as expressed in (2.34) is a system of six independent equations in terms of eight variables $(\rho, p_r, p_t, \Delta, E, \sigma, y, Z)$. To solve the system we require that two quantities involved in the integration process should be specified.

2.5 Physical properties

There are a number of solutions to the Einstein and Einstein-Maxwell equations that could be used to describe the interior spacetime of stellar objects in relativistic astrophysics. The physical criteria require that any new interior solution applicable to the stellar body should be matched smoothly to the appropriate exterior exact solution. In this regard, we mention two important exterior solutions of astrophysical relevance. The static spherically symmetric spacetime surrounding the body of mass M is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.35}$$

which is the well known Schwarzschild exterior line element (Schwarzschild 1916). The Schwarzschild exterior solution (2.35) is essential for discussion of the classical tests of general relativity: the bending of light, perihelion advance of mercury, gravitational red shift and the time delay in radar propagation. For a thorough treatment of these classical tests see D'Inverno (1992), Wald (1984) and Will (1981).

The gravitational field outside a static charged spherically symmetric body is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.36}$$

Here M is the mass of the body and Q is a constant related to the total charge of the sphere. The line element (2.36) is the Reissner-Nordstrom exterior solution (Reissner 1916). When $Q = 0$, (2.36) reduces to the exterior Schwarzschild line element (2.35). The Einstein and Einstein-Maxwell equations (2.23) and (2.28) respectively, admit a variety of exact solutions. However many solutions do not correspond to a physical matter distribution. For a relativistic stellar model we need to impose conditions on the solutions of the Einstein and Einstein-Maxwell equations for a physically reasonable model. To this end we stipulate a number of criteria for physical acceptability that have been used in previous investigations. Here we briefly outline a number of conditions which realistic stellar models should satisfy:

(a) The energy density ρ and fluid radial pressure p_r should be positive and finite everywhere in interior of the star. At the boundary the radial pressure must vanish.

(b) The gradients $\frac{d\rho}{dr}$ and $\frac{dp_r}{dr}$ should be negative.

(c) The speed of sound should be less than the speed of light so that $0 \leqslant \dfrac{dp_r}{dr} \leqslant 1$.

(d) The interior metric should match continuously with the Reissner-Nordstrom metric and the exterior Schwarzschild metric for the case of charged and neutral matter respectively.

(e) In the case of charged solutions, the electric field intensity $E(r)$ must be continuous at the boundary.

(f) The metric functions $e^{2\lambda}$, $e^{2\nu}$ and the electric field intensity E should be positive and nonsingular throughout the interior of the star.

(g) The solutions should be stable with respect to radial perturbations.

It should be mentioned that most of the solutions that have been found do not satisfy all the conditions (a) to (g) throughout the interior of the star, and may be valid only for some regions of spacetime. For example the solutions of Herrera and Ponce de Leon (1985a, 1985b, 1985c), Tikekar (1984), Tolman (1939) and Whitman and Burch (1981) become singular at the centre. Such solutions have to be treated as an envelope of the core of the star and have to be matched to another model valid for the core. Note that these singular solutions may be useful in providing qualitative features which may be present in particular stars. Some of the conditions (a) to (g) are too stringent; for example condition (b) requiring that the pressure and the energy density be strictly decreasing outwards to the surface may not be satisfied for some physical applications space (Maharaj and Maartens 1989). A comprehensive treatment of perfect fluid solutions to the Einstein field equations for static spherically symmetric models was undertaken by Delgaty and Lake (1998). Any new solution that we find should be checked against their results to confirm that it is not an existing model.

Chapter 3

Solutions with linear equation of state

3.1 Introduction

It is known that both anisotropic matter and the electromagnetic field are important in astrophysical processes. For physically relevant stellar bodies the matter distribution should satisfy a barotropic equation of state. In general, it is difficult to find solutions to the Einstein-Maxwell system with a polytropic equation of state. Solutions can be found with a linear equation of state. In this chapter, we review and study the Thirukkanesh and Maharaj (2008) models. By making choices for one of gravitational potentials, the electromagnetic field and fixing a linear equation of state a class of solutions may be generated. In §3.2 we present three classes of exact solutions to the Einstein-Maxwell system in terms of elementary functions. The mass function corresponds to physically acceptable matter distributions. In §3.3 we observe that the charge density is singular at the origin. This is demonstrated in Figure 3.1 which plots the charge density close to the stellar centre. For a realistic charge distribution we require regularity at the origin.

3.2 Thirukkanesh and Maharaj models

To solve the Einstein-Maxwell system of equations (2.34) it is necessary to make choices for two of the independent variables. Thirukkanesh and Maharaj (2008) made the specific choices

$$Z = \frac{1+(a-b)x}{1+ax} \tag{3.1}$$

$$\frac{E^2}{C} = \frac{k(3+ax)}{(1+ax)^2} \tag{3.2}$$

for the gravitational potential Z and the electric field intensity E. The quantities a, b, and k are real constants. On substituting (3.1) and (3.2) into (2.34e) they obtained

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{(1+\alpha)b}{4[1+(a-b)x]} + \frac{\alpha b}{2(1+ax)[1+(a-b)x]} - \frac{\beta(1+ax)}{4C[1+(a-b)x]} \\ &\quad - \frac{(1+\alpha)k(3+ax)}{8(1+ax)[1+(a-b)x]} \end{aligned} \tag{3.3}$$

For the integration of equation (3.3) we consider three cases: $b=0$, $a=b$ and $a \neq b$.

3.2.1 The case $b = 0$

When $b=0$, (3.3) gives the solution

$$y = D\,(1+ax)^{-k(1+\alpha)/(8a)} \exp\left[\frac{k(1+\alpha)}{4a(1+ax)} - \frac{\beta x}{4C}\right] \tag{3.4}$$

where D is the constant of integration. The potential y in (3.4) generates a negative density $\rho = -\dfrac{E^2}{2}$ which is not physical. Note that (3.4) is a correction of the corresponding expression in Thirukkanesh and Maharaj (2008) model. We have confirmed the validity of our corrected form in (3.4) with the package Mathematica (Wolfram 1999).

3.2.2 The case $b = a$

When $a=b$, (3.3) gives the solution

$$y = D\,(1+ax)^{(2a\alpha - k(1+\alpha))/(4a)} \exp\left[F(x)\right] \tag{3.5}$$

where

$$F(x) = \frac{x}{8C}\left[-kC(1+\alpha) - 2\beta + a(2C(1+\alpha) - \beta x)\right]$$

and D is the constant of integration. Then we can generate an exact model for the system (2.34) as follows

$$\begin{aligned}
e^{2\lambda} &= 1 + ax && \text{(3.6a)}\\
e^{2\nu} &= A^2D^2\,(1+ax)^{(2a\alpha - k(1+\alpha))/(2a)}\exp\left[2F(x)\right] && \text{(3.6b)}\\
\frac{\rho}{C} &= \frac{(2a-k)}{2}\frac{(3+ax)}{(1+ax)^2} && \text{(3.6c)}\\
p_r &= \alpha\rho - \beta && \text{(3.6d)}\\
p_t &= p_r + \Delta && \text{(3.6e)}\\
\Delta &= \frac{1}{16C(1+ax)^3}\{C^2[k^2(1+\alpha)^2x(3+ax)^2\\
&\quad +4a^2x(3 - 8\alpha + 9\alpha^2 + a^2(1+\alpha)^2x^2\\
&\quad +2ax(2+3\alpha+3\alpha^2)) - 4k(12 + a^3(1+\alpha)^2x^3 + a^2x^2(7+9\alpha+6\alpha^2)\\
&\quad -ax(12+5\alpha+9\alpha^2))] - 4Cx(1+ax)^2[(1+\alpha)(2a^2x - 3k)\\
&\quad -a\beta(k(1+\alpha) - 6\alpha - 4)] + 4\beta^2x(1+ax)^4\} && \text{(3.6f)}\\
\frac{E^2}{C} &= \frac{k(3+ax)}{(1+ax)^2} && \text{(3.6g)}\\
\frac{\sigma^2}{C} &= \frac{k(a^2x^2+3ax+6)^2}{x(3+ax)(1+ax)^5} && \text{(3.6h)}
\end{aligned}$$

Note that the exact solution (3.6) of the Einstein-Maxwell system is given in terms of elementary functions when $b = a$.

The solution (3.6) may be used to describe a charged anisotropic stellar object with a linear equation of state. In this case the mass function is

$$m(x) = \frac{(2a-k)x^{3/2}}{4C^{3/2}(1+ax)} \tag{3.7}$$

which is similar to forms used by other researchers. If $k = 0$ then $E = \sigma = 0$ and the charge vanishes; we have an uncharged anisotropic star. Then $(3.6f)$ gives the following expression

$$\Delta = \frac{1}{4C(1+ax)}\{C^2a^2x[3-8\alpha+9\alpha^2+a^2(1+\alpha)^2x^2+2ax(2+3\alpha+3\alpha^2)]$$

$$-2Cx(1+ax)^2[(1+\alpha)a^2x+a\beta(3\alpha+2)]+4\beta^2x(1+ax)^2\} \qquad (3.8)$$

for the anisotropy. Consequently when $k = 0$ the model is necessarily anisotropic with $\Delta \neq 0$ in general even in the simple case of neutral matter. Models with $\Delta \neq 0$ are important in describing physical processes in anisotropic relativistic compact objects as demonstrated by Dev and Gleiser (2002, 2003), Chaisi and Maharaj (2005, 2006) and Mak and Harko (2002, 2003).

