UNIVERSITY OF KWAZULU-NATAL

# ON THE PHYSICAL VIABILITY OF HORIZON–FREE COLLAPSE

MLUNGISI ALEX DOCTOR NTSHANGASE

# On The Physical Viability of Horizon-free Collapse

by

Mlungisi Alex Doctor Ntshangase

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As the candidate's supervisor I have approved this dissertation for submission.

Signed: .......................... By: Dr. M Govender, On the: ...........................

Signed: .......................... By: Prof. K Govinder, On the: ...........................

# Abstract

The so-called Cosmic Censorship Conjecture has drawn widespread attention amongst astrophysicists and particle physicists. In particular, the end-state of gravitational collapse of a bounded matter distribution is a source of much debate with the discovery of naked singularities resulting from the continued gravitational collapse of reasonable matter distributions. One of the first attempts at investigating the final outcome of gravitational collapse of a stellar object was undertaken by Oppenheimer and Snyder in 1939. Their model was highly idealised and focussed on a dust sphere contracting under its own gravity. With the discovery of the Vaidya solution, it became possible to model stars emitting energy to the exterior spacetime. In this dissipative model, the exterior spacetime is nonempty and the collapsing stellar body is enveloped by a zone of null radiation. The smooth matching of the interior spacetime to the Vaidya exterior was achieved by Santos in 1985. It was then possible to model radiating stars undergoing gravitational collapse. The energy momentum tensor for the interior stellar fluid was modelled on more realistic physics and was extended to include heat flux, neutrino transport, shear, pressure anisotropy, bulk viscosity and the electromagnetic field. It has been shown that the collapse of reasonable matter distributions always lead to the formation of a black hole in the absence of shear or in the case of homogeneous densities.

In this study we investigate a radiating stellar model proposed by Banerjee *et al*

(BCD model) in which the horizon is never encountered. The interior matter distribution is that of an imperfect fluid with heat flux and the exterior spacetime is described by the radiating Vaidya metric. Our approach is more general than the one proposed by Banerjee *et al* as they fix the gravitational potentials for the interior line element by making *ad-hoc* assumptions. A consequence of their model is that it undergoes horizon–free collapse. We start off with the fact that the horizon never forms throughout the collapse process. This restricts the gravitational behaviour of the model. We utilise the boundary condition to determine the temporal evolution of the model. As a result, we obtain new collapsing models in which the horizon never forms.

In order to investigate the physical viability of our generalised *BCD* model we analyse the luminosity profile and the temperature profiles within the framework of extended irreversible thermodynamics. We highlight interesting physical features of our results.

**To**

**Great Laws of Chance**

# Preface and Declaration

The study described in this dissertation was carried out in the School of Mathematics, Statistics, and Computer Science, Westville Campus, University of KwaZulu-Natal, Durban. This dissertation was completed under the supervision of Dr. M Govender and Prof. K Govinder.

The research contained in this dissertation represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

Signed: ...................... (Mlungisi Alex Doctor Ntshangase)

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# Chapter 1

# Introduction

In the late 1930s, Oppenheimer and Snyder developed a model to study the continued gravitational collapse of a star (Oppenheimer and Snyder 1939). They were able to show that for a large star the continued contraction leads to the formation of an event horizon such that no material particles or photons can escape to an external observer. The Oppenheimer-Snyder (OS) model was highly idealised in the sense that they considered a dust sphere collapsing under its own gravity. There was a need to generalise the OS model to include more realistic matter distributions. Observations indicate that gravitational collapse is a highly dissipative process, particularly in the latter stages where energy is carried away from the stellar core via neutrinos and photons. This would then mean that the exterior spacetime of a radiating collapsing star would be nonempty.

Vaidya (1951, 1953) derived the solution that describes the exterior spacetime of a radiating, spherically symmetric mass distribution emitting energy in the form of null radiation. It then became possible to model dissipative collapse in which the heat generated within the stellar core is transported to the exterior spacetime across the boundary of the star. The Vaidya solution was interpreted as an atmosphere composed of a null fluid enveloping the collapsing core. In this collapse scenario, the interior spacetime must be matched to the exterior Vaidya solution to ensure continuity of the gravitational potentials and the radiation flux.

Santos (1985), derived a set of junction conditions for a spherically symmetric radiating star which allowed for the smooth matching of a spherically symmetric line element to the outgoing Vaidya solution across a time-like hypersurface. This paved the way for researchers to model radiating stars where energy loss was due to a radial heat flux from the central regions of the star to the exterior spacetime. Since the presentation of the junction conditions by Santos there has been a proliferation of models of radiative collapse in which the core included bulk viscosity, shear, electromagnetic field, neutrino generation, pressure anisotropy and the cosmological constant (Maharaj *et al* 2013; Sharma and Das 2013). A number of gravitational collapse models, which assume different forms of the stress-energy-momentum tensor, have been studied (Bonnor *et al* 1989; Herrera and Santos 1997a; Naidu *et al* 2006). These gravitational collapse studies have revealed that the presence of the shearing effects delays the formation of the apparent horizon. This occurs by making the final stages of collapse incoherent which leads to the formation of the naked singularity (Joshi *et al* 2002). In the ongoing study of gravitational collapse of massive stars mathematical support for the existence of naked singularities has been shown (Harada *et al* 1998; Kudoh *et al* 2000; Singh *et al* 1996; Dwivedi and Joshi 1997; Herrera *et al* 1997).

Ori and Piran (1990) modelled a self-similar gravitationally collapsing object whose stress-energy-momentum tensor obeyed the principles of perfect fluid motion. Their work showed that the evolution of such a gravitationally collapsing body leads to the formation of a curvature singularity. The imposed self-similarity assumption was shown to have a great effect in reducing the Einstein field equations to a rather simple form of ordinary differential equations (which are solved numerically) along with the equations for radial and non-radial null geodesics (Singh *et al* 1999). Joshi and Dwivedi (1992) further developed the Ori and Piran model. They derived the Einstein field equations for the collapsing self-similar perfect fluid object and reduced the geodesic equation in the neighborhood of the singularity to an algebraic equation. In their calculations it appeared that even massive stars do form naked singularities during the collapse process (Singh *et al* 1999).

With a huge body of literature which continues to grow on a daily basis the Cosmic Censorship Conjecture is still not proved or disproved. This has led researchers to believe that the final outcome of gravitational collapse can only be adequately described by quantum gravity or modified gravity theories such as $f(R)$ gravity ($f(R)$ comes from the modification of general relativity by abandoning the simplicity assumption that the action should be linear in the scalar curvature R. This model uses a more general assumption which depends on a generic function $f(R)$ (Sotiriou 2006)), Einstein-Gauss-Bonnet gravity, Lovelock gravity, Brane-world scenarios, etc. Briefly, the $f(R)$ gravity theory is divided into three parts, viz, metric formalism, Palatini formalism, and metric-affine formalism. The interesting characteristic about these modified gravity theories is that they allow more dimensions. In particular, the field equations of $f(R)$ gravity theories admit a larger variety of solutions than Einstein's theory of gravity (Capozziello *et al* 2010). These facts make such gravity theories more appealing in the context of gravitation. Much work has been done in these gravitation models and in recent years there has been a renewed interest in the study of gravity theories of higher dimensions (Ghosh and Deshkar (2008)). However, even though several models of $f(R)$ gravity in the Palatini formalism have been studied, most interest was concentrated on those having terms inversely proportional to the scalar curvature (Sotiriou 2006). Moreover, even though such gravity theories seem to be capable of giving more gravity explanations they are unfortunately not free of problems. First of all they lead to fourth-order differential equations which are difficult to solve. Additionally, it is doubtful whether they can pass the known solar system tests or any other observational tests and whether they have the correct Newtonian limit (Sotiriou 2006). Furthermore, the definition of the matter content in higher order gravitational models gives instabilities which raises doubt about the physical viability of these models (Sotiriou 2006).

In classical general relativity researchers have attempted to incorporate more general matter distributions so that the initial conditions, at the onset of collapse, are more realistic. In recent gravitational collapse studies different authors have showed that the Einstein field equations which result from the collapse model whose matter content only possess tangential pressure are simpler than those obtained in the matter content which include radial pressure (Magli 1997; Singh and Witten 1997; Barve *et* *al* 1999). In this regard, only tangential pressure fluid models in the gravitational collapse studies seem to give pathways to studying the stability of dust naked singularities against the introduction of pressure (Singh *et al* 1999).

The existence of naked singularities has been shown many times by different authors. Banerjee *et al* (2002), proposed a new model (BCD model) to study gravitational collapse of stars. In their study, the interior of the collapsing sphere is uniformly filled with a heat conducting perfect fluid. The sphere radiates energy in the form of radial heat flux from the beginning of the collapse. Due to dissipation of energy the exterior spacetime is represented by the Vaidya metric. The junction conditions are used to match the interior with the exterior metric throughout the boundary. The most interesting feature of this model is that the rate of mass loss is balanced by the fall of the boundary radius. Hence the horizon never forms and the collapse results in a naked singularity. We will analyse this scenario in greater detail later.

This thesis is organised as follows:

- In Chapter 2 we provide a more detailed discussion on the geometry of curved surfaces which is used as the most important tool in general relativity. We only cover those aspects of differential geometry which are helpful for the completion of this research.

- In Chapter 3 the Einstein field equations for the interior of the collapsing star are presented. The Vaidya solution which describes the exterior gravitational field of our collapsing star is also presented here. We also state the main junction conditions required for the smooth matching of the interior spacetime to the

Vaidya solution.

- In Chapter 4 we highlight the main elements of extended irreversible thermodynamics that will be utilised in studying the temperature profiles of our radiating stellar models. We also review the energy conditions.

- In Chapter 5 we review horizon-free collapse and give a brief but more clarifying discussion on the BCD model of the gravitational collapse of stars. We further attempt to generalise the BCD model by assuming at the very onset of collapse that the horizon will not be encountered throughout the collapse process. A detailed physical analysis of our results is provided here. These include the investigation of luminosity, redshift, and temperature profiles of the collapsing star.

- In Chapter 6 we give an overall conclusion of our work and highlight the importance of pursuing this study further.

# Chapter 2

# Preliminaries

## 2.1 Introduction

In 1916, Albert Einstein (1879 – 1955) published a well appreciated theory in physics referred to as the general theory of relativity (Einstein 1916). This generalised his special theory of relativity published in 1905 (Einstein 1905). Unlike classical physics, general relativity recognises time as the fourth relative coordinate. This realization gave birth to the four dimensional coordinate system. The Einstein field equations presented in general relativity reveal a strong connection between matter and the geometry of spacetime. After Euclid's Geometry, Riemann developed the geometry of curved surfaces which served as one of the most important bases in the development of general relativity. With differential geometry, tensor calculus, and physics Einstein developed a set of equations which give a strong connection between matter (this includes gravitation) and spacetime. These equations are very important in the study of gravitational collapse of stars as they give a full understanding of gravitational strength, curvature changes, and the rate of energy changes as the star undergoes dissipative gravitational collapse. In this chapter we give a clarifying overview on these aspects which forms a working basis for our study. In §2.2 we provide an overview of the differential geometry which forms the basis for describing curved spacetimes. We present the Christoffel symbols and their link to the Riemannian curvature tensor, Ricci tensor and finally the construction of the Einstein tensor. We also show how to use these elements in combination with the stress-energy-momentum tensor to derive the Einstein field equations.

## 2.2 Differential Geometry

To understand the geometry of spacetime, either curved or flat, one needs to understand the distance between any two neighbouring points in that spacetime. In flat space the distance between neighbouring points is given by the following formula

$$ds^2 = dx^2 + dy^2.$$

For any arbitrary coordinate system where the curvature is nonzero, the distance between these two points is given by

$$ds^2 = g_{ab} dx^a dx^b, \tag{2.2.1}$$

where $g_{ab}$ is the field metric function known as the metric tensor. In general relativity every vector is accompanied by a basis vector. Therefore the covariant derivative of any vector includes an additional term which gives the derivative of the corresponding basis vector. Christoffel derived a single field function referred to as a Christoffel symbol completely defined in terms of the metric field function and its derivatives (Schutz

1985). The Christoffel symbols are used instead of directly finding the change of a basis vector. The general form of the Christoffel symbols are as follows

$$\Gamma^a_{bc} = \frac{1}{2} g^{da}(g_{bd,c} - g_{bc,d} + g_{cd,b}), \tag{2.2.2}$$

where the indices read as follows: $a$ gives the components of the resulting vector, $b$ gives the basis vector to be differentiated, and $c$ is the coordinate with respect to which $b$ is being differentiated. The Christoffel symbols are very useful in the context of general relativity as they are also used to define many important functions.

