

Connectedness and the Hyperspace of Metric Spaces

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Abstract

One of the prime motivations for studying hyperspaces of a metric space is to understand the original space itself. The hyperspace of a metric space X is the space 2^X of all non-empty closed bounded subsets of it, endowed with the Hausdorff metric. Our purpose is to study, in particular, connectedness properties of X and its hyperspace. We shall be concerned with knowing if a property P is extensional, that is, if X has property P then so does the hyperspace, or if a property P is reflective, that is, if the hyperspace has property P then so does X itself.

The hyperspace 2^X and its subspace $\mathcal{C}(X)$ will be the focus of our study. First the Hausdorff metric, ρ , is considered and introduced for the hyperspace 2^X which is also inherited by $\mathcal{C}(X)$. As in (Nadler; [8]), when X is a continuum, the property of compactness is shown to be extensional to 2^X and $\mathcal{C}(X)$. This is further generalised, when it is shown that each of 2^X and $\mathcal{C}(X)$ is arcwise connected and hence are each arcwise connected continua, when X is a continuum. The classical results, the Boundary Bumping Theorems (due to Janiszewski [4]), which provide the required conditions under which the component of a set intersects its boundary, is proved using the Cut Wire Theorem (Whyburn; [13]). As an application, the Boundary Bumping Theorem (for open sets) is used to show the existence of continua arising out of convergence, in the Continuum of Convergence Theorem (Nadler; [8]).

Using a construction of Whitney ([12]), the existence of a Whitney map, μ , for 2^X and ω for $\mathcal{C}(X)$ are given. Using μ , a special function $\sigma : [0, 1] \rightarrow 2^X$ (due to Kelley [3]) called a segment is considered in the study of the arc structure of 2^X and $\mathcal{C}(X)$. The equivalence of the existence of an order arc in 2^X and the existence