3.2.3 The case $b \neq a$

On integrating (3.3), with $b \neq a$, we get

$$y = D(1+ax)^m[1+(a-b)x]^n \exp\left[\frac{-a\beta x}{4C(a-b)}\right] \qquad (3.9)$$

where D is the constant of integration. The constants m and n are given by

$$m = \frac{2\alpha b-(1+\alpha)k}{4b}$$

$$n = \frac{1}{8bC(a-b)^2}[2a^2C(k(1+\alpha)-2\alpha b)-abC(5k(1+\alpha)-2b(1+5\alpha))$$

$$+b^2(3kC(1+\alpha)-2bC(1+3\alpha)+2\beta)]$$

Then we can find an exact model for the system (2.34) in the form

$$
\begin{aligned}
e^{2\lambda} &= \frac{1+ax}{1+(a-b)x} && \text{(3.10a)}\\
e^{2\nu} &= A^2D^2\,(1+ax)^{2m}\,[1+(a-b)x]^{2n}\exp\left[\frac{-a\beta x}{2C(a-b)}\right] && \text{(3.10b)}\\
\frac{\rho}{C} &= \frac{(2b-k)}{2}\frac{(3+ax)}{(1+ax)^2} && \text{(3.10c)}\\
p_r &= \alpha\rho-\beta && \text{(3.10d)}\\
p_t &= p_r+\Delta && \text{(3.10e)}\\
\Delta &= \frac{-bC}{(1+ax)}-\frac{bC(1+5\alpha)}{(1+\alpha)(1+ax)^2}+\frac{2\beta}{1+\alpha}+\frac{Cx[1+(a-b)x]}{(1+ax)}\\
&\quad\times\left[4\left(\frac{a^2m(m-1)}{(1+ax)^2}+\frac{2a(a-b)mn}{(1+ax)[1+(a-b)x]}+\frac{(a-b)^2n(n-1)}{[1+(a-b)x]^2}\right)\right.\\
&\quad\left.-\frac{2a\beta(a(m+n)[1+(a-b)x]-bn)}{(a-b)C(1+ax)[1+(a-b)x]}+\frac{a^2\beta^2}{4C^2(a-b)^2}\right]\\
&\quad-\frac{4[1+ax(2+(a-b)x)]-b(5+\alpha)x}{2(a-b)(1+\alpha)(1+ax)^3[1+(a-b)x]}\\
&\quad\times[-4b^2Cn+a^3x(-4C(m+n)+\beta x)+a^2(4C(m+n)(2bx-1)\\
&\quad+\beta(2-bx)x)+a(-4b^2C(m+n)x+\beta+b(4Cm+8Cn-\beta x))] && \text{(3.10f)}\\
\frac{E^2}{C} &= \frac{k(3+ax)}{(1+ax)^2} && \text{(3.10g)}\\
\frac{\sigma^2}{C} &= \frac{k[1+(a-b)x](a^2x^2+3ax+6)^2}{x(3+ax)(1+ax)^5} && \text{(3.10h)}
\end{aligned}
$$

Again the exact solution (3.10) of the Einstein-Maxwell system can be given solely in terms of elementary functions when $b \neq 0$.

The solution (3.10) may be used to model a charged anisotropic star with a linear equation of state. For this case the mass function is

$$
m(x) = \frac{(2b-k)x^{3/2}}{4C^{3/2}(1+ax)} \qquad (3.11)
$$

which is similar to (3.7). The mass function (3.11) has been used to study several physical scenarios in relativistic astrophysics: Finch and Skea (1989) generated neutron star models, Lobo (2006) showed that dark energy stars are stable, Mark and Harko (2004) analysed anisotropic relativistic matter and Matese and Whitman (1980) considered equilibrium stellar configurations. It is interesting to note that Sharma and Maharaj

(2007) showed that this mass distribution may be applied to strange stars with quark matter. Thirukkanesh and Maharaj (2008) showed that (3.11) describes relativistic charged compact spheres with anisotropic matter distribution in general relativity. The linear equation of state is consistent with models of dark energy stars and charged quark distributions. In particular the results of Thirukkanesh and Maharaj (2008) are consistent with the equation of state for strange matter formulated by Dey *et al* (1998, 1999). The model of Dey *et al* (1998, 1999) generates masses for the X-ray binary pulsar SAX J1808.4-3658 consistent with observational data. Thus the class of solutions in (3.10) is of astrophysical importance.

3.3 Singularity in charge density

The gravitational potentials ν and λ are continuous and well-behaved in the stellar interior. The matter variables ρ, p_r and $p_\perp$ are physically reasonable and there is a linear barotropic equation of state. The speed of sound is less than the speed of light. The electric field intensity E is finite and continuous. However the charge density σ in (3.10h) becomes singular at the stellar origin $x = 0$. In Figure 3.1 we have plotted the charge density for the values $a = 3$, $b = 2.15$, $k = 0.2$ and $C = 1$ which highlights this feature. Valera *et al* (2010) have pointed out the need for the regularity of the charge distribution at the centre of the sphere. The vanishing of the electric field at the centre of a spherically symmetric charge distribution should be a condition for the physical relevance of the solution. Consequently there is a need to find new solutions of the Einstein-Maxwell system of equations (2.34) which ensure that the electric field intensity E and charge density σ remain finite at the stellar centre. This is the subject of the next two chapters.

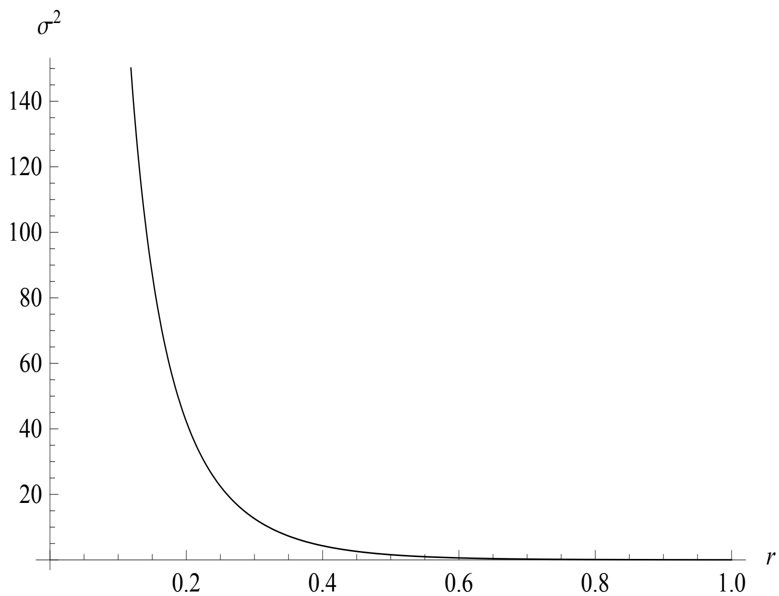

Figure 3.1: Charge density.

Chapter 4

New solutions I

4.1 Introduction

In chapter 3 we concluded that the class of solutions is singular in the charge density at the stellar origin. In this chapter we seek new solutions to the Einstein-Maxwell system, with no singularity in the charge distribution at the centre. We achieve this requiring that the electric field vanishes at the centre. We consider a new form for the electric field intensity but we keep the same form utilised in chapter 3 for the gravitational potential in the integration process. In §4.2 three new classes of exact solutions to the Einstein-Maxwell system are presented in terms of elementary functions. The first class is unphysical but the other two classes may be used to model a relativistic star. We show that the mass functions are similar to the forms in the models of Thirukkanesh and Maharaj (2008). In §4.3 we regain particular solutions found previously. We study the physical features of the results in §4.4 and provide plots for the energy density, radial pressure, electric field intensity, charge density and mass.

4.2 Physical models

In this chapter we make the choices

$$Z = \frac{1 + (a - b)x}{1 + ax} \tag{4.1}$$

$$\frac{E^2}{C} = \frac{sa^2x^2}{(1 + ax)^2} \tag{4.2}$$

for the gravitational potential Z and the electric field intensity E. Note that the form (4.1) for the potential Z is the same as in chapter 3 but the electric field E in (4.2) is different. The quantities a, b, and s are real constants. With these choices, equation (2.34e) becomes

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{(1+\alpha)b}{4[1+(a-b)x]} + \frac{\alpha b}{2(1+ax)[1+(a-b)x]} - \frac{\beta(1+ax)}{4C[1+(a-b)x]} \\ &\quad - \frac{(1+\alpha)sa^2x^2}{8(1+ax)[1+(a-b)x]} \end{aligned} \tag{4.3}$$

Three classes of exact solutions are presented: $b = 0$, $a = b$ and $a \neq b$.

4.2.1 The case $b = 0$

For $b = 0$, (4.3) becomes

$$\frac{\dot{y}}{y} = -\frac{\beta}{4C} - \frac{(1+\alpha)sa^2x^2}{8(1+ax)^2} \tag{4.4}$$

with the solution

$$y = D\,(1+ax)^{s(1+\alpha)/4a} \exp\left[-\frac{sax(1+\alpha)(2+ax)}{8a(1+ax)} - \frac{\beta x}{4C}\right] \tag{4.5}$$

where D is the constant of integration. We realise for this case that the density is $\rho = -\dfrac{E^2}{2}$ which is negative and therefore this case is not physical.