General Relativity dwells in curved spacetime. The degree of curvature, given by

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}, \tag{2.2.3}$$

is called the Riemann curvature tensor. It is defined in terms of the first and second derivatives of the field metric tensor encapsulated by the Christoffel symbols. The Riemann tensor vanishes in a flat spacetime ($R^a_{bcd} = 0$) and in local frames. The contraction of the Riemann tensor on the first and third indices gives the Ricci tensor

$$R_{ab} = \Gamma^d_{ab,d} - \Gamma^d_{ad,b} + \Gamma^d_{ed}\Gamma^e_{ab} - \Gamma^d_{eb}\Gamma^e_{ad}. \tag{2.2.4}$$

A further contraction yields the Ricci scalar

$$R = \Gamma^d_{aa,d} - \Gamma^d_{ad,a} + \Gamma^d_{ed}\Gamma^e_{aa} - \Gamma^d_{ea}\Gamma^e_{ad}. \tag{2.2.5}$$

The Einstein field equations use the stress-energy-momentum tensor

$$T_{ab} = (\mu + p_\perp)u_a u_b + p_\perp g_{ab} + (p_r - p_\perp)x_a x_b + q_a u_b + q_b u_a \tag{2.2.6}$$

where $\mu$ is the energy density, $p_r$ and $p_\perp$ are the radial and tangential pressure, respectively, $q_a$ is the heat flow vector, $x_a$ is a unit spacelike 4-vector along the radial direction and $\mathbf{u}$ is the fluid four-velocity to represent matter.

The General Theory of Relativity is basically the study of the distribution and behaviour of physical quantities in the spacetime manifold. This theory actually takes over from Newtonian physics and tries to explain the physics of intermediate celestial dimensions.

In Newtonian Physics,

$$\nabla^2 \phi = 4\pi G \rho \tag{2.2.7}$$

is the equation governing the gravitational potential. In (2.2.7), $\phi$ gives the gravitational potential, $G$ is the gravitational constant and $\rho$ is the energy density. To find Einstein's field equations one can simply perform contractions on the Riemann tensor and Ricci tensor until the twice contracted Bianchi identities are reached. Simple manipulations of these identities yield the Einstein tensor. Another way to find the Einstein tensor is by writing

$$R_{ab} = \kappa T_{ab}$$

prompted by (2.2.7). As we shall show, this prescription does not work, and needs to be modified. In this equation $\kappa$ is a scaling constant (established to equal $8\pi$). For this expression to generate conservation laws, it is required that

$$R_{ab;c} = \kappa T_{ab;c} = 0.$$

However, it turns out that $R_{ab;c} \neq 0$ but rather is given by

$$R_{ab;c} = \frac{1}{2} g_{ab} R_{,c}.$$

Given that the covariant derivative of a scalar is the same as its partial derivative and the divergence of the metric tensor is zero, we can rewrite this expression as

$$R_{ab;c} = \left(\frac{1}{2}g_{ab}R\right)_{;c}. \tag{2.2.8}$$

Grouping terms on the left hand side and letting

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R \tag{2.2.9}$$

(2.2.8) becomes

$$(G_{ab})_{;b} = 0. \tag{2.2.10}$$

Therefore with $G_{ab}$ and $T_{ab}$ conservation laws could be satisfied. This triumph of discovering $G_{ab}$ marked the birth of the Einstein field equations given as

$$G_{ab} = \kappa T_{ab}, \tag{2.2.11}$$

where $G_{ab}$ is the Einstein tensor and $\kappa$ is a coupling constant.

In physics, constants only refine calculations and do not bring about any change in the behaviour of the physical elements. As a result, for simplicity we set $\kappa = 1$ without any loss of generality.

# Chapter 3

# Radiating Stars

## 3.1 Introduction

In studying the models of gravitational collapse of stars, astrophysicists and general relativists had difficulties in finding a way to connect the interior and exterior spacetime of a collapsing sphere. Further, even if there was a way to connect these geometries they also did not have a good exterior spacetime metric to model the exterior of a radiating gravitational body. The exterior spacetime metric was found by Vaidya (1951) while Santos (1985) obtained a set of junction conditions which ensures a smooth connection between the interior and exterior spacetime of a gravitationally collapsing dust body. With these discoveries it became possible for scientists to model gravitationally collapsing spherical bodies which emit energy in the form of heat and other forms of radiation. However, even with these solutions the study of collapse models involving vorticity effects is still underdeveloped. In §3.2 we show how Vaidya derived a mathematical equation which represents the atmosphere of a spherically symmetric collapsing object using the Schwarzschild exterior solution. We then use the Vaidya solution to represent the exterior spacetime of a gravitationally collapsing spherically symmetric object dissipating energy in the form of a radial heat flux. As we did with the interior spacetime metric in Chapter 2, we also determine the values of the Ricci scalar, Einstein tensor, etc. and derive the exterior field equations. In §3.3 we give a general overview of shear-free junction conditions.

## 3.2 Spherical Symmetry

The interior spacetime for our collapsing body is described by the spherically symmetric, shear-free line element in comoving isotropic coordinates:

$$ds^2 = -A^2 dt^2 + B^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{3.2.1}$$

where $A = A(r,t)$ and $B = B(r,t)$ are the metric functions yet to be determined. The line element (3.2.1) obeys quadratic function laws and so can be written as

$$g_{ab} = \begin{pmatrix} -A^2 & 0 & 0 & 0 \\ 0 & B^2 & 0 & 0 \\ 0 & 0 & B^2r^2 & 0 \\ 0 & 0 & 0 & B^2r^2\sin^2\theta \end{pmatrix}. \tag{3.2.2}$$

Since this is a diagonal metric, its inverse is simply

$$g^{ab} = diag\left(\frac{1}{-A^2}, \frac{1}{B^2}, \frac{1}{B^2r^2}, \frac{1}{B^2r^2\sin^2\theta}\right). \tag{3.2.3}$$

Using (2.2.2) we obtain the following non-zero Christoffel symbols

$$\Gamma^0_{00} = \frac{A_t}{A} \qquad \Gamma^0_{01} = \frac{A_r}{A}$$

$$\Gamma^0_{11} = \frac{BB_t}{A^2} \qquad \Gamma^0_{22} = \frac{r^2BB_t}{A^2}$$

$$\Gamma^0_{33} = \frac{r^2BB_t\sin^2\theta}{A^2} \qquad \Gamma^1_{00} = \frac{AA_r}{B^2}$$

$$\Gamma^1_{10} = \frac{B_t}{B} \qquad \Gamma^1_{11} = \frac{B_r}{B}$$

$$\Gamma^1_{33} = -r\sin^2\theta\left(\frac{rB_r}{B} + 1\right) \qquad \Gamma^1_{22} = -r\left(\frac{rB_r}{B} + 1\right)$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^2_{20} = \frac{B_t}{B}$$

$$\Gamma^2_{21} = \frac{B_r}{B} + \frac{1}{r} \qquad \Gamma^3_{30} = \frac{B_t}{B}$$

$$\Gamma^3_{31} = \frac{B_r}{B} + \frac{1}{r} \qquad \Gamma^3_{32} = \cot\theta$$

for the line element (3.2.1).

Using (2.2.4), we obtain the corresponding non-zero components of the Ricci tensor

$$R_{00} = \frac{AA_rB_r}{B^3} + \frac{3A_tB_t}{AB} + \frac{2AA_r}{rB^2} - \frac{3B_{tt}}{B} + \frac{AA_{rr}}{B^2} \tag{3.2.4a}$$

$$R_{01} = \frac{2A_rB_t}{AB} + \frac{2B_rB_t}{B^2} - \frac{2B_{tr}}{B} \tag{3.2.4b}$$

$$R_{11} = \frac{BB_{tt}}{A^2} - \frac{A_tB_tB}{A^3} - \frac{A_{rr}}{A} - \frac{2B_{rr}}{B} + 2\left(\frac{B_t}{A}\right)^2 + \frac{A_rB_r}{AB}$$

$$+ 2\left(\frac{B_r}{B}\right)^2 - \frac{2B_r}{rB} \tag{3.2.4c}$$

$$R_{22} = \frac{r^2BB_{tt}}{A^2} + 2r^2\left(\frac{B_t}{A}\right)^2 - \frac{r^2B_{rr}}{B} - \frac{r^2A_tB_tB}{A^3} - \frac{3rB_r}{B}$$

$$- \frac{r^2A_rB_r}{AB} - \frac{rA_r}{A} \tag{3.2.4d}$$

Then, (3.2.3) together with (3.2.4), give the Ricci scalar as

$$R = \frac{-2A_{rr}}{AB^2} - \frac{6A_tB_t}{A^3B} + \frac{6B_{tt}}{A^2B} - \frac{4A_r}{rAB^2} + 6\left(\frac{B_t}{AB}\right)^2 - \frac{8B_r}{rB^3} + \frac{2B_r^2}{B^4} - \frac{4B_{rr}}{B^3} - \frac{2A_rB_r}{AB^3}. \tag{3.2.5}$$

Since the Einstein tensor shares exactly the same indices as the Ricci and metric tensor, the number of non-vanishing components of the Einstein tensor is equal to the number of non-vanishing components of the Ricci tensor. These components are given by

$$G_{00} = \frac{3B_t^2}{B^2} + \frac{A^2}{B^2}\left(\frac{B_r^2}{B^2} - \frac{4B_r}{rB} - \frac{2B_{rr}}{B}\right) \tag{3.2.6a}$$

$$G_{01} = \frac{2A_rB_t}{AB} + \frac{2B_rB_t}{B^2} - \frac{2B_{tr}}{B} \tag{3.2.6b}$$

$$\begin{aligned} G_{11} &= -\frac{B_t^2}{A^2} + \frac{2A_rB_r}{AB} + \frac{2B_r}{rB} + \frac{2A_tB_tB}{A^3} - \frac{2BBtt}{A^2} \\ &+ \frac{B_r^2}{B^2} + \frac{2A_r}{rA} \end{aligned} \tag{3.2.6c}$$

$$\begin{aligned} G_{22} &= -\frac{2r^2BB_{tt}}{A^2} + \frac{2r^2A_tB_tB}{A^3} - \frac{r^2B_t^2}{A^2} + \frac{rA_r}{A} + \frac{rB_r}{B} \\ &+ \frac{r^2B_{rr}}{B} + \frac{r^2A_{rr}}{A} - \left(\frac{rB_r}{B}\right)^2 \end{aligned} \tag{3.2.6d}$$

Using (2.2.6), (2.2.11), (3.2.6) and the following conditions

$$u^a = \frac{1}{A}\delta_0^a \tag{3.2.7a}$$

$$q^a = (0, q, 0, 0) \tag{3.2.7b}$$

$$q_au^a = 0 \tag{3.2.7c}$$

$$u^au_a = -1, \tag{3.2.7d}$$

where $\delta_0^a$ is the Kronecker delta, $q^a$ is the radial heat flux, and $u^a$ is a timelike 4-vector, the Einstein field equations are easily obtained to be

$$\mu = \frac{3B_t^2}{A^2B^2} + \frac{1}{B^2}\left(\frac{B_r^2}{B^2} - \frac{4B_r}{rB} - \frac{2Brr}{B}\right) \tag{3.2.8a}$$

$$p_r = \frac{2A_rB_r}{AB^3} - \frac{B_t^2}{A^2B^2} + \frac{2B_r}{rB^3} + \frac{2A_tB_t}{A^3B} - \frac{2B_{tt}}{A^2B} + \frac{B_r^2}{B^4} + \frac{2A_r}{rAB^2} \tag{3.2.8b}$$

$$p_\perp = \frac{2A_tB_t}{A^3B} - \frac{2B_{tt}}{A^2B} - \frac{B_t^2}{A^2B^2} + \frac{A_r}{rAB^2} + \frac{B_r}{rB^3} + \frac{B_{rr}}{B^3} + \frac{A_{rr}}{AB^2} - \frac{B_r^2}{B^4} \tag{3.2.8c}$$

$$q = \frac{1}{AB^3}\left(2B_{tr} - \frac{2A_rB_t}{A} - \frac{2B_rB_t}{B}\right) \tag{3.2.8d}$$

By equating the radial and tangential pressure components, one obtain the following isotropic pressure condition

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \frac{2A_rB_r}{AB} + \frac{B_r}{rB} + \frac{2B_r^2}{B^2} + \frac{A_r}{rA} \tag{3.2.9}$$

It is remarkable that this equation has no explicit temporal dependence. The forms of the interior field equations which obey the condition of pressure isotropy will be determined in chapter 5 and further discussions will be undertaken in chapter 5 and chapter 6.

## 3.3 The Exterior Vaidya Spacetime

The Vaidya solution is derived from the Schwarzschild static solution

$$ds^2 = -\left(1 - \frac{2M}{\bar{r}}\right)dt^2 + \left(1 - \frac{2M}{\bar{r}}\right)^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{3.3.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $M$ is a constant, and $\bar{r}$ is the radial coordinate. In this thesis we study the gravitational collapse with outgoing radiation. The exterior Vaidya spacetime for outgoing radiation is derived using the following coordinate transformation:

$$t = v + \bar{r} + 2M \ln\left(\frac{\bar{r}}{2M} - 1\right) \tag{3.3.2}$$

where $v$ denotes the retarded time. Differentiating (3.3.2) with respect to $\bar{r}$ gives

$$dt = dv + \left(1 - \frac{2M}{\bar{r}}\right)^{-1} d\bar{r}. \tag{3.3.3}$$

Substituting (3.3.3) in (3.3.1) with some manipulations one obtains the exterior Vaidya metric

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right) dv^2 - 2dvd\bar{r} + \bar{r}^2 d\Omega^2, \tag{3.3.4}$$

where $m = m(v)$ is the mass function of the collapsing star. In metric notation (3.3.4) takes the form

$$g_{ab} = \begin{pmatrix} -\left(1 - \frac{2m}{\bar{r}}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \bar{r}^2 & 0 \\ 0 & 0 & 0 & \bar{r}^2 \sin^2\theta \end{pmatrix} \tag{3.3.5}$$

with inverse

$$g^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & \left(1 - \frac{2m}{\bar{r}}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{\bar{r}^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\bar{r}^2 \sin^2\theta} \end{pmatrix} \tag{3.3.6}$$

The corresponding non-vanishing Christoffel symbols are

$$\Gamma^0_{00} = -\frac{m}{\bar{r}^2} \qquad \Gamma^0_{22} = \bar{r}$$

$$\Gamma^0_{33} = \bar{r}\sin^2\theta \qquad \Gamma^1_{00} = -\frac{1}{\bar{r}}\frac{dm}{dv} + \frac{m}{\bar{r}^3}(\bar{r} - 2m)$$

$$\Gamma^1_{22} = 2m - \bar{r} \qquad \Gamma^1_{01} = \frac{m}{\bar{r}^2}$$

$$\Gamma^1_{33} = (2m - \bar{r})sin^2\theta \qquad \Gamma^2_{33} = -\cos\theta\sin\theta$$

$$\Gamma^2_{12} = \frac{1}{\bar{r}} \qquad \Gamma^3_{23} = \cot\theta$$

$$\Gamma^3_{13} = \frac{1}{\bar{r}}$$

The corresponding Ricci tensor for this metric has only one non-vanishing component

$$R_{00} = -\frac{2}{\bar{r}^2}\frac{dm}{dv} \tag{3.3.7}$$

and due to $g^{00}R_{00} = 0$ the Ricci scalar vanishes.

Since there is only one Ricci tensor component with $R = 0$, there is only one non-zero component of the Einstein tensor,

$$G_{00} = -\frac{2}{\bar{r}^2}\frac{dm}{dv},$$

where in general this is written as

$$G_{ab} = -\frac{2}{\bar{r}^2}\frac{dm}{dv}\delta^0_a\delta^0_b. \tag{3.3.8}$$

Since only one component ($G_{00}$) is non-zero and $g^{00} = 0$, the field equations will only give the energy density of the emitted radiation

$$E = -\frac{2}{\bar{r}^2}\frac{dm}{dv}. \tag{3.3.9}$$

Due to energy laws (3.3.9) is a valid equation only if $\frac{dm}{dv} \leq 0$ throughout the collapse period. This restriction in the change of mass supports the loss of star's energy from the beginning to the end of the collapse. We will discuss this further in chapters five and six.

## 3.4 Summary of Junction Conditions

In general relativity matter has a great effect on the geometry of spacetime. When studying the models of gravitational collapse this geometry-matter connection is always considered. This is due to the fact that, in every collapse model, the interior distribution of matter can never be the same as the matter content on the other side of the star's surface. As a result the interior and exterior spacetimes are always considered as two distinct regions separated by a three dimensional hypersurface.

The interior spacetime gives the interior geometry of the collapsing cloud while the exterior spacetime gives the corresponding geometry for the outer atmospheres. Both interior and exterior spacetime manifolds vary depending on the physical conditions assumed in the collapsing cloud model. For the case of the present study, the assumed spherically symmetric heat conducting dust-ball of uniform distribution of matter is non-rotating and shear-free. Hence the interior spacetime manifold is given by the following vorticity-and-shear-free interior spacetime metric

$$ds^2_- = -A^2dt^2 + B^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{3.4.1}$$

The interior distribution of matter is given by the stress-energy-momentum tensor as follows:

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a. \tag{3.4.2}$$

Since the exterior spacetime is filled with radiation resulting from the collapse, the exterior spacetime is represented by the Vaidya metric for outgoing radiation given as

$$ds^2_+ = -\left(1 - \frac{2m}{\bar{r}}\right) dv^2 - 2dv d\bar{r} + \bar{r}^2 d\Omega^2 \tag{3.4.3}$$

The interior and exterior spacetimes are generally separated by a time-like hypersurface which also marks the boundary of the collapsing cloud and is given as

$$ds^2_\Sigma = -d\tau^2 + \Re^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.4.4}$$

where $\Sigma$ denotes the boundary. The interior boundary surface is given by

$$f(r,t) = r - r_\Sigma = 0$$

while the exterior surface is

$$f(r,v) = \bar{r} - r_\Sigma = 0.$$

For a complete model of collapse the geometries need to be connected. Santos (1985), derived a set of junction conditions. However, here we only give a summary (detailed elaborations on these conditions can be found in (Israel 1967, Bonnor *et al* 1989, Santos 1985, Lichnerowicz 1955, O Brien and Synge 1952, Lake 1987)).

Realizing that the interior and exterior spacetimes are separated, the first fundamental form requires the continuity of the exterior and interior regions over the time-like hypersurface as follows:

$$ds_-^2 = ds_+^2 = ds_\Sigma^2. \tag{3.4.5}$$

Under this restriction, equating the interior and intrinsic metric gives

$$A(r_\Sigma, t)\dot{t} \quad = \quad 1 \tag{3.4.6a}$$

$$r_\Sigma B(r_\Sigma, t) \quad = \quad \Re(\tau), \tag{3.4.6b}$$

where dot represent differentiation with respect to time on the surface $\Sigma$. For the exterior metric the following is obtained

$$\bar{r}_\Sigma(v) \quad = \quad \Re(\tau) \tag{3.4.7a}$$

$$\left(1 - \frac{2m}{\bar{r}} + 2\frac{d\bar{r}}{dv}\right)_\Sigma \quad = \quad \left(\frac{1}{\dot{v}^2}\right)_\Sigma \tag{3.4.7b}$$

Hence the continuity of the metric functions on the boundary ($\Sigma$) requires the following boundary conditions

$$A(r_\Sigma, t)dt \quad = \quad \left(1 - \frac{2m}{\bar{r}_\Sigma} + 2\frac{d\bar{r}_\Sigma}{dv}\right)_\Sigma^{\frac{1}{2}} dv \tag{3.4.8a}$$

$$r_\Sigma B(r_\Sigma, t) \quad = \quad \bar{r}_\Sigma(v). \tag{3.4.8b}$$

These conditions are easily obtained by further finding equality between (3.4.6) and (3.4.7).

Secondly, the second fundamental form requires the continuity of the extrinsic curvatures on the hypersurface as follows

$$K_{ij}^{+} - K_{ij}^{-} = 0 \tag{3.4.9}$$

where

$$K_{ij}^{\pm} \equiv -n_a^{\pm}\frac{\partial^2 X_{\pm}^a}{\partial\xi^i\partial\xi^j} - n_a^{\pm}\Gamma_{cd}^a\frac{\partial X_{\pm}^c \partial X_{\pm}^d}{\partial\xi^i\partial\xi^j} \tag{3.4.10}$$

The nonvanishing components of $K_{ij}^{-}$ are

$$K_{\tau\tau}^{-} = \left(-\frac{A_r}{AB}\right)_{\Sigma} \tag{3.4.11a}$$

$$K_{\theta\theta}^{-} = (r(rB)_r)_{\Sigma} \tag{3.4.11b}$$

while for the exterior we have

$$K_{\tau\tau}^{+} = \left(\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\bar{r}^2}\right)_{\Sigma} \tag{3.4.12a}$$

$$K_{\theta\theta}^{+} = (\dot{v}(\bar{r}-2m) + \bar{r}\dot{\bar{r}})_{\Sigma} \tag{3.4.12b}$$

Hence, the general conditions of the second fundamental form are

$$\left(-\frac{A_r}{AB}\right)_{\Sigma} = \left(\frac{\ddot{v}}{\dot{v}} - \dot{v}\frac{m}{\bar{r}^2}\right)_{\Sigma} \tag{3.4.13a}$$

$$(r(rB)_r)_{\Sigma} = (\dot{v}(\bar{r}-2m) + \bar{r}\dot{\bar{r}})_{\Sigma} \tag{3.4.13b}$$

For the interest of the physics of the model, the junction conditions are further developed and expressed in terms of physical quantities. For instance, using (3.4.6) and

(3.4.7) in the continuity of spacetime condition and (3.4.13b) in the extrinsic curvature condition, the total mass of the collapsing cloud as read on the boundary ($\Sigma$) is given as

$$m_\Sigma = \left[\frac{r^3 B B_t^2}{2A^2} - r^2 B_r - \frac{r^3 B_r^2}{2B}\right]. \qquad (3.4.14)$$

Since mass is strongly related to the gravity of the object, equation (3.4.14) gives a strong insight into the gravitational strength associated with the assumed collapsing dust cloud. For more information the reader is referred to (Cahill and McVittie 1970, Hernandez and Misner 1966). There are other boundary conditions which may be obtained from mathematical manipulations of the above mentioned conditions. The most important boundary condition is the connection between pressure and the heat flux:

$$p_\Sigma = (qB)_\Sigma. \qquad (3.4.15)$$

This condition guarantees that, for every collapsing dust cloud radiating energy in the form of heat flux, the pressure at the surface of such an object is never zero. This finding by Santos reclaimed correctness in the study of the collapsing objects as other scientists had previously made some errors in pressure and heat flux assumptions (Glass 1981).

# Chapter 4

# Review of Thermodynamics

## 4.1 Introduction

In this chapter we review the fundamentals of extended irreversible thermodynamics and how it governs the temperature profiles of radiating stars. The classic approach by Eckart (1940), and Landau and Lifschitz (1959) had a few shortcomings especially from a relativistic point of view. For instance, the Eckart tranport equation suffers various pathologies such as the prediction of infinite propagation velocities for the thermal signals and unstable equilibrium states (Govender 2014 and references therein). The Eckart postulate of the scalar curvature being linear is too simple and as a result removes vital terms that are the key in order to preserve causality and stability. However, even with such drawbacks these models formed a crucial stage for later work. Mueller (1967) developed an extended non-relativistic version of irreversible thermodynamics that obeyed causality. The Mueller version was later developed by Israel and Stewart to a relativistic form (Israel (1976), Stewart (1977)). The Israel-Stewart theory developed from the theory of Grad and Mueller (Grad 1949, Mueller 1967) is largely accepted in the context of relativistic theories. One of the notable features of their work is that they impose the requirement of non-negativeness of the divergence of the entropy current. From this condition Israel and Stewart obtained equations which can be easily derived from the divergence of the off-equilibrium entropy current satisfying the second law of thermodynamics (Andersson and Comer 2006, El *et al* 2010). The Israel–Stewart equations have been successfully applied in different models. However, the study of extended irreverible thermodynamics is still under development (Carter 1991, El *et al* 2010). The recent work of the extended irreversible thermodynamic theory (also by Israel and Stewart) takes into account causality in thermodynamics by treating dissipative perturbations as being non-instantaneous (Govender and Thirukannesh 2014).

The Einstein field equations in general relativity give a very complex set of equations to describe spacetime and its matter content. The field equations are very general in a sense that this one set of equations is easily manipulated to study different models of gravity. In these equations the left-hand-side gives the structure of spacetime geometry through the Einstein tensor and the right-hand-side gives the matter content existing in such a spacetime (through the stress-energy-momentum tensor) therefore representing the actual strength of gravity. These equations are very general. However, universality between the left-hand-side and right-hand-side is not balanced. This is clear in a sense that one spacetime geometry model can be used in different models of matter content. Briefly, the stress-energy-momentum tensor depends on the particular type of matter and interactions one chooses to use to represent the matter content of the model (Visser and Barcelo 2008). Therefore the stress-energy-momentum tensor faces a serious challenge that in every model it undergoes a series of tests which evaluates its reliability. One key generic feature that almost all physically reasonable matter seems to satisfy is that the energy density is always positive or zero, at least in Minkowski spacetime. However, energy density is not the only feature of the matter content that needs to be always validated. Faced with such challenges, scientists developed a set of mathematical theorems which every matter model should satisfy in order to be a physically reasonable model of matter. These theorems are known as the energy conditions and give a required test in every gravitationally collapsing model. Beyond just checking positivity or non-negativity of the energy density, the energy conditions give a form of affirmation that various linear combinations of the components of the stress-energy-momentum tensor at any locally specified point in spacetime should be positive, or at least non-negative (Visser and Barcelo 2008). The energy conditions in essence can be viewed as a set of powerful mathematical theorems similar to the singularity theorems which under certain circumstances guarantee gravitational collapse and/or the existence of a certain type of singularity (Visser and Barcelo 2008). A number of theoretical studies in physics, cosmology, astrophysics, and general relativity have used the concept of energy conditions to affirm different forms of theorems (Sharif and Waheed 2013). The significant theoretical contribution of these conditions can be seen in the following notions: Hawking–Penrose singularity conjecture, positive mass theorem, derivation of second law of black hole thermodynamics (Sharif and Waheed 2013), Hawking-Ellis conservation theorem (Hossain 2005), etc. The energy conditions exist in different forms. In this thesis we only focus on the weak energy condition, strong energy condition, and dominant energy condition.