4.2.2 The case $b = a$

When $a = b$, (4.3) becomes

$$\begin{aligned}\frac{\dot{y}}{y} &= \frac{(1+\alpha)a}{4} + \frac{\alpha a}{2(1+ax)} - \frac{\beta(1+ax)}{4C} \\ &\quad -\frac{sa^2x^2(1+\alpha)}{8(1+ax)} \end{aligned} \tag{4.6}$$

On integrating (4.3) we get

$$y = D\,(1+ax)^{(4a\alpha - s(1+\alpha))/(8a)} \exp\left[F(x)\right] \tag{4.7}$$

where

$$F(x) = \frac{x}{16C}\left[2Cs(1+\alpha) - 4\beta - a(C(-4+sx)(1+\alpha) + 2\beta x)\right]$$

and D is the constant of integration. Therefore, we can generate a new exact model for the system (2.34) in the form

$$e^{2\lambda} = 1 + ax \tag{4.8a}$$

$$e^{2\nu} = A^2D^2\,(1+ax)^{(4a\alpha - s(1+\alpha))/(4a)} \exp\left[2F(x)\right] \tag{4.8b}$$

$$\frac{\rho}{C} = \frac{6a + ax(2a - sax)}{2(1+ax)^2} \tag{4.8c}$$

$$p_r = \alpha\rho - \beta \tag{4.8d}$$

$$p_t = p_r + \Delta \tag{4.8e}$$

$$\begin{aligned}\Delta &= \frac{1}{16C(1+ax)^3}\{C^2[4a^2x(3 - 8\alpha + 9\alpha^2 + a^2(1+\alpha)^2x^2 \\ &\quad +2ax(2+3\alpha+3\alpha^2)) + s^2(2x(a^2x^2-1)(-asx+s) \\ &\quad +a^2x^2 - 6ax + 1) - 2s(ax(6\alpha^2 + 6\alpha + 2a(2\alpha^2 + 2\alpha + 1)) \\ &\quad +a^2x^2(\alpha^2 + 2\alpha - 2a(1+\alpha+x) + 17) + a(\alpha^2 - 2\alpha) + 7)] \\ &\quad -4Cx(1+ax)^2[(1+\alpha)(2a^2x - 2s) \\ &\quad -a\beta(sax(1+\alpha) - 6\alpha - 4)] + 4\beta^2x(1+ax)^4\} \end{aligned} \tag{4.8f}$$

$$\frac{E^2}{C} = \frac{sa^2x^2}{(1+ax)^2} \tag{4.8g}$$

$$\frac{\sigma^2}{C} = \frac{4sa^2x(2+ax)^2}{(1+ax)^5} \tag{4.8h}$$

The exact solution (4.8) of the Einstein-Maxwell system is given in terms of elementary functions.

In this case, the mass function has the form

$$m(x) = \frac{1}{8C^{3/2}} \left[\frac{(12a^2x + s(15 + 10ax - 2a^2x^2))x^{1/2}}{3a(1+ax)} - \frac{5s \arctan(\sqrt{ax})}{a^{3/2}} \right] \tag{4.9}$$

If the charge vanishes ($s = 0$) then (4.9) becomes

$$m(x) = \frac{2ax^{3/2}}{4C^{3/2}(1+ax)}$$

which is the same form as (3.7) with $k = 0$ in §3.2.2. Therefore the solution in this section contain an anisotropic model, with a linear equation of state, in the uncharged limit which is the same as that in §3.2.2. At the stellar origin $x = 0$ we have

$$E = 0, \qquad \sigma^2 = 0$$

and there is no singularity at the centre of sphere in the electromagnetic field.

4.2.3 The case $b \neq a$

On integrating (4.3), with $b \neq a$, we get

$$y = D(1+ax)^m [1 + (a-b)x]^n \exp\left[-\frac{ax[Cs(1+\alpha) + 2\beta]}{8C(a-b)} \right] \tag{4.10}$$

where D is the constant of integration. The constants m and n are given by

$$\begin{aligned} m &= \frac{4b\alpha - s(1+\alpha)}{8b} \\ n &= \frac{1}{8bC(a-b)^2}[a^2C(s(1+\alpha) - 4b\alpha) + 2ab^2C(1+5\alpha) \\ &\quad +2b^2(-bC(1+3\alpha) + \beta)] \end{aligned}$$

Then we can find a new exact solution for the system (2.34) in the form

$$
\begin{aligned}
e^{2\lambda} &= \frac{1+ax}{1+(a-b)x} && \text{(4.11a)}\\
e^{2\nu} &= A^2D^2\,(1+ax)^{2m}\,[1+(a-b)x]^{2n}\exp\left[-\frac{ax[Cs(1+\alpha)+2\beta]}{4C(a-b)}\right] && \text{(4.11b)}\\
\frac{\rho}{C} &= \frac{6b+ax(2b-sax)}{2(1+ax)^2} && \text{(4.11c)}\\
p_r &= \alpha\rho-\beta && \text{(4.11d)}\\
p_t &= p_r+\Delta && \text{(4.11e)}\\
\Delta &= \frac{-bC}{(1+ax)}-\frac{bC(1+5\alpha)}{(1+\alpha)(1+ax)^2}+\frac{2\beta}{1+\alpha}+\frac{Cx[1+(a-b)x]}{(1+ax)}\\
&\quad\times\left[4\left(\frac{a^2m(m-1)}{(1+ax)^2}+\frac{2a(a-b)mn}{(1+ax)[1+(a-b)x]}+\frac{(a-b)^2n(n-1)}{[1+(a-b)x]^2}\right)\right.\\
&\quad-\frac{a[Cs(1+\alpha)+2\beta](a(m+n)[1+(a-b)x]-bn)}{(a-b)C(1+ax)[1+(a-b)x]}\\
&\quad\left.+\frac{a^2[Cs(1+\alpha)+2\beta]^2}{16C^2(a-b)^2}\right]\\
&\quad-\frac{4[1+ax(2+(a-b)x)]-b(5+\alpha)x}{4(a-b)(1+\alpha)(1+ax)^3[1+(a-b)x]}\\
&\quad\times[-8b^2Cn+a^3x(-8C(m+n)+[Cs(1+\alpha)+2\beta]x)\\
&\quad+a^2(8C(m+n)(2bx-1)+[Cs(1+\alpha)+2\beta](2-bx)x)\\
&\quad+a(-8b^2C(m+n)x+[Cs(1+\alpha)+2\beta]\\
&\quad+b(8Cm+16Cn-[Cs(1+\alpha)+2\beta]x))] && \text{(4.11f)}\\
\frac{E^2}{C} &= \frac{sa^2x^2}{(1+ax)^2} && \text{(4.11g)}\\
\frac{\sigma^2}{C} &= \frac{4sa^2x[1+(a-b)x](2+ax)^2}{(1+ax)^5} && \text{(4.11h)}
\end{aligned}
$$

The exact solution (4.11) of the Einstein-Maxwell system is given in terms of elementary functions.

We have generated a second class of exact solution (4.11) that model a charged anisotropic star with a linear equation of state. In this case the mass function is given by

$$
m(x)=\frac{1}{8C^{3/2}}\left[\frac{(12abx+s(15+10ax-2a^2x^2))x^{1/2}}{3a(1+ax)}-\frac{5s\arctan(\sqrt{ax})}{a^{3/2}}\right] \qquad (4.12)
$$

If the charge vanishes ($s = 0$) then (4.12) becomes

$$m(x) = \frac{2bx^{3/2}}{4C^{3/2}(1 + ax)}$$

which is the same form as (3.11) when $k = 0$ in §3.2.3. We observe that the solution in this section contains an anisotropic model, with a linear equation of state, in the uncharged limit which is the same as in §3.2.3. It is important to note that when $x = 0$ at the origin we get

$$E = 0, \qquad \sigma^2 = 0$$

so that there is no singularity in the electric field intensity E and the charge density σ for the electromagnetic field.

4.3 Known solutions

We show that particular solutions found in the past are contained in our new class of models.

4.3.1 Sharma and Maharaj model

If we set $\beta = \alpha\tilde{\rho}$, $s = 0$, then

$$p_r = \alpha(\rho - \tilde{\rho})$$

where $\tilde{\rho}$ is the density at the surface. Therefore we regain the linear equation of state of Sharma and Maharaj (2007) for strange matter stars. By setting $C = 1$ and $A^2D^2 = B$ we find the following form of the line element

$$\begin{aligned} ds^2 &= -B(1 + ar^2)^{\alpha}[1 + (a - b)r^2]^{\gamma} \exp\left(\frac{-a\beta r^2}{2(a - b)}\right) dt^2 \\ &\quad + \frac{1 + ar^2}{1 + (a - b)r^2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{4.13}$$

where

$$\gamma = \frac{5ab\alpha - 2a^2\alpha - 3b^2\alpha + ab - b^2 + b\beta}{2(a - b)^2}$$

is a constant. The line (4.13) corresponds to the uncharged anisotropic model of Sharma and Maharaj (2007) and is consistent with the equation of state for quark matter.

4.3.2 Lobo model

If we set $\beta = 0$ and $s = 0$ then p_r becomes

$$p_r = \alpha\rho$$

and we regain the equation of state studied by Lobo (2006). On setting $a = 2b$, $C = 1$ and $A^2D^2 = B$ we generate the metric

$$\begin{aligned} ds^2 &= -(1+br^2)^{(1-\alpha)/2}(1+2br^2)^{\alpha}dt^2 + \left(\frac{1+2br^2}{1+br^2}\right)dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{4.14}$$

The line element (4.14) corresponds to the anisotropic uncharged model of Lobo (2006) who showed that there are regions of spacetime in which dark energy stars are stable.

4.3.3 Isotropic models

In general we observe that $\Delta \neq 0$ and the model remains anisotropic. However, we can show for particular parameter values that $\Delta = 0$ in the general solution (4.11). If we set $s = 0$ and $a = 0$, $b = 1$ then we obtain

$$\begin{aligned} m &= \frac{\alpha}{2} \\ n &= \frac{1}{4C}[\beta - (1+3\alpha)C] \\ \Delta &= \frac{x}{4C(1-x)}[\beta - 3(1+\alpha)C][\beta - (1+3\alpha)C] \end{aligned} \tag{4.15}$$

Two different cases arise as a consequence of (4.15) by setting $\Delta = 0$. Firstly, we observe that when $\beta = 0$ and $\alpha = -1$ then $\Delta = 0$. The equation of state becomes $p_r (= p_t) = -\rho$. In this case the line element becomes

$$ds^2 = -\left(1 + \frac{r^2}{R^2}\right)dt^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.16}$$

where we have set $A = D = 1$ and $C = \frac{1}{R^2}$. We mention that the metric (4.16) corresponds to the familiar isotropic uncharged de Sitter model. Secondly, we observe that when $\beta = 0$ and $\alpha = -\frac{1}{3}$ then $\Delta = 0$. The equation of state becomes $p_r(= p_t) = -\frac{1}{3}\rho$. In this present case the line becomes

$$ds^2 = -A^2dt^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (4.17)$$

where we have set $D = 1$ and $C = \frac{1}{R^2}$. The metric (4.17) corresponds to the well-known isotropic uncharged Einstein model.