## 4.2 Extended Irreversible Thermodynamics

We begin our discussion by noting that, unlike the classical models of thermodynamics which imposed conservation of particle number and the energy momentum tensor

$$n^{\alpha}{}_{;\alpha} = 0, \qquad T^{\alpha\beta}{}_{;\beta} = 0 \tag{4.2.1}$$

where $n$ is the number density, the Stewart and Israel irreversible thermodynamic theory only restricts the negativity of the entropy and allows it to increase with time. This requirement is nothing more than the second law of thermodynamics:

$$S^{\alpha}_{;\alpha} \geq 0 \tag{4.2.2}$$

where $S$ is the entropy. The expression (4.2.2) is known as the covariant form of the second law of thermodynamics. The entropy itself with the inclusion of the dissipative term $R^{\alpha}$ is given as

$$S^{\alpha} = Snu^{\alpha} + \frac{R^{\alpha}}{T} \tag{4.2.3}$$

and $T$ represents the local temperature. If linearity is imposed between the thermodynamical fluxes and the thermodynamical forces the transport equations reduce to

$$\Pi = -3\zeta H \tag{4.2.4a}$$

$$q_\alpha = -\lambda \left(D_\alpha T + T\dot{u}_\alpha\right) \tag{4.2.4b}$$

$$\pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta}, \tag{4.2.4c}$$

where $\Pi$ is the bulk viscosity, $\pi_{\alpha\beta}$ the shear viscosity tensor, $\zeta$, $\lambda$, and $\eta$ respectively denote the coefficient of viscosity, heat, and shear, $H$ is the expansion and $\sigma_{\alpha\beta}$ is the shear tensor.

The entropy production rate becomes

$$S^\alpha_{;\alpha} = \frac{\Pi^2}{\zeta T} + \frac{q_\alpha q^\alpha}{\lambda T^2} + \frac{\pi_{\alpha\beta}\pi^{\alpha\beta}}{2\eta T}. \tag{4.2.5}$$

From (4.2.5), (4.2.2) is satisfied only if

$$\zeta \geq 0, \; \lambda \geq 0, \; \eta \geq 0$$

The increase in entropy is meaningful in the sense that the rise in temperature (which may be always associated with the kinetic energy) gives rise to the drift velocities of particles within the fluid, resulting in more chaos or disorder in the physical structures of the modelled object. The relationship between the temperature and entropy is given by

$$TdS = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) \tag{4.2.6}$$

which is known as the Gibbs equation. The extended irreversible thermodynamic theory uses a more general version of the scalar curvature giving rise to a more general expression of the entropy four-current, which in the context of the Mueller–Israel–Stewart theory reads as

$$S^{\mu} = Snu^{\mu} + \frac{q^{\mu}}{T} - \left(\gamma_0 \Pi^2 + \gamma_1 q_{\upsilon} q^{\upsilon} + \gamma_2 \pi_{\upsilon\kappa} \pi^{\upsilon\kappa}\right) \frac{u^{\mu}}{2T} + \frac{\alpha_0 \Pi q^{\mu}}{T} + \frac{\alpha_1 \pi^{\mu\upsilon} q_{\upsilon}}{T}. \quad (4.2.7)$$

Note that $\gamma_A(\rho, n)$ are thermodynamic coefficients for different contributions to the entropy density, and $\alpha_A(\rho, n)$ are thermodynamic viscous/heat coupling coefficients (Herrera *et al* 2009). If the thermodynamic viscous/heat coupling coefficients are set to zero, (4.2.7) becomes

$$S^{\mu} = Snu^{\mu} + \frac{q^{\mu}}{T} - \left(\gamma_0 \Pi^2 + \gamma_1 q_{\upsilon} q^{\upsilon} + \gamma_2 \pi_{\upsilon\kappa} \pi^{\upsilon\kappa}\right) \frac{u^{\mu}}{2T} \quad (4.2.8)$$

Multipying throughout by $-u_{\mu}$ gives

$$-u_{\mu} S^{\mu} = Sn - \left(\gamma_0 \Pi^2 + \gamma_1 q_{\upsilon} q^{\upsilon} + \gamma_2 \pi_{\upsilon\kappa} \pi^{\upsilon\kappa}\right) \frac{1}{2T} \quad (4.2.9)$$

which yields the entropy density as measured by a comoving observer. From the Bianchi identities and the Gibbs equation, the divergence of (4.2.8) is given as

$$\begin{aligned} TS^{\alpha}{}_{;\alpha} &= -\Pi \left[3H + \gamma_0 \dot{\Pi} + \frac{1}{2} T \left(\frac{\gamma_1}{T} u^{\alpha}\right)_{;\alpha} \Pi\right] \\ &\quad - q^{\alpha} \left[D_{\alpha} \ln T + \dot{u}_{\alpha} + \gamma_1 \dot{q}_{\alpha} + \frac{1}{2} \left(\frac{\gamma_1}{T} u^{\mu}\right)_{;\mu} q_{\alpha}\right] \\ &\quad - \pi^{\alpha\mu} \left[\sigma_{\alpha\mu} + \gamma_2 \dot{\pi}_{\alpha\mu} + \frac{1}{2} T \left(\frac{\gamma_2}{T} u^{\nu}\right)_{;\nu} \pi_{\alpha\mu}\right] \qquad (4.2.10) \end{aligned}$$

Guided by the second law of thermodynamics (4.2.2) and following the same assumptions as in (4.2.4) the transport equations in the extended theory are obtained as follows:

$$\tau_0\dot{\Pi} + \Pi = -3\zeta H - \left[\zeta T \left(\frac{\tau_0}{\frac{1}{2}\zeta T}u^\alpha\right)_{;\alpha} \Pi\right] \tag{4.2.11a}$$

$$\tau_1 h^\beta{}_\alpha \dot{q}_\beta + q_\alpha = -\lambda(D_\alpha T + T\dot{u}_\alpha) - \left[\frac{1}{2}\lambda T^2 \left(\frac{\tau_1}{\lambda T}u^\beta\right)_{;\beta} q_\alpha\right] \tag{4.2.11b}$$

$$\tau_2 h^\mu{}_\alpha h^\nu{}_\beta \dot{\pi}_{\mu\nu} + \pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta} - \left[\eta T \left(\frac{\tau_2}{2\eta T}u^\nu\right)_{;\nu} \pi_{\alpha\beta}\right]. \tag{4.2.11c}$$

Ignoring the terms in the square brackets yields the Israel–Stewart truncated versions of the transport equations:

$$\tau_0\dot{\Pi} + \Pi = -3\zeta H \tag{4.2.12a}$$

$$\tau_1 h^\beta{}_\alpha \dot{q}_\beta + q_\alpha = -\lambda(D_\alpha T + T\dot{u}_\alpha) \tag{4.2.12b}$$

$$\tau_2 h^\mu{}_\alpha h^\nu{}_\beta \dot{\pi}_{\mu\nu} + \pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta}, \tag{4.2.12c}$$

where the relaxation times $\tau_A(\rho, n)$ are given by

$$\tau_0 = \zeta\gamma_0 \tag{4.2.13a}$$

$$\tau_1 = \lambda T\gamma_1 \tag{4.2.13b}$$

$$\tau_2 = 2\eta\gamma_2. \tag{4.2.13c}$$

Now that we have obtained the truncated versions of the transport equation, we determine the actual equation that will model the temperature profiles of a radiating collapsing stellar model. For our case we seek to determine a causal heat transport equation of a spherically symmetric collapsing cloud whose interior matter is that of imperfect fluid. This spherical cloud dissipates energy in the form of radial heat flux during the collapse. Hence, for further discussions on the temperature profiles of this collapse model we utilise the transport equation developed in the extended thermodynamic theory, viz. (4.2.12b). For this thesis we can write (4.2.12b) as

$$\tau h^{\beta}{}_{\alpha}\dot{q}^{\beta} + q^{\alpha} = -\lambda\left(h^{\beta\alpha}T_{,\beta} + Tu^{\alpha}_{;\beta}u^{\beta}\right), \tag{4.2.14}$$

where (3.2.1) has been used to raise the indices in (4.2.14).

In (4.2.14), $h^{\beta}{}_{\alpha}$ is the projection tensor, $\tau$ and $\lambda$ are the relaxation time and thermal conductivity, respectively. The heat flux is given as

$$q^{\alpha} = q\frac{1}{B}\delta^{\alpha}_{1}. \tag{4.2.15}$$

The projection tensor is given as

$$h^{\beta}_{\alpha} = g^{\beta}_{\alpha} + u^{\beta}u_{\alpha}. \tag{4.2.16}$$

The four-acceleration, in component form, is given as $a^\alpha = (a^0, a^1, a^2, a^3)$ and the only non-zero component of this vector is the radial component given, in a general form, as

$$a^\alpha = \frac{A_r}{AB^2}\delta_1^\alpha. \tag{4.2.17}$$

Since

$$\dot{q}^\beta = q^\beta_{;\mu}V^\mu$$

and only $\beta = 1$ and $\mu = 0$ give a non-zero component of $q$, this implies

$$\dot{q} = \frac{(qB)_{,t}}{AB} \tag{4.2.18}$$

is the only form of $q$ used in (4.2.14).

Using (4.2.15), (4.2.16), (4.2.17), and (4.2.18), (4.2.14) becomes

$$\tau(qB)_{,t} + AqB = -\lambda\frac{(AT)_{,r}}{B}. \tag{4.2.19}$$

Now, adopting the following values and assumptions (Naidu and Govender 2007):

$$\lambda = \gamma T^3 \tau_c$$

$$\tau_c = \left(\frac{\alpha}{\gamma}\right) T^{-\omega}$$

and

$$\tau = \left(\frac{\beta\gamma}{\alpha}\right)\tau_c,$$

where $\tau_c$ is the mean collision time, (4.2.19) becomes

$$\beta T^{-\omega}(qB)_{,t} + AqB = -\alpha T^{3-\omega}\frac{(AT)_{,r}}{B}. \tag{4.2.20}$$

Equation (4.2.20) is a very important equation in the study of temperature profiles of radiating stars. This is proved by its versatility to give a number of different expressions governing the temperature of a collapsing sphere in different physical conditions. For instance, for a simple case of constant collision time $\omega = 0$ (which may be the case for perfect fluids with uniform flow of fluid particles), (4.2.20) after some manipulations gives

$$(AT)^4 = -\frac{4}{\alpha}\left[\beta \int A^3 B(qB)_{,t}\mathrm{d}r + \int A^4 qB^2 \mathrm{d}r\right] + F(t) \tag{4.2.21}$$

as the equation modelling the temperature. On the other hand if the interior matter is exposed to random changes of particle drift velocities resulting in arbitrary random collisions ($\omega = 4$), the same equation yields

$$\begin{aligned}(AT)^4 &= -\frac{4\beta}{\alpha}\exp\left(-\int \frac{4qB^2}{\alpha}\mathrm{d}r\right)\int A^3 B(qB)_{,t}\exp\left(\int \frac{4qB^2}{\alpha}\mathrm{d}r\right)\mathrm{d}r \\ &\quad + F(t)\exp\left(-\int \frac{4qB^2}{\alpha}\mathrm{d}r\right)\end{aligned} \tag{4.2.22}$$

as the equation representing the temperature of such a general case (Govinder and Govender 2001). In (4.2.20) $\beta$ gives the strength of the relaxational effects. By setting $\beta = 0$, equation (4.2.20) is able to yield all the noncausal temperature profiles

$$\begin{aligned}(AT)^{4-\omega} &= \frac{\omega - 4}{\alpha}\int A^{4-\omega} qB^2 dr + F(t),\ \omega \neq 4 \\ \ln(AT) &= -\frac{1}{\alpha}\int qB^2 dr + F(t),\ \omega = 4\end{aligned} \tag{4.2.23}$$

for all collision time models. In the above equations $F(t)$ is an arbitary function of integration fixed by the expression of the surface temperature of the star given as

$$(T^4)_\Sigma = \frac{L_\infty}{4\pi r^2 \delta B^2}, \tag{4.2.24}$$

where $L_\infty$ gives the star's luminosity as measured by a distant observer.

The luminosity we discuss in this thesis gives the total energy of radiation on the surface of the collapsing star as measured by a distant observer. The luminosity, as seen in (4.2.24) is directly affected by the star's surface temperature. However, temperature is not the only factor that affects the magnitude of the luminosity. The redshift and the total surface area of the star has a tremendous effect on the star's luminosity. The luminosity is given as

$$L = -\frac{dm}{dv}, \tag{4.2.25}$$

where

$$\frac{dm}{dv} = \frac{dm}{dt}\frac{dt}{d\tau}\frac{d\tau}{dv}. \tag{4.2.26}$$

Luminosity and redshift are connected by the following equation

$$(1 + z_\Sigma)^2 = \frac{L_\Sigma}{L_\infty}, \tag{4.2.27}$$

where $z_\Sigma$, and $L_\Sigma$ are the surface redshift and surface luminosity, respectively.

## 4.3 Energy Conditions

In this section we review the energy conditions. The energy conditions have the following structure:

- **Weak energy condition**

The weak energy condition is simply the requirement that the classical energy density should always be non-negative. In mathematical form this condition is stated as

$$T_{ab}\xi^a\xi^b \geq 0 \tag{4.3.1}$$

for all time-like 4-velocity vectors $\xi^b$. Basically, the weak energy condition can be interpreted as the condition which asserts that the energy density of the fluid as measured by any observer travelling at $\xi^b$ in spacetime is non-negative. Therefore this condition requires $\mu \geq 0$ and $\mu + p_i \geq 0$, where $i = 1, 2, 3$ because the $i = 0$ term is always taken by the energy density, and $p_i$ is the principal pressure (Hossain 2005).

- **Strong energy condition**

The strong energy condition is a condition that requires the stress-energy-momentum tensor to satisfy

$$T_{ab}\xi^a\xi^b \geq -\frac{1}{2}T \tag{4.3.2}$$

for all unit time-like 4-velocity vectors $\xi^b$. In (4.3.2) $T$ is the trace of $T_{ab}$ . In a simpler sense, the strong energy condition requires $\mu + \Sigma_{j=1}^{3} p_j \geq 0$ and $\mu + p_i \geq 0$ for all $i = 1, 2, 3$ components (Hossain 2005). This energy condition can be violated only if the total energy density $T_{ab}\xi^a\xi^b$ is negative or if, for $T_{ab}\xi^a\xi^b > 0$, there exists a large negative pressure of $T_{ab}$.

- **Dominant energy condition**

The dominant energy condition is a requirement that the local energy density must always be non-negative for all time-like observers and the local energy-momentum 4-current, i.e. $T_{ab}\xi^b$, to be future directed, non-spacelike for all future directed, time-like 4-velocity vectors $\xi^b$. Therefore, the mathematical form of the dominant energy condition is stated as

$$T_{ab}\xi^a\xi^b \geq 0; \quad T_{ab}\xi^b T^a_c \xi^c \leq 0. \tag{4.3.3}$$

Furthermore, the dominant energy condition restricts the definition of the stress-energy-momentum tensor to be of the form in which the speed of energy-flow does not exceed the speed of light. This energy condition also requires the dominance of the energy density over pressure, $\mu \geq |p_i|$ for all $i = 1, 2, 3$. Naturally, the dominant energy condition implies the weak energy condition and violation of this energy condition raises concern about the causality and the stability of the system (Hossain 2005).

In a spacetime with metric $g_{ab}$ the stress-energy-momentum tensor can be given as

$$T_{ab} = (\mu + p_\perp)u_a u_b + p_\perp g_{ab} + (p_r - p_\perp)x_a x_b + q_a u_b + q_b u_a \tag{4.3.4}$$

where $\mu$ is the rest energy density, $p_r$ is the radial pressure, $p_\perp$ is the tangential pressure, and $q_a$ is the heat flux. An investigation of the energy conditions is closely related to the eigenvalue problem of $T_{ab}$ and therefore, on a four-dimensional spacetime manifold, it leads to the search for the roots of a polynomial of degree four (Kolassis *et al* 1988). Therefore, the easiest way to write down the energy conditions is to calculate the eigenvalues of the stress-energy-momentum tensor which, for the tensor type in (4.3.4), must be real. The eigenvalues of the stress-energy-momentum tensor (4.3.4) are the roots of the equation

$$|T_{ab} - \lambda g_{ab}| = 0, \tag{4.3.5}$$

where $\lambda$ is a scalar giving the desired eigenvalues. For (4.3.4) and the line element (3.2.1), equation (4.3.5) can be rewritten as

$$\begin{vmatrix} (\mu + \lambda)A^2 & -qAB & 0 & 0 \\ -qAB & (p_r - \lambda)B^2 & 0 & 0 \\ 0 & 0 & (p_\perp - \lambda)B^2 r^2 & 0 \\ 0 & 0 & 0 & (p_\perp - \lambda)B^2 r^2 \sin^2\theta \end{vmatrix} = 0. \tag{4.3.6}$$

After lengthy calculations, the determinant of equation (4.3.6) gives a long and complicated characteristic equation which reduces to the following expression

$$\left[(p_r-\lambda)(\mu+\lambda)-q^2\right]\times\left[(p_\perp-\lambda)^2A^2B^2r^2\sin^2\theta\right]=0. \tag{4.3.7}$$

After some manipulations, (4.3.7) give the following possible roots:

$$\lambda_0 = \frac{1}{2}(\mu-p_r+\Delta) \tag{4.3.8a}$$

$$\lambda_1 = \frac{1}{2}(\mu-p_r-\Delta) \tag{4.3.8b}$$

$$\lambda_2 = p_\perp \tag{4.3.8c}$$

$$\lambda_3 = p_\perp \tag{4.3.8d}$$

where

$$\Delta=\sqrt{(p_r+\mu)^2-(2q)^2} \tag{4.3.9}$$

Requiring these roots to fall into the category of real numbers requires

$$(p_r+\mu)^2\geq(2q)^2$$

in equation (4.3.9) for every $p_r,\ \mu$, and $q$ in the stress-energy-momentum tensor (4.3.4). The energy conditions associated with the stress-energy-momentum tensor (4.3.4) are equivalent to the following eigenvalue relations (Kolassis *et al* 1988):

- Weak energy condition:

$$-\lambda_0\geq 0,\ \ -\lambda_0+\lambda_i\geq 0$$

•Strong energy condition:

$$-\lambda_0 + \Sigma_i \lambda_i \geq 0, \quad -\lambda_0 + \lambda_i \geq 0$$

•Dominant energy condition:

$$-\lambda_0 \geq 0, \ \lambda_0 \leq \lambda_i \leq -\lambda_0$$

where $\lambda_0$ denotes the eigenvalue corresponding to the timelike eigenvector, and $\lambda_i$ $(i = 1, 2, 3)$ denotes the eigenvalues corresponding to the spacelike eigenvectors. Since these eigenvalues are given in terms of $p_r, \ p_\perp, \ \mu$, and $q$, the above energy conditions can be rewritten and take the following more compact form:

•**Weak energy condition**:

$$\mu - p_r + \sqrt{(p_r + \mu)^2 - (2q)^2} \geq 0 \tag{4.3.10}$$

$$\mu + p_r + \sqrt{(p_r + \mu)^2 - (2q)^2} \geq 0 \tag{4.3.11}$$

•**Strong energy condition**:

$$2p_\perp + \sqrt{(p_r + \mu)^2 - (2q)^2} \geq 0 \tag{4.3.12}$$

•**Dominant energy condition**:

$$\mu - p_r \geq 0 \tag{4.3.13}$$

$$\mu - p_r - 2p_\perp + \sqrt{(p_r + \mu)^2 - (2q)^2} \geq 0. \tag{4.3.14}$$

It should be noted that this discussion of the energy conditions tries to give a corollary of the energy condition theorem as noted in (Kolassis *et al* 1988).

# Chapter 5

# Review of Horizon-free Collapse

## 5.1 Introduction

The end-state of gravitational collapse of massive stars within the framework of Einstein's theory of general relativity is a much debated topic amongst astrophysicists and relativists. While the singularity theorems predict that the gravitational collapse of physically reasonable matter fields may lead to the formation of closed trapped surfaces, it is the nature of these surfaces (black holes or naked singularities) which is called into question. Various counter-examples to the Cosmic Censorship Hypothesis have been presented in the literature in which the initial conditions before the onset of collapse were physically reasonable. In the case of dissipative gravitational collapse, the so-called horizon-free collapse first proposed by Banerjee *et al* (2002) addressed the scenario in which the energy radiated by a shear-free collapsing fluid is balanced by the rate of collapse of the gravitating sphere. In such a model the horizon never forms and the collapse may proceed until the body evaporates leaving behind Minkowski spacetime. In §4.2 we revisit the BCD model and highlight its importance within the context of gravitational collapse. In §4.3 we generalise the BCD model by simultaneously invoking the horizon-free condition and the boundary condition for a general spherically symmetric, shear-free star undergoing dissipative collapse. We show for the first time that the horizon-free condition provides a definition for one of the metric functions which, when substituted into the boundary condition, leads to an algebraic equation yielding the radius of the collapsing sphere. We further investigate the physical viability of a particular model by analysing the behaviour of the density, radial and tangential stresses, mass profile, luminosity and the temperature in both the causal and noncausal theories.

## 5.2 Overview of the BCD Model

In this section we briefly review the horizon-free collapse model first proposed by Banerjee *et al* (2002). This will set the scene for our investigation into a more general formulation of horizon-free collapse. The interior of the spacetime for the BCD model is described by a spherically symmetric, shear free line element given by

$$ds^2 = -A^2 dt^2 + B^2 \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{5.2.1}$$

where the metric functions $A$ and $B$ are determined by making assumptions that constrain the physics of the model. Since the star is undergoing radiative collapse its exterior spacetime is nonempty and is described by the Vaidya solution which we repeat here for easy reference:

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right) dv^2 - 2dv d\bar{r} + \bar{r}^2 d\Omega^2. \tag{5.2.2}$$

The stellar fluid for the interior of the BCD model is assumed to be a perfect fluid with heat conduction. The stress-energy-momentum tensor takes the form

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a, \tag{5.2.3}$$

where $\mu$ is the energy density and $p$ the isotropic pressure of the stellar fluid. It is assumed that the collapsing body dissipates energy in the form of a radial heat flux so that $q^a = (0, q^1, 0, 0)$ is the heat flux vector. The Einstein field equations for the interior spacetime are

$$\mu = \frac{3B_t^2}{A^2B^2} + \frac{1}{B^2}\left(\frac{B_r^2}{B^2} - \frac{4B_r}{rB} - \frac{2Brr}{B}\right) \tag{5.2.4a}$$

$$p_r = \frac{2A_rB_r}{AB^3} - \frac{B_t^2}{A^2B^2} + \frac{2B_r}{rB^3} + \frac{2A_tB_t}{A^3B} - \frac{2B_{tt}}{A^2B} + \frac{B_r^2}{B^4} + \frac{2A_r}{rAB^2} \tag{5.2.4b}$$

$$p_\perp = \frac{2A_tB_t}{A^3B} - \frac{2B_{tt}}{A^2B} - \frac{B_t^2}{A^2B^2} + \frac{A_r}{rAB^2} + \frac{B_r}{rB^3} + \frac{B_{rr}}{B^3} + \frac{A_{rr}}{AB^2} - \frac{B_r^2}{B^4} \tag{5.2.4c}$$

$$q = \frac{1}{AB^3}\left(2B_{tr} - \frac{2A_rB_t}{A} - \frac{2B_rB_t}{B}\right), \tag{5.2.4d}$$

where the radial and tangential stresses are assumed to be equal. The pressure isotropy condition is obtained by equating (5.2.4b) and (5.2.4c) to yield

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \frac{2A_rB_r}{AB} + \frac{B_r}{rB} + \frac{2B_r^2}{B^2} + \frac{A_r}{rA}. \tag{5.2.5}$$

The BCD model is based on an ad-hoc assumption on the behaviour of the metric functions. Assuming separability of the metric functions $A$ and $B$ into the radial and time coordinates such that $A(r,t) = a(r), B(r,t) = b(r)R(t)$, the Einstein field equations (5.2.4) and isotropy condition (5.2.5) respectively become

$$\mu = \frac{1}{R^2}\left[\frac{3\dot{R}^2}{a^2} + \frac{1}{b^2}\left(\frac{b'^2}{b^2} - \frac{4b'}{rb} - \frac{2b''}{b}\right)\right] \tag{5.2.6a}$$

$$p_r = \frac{1}{R^2}\left[\frac{1}{b^2}\left(\frac{2a'b'}{ab} + \frac{2}{r}\left(\frac{b'}{b} + \frac{a'}{a}\right) + \frac{b'^2}{b^2}\right) - \frac{1}{a^2}(2R\ddot{R} + \dot{R}^2)\right] \tag{5.2.6b}$$

$$q = -\frac{2a'\dot{R}}{a^2b^2R^3} \tag{5.2.6c}$$

and

$$\frac{a''}{a} + \frac{b''}{b} - 2\frac{a'b'}{ab} - \frac{a'}{ra} - 2\frac{b'^2}{b^2} - \frac{b'}{rb} = 0, \tag{5.2.7}$$

where primes denote differentiation with respect to $r$ and dots represent differentiation with respect to $t$. The isotropy condition does not contain time explicitly and so for this condition to hold throughout the collapse one only needs to specify the radial dependence of the metric functions which will satisfy (5.2.7). The temporal dependence follows via the "constants" of integration. In the BCD model it was assumed that $b(r) = 1$ and $a(r) = 1 + \xi_0 r^2$. Therefore the isotropy condition is always satisfied if $A(r,t) = 1 + \xi_0 r^2$ and $B(r,t) = R(t)$. The explicit time dependence of the model is obtained by solving the junction conditions valid on $\Sigma$. (The junction conditions required for the smooth matching of the line element (5.2.1) and the Vaidya exterior were discussed in