4.4 Physical analysis

In this section we demonstrate that the exact solutions found in §4.2 for $a \neq b$ are physically reasonable. We utilised the software package Mathematica (Wolfram 1999) to generate these plots, and we made the choices $a = 2$, $b = 2.15$, $\alpha = 0.33$, $\beta = \alpha\tilde{\rho} = 0.198$, $C = 1$ and $s = 2.5$, where $\tilde{\rho}$ is the density at the boundary $r = 1.0$. We generated the following plots:

- Figure 4.1: Energy density.
- Figure 4.2: Radial pressure.
- Figure 4.3: Electric field intensity.
- Figure 4.4: Charge density.
- Figure 4.5: Mass.

The energy density ρ (in Figure 4.1) is positive, finite and monotonically decreasing. The radial pressure p_r in (in Figure 4.2) is similar to ρ since p_r and ρ are related by a linear equation of state. The values of ρ and p_r are lower in the presence of the electric field $E \neq 0$. The form chosen for E (Figure 4.3) is physically reasonable and describes an increasing function. The charge density (in Figure 4.4) describes a functions which initially increases, reaches a maximum and then decreases as we approach the boundary. The mass function (in Figure 4.5) is a strictly increasing function and is continuous and finite. We observe that the mass, in the presence of charge, has lower values that the corresponding uncharged case. This is consistent as $E \neq 0$ generates lower densities which represents a weaker total field since the electromagnetic field is repulsive. These profiles replicate the behaviour of the Thirukkanesh and Maharaj (2008) model. However we emphasise that the charge density σ is regular at the stellar origin $x = 0$ as can be clearly seen in Figure 4.4. Thus all matter variables, electromagnetic quantities and gravitational potentials are nonsingular and well-behaved in a region containing the stellar centre.

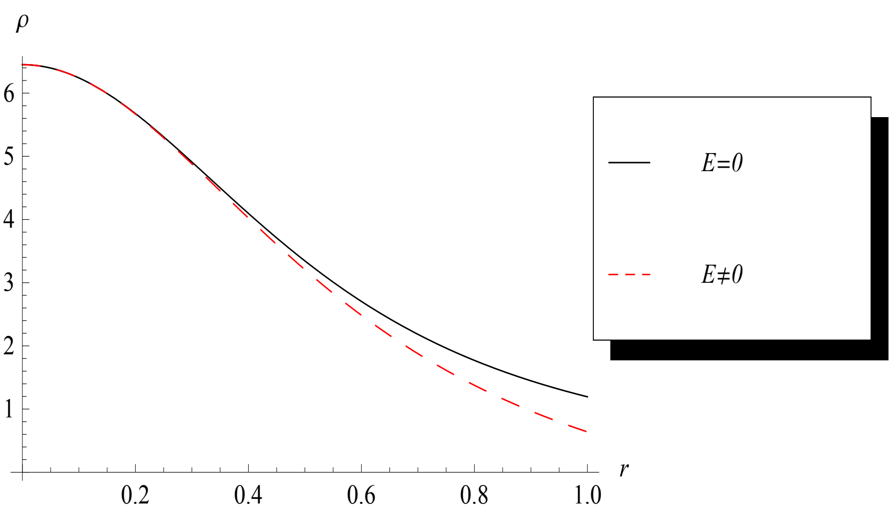

Figure 4.1: Energy density.

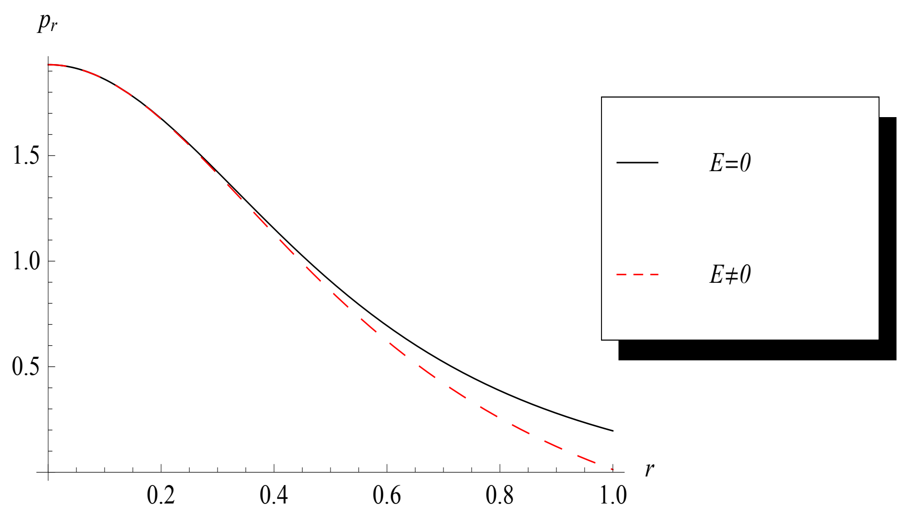

Figure 4.2: Radial pressure.

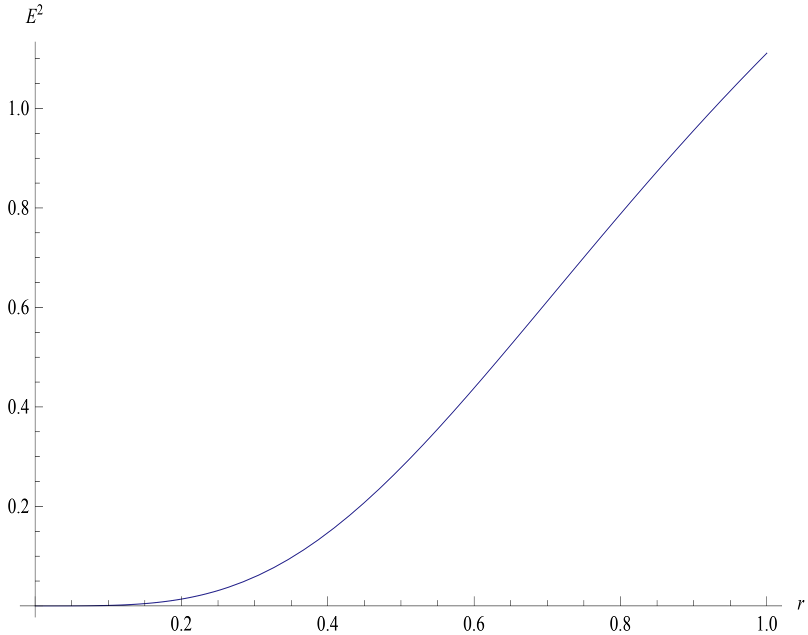

Figure 4.3: Electric field intensity.

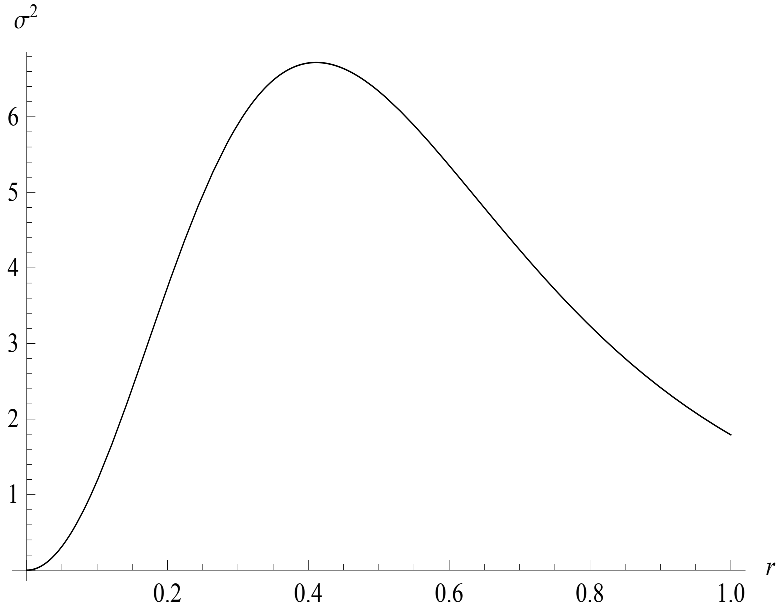

Figure 4.4: Charge density.

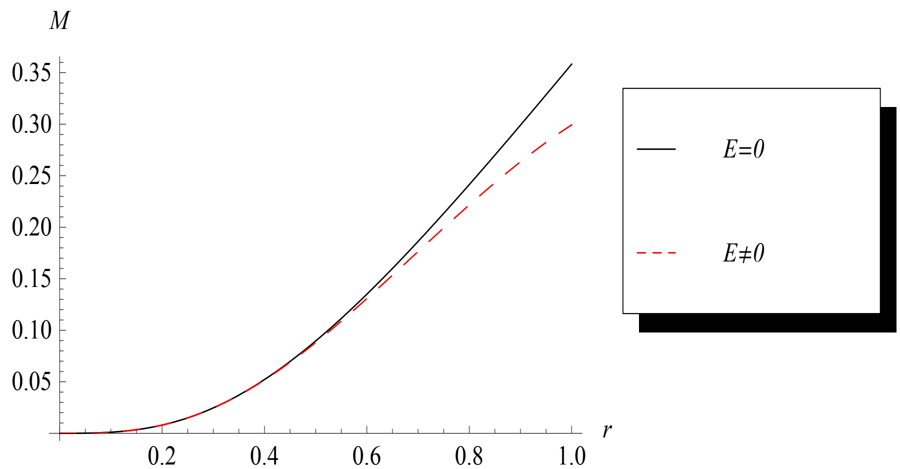

Figure 4.5: Mass.