Chapter 3.) For the BCD model these become

$$p_\Sigma = (qB)_\Sigma \tag{5.2.8a}$$

$$r_\Sigma = (rB)_\Sigma \tag{5.2.8b}$$

$$m_\Sigma = \left[\frac{r^3 B B_t^2}{2A^2} - r^2 B_r - \frac{r^3 B_r^2}{2B}\right], \tag{5.2.8c}$$

where $m$ is the total mass enclosed within a sphere of radius $r$. Substituting (5.2.6b) and (5.2.6c) into (5.2.8a)we obtain the temporal evolution equation for the BCD model

$$2R\ddot{R} + \dot{R}^2 + m\dot{R} = n, \tag{5.2.9}$$

where at the boundary both $m(r)$ and $n(r)$ are constants respectively given as

$$m = \frac{-2a'}{b} \tag{5.2.10a}$$

$$n = \frac{a^2}{b^2}\left[\frac{2a'b'}{ab} + \frac{2}{r}\left(\frac{a'}{a} + \frac{b'}{b}\right) + \frac{b'^2}{b^2}\right] \tag{5.2.10b}$$

A simple solution of (5.2.9) is $R = -Ct$, where $C$ is an integration constant. With this solution the Einstein field equations become

$$\mu = \frac{3}{t^2(1+\xi_0 r^2)^2} \tag{5.2.11a}$$

$$p = \frac{1}{t^2(1+\xi_0 r^2)^2}\left[\frac{4\xi_0}{C^2}(1+\xi_0 r^2) - 1\right] \tag{5.2.11b}$$

$$q = -\frac{4\xi_0 r}{C^2 t^3 (1+\xi_0 r^2)^2} \tag{5.2.11c}$$

and

$$m = -4\xi_0 r_0, \quad n = 4\xi_0(1+\xi_0 r_0^2), \quad C = \frac{1}{2}\left[-|m| + (m^2+4n)^{\frac{1}{2}}\right].$$

With the $m$, $n$ values and (5.2.9), the following was obtained

$$C^2 < 4\xi_0(1+\xi_0 r_0^2).$$

Since the fluid is collapsing, the volume expansion rate is required to be negative, this is fulfilled if $C > 0$. For the BCD model to be physically true it requires the following conditions which give some restrictions on the value of $C$:

$$2\xi_0 > C^2 > \xi(1+\xi_0 r^2) \tag{5.2.12a}$$

$$\left[1 - \frac{2\xi_0 r}{C}\right]^2 > -\frac{2\xi_0}{C^2}(1-\xi_0 r^2). \tag{5.2.12b}$$

In order to avoid the horizon during the collapse history of the star we require

$$1 - \frac{2m_\Sigma}{r_\Sigma} > 0.$$

Using the above metric functions this condition can be rewritten as follows:

$$1 - \frac{C^2 r_0^2}{(1+\xi_0 r_0^2)} > 0. \tag{5.2.13}$$

The horizon-free condition for a collapsing star can be dynamically achieved if the rate of energy loss is balanced by the collapse rate of the shrinking core. This balance in the rate changes of the two components further ensures the loss of mass which prevents the consequence of trapped surfaces due to strong gravitation. Another noted feature in this model is that the occurence or non-occurence of the horizon is independent of time. From the horizon-free condition it is possible to set $\xi_0 = 0$, with the condition still holding. However, setting $\xi_0 = 0$ results in the vanishing of the heat flux at the boundary.

## 5.3 General Horizon-free Collapse

In this section we study horizon-free collapse by imposing the horizon-free condition and invoking the matching condition for a shear-free, spherically symmetric star undergoing dissipative collapse. The interior spacetime for the collapsing body is given by

$$ds^2 = -A^2 dt^2 + B^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{5.3.1}$$

where the metric functions $A(r,t)$ and $B(r,t)$ are undetermined. The stellar fluid is described by the stress-energy-momentum tensor of the form

$$T_{ab} = (\mu + p_\perp)u_a u_b + p_\perp g_{ab} + (p_r - p_\perp)x_a x_b + q_a u_b + q_b u_a, \tag{5.3.2}$$

where $x_a$ is a unit spacelike 4-vector along the radial direction. The corresponding field equations of this collapse model are as follows:

$$\mu = \frac{3B_t^2}{A^2B^2} + \frac{1}{B^2}\left(\frac{B_r^2}{B^2} - \frac{4B_r}{rB} - \frac{2Brr}{B}\right) \tag{5.3.3a}$$

$$p_r = \frac{2A_rB_r}{AB^3} - \frac{B_t^2}{A^2B^2} + \frac{2B_r}{rB^3} + \frac{2A_tB_t}{A^3B} - \frac{2B_{tt}}{A^2B} + \frac{B_r^2}{B^4} + \frac{2A_r}{rAB^2} \tag{5.3.3b}$$

$$p_\perp = \frac{2A_tB_t}{A^3B} - \frac{2B_{tt}}{A^2B} - \frac{B_t^2}{A^2B^2} + \frac{A_r}{rAB^2} + \frac{B_r}{rB^3} + \frac{B_{rr}}{B^3} + \frac{A_{rr}}{AB^2} - \frac{B_r^2}{B^4} \tag{5.3.3c}$$

$$q = \frac{1}{AB^3}\left(2B_{tr} - \frac{2A_rB_t}{A} - \frac{2B_rB_t}{B}\right) \tag{5.3.3d}$$

From (3.4.15), (5.3.3b), and (5.3.3d) we obtained the following boundary condition

$$B^2\left[-2\frac{B_{tt}}{B}-\left(\frac{B_t}{B}\right)^2+2\frac{A_t}{A}\frac{B_t}{B}\right]+2AB\left[-\frac{B_{rt}}{B}+\frac{B_rB_t}{B^2}+\frac{A_r}{A}\frac{B_t}{B}\right]$$

$$+A^2\left[\left(\frac{B_r}{B}\right)^2+2\frac{A_r}{A}\frac{B_r}{B}+\frac{2}{r}\left(\frac{A_r}{A}+\frac{B_r}{B}\right)\right]=0 \tag{5.3.4}$$

valid on $\Sigma$. The horizon-free condition is given as

$$\left[\left(\frac{rB_t}{2A}\right)^2-r\frac{B_r}{2B}-\left(\frac{rB_r}{2B}\right)^2\right]_\Sigma=\alpha, \tag{5.3.5}$$

where $\alpha$ is a constant.

In order to ensure that (5.3.4) and (5.3.5) are simultaneously satisfied, we have the following constraint equation

$$\frac{1}{2}\left(\frac{rB_r}{B}\right)^2+rB\left(\frac{B_r}{B^2}+\frac{1}{rB}\right)\left[\alpha+\frac{rB_r}{2B}+\left(\frac{rB_r}{2B}\right)^2\right]^{\frac{1}{2}}-\alpha+\frac{rB_r}{B}+\frac{1}{2}$$

$$-\frac{r}{2}\left[\alpha+\frac{rB_r}{2B}+\left(\frac{rB_r}{2B}\right)^2\right]^{-\frac{1}{2}}\left[\frac{B_r}{2B}+\frac{rB_{rr}}{2B}+\frac{r^2B_rB_{rr}}{2B^2}-\frac{r^2B_r^3}{2B^3}\right]$$

$$-\left[\alpha+\frac{rB_r}{2B}+\left(\frac{rB_r}{2B}\right)^2\right]^{-1}\left[\frac{rB_r}{8B}+\frac{r^2B_{rr}}{8B}+\frac{r^3B_rB_{rr}}{4B^2}-\frac{r^3B_r^3}{8B^3}+\frac{r^2B_r^2}{8B^2}\right.$$

$$\left.+\frac{r^4B_r^2B_{rr}}{8B^3}-\frac{r^4B_r^4}{8B^4}\right]=0 \tag{5.3.6}$$

which is an algebraic equation on $\Sigma$. If we now demand that the metric function $B(r,t)$ is separable in $r$ and $t$, i.e. $B(r,t)=f(t)g(r)$, (5.3.6) will be independent of time. This means that we have total freedom in choosing the temporal dependence of our model. The form of $g(r)$ will restrict the value of $\alpha$ or the radius of the star.

We will now consider a specific model in which

$$B = (1 + at^2) \times (\beta + b/r^3) \tag{5.3.7}$$

which, when substituted into (5.3.5), yields

$$A = \frac{art\left(\frac{b}{r^3} + \beta\right)}{\sqrt{\alpha + \frac{9b^2}{4r^6\left(\frac{b}{r^3}+\beta\right)^2} - \frac{3b}{2r^3\left(\frac{b}{r^3}+\beta\right)}}}. \tag{5.3.8}$$

The Einstein field equations (5.3.3) for the metric functions (5.3.7) and (5.3.8) become

$$\rho \quad = \quad \frac{6r^4\left(b^2(1+2\alpha) + br^3(-5+4\alpha)\beta + 2r^6\alpha\beta^2\right)}{\left(1+at^2\right)^2\left(b+r^3\beta\right)^4} \tag{5.3.9a}$$

$$p_r \quad = \quad -\frac{2r^4\left(2b^4\left(-6-5\alpha+4\alpha^2\right) + 8b^3r^3\left(9-2\alpha+4\alpha^2\right)\right.}{\left(1+at^2\right)^2\left(b+r^3\beta\right)^4\left(b^2(3+4\alpha) + 2br^3(-3+4\alpha)\beta + 4r^6\alpha\beta^2\right)}$$

$$\times\ \beta + 3b^2r^6\left(-21-2\alpha+16\alpha^2\right)\beta^2 + br^9\left(15-4\alpha+32\alpha^2\right)$$

$$\left.\times\ \beta^3 + 4r^{12}\alpha(-1+2\alpha)\beta^4\right) \tag{5.3.9b}$$

$$p_\perp \quad = \quad \frac{r^4\left(-b^5(-1+4\alpha)(3+4\alpha)^2 + N\right)}{\left(1+at^2\right)^2\left(b+r^3\beta\right)^3\left(b^2(3+4\alpha) + 2br^3(-3+4\alpha)\beta + 4r^6\alpha\beta^2\right)^2} \tag{5.3.9c}$$

$$q \quad = \quad \frac{2r^7\left(2b-r^3\beta\right)\left(b^2(3+4\alpha) + br^3(-15+8\alpha)\beta + 4r^6\alpha\beta^2\right)}{\left(1+at^2\right)^3\left(b+r^3\beta\right)^6\sqrt{\frac{b^2(3+4\alpha)+2br^3(-3+4\alpha)\beta+4r^6\alpha\beta^2}{(b+r^3\beta)^2}}}, \tag{5.3.9d}$$

where

$$N = b^4r^3\left(81+4\alpha\left(123+68\alpha-80\alpha^2\right)\right)\beta+2b^3r^6(27-4\alpha(117+4\alpha(-41+20\alpha)))\beta^2$$
$$+\,b^2r^9(225-4\alpha(363+8\alpha(-47+20\alpha)))\beta^3$$
$$-\,4br^{12}\alpha(3+20\alpha(-7+4\alpha))\beta^4+16r^{15}(1-4\alpha)\alpha^2\beta^5.$$

The mass of this star as measured at the boundary for solutions (5.3.7) and (5.3.8) is

$$m_\Sigma = \frac{2\left(1+at^2\right)\alpha\left(b+r^3\beta\right)}{r^2}. \tag{5.3.10}$$

On checking the physical viability of this model we considered the luminosity, redshift, and temperature profile. The star's luminosity is given as

$$L = -\frac{(2(\alpha-2)(4\alpha+3)b^4+8\left(4\alpha^2-2\alpha+9\right)b^3\beta r^3+W_1)\times W_2}{\left(b+\beta r^3\right)^4\left((4\alpha+3)b^2+2(4\alpha-3)b\beta r^3+4\alpha\beta^2r^6\right)}, \tag{5.3.11}$$

where

$$W_1 = 3(2\alpha(8\alpha-1)-21)b^2\beta^2r^6+(4\alpha(8\alpha-1)+15)b\beta^3r^9+4\alpha(2\alpha-1)\beta^4r^{12}$$

and

$$W_2 = \left(\beta r^3\left(\sqrt{4\alpha+\frac{3b\left(b-2\beta r^3\right)}{\left(b+\beta r^3\right)^2}}+1\right)+b\left(\sqrt{4\alpha+\frac{3b\left(b-2\beta r^3\right)}{\left(b+\beta r^3\right)^2}}-2\right)\right)^2.$$

The surface redshift is calculated as follows

$$Z = \frac{1}{1-\frac{3b}{b+r^3\beta}+2\sqrt{\alpha+\frac{3b(b-2r^3\beta)}{4(b+r^3\beta)^2}}}. \tag{5.3.12}$$

The temperature profiles are obtained using the following expression

$$T^4 = \frac{\left(\alpha+\frac{3b\left(b-2\beta r^3\right)}{4(b+\beta r^3)^2}\right)^2\left(F(t)-\frac{128a^4t^4\left(b+\beta r^3\right)^4\times\Psi}{3r^6(at^2+1)^2((4\alpha+3)b^2+2(4\alpha-3)b\beta r^3+4\alpha\beta^2r^6)^2}\right)}{a^4r^4t^4\left(\frac{b}{r^3}+\beta\right)^4}, \tag{5.3.13}$$

where $\Psi$ is given by

$$\Psi = 3\gamma r^2 \left((4\alpha+3)b^2 + 2(4\alpha-3)b\beta r^3 + 4\alpha\beta^2 r^6\right) - \left(at^2+1\right)\left(b+\beta r^3\right)^3$$

$$\times \sqrt{\frac{(4\alpha+3)b^2 + 2(4\alpha-3)b\beta r^3 + 4\alpha\beta^2 r^6}{\left(b+\beta r^3\right)^2}}$$