Chapter 5

New solutions II

5.1 Introduction

We found a new class of exact solutions to the Einstein-Maxwell system in the previous chapter. We show in this chapter that it is possible to find further solutions for a charged relativistic star by selecting a different form for electric field intensity. In §5.2 we present three new classes of exact models to the Einstein-Maxwell system. For particular parameter values we regain earlier results. The key insight is to combine both choices of the electromagnetic field used in previous chapters and keep the same form for the gravitational potential. A physical analysis of the new solutions is performed in §5.3 and we plot the energy density, the radial pressure, the electric field intensity, the charge density and the mass. We generate values for mass for charged and uncharged matter in §5.4. This analysis extends the treatment of Sharma and Maharaj (2007) and Thirukkanesh and Maharaj (2008) and confirms that the new exact solutions found are physically reasonable.

5.2 New general models

In chapter 4 we found a new class of solutions to the Einstein-Maxwell system. Others choices of Z and E may also lead to new solutions. We demonstrate this by making the choices

$$Z = \frac{1 + (a-b)x}{1 + ax} \tag{5.1}$$

$$\frac{E^2}{C} = \frac{k(3 + ax) + sa^2x^2}{(1 + ax)^2} \tag{5.2}$$

for the gravitational potential Z and the electric field intensity E. We observe that Z is the same form as that used by Thirukkanesh and Maharaj (2008). The form for E is different from previous analyses. However note that when $k = 0$ we regain the field intensity E considered in chapter 4, and when $s = 0$ we regain E studied by Thirukkanesh and Maharaj (2008). Therefore the choice in this chapter is a generalisation of previous studies. The quantities a, b, k and s are real constants. On substituting (5.1) and (5.2) into (2.34e) we get

$$\begin{aligned}\frac{\dot{y}}{y} &= \frac{(1+\alpha)b}{4[1+(a-b)x]} + \frac{\alpha b}{2(1+ax)[1+(a-b)x]} - \frac{\beta(1+ax)}{4C[1+(a-b)x]} \\ &\quad - \frac{(1+\alpha)[k(3+ax)+sa^2x^2]}{8(1+ax)[1+(a-b)x]}\end{aligned} \tag{5.3}$$

For the integration of equation (5.3) it is convenient to consider three cases: $b = 0$, $a = b$ and $a \neq b$.

5.2.1 The case $b = 0$

When $b = 0$, (5.3) becomes

$$\frac{\dot{y}}{y} = -\frac{\beta}{4C} - \frac{(1+\alpha)(k(3+ax)+sa^2x^2)}{8(1+ax)^2} \tag{5.4}$$

which gives the solution

$$y = D\,(1+ax)^{(-(k-2s)(1+\alpha))/(8a)} \exp\left[\frac{(1+\alpha)[2k - sax(2+ax)]}{8a(1+ax)} - \frac{\beta x}{4C}\right] \tag{5.5}$$

where D is the constant of integration. The potential y in (5.5) generates a negative density $\rho = -\frac{E^2}{2}$ which is physically undesirable.

5.2.2 The case $b = a$

When $a = b$, (5.3) becomes

$$\begin{aligned}\frac{\dot{y}}{y} &= \frac{(1+\alpha)a}{4} + \frac{\alpha a}{2(1+ax)} - \frac{\beta(1+ax)}{4C} \\ &\quad -\frac{(1+\alpha)[k(3+ax)+sa^2x^2]}{8(1+ax)}\end{aligned} \tag{5.6}$$

On integrating (5.6) we get

$$y = D\,(1+ax)^{(4a\alpha-(2k+s)(1+\alpha))/(8a)} \exp\left[F(x)\right] \tag{5.7}$$

where

$$F(x) = \frac{x}{16C}\left[2C(s-k)(1+\alpha) - 4\beta - a(C(-4+sx)(1+\alpha)+2\beta x)\right]$$

and D is the constant of integration. Then we can generate an exact model for the system (2.34) in the form

$$e^{2\lambda} = 1 + ax \tag{5.8a}$$

$$e^{2\nu} = A^2 D^2 (1+ax)^{(4a\alpha-(s+2k)(1+\alpha))/(4a)} \exp\left[2F(x)\right] \tag{5.8b}$$

$$\frac{\rho}{C} = \frac{(2a-k)(3+ax) - sa^2x^2}{2(1+ax)^2} \tag{5.8c}$$

$$p_r = \alpha\rho - \beta \tag{5.8d}$$

$$p_t = p_r + \Delta \tag{5.8e}$$

$$\begin{aligned}\Delta = & \frac{1}{16C(1+ax)^3}\{C^2[k^2((1+\alpha)^2x(3+ax)^2 + 2sx(1+ax)) \\ & +4a^2x(3 - 8\alpha + 9\alpha^2 + a^2(1+\alpha)^2x^2 + 2ax(2+3\alpha+3\alpha^2)) \\ & -4k(12 + a^3(1+\alpha)^2x^3 + a^2x^2(7+9\alpha+6\alpha^2) \\ & +ax(12+5\alpha+9\alpha^2+4s)) \\ & +s^2(2x(a^2x^2-1)(2k - asx + s) + a^2x^2 - 6ax + 1) \\ & -2s(ax(6\alpha^2 + 6\alpha + 2a(2\alpha^2+2\alpha+1) + k(7-2\alpha-\alpha^2)) \\ & +a^2x^2(\alpha^2 + 2\alpha - 2a(1+\alpha+x) + 17) + a(\alpha^2 - 2\alpha) \\ & -k(1+\alpha)^2 + 7)] - 4Cx(1+ax)^2[(1+\alpha)(2a^2x - 3k - 2s) \\ & -a\beta(k(1+\alpha) + sax(1+\alpha) - 6\alpha - 4)] + 4\beta^2x(1+ax)^4\}\end{aligned} \tag{5.8f}$$

$$\frac{E^2}{C} = \frac{k(3+ax) + sa^2x^2}{(1+ax)^2} \tag{5.8g}$$

$$\frac{\sigma^2}{C} = \frac{\left(\sqrt{k}(a^2x^2 + 3ax + 6) + 2\sqrt{s}ax\sqrt{3+ax}(2+ax)\right)^2}{x(3+ax)(1+ax)^5} \tag{5.8h}$$

The new exact solution (5.8) of Einstein-Maxwell system is presented in terms of elementary functions. When $s = 0$ then (5.8) reduces to equations (3.6) in chapter 3. If $k = 0$ then (5.8) is the same as (4.8) in chapter 4. Consequently the new solution (5.8) represents a generalisation of our earlier results. In this case the mass function is given by

$$m(x) = \frac{1}{8C^{3/2}}\left[\frac{((12a^2-6ak)x+s(15+10ax-2a^2x^2))x^{1/2}}{3a(1+ax)} - \frac{5s\arctan(\sqrt{ax})}{a^{3/2}}\right] \quad (5.9)$$

which is similar to earlier forms. The singularity in the charge distribution at the centre is still present in general, but can be eliminated when $k = 0$. Then equation (5.8h) becomes

$$\frac{\sigma^2}{C} = \frac{4sa^2x(2+ax)^2}{(1+ax)^5} \quad (5.10)$$

which is the same result as in chapter 4. At the stellar centre $x = 0$ and the charge density vanishes.

5.2.3 The case $b \neq a$

On integrating (5.3), with $b \neq a$ we obtain

$$y = D(1+ax)^m[1+(a-b)x]^n \exp\left[-\frac{ax[Cs(1+\alpha)+2\beta]}{8C(a-b)}\right] \quad (5.11)$$

where D is the constant of integration. The constants m and n are given

$$\begin{aligned} m &= \frac{4\alpha b-(1+\alpha)(s+2k)}{8b} \\ n &= \frac{1}{8bC(a-b)^2}[a^2C((1+\alpha)(s+2k)-4\alpha b)-abC(5k(1+\alpha)-2b(1+5\alpha)) \\ &\quad +b^2(3kC(1+\alpha)-2bC(1+3\alpha)+2\beta)] \end{aligned}$$