For validation of these results we carry out a graphical analysis of the thermodynamical variables, luminosity, redshift, and temperature profiles. For all the graphical analysis below we used the following parameter values

$$b \to 1,\ \alpha \to \frac{1}{6},\ a \to 1,\ C \to 1,\ t \to -10,\ \beta \to 1$$

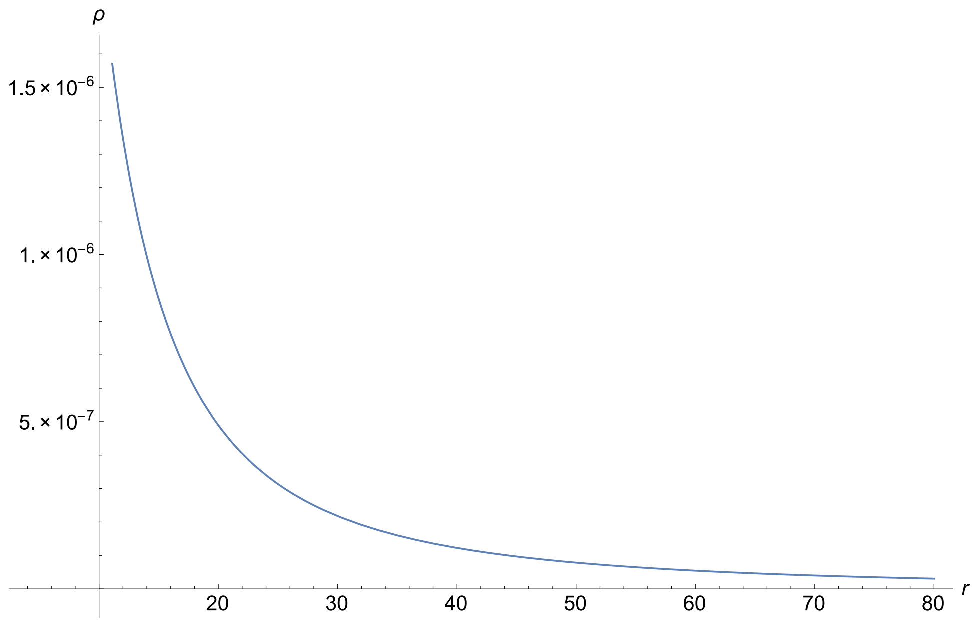


Figure 5.1: Energy density vs radial coordinate

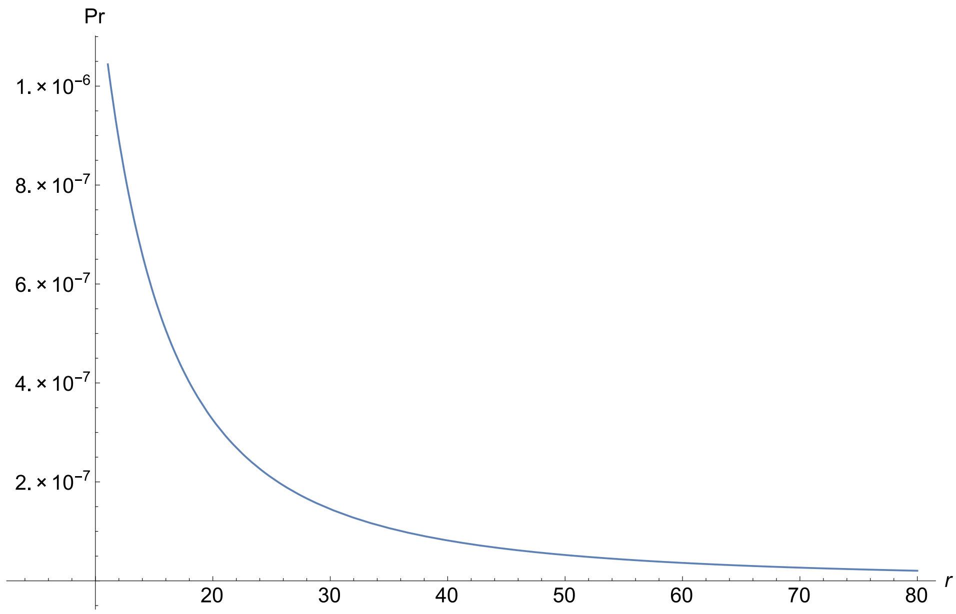


Figure 5.2: Radial pressure vs radial coordinate

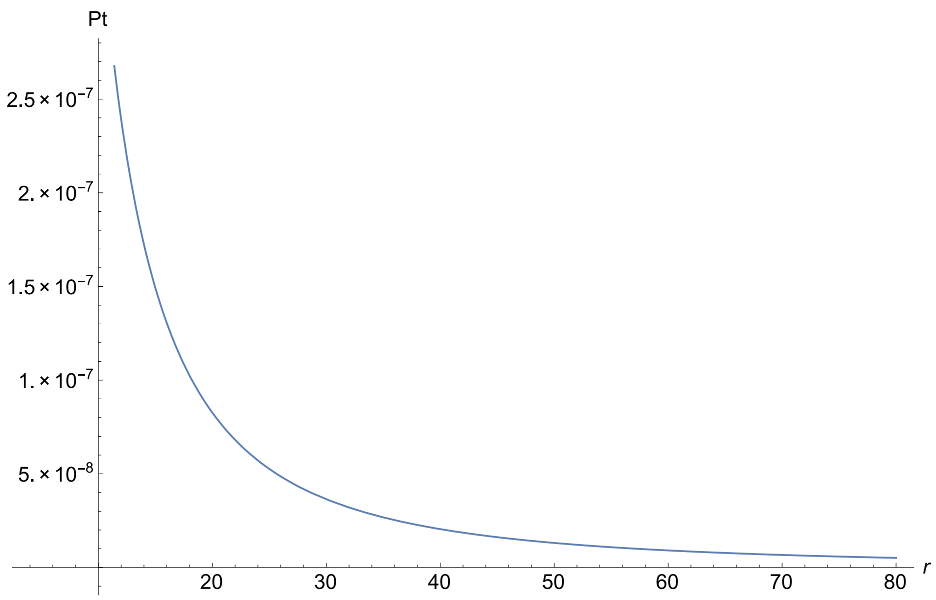


Figure 5.3: Tangential pressure vs radial coordinate

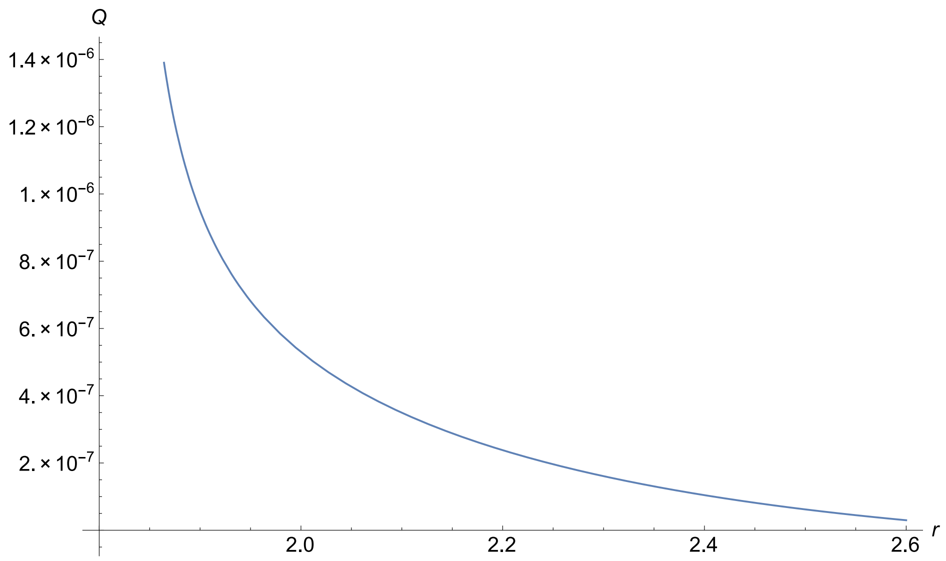


Figure 5.4: Heat flux vs radial coordinate

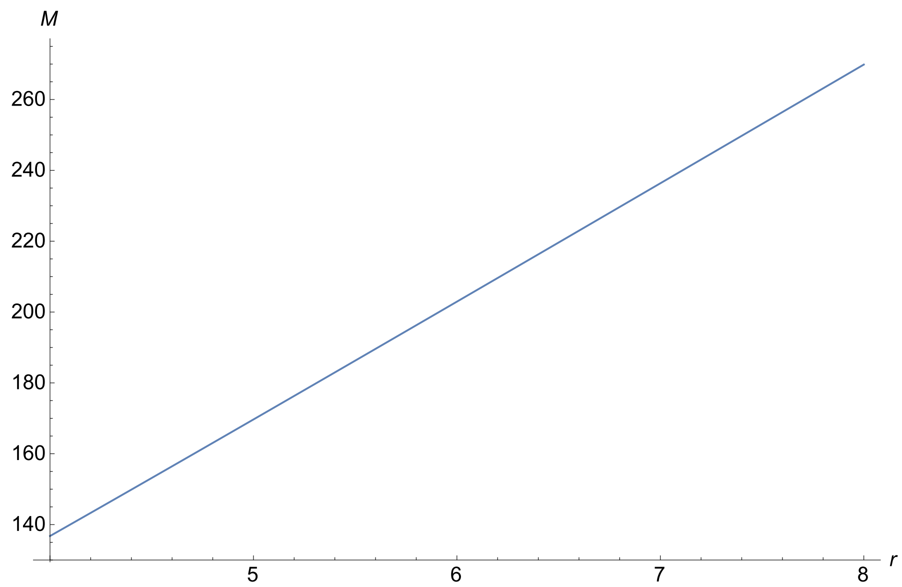


Figure 5.5: Star's mass vs radial coordinate

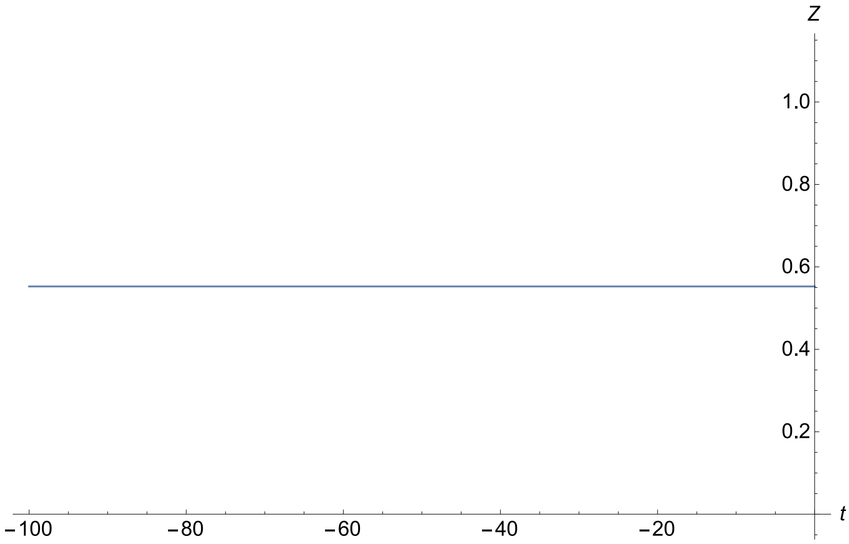


Figure 5.6: Redshift vs time coordinate

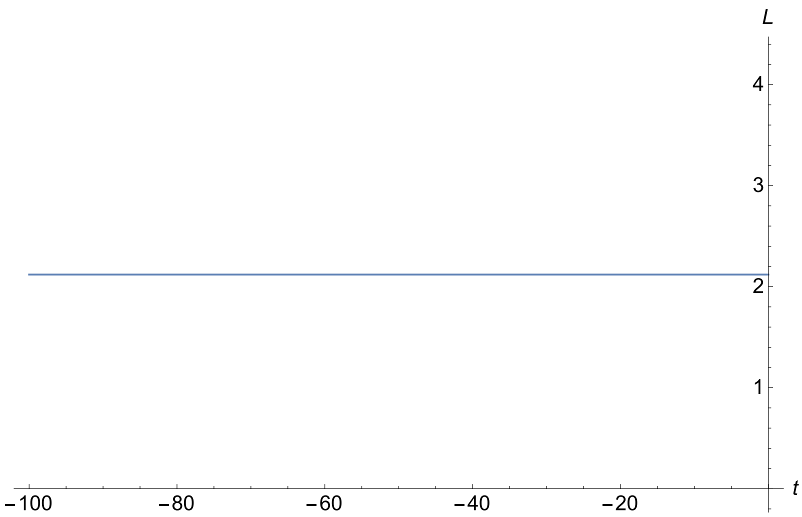


Figure 5.7: Luminosity vs time coordinate

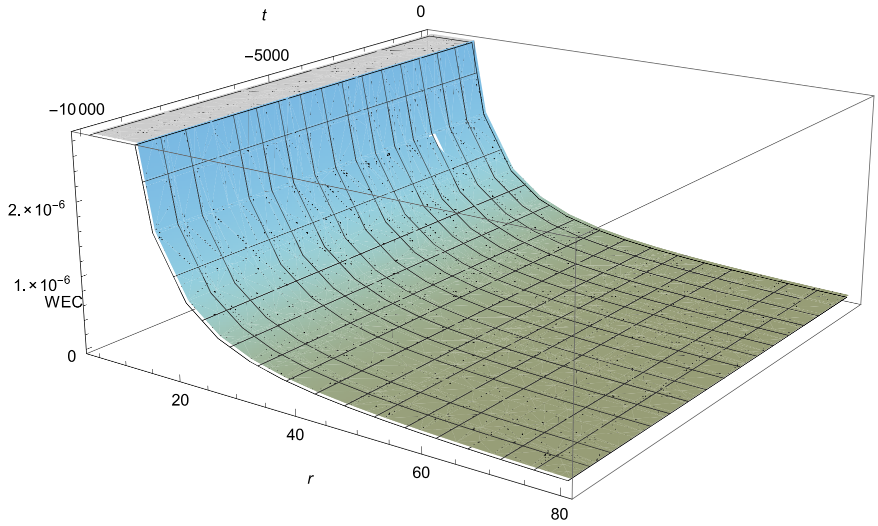


Figure 5.8: $X = \sqrt{(\mu + p_r)^2 - 4q^2} + (\mu - p_r) > 0$ as a function of $r$ and $t$.