Then we can generate an exact model for the system (2.34) in the form

$$
\begin{aligned}
e^{2\lambda} &= \frac{1+ax}{1+(a-b)x} && \text{(5.12a)}\\
e^{2\nu} &= A^2D^2\left(1+ax\right)^{2m}\left[1+(a-b)x\right]^{2n}\exp\left[-\frac{ax[Cs(1+\alpha)+2\beta]}{4C(a-b)}\right] && \text{(5.12b)}\\
\frac{\rho}{C} &= \frac{(2b-k)(3+ax)-sa^2x^2}{2(1+ax)^2} && \text{(5.12c)}\\
p_r &= \alpha\rho-\beta && \text{(5.12d)}\\
p_t &= p_r+\Delta && \text{(5.12e)}\\
\Delta &= \frac{-bC}{(1+ax)}-\frac{bC(1+5\alpha)}{(1+\alpha)(1+ax)^2}+\frac{2\beta}{1+\alpha}+\frac{Cx[1+(a-b)x]}{(1+ax)}\\
&\times\left[4\left(\frac{a^2m(m-1)}{(1+ax)^2}+\frac{2a(a-b)mn}{(1+ax)[1+(a-b)x]}+\frac{(a-b)^2n(n-1)}{[1+(a-b)x]^2}\right)\right.\\
&-\frac{a[Cs(1+\alpha)+2\beta](a(m+n)[1+(a-b)x]-bn)}{(a-b)C(1+ax)[1+(a-b)x]}\\
&\left.+\frac{a^2[Cs(1+\alpha)+2\beta]^2}{16C^2(a-b)^2}\right]-\frac{4[1+ax(2+(a-b)x)]-b(5+\alpha)x}{4(a-b)(1+\alpha)(1+ax)^3[1+(a-b)x]}\\
&\times[-8b^2Cn+a^3x(-8C(m+n)+[Cs(1+\alpha)+2\beta]x)\\
&+a^2(8C(m+n)(2bx-1)+[Cs(1+\alpha)+2\beta](2-bx)x)\\
&+a(-8b^2C(m+n)x+[Cs(1+\alpha)+2\beta]\\
&+b(8Cm+16Cn-[Cs(1+\alpha)+2\beta]x))] && \text{(5.12f)}\\
\frac{E^2}{C} &= \frac{k(3+ax)+sa^2x^2}{(1+ax)^2} && \text{(5.12g)}\\
\frac{\sigma^2}{C} &= \frac{[1+(a-b)x]\left(\sqrt{k}(a^2x^2+3ax+6)+2\sqrt{s}ax\sqrt{3+ax}(2+ax)\right)^2}{x(3+ax)(1+ax)^5} && \text{(5.12h)}
\end{aligned}
$$

The exact solution (5.12) of the Einstein-Maxwell system can be written in terms of elementary functions. When $s=0$ then (5.12) reduces to equations (3.10) derived in chapter 3. If $k=0$ then (5.12) becomes (4.8) obtained in chapter 4. Therefore the new exact solution (4.11) of the Einstein-Maxwell system is a generalisation of our earlier results. For this case the mass function is given by

$$
\begin{aligned}
m(x) &= \frac{1}{8C^{3/2}}\left[\frac{((12ab-6ak)x+s(15+10ax-2a^2x^2))x^{1/2}}{3a(1+ax)}\right.\\
&\left.-\frac{5s\arctan(\sqrt{ax})}{a^{3/2}}\right] && \text{(5.13)}
\end{aligned}
$$

which contains earlier forms. In general there is a singularity in the charge density at the centre from (5.12h). This singularity is eliminated when $k = 0$ so that

$$\frac{\sigma^2}{C} = \frac{4sa^2x[1+(a-b)x](2+ax)^2}{(1+ax)^5} \tag{5.14}$$

which corresponds to the result in chapter 4. At the centre of the star $x = 0$ and the charge density vanishes.

5.3 Physical analysis

In this section we show that the exact solutions found in §5.2 for $a \neq b$ are physically reasonable. We utilised the software package Mathematica (Wolfram 1999) to generate these plots, and we made the choices $a = 2$, $b = 2.5$, $\alpha = 0.33$, $\beta = \alpha\tilde{\rho} = 0.198$, $C = 1$ and $s = 2.5$, $k = 0.2$ where $\tilde{\rho}$ is the density at the boundary $r = 1.0$. We generated the following plots:

- Figure 5.1: Energy density.
- Figure 5.2: Radial pressure.
- Figure 5.3: Electric field intensity.
- Figure 5.4: Charge density.
- Figure 5.5: Mass.

The energy density ρ (in Figure 5.1) is positive, finite and monotonically decreasing. In Figure 5.2 the radial pressure p_r is similar to ρ since p_r and ρ are related by a linear equation of state. The values of ρ and p_r are lower in the presence of the electric field $E \neq 0$. The form chosen for E (Figure 5.3) is physically reasonable and describes a functions which initially decreases, reaches a minimum and then increases as we approach the boundary. The charge density (in Figure 5.4) in general is a continuous and decreasing function. The singularity at the stellar centre is eliminated when $k = 0$. The mass function (in Figure 5.5) is strictly increasing function which is continuous and finite. We observe that the mass, in the presence of charge, has lower values that the corresponding uncharged case as was the case for plots in chapter 4. This is consistent as $E \neq 0$ generates lower densities which produces a weaker total field since the electromagnetic field is repulsive. Thus all matter variables, electromagnetic quantities and gravitational potentials are nonsingular and well-behaved in a region away from the stellar centre. At the stellar centre we need to set $k = 0$ to have finite values for the charge density.

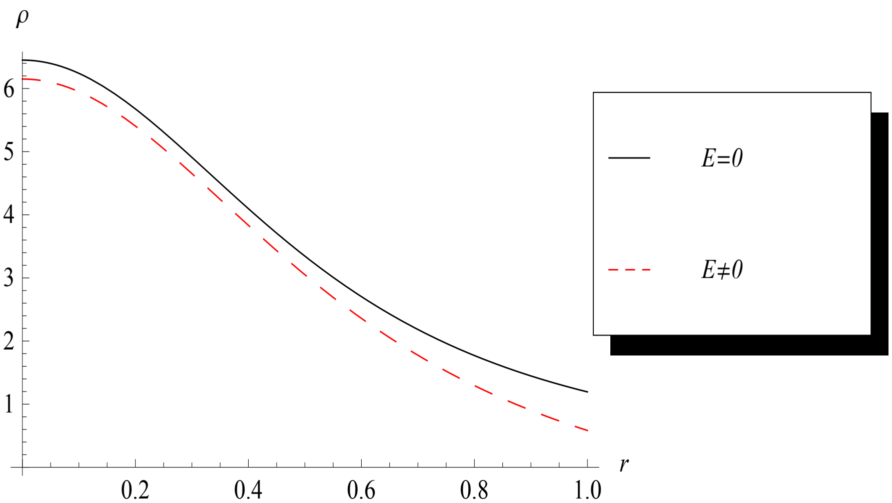

Figure 5.1: Energy density.

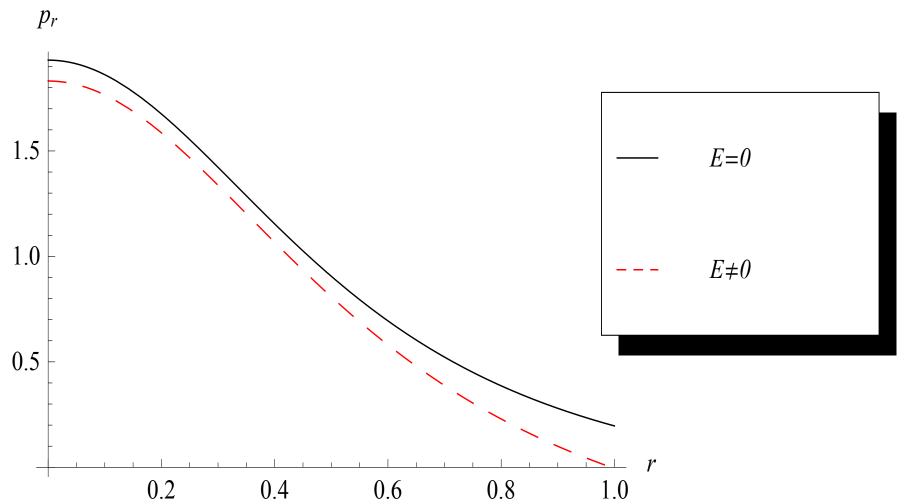

Figure 5.2: Radial pressure.

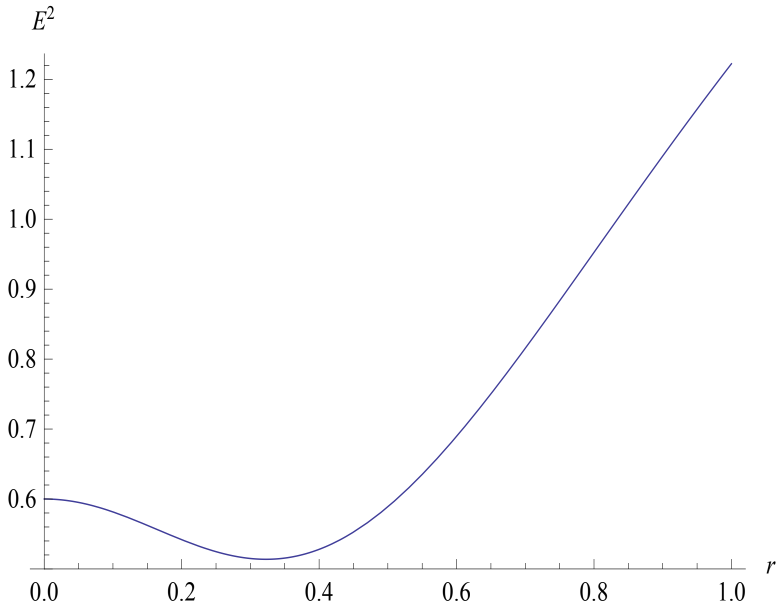

Figure 5.3: Electric field intensity.

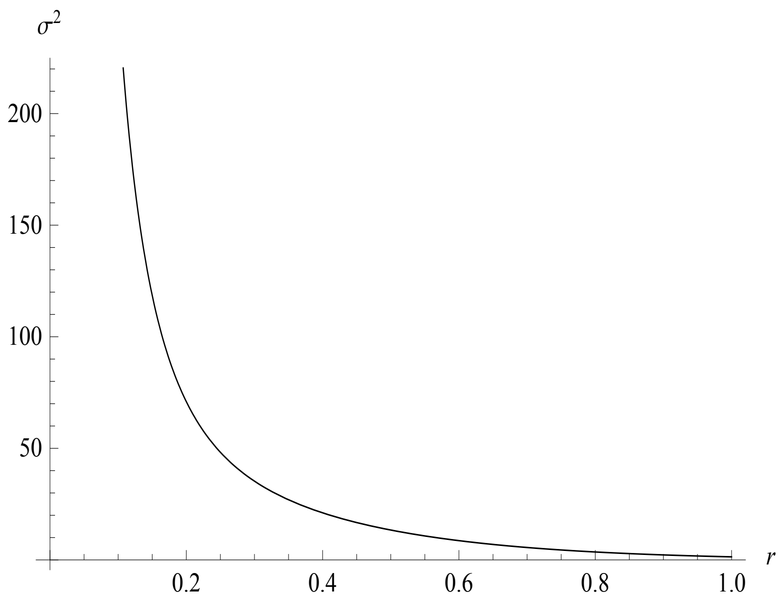

Figure 5.4: Charge density.

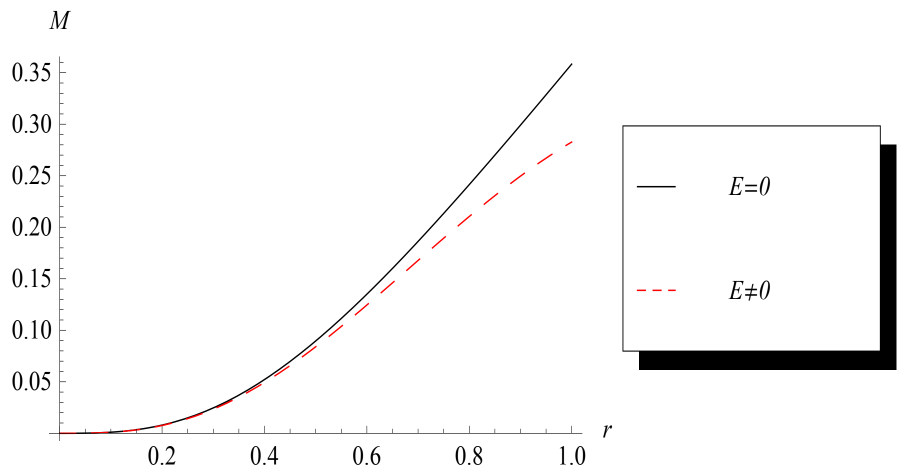

Figure 5.5: Mass.

5.4 Stellar structure

In this section we demonstrate that the solutions found in this thesis could be used to model realistic stellar bodies. For the astrophysical importance of our solutions, we try to compare them to those found by Sharma and Maharaj (2007) and Thirukkanesh and Maharaj (2008). We introduce the transformations:

$$\tilde{a} = aR^2, \ \tilde{b} = bR^2, \ \tilde{\beta} = \beta R^2, \ \tilde{k} = kR^2, \ \tilde{s} = sR^2$$

Using these transformations the energy density in (5.12c) becomes

$$\rho = \frac{(2\tilde{b} - \tilde{k})(3 + \tilde{a}y) - \tilde{s}\tilde{a}^2 y^2}{2R^2(1 + \tilde{a}y)^2} \tag{5.15}$$

The mass contained within a radius r is given by

$$M = \frac{r^3(6\tilde{b} - 3\tilde{k} + 5\tilde{s})}{12R^2(1 + \tilde{a}y)} + \frac{\tilde{s}r(15 - 2\tilde{a}y^2)}{24\tilde{a}(1 + \tilde{a}^2 y)} - \frac{5\tilde{s}R \arctan[\sqrt{\tilde{a}y}]}{8\tilde{a}^{3/2}} \tag{5.16}$$

where we have set $C = 1$ and $y = \frac{r^2}{R^2}$. These expressions are relevant to the solutions presented in this chapter 5.

When $\tilde{k} = 0$ and $\tilde{s} \neq 0$ ($E \neq 0$), we obtain

$$\rho = \frac{2\tilde{b}(3 + \tilde{a}y) - \tilde{s}\tilde{a}^2 y^2}{2R^2(1 + \tilde{a}y)^2} \tag{5.17a}$$

$$M = \frac{r^3(6\tilde{b} + 5\tilde{s})}{12R^2(1 + \tilde{a}y)} + \frac{\tilde{s}r(15 - 2\tilde{a}^2 y^2)}{24\tilde{a}(1 + \tilde{a}y)} - \frac{5\tilde{s}R \arctan[\sqrt{\tilde{a}y}]}{8\tilde{a}^{3/2}} \tag{5.17b}$$

This correspond to the solutions presented in Chapter 4. When $\tilde{k} \neq 0$ and $\tilde{s} = 0$ ($E \neq 0$), we obtain

$$\rho = \frac{(2\tilde{b} - \tilde{k})(3 + \tilde{a}y)}{2R^2(1 + \tilde{a}y)^2} \tag{5.18a}$$

$$M = \frac{r^3(2\tilde{b} - \tilde{k})}{4R^2(1 + \tilde{a}y)} \tag{5.18b}$$

This correspond to the solutions presented in Chapter 3 and relate to the results of Thirukkanesh and Maharaj (2008).

When $\tilde{k} = 0$, $\tilde{s} = 0$ $(E = 0)$, then we have

$$\rho = \frac{\tilde{b}(3 + \tilde{a}y)}{R^2(1 + \tilde{a}y)^2} \tag{5.19a}$$

$$M = \frac{\tilde{b}r^3}{2R^2(1 + \tilde{a}y)} \tag{5.19b}$$

In this case there is no charge and we obtain the expressions of Sharma and Maharaj (2007).

We calculate the masses for the various cases listed above. We set $r = 7.07$ km, $R = 43.245$ km, $\tilde{k} = 37.403$ and $\tilde{s} = 0.137$. We tabulate the information in the following tables:

- Table 5.1: Masses of earlier models.
- Table 5.2: Masses for solutions I models.
- Table 5.3: Masses for solutions II models.
- Table 5.4: Comparative masses.

Note that when $k = 0$, $s = 0$ we have an uncharged stellar body and we regain masses generated by Sharma and Maharaj (2007). When $k \neq 0$ and $s = 0$ we find masses for a charged relativistic star in the Thirukkanesh and Maharaj (2008) models. We have included those two sets of values for consistency and to demonstrate that our general results contain the special cases considered previously. Table 5.1 contains these earlier results. In Table 5.2 we have $s \neq 0$ and the masses found correspond to new charged solutions, with nonsingular charge densities at the origin, considered in chapter 4. In Table 5.3 we have both $k \neq 0$ and $s \neq 0$ in the masses for the solutions considered here in chapter 5. Table 5.4 is essentially a consolidated set of masses from the previous tables. In all cases we obtain stellar masses which are physically reasonable. We observe that the presence of charge generates a lower mass M because of the repulsive electromagnetic field which corresponds to a weaker field. Observe that our masses are consistent with the results of Dey *et al* (1998, 1999) with an equation of state for strange matter. When the charge is absent the mass $M = 1.434M_{\odot}$; the presence of charge in the different solutions affects this value. It is interesting to observe that the nonsingular charge density of Table 5.2 has the smallest charge. Dey *et al* (1998, 1999) have shown that these values are consistent with observations for the X-ray binary pulsar SAX J1808.4-3658. Consequently these charged general relativistic models have astrophysical significance.

Table 5.1: Masses of earlier models

$\tilde{b}$	$\tilde{a}$	$M(M_\odot)$ $k=0, s=0$ $E=0$	$M(M_\odot)$ $k\neq 0, s=0$ $E\neq 0$
30	23.681	1.176	0.442
40	36.346	1.298	0.691
50	48.307	1.396	0.874
54.34	53.340	1.434	0.940
60	59.788	1.478	1.017
70	70.920	1.547	1.134
80	81.786	1.607	1.231
90	92.442	1.659	1.314
100	102.929	1.706	1.387

Table 5.2: Masses for solutions I models

$\tilde{b}$	$\tilde{a}$	$M(M_{\odot})$ $k=0, s=0$ $E=0$	$M(M_{\odot})$ $k=0, s\neq 0$ $E\neq 0$
30	23.681	1.176	1.132
40	36.346	1.298	1.258
50	48.307	1.396	1.348
54.34	53.340	1.434	1.381
60	59.788	1.478	1.418
70	70.920	1.547	1.475
80	81.786	1.607	1.522
90	92.442	1.659	1.561
100	102.929	1.706	1.387

Table 5.3: Masses for solutions II models

$\tilde{b}$	$\tilde{a}$	$M(M_\odot)$ $k=0, s=0$ $E=0$	$M(M_\odot)$ $k\neq 0, s\neq 0$ $E\neq 0$
30	23.681	1.176	0.381
40	36.346	1.298	0.651
50	48.307	1.396	0.826
54.34	53.340	1.434	0.887
60	59.788	1.478	0.958
70	70.920	1.547	1.062
80	81.786	1.607	1.146
90	92.442	1.659	1.216
100	102.929	1.706	1.275

Table 5.4: Comparative masses

$\tilde{b}$	$\tilde{a}$	$M(M_\odot)$ $k=0, s=0$ $E=0$	$M(M_\odot)$ $k\neq 0, s=0$ $E\neq 0$	$M(M_\odot)$ $k=0, s\neq 0$ $E\neq 0$	$M(M_\odot)$ $(k\neq 0, s\neq 0)$ $E\neq 0$
30	23.681	1.176	0.442	1.132	0.381
40	36.346	1.298	0.691	1.258	0.651
50	48.307	1.396	0.874	1.348	0.826
54.34	53.340	1.434	0.940	1.381	0.887
60	59.788	1.478	1.017	1.418	0.958
70	70.920	1.547	1.134	1.475	1.062
80	81.786	1.607	1.231	1.522	1.146
90	92.442	1.659	1.314	1.561	1.216
100	102.929	1.706	1.387	1.594	1.275

Chapter 6

Conclusion

Our purpose in this thesis was to find new exact solutions to the Einstein-Maxwell systems with a barotropic equation of state for static spherically symmetric gravitational fields. In particular we chose a linear equation of state relating the energy density to the radial pressure. Such models may be used to model relativistic stars in astrophysical situations. We believe that the new solutions to the Einstein-Maxwell systems presented are physically reasonable. We expect that a detailed physical analysis of these solutions could be useful in describing realistic bodies (dense stars) in general relativity.