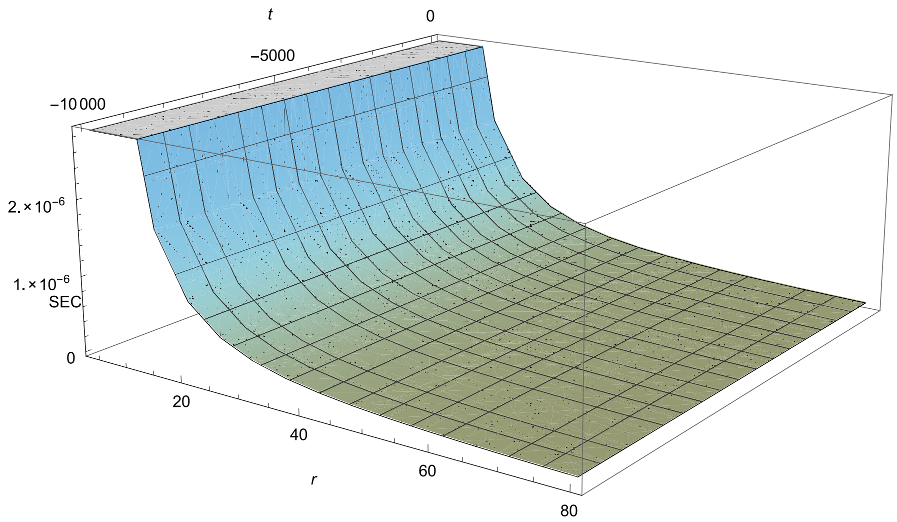


Figure 5.9: $Z = \sqrt{(\mu + p_r)^2 - 4q^2} + 2p_\perp > 0$ as a function of $r$ and $t$.

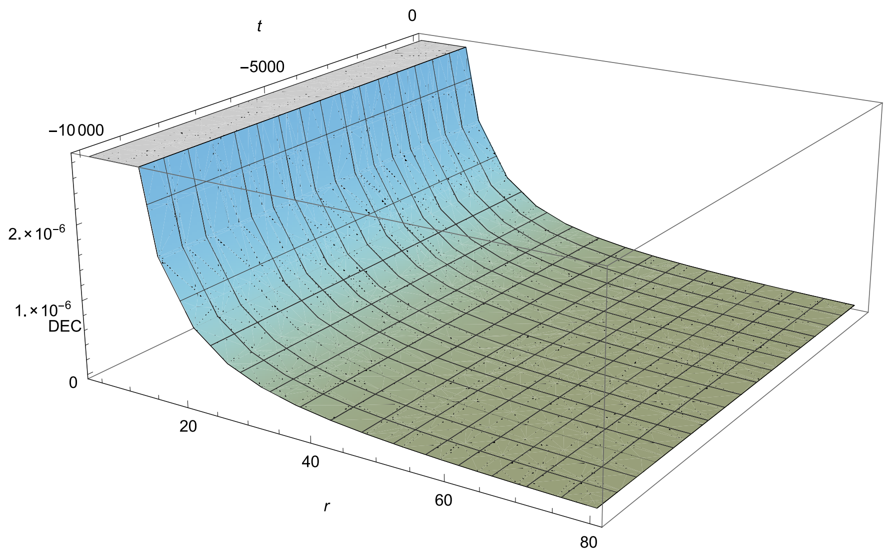


Figure 5.10: $\Upsilon = \mu - p_r - 2p_\perp + \sqrt{(\mu + p_r)^2 - 4q^2} > 0$ as a function of $r$ and $t$.

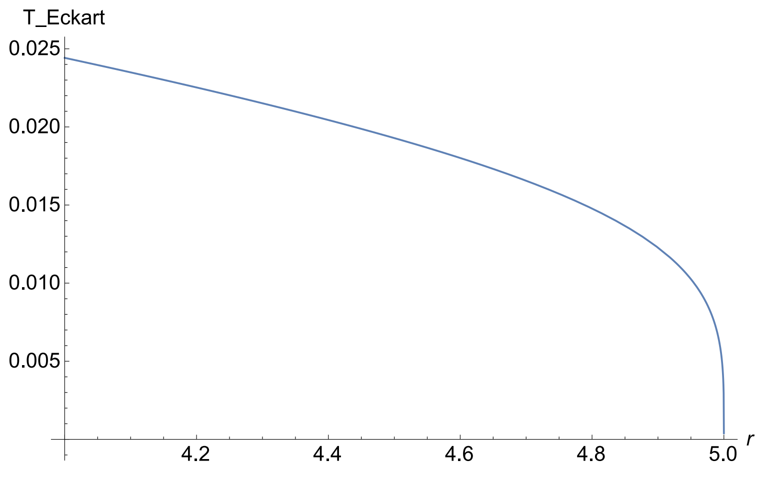


Figure 5.11: Eckart temperature as a function of the radial coordinate

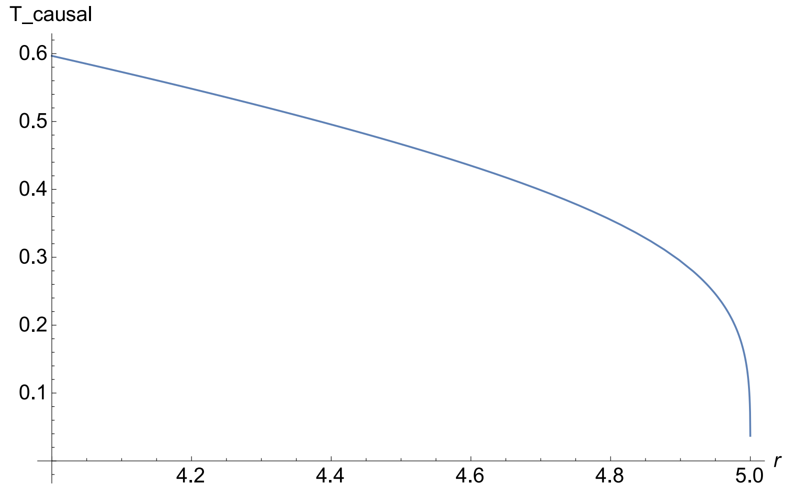


Figure 5.12: Causal temperature as a function of the radial coordinate

## 5.4 Discussion of Results

Figure 5.1 shows that the energy density is a monotonically decreasing function of the radial coordinate. This is expected for a collapsing sphere where the density is highest at the center of the core and decreases towards the surface of the star. We observe similar behaviour in the radial pressure within the stellar interior. The high central pressure results from a dense core as well as the high generation of heat energy in this region. The radial pressure drops off as one moves away from the hotter core region towards the cooler surface layers of the star. The radial pressure is exhibited in Figure 5.2 which shows a monotonically decreasing function of the radial coordinate. The hotter regions of the core (centre) has a higher radial pressure associated with it compared to the cooler surface layers. The tangential pressure (Figure 5.3) is positive at each interior point of the star, diverging at the centre and decreasing as one moves towards the stellar boundary. The heat flux diverges as we approach the centre and decreases towards the surface of the star as illustrated in Figure 5.4. Figure 5.5 illustrates the behaviour of the mass as a function of the radial coordinate. We note that the mass increases linearly with $r$ and vanishes at the centre as expected. Redshift and luminosity being exterior elements of the collapsing system appear on the second quadrant of their graphs. On average, the total redshift (Figure 5.6) in emitted radiation is a constant. This may be due to low redshift (measured per unit square meter) of the larger surface area summing up to balance the higher redshift of the smaller surface area as the star shrinks during collapse. This constant difference in emitted radiation is also a good indication of a horizon-free collapse with the final fate of collapse being a naked singularity. The total luminosity (Figure 5.7) is also a constant. This is no surprise because the redshift has a direct influence on the luminosity. Figures 5.8–5.10 illustrate the weak, strong and dominant energy conditions as functions of the radial and temporal coordinates. Observations of these figures indicate that all energy conditions are satisfied throughout the stellar interior during the course of collapse.

Figure 5.11 displays the Eckart temperature profile for the interior matter distribution. We note that the temperature is a maximum at the centre of the star and drops off smoothly towards the boundary. The behaviour of the temperature gradient is associated with the production of heat energy within the stellar core. Relaxational effects are clearly illustrated in Figure 5.12 in which the causal temperature profile is plotted as a function of the radial coordinate. A comparison of Figures 5.11 and 5.12 indicates that the causal temperature is everywhere greater than its noncausal counterpart. Our results confirm earlier findings in both perturbative and non-perturbative models (Govender *et al* 1999, Reddy *et al* 2014, Thirukkanesh and Govender 2014, Thirukkanesh *et al* 2012). We note that from figure 5.11, the temperature at the surface is non-zero and is proportional to the surface luminosity.

The horizon-free condition has been displayed in many other models of dissipative gravitational collapse, ranging from collapse with an equation of state (Wagh *et al* 1999), Euclidean stars (Herrera *et al* 2006, Govender *et al* 2010, Govinder and Govender 2012) through to higher dimensional collapse. We note that in the BCD model, a singularity is encountered as $t \to 0$. Our model does not suffer from this pathology. However, the class of models found by our approach all possess a singularity at $r = 0$. A pleasing feature of our model is that the temporal dependence of the metric functions and thermodynamical variables is completely arbitrary. This means that our model of a radiating star can be treated as a core-envelope model which is valid in the region $r_1 \leq r \leq r_0$ where $r_1$ is some finite radius from the center of the star and $r_0$ is the boundary of the collapsing star at some snapshot in time. In order to obtain a complete model of a radiating star whose interior is matched to the exterior Vaidya solution we need to match a core solution valid in the region $0 \leq r \leq r_1$ which matches to our solution at $r_1$. This is possible because of the freedom we have in the temporal evolution of our model.

# Chapter 6

# Conclusion

In this investigation we considered a spherically symmetric star executing shear-free motion and dissipating energy in the form of a radial heat flux to the exterior spacetime. At the onset of collapse we demanded that the stellar fluid obeys the horizon-free condition. The horizon-free condition together with the boundary condition leads to an algebraic equation valid at the boundary of the collapsing star. A remarkable feature of our approach is that the temporal evolution of our models is completely free. We now provide an overview of our investigation.

- In chapter one we reviewed the available literature on dissipative gravitational collapse. In particular, we highlighted the research conducted into the end-states of collapse. We also reviewed the work done in higher-order theories of gravity and their connection to the outcome of the collapse process.

- In chapter two we provided the fundamental tools of differential geometry and tensor calculus required for the completion of our study. The reader is introduced to the notion of curved spacetime and its link to the matter content via the

Einstein field equations.

- The framework for studying radiating stars undergoing dissipative collapse is established in chapter three. The interior spacetime described by a shear-free spherically symmetric line element is introduced. A discussion of the matter content of the interior stellar fluid is provided and its connection to the geometry is established via the Einstein field equations. We introduce the Vaidya solution which describes the atmosphere of the radiating collapsing body. We present the junction conditions for the smooth matching of the interior of the stellar fluid to Vaidya outgoing solution. A general discussion on irreversible thermodynamics in curved spacetime is also presented.

- In chapter four we review the shortcomings of the Eckart theory of thermodynamics and motivated for the need to employ causal thermodynamics to study the heat transport in relativistic collapse models. We also derived the energy conditions in the presence of dissipation for spherically symmetric, radiating stars in which the radial and transverse pressures are unequal.

- In section five we provide a review of horizon-free collapse from the point of view of the BCD model. We generalise the *BCD* ansatz to a more general collapse scenario. We showed for the first that the boundary condition leads to an algebraic constraint thus allowing for complete freedom in the temporal behavior of the model. This is a new result in horizon-free collapse. We show that the model we obtained via our algorithm is physically reasonable and provides reasonably behaved temperature profiles in both the noncausal and causal regimes.

Looking to the future we wish to extend our algorithm to include shear as well as to higher dimensional spacetimes. To date the causal temperature profiles for radiative collapse have been obtained via a truncated causal transport equation. We want to investigate the temperature profiles by employing the full causal heat transport equation.

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