We briefly review the work carried out in this thesis:

- We give in chapter 2 a review of differential geometry used in general relativity. For uncharged and charged matter, we assume that the line element is static spherically symmetric with both perfect fluids and imperfect fluids. We present a more tractable form of the Einstein-Maxwell field equations due to Durgapal and Bannerji (1983). We review, for a realistic relativistic stellar model, the physical properties required for the interior solutions to the Einstein-Maxwell field equations.

- In chapter 3 we make physical reasonable choices for the gravitational potential and the electric field intensity. These have the form

$$\begin{aligned} Z &= \frac{1+(a-b)x}{1+ax} \\ \frac{E^2}{C} &= \frac{k(3+ax)}{(1+ax)^2} \end{aligned}$$

We regain the models of Thirukkanesh and Maharaj (2008) and indicate that the charge density is singular at the centre of the sphere.

- In chapter 4 we select the analytic forms

$$\begin{aligned} Z &= \frac{1+(a-b)x}{1+ax} \\ \frac{E^2}{C} &= \frac{sa^2x^2}{(1+ax)^2} \end{aligned}$$

New solutions to the Einstein-Maxwell system are presented. A detailed physical analysis indicates that this class of models are well-behaved and there is no singularity in the electromagnetic field at the stellar origin.

- We choose the generalised forms

$$\begin{aligned} Z &= \frac{1+(a-b)x}{1+ax} \\ \frac{E^2}{C} &= \frac{k(3+ax)+sa^2x^2}{(1+ax)^2} \end{aligned}$$

in chapter 5 which contains previous cases studied. Exact solutions are presented to the Einstein-Maxwell equations, and the matter and electrical variables are plotted. We include tables listing stellar masses for particular values for both neutral and charged matter. The presence of the electric field reduces the stellar mass in general. We show that the masses, with the electromagnetic field, are consistent with values obtained by Dey *et al* (1998, 1999) for quark stars with strange matter.

In summary we have found new classes of exact solutions to the Einstein-Maxwell system of equations. The solutions obtained contain the results of previous investigations and may be used to model a charged relativistic sphere with anisotropic pressures. Note that we have found models which are regular in the centre for the charge density which is an improvement on the results of Thirukkanesh and Maharaj (2008).

Bibliography

[1] Chaisi M and Maharaj S D, Compact anisotropic spheres with prescribed energy density, *Gen. Relativ. Gravit.* **37**, 1177 (2005).

[2] Chaisi M and Maharaj S D, Anisotropic static solutions in modelling highly compact bodies, *Gen. Relativ. Gravit.* **66**, 609 (2006).

[3] Choquet-Bruhat Y, DeWitt-Morette C and Dillard-Bleick M, Analysis, Manifolds and Physics (Princeton University Press: Princeton) (1982).

[4] Davies P, The New Physics (Cambridge University Press: Cambridge) (1989).

[5] de Felice F and Clark C J S, Relativity on Manifolds (Cambridge University Press: Cambridge) (1990).

[6] Delgaty M S R and Lake K, Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein's equations, *Comput. Phys. Commun.* **115**, 395 (1998).

[7] Dev K and Gleiser M, Anisotropic stars: exact solutions, *Gen. Relativ. Gravit.* **34**, 1793 (2002).

[8] Dev K and Gleiser M, Anisotropic stars II: stability, *Gen. Relativ. Gravit.* **35**, 1435 (2003).

[9] Dey M, Bombaci I, Dey J, Ray S and Samanta B C, Strange stars with realistic quark vector interaction and phenomenological density-dependent scalar potential, *Phys. Lett. B* **438**, 123 (1998).

[10] Dey M, Bombaci I, Dey J, Ray S and Samanta B C, *Phys. Lett. B* **447**, 352 (addendum) (1999a).

[11] Dey M, Bombaci I, Dey J, Ray S and Samanta B C, *Phys. Lett. B* **467**, 303 (erratum) (1999b).

[12] D'Inverno R, Introducing Einstein's Relativity (Oxford University Press: New-York) (1992).

[13] Durpagal M C and Banerji R, New analytical stellar model in general relativity, *Phys. Rev.* **27**, 328 (1983).

[14] Finch M R and Skea J E F, A relativistic stellar model based on an ansatz of Duorah and Ray, *Class. Quantum Grav.* **6**, 467 (1989).

[15] Gron O and Hervik S, Einstein's general relativity with modern applications in cosmology (Springer Verlag: Berlin) (1990).

[16] Hansraj S and Maharaj S D, Charged analogue of Finch-Skea stars, *Int. J. Mod. Phys. D* **15**, 1311 (2006).

[17] Herrera L and Ponce de Leon J, Perfect fluid spheres admitting a one-parameter of group conformal motions, *J. Math. Phys.* **26**, 778 (1985a).

[18] Herrera L and Ponce de Leon J, Anisotropic spheres admitting a one-parameter of group conformal motions, *J. Math. Phys.* **26**, 2018 (1985b).

[19] Herrera L and Ponce de Leon J, Confined gravitational field provided by anisotropic fluids, *J. Math. Phys.* **26**, 2847 (1985c).

[20] Ivanov B V, Static charged perfect fluid spheres in general relativity, *Phys. Rev. D* **65**, 104001 (2002).

[21] Karmakar S, Mukherjee R, Sharma R and Maharaj S D, The role of pressure anisotropiy on the maximum mass of cold compact stars, *Pramana - J. Phys.* **68**, 881 (2007).

[22] Komathiraj K and Maharaj S D, Classes of exact Einstein-Maxwell solutions, *Gen. Relativ. Gravit.* **39**, 2079 (2007a).

[23] Komathiraj K and Maharaj S D, Tikekar superdense stars in electric fields, *J. Math. Phys.* **48**, 042501 (2007b).

[24] Krasinki A, Inhomogeneous cosmological models (Cambridge University Press: Cambridge) (1997).

[25] Lobo F S N, Stable dark energy stars, *Class. Quantum Grav.* **23**, 1525 (2006).

[26] Maharaj S D and Komathiraj K, Generalized compact spheres in electric fields, *Class. Quantum Grav.* **24**, 4513 (2007).

[27] Maharaj S D and Thirukkanesh S, Some new static charged spheres, *Nonlinear Analysis: Real World Applications* **10**, 3396 (2009).

[28] Maharaj S D and Maartens R, Static anisotropic fluid spheres in general relativity with nonuniform density, *Gen. Relativ. Gravit.* **21**, 899 (1989).

[29] Mak M K and Harko T, An exact anisotropic quark star model, *Chin. J. Astron. Astrophys* **2**, 248 (2002).

[30] Mak M K and Harko T, Anisotropic stars in general relativity, *Proc. Roy. Soc. Lond A* **459**, 393 (2003).

[31] Mak M K and Harko T, Quark stars admitting a one-parameter group of conformal motion, *Int. J. Mod. Phys. D* **13**, 149 (2004).

[32] Matese J J and Whitman P G, New method for extracting static equilibrium configurations in general relativity, *Phys. Rev. D* **22**, 1270 (1980).

[33] Misner C W, Thome K S, Wheeler J A, Gravitation (Freeman: San Francisco) (1973).

[34] Narlikar J V, An introduction of cosmology (Cambridge University Press: Cambridge) (2002).

[35] Reissner H, Uber die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie, *Ann. Phys. Lpz.* **50**, 106 (1916).

[36] Schwarzchild K, Uber das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einstein Theorie, *Sitz. Deut. Akad. Wiss. Berlin, Kl. Math.Phys.* **24**, 424 (1916).

[37] Shapiro S L and Teukolsky S A, Black holes, white dwarfs and neutron stars (Wiley: New York) (1983).

[38] Sharma R and Maharaj S D, A class of relativistic stars with a linear equation of state, *Mon. Not. R. Astron. Soc.* **375**, 1265 (2007).

[39] Sharma R, Mukherjee S and Maharaj S D, General solution for a class of static charged spheres, *Gen. Relativ. Gravit.* **33**, 999 (2001).

[40] Stephani H, General Relativity: An introduction to the theory of gravitational field (Cambridge University Press: Cambridge) (1990).

[41] Stephani H, Kramer D, MacCallum M A H and Herlt E, Exact solutions of Einstein's field equations (Cambridge University Press: Cambridge) (2003).

[42] Straumann N, General relativity with applications to astrophysics (Springer: Berlin) (2004).

[43] Tikekar R, Spherical charged fluid distribution in general relativity, *J. Math. Phys.* **25**, 1481 (1984).

[44] Tolman R C, Static solutions of Einstein's field equations for spheres of fluid, *Phys. Rev.* **55**, 364 (1939).

[45] Thirukkanesh S and Maharaj S D, Charged anisotropic matter with a linear equation of state, *Class. Quantum Grav.* **25**, 235001 (2008).

[46] Varela V, Rahaman F, Ray S, Chakraborty K and Kalam M, Charged anisotropic matter with linear or nonlinear equation, *Phys. Rev. D* **82**, 044052 (2010).

[47] Wald R, General relativity (University of Chicago Press: Chicago) (1984).

[48] Whitman P G and Burch R C, Charged spheres in general relativity, *Phys. Rev. D* **24**, 2049 (1981).

[49] Will C M, The Theory and Experimental in Gravitational Physics (Cambridge University Press: Cambridge) (1981).

[50] Wolfram S, Mathematica (Cambridge University Press: Cambridge) (1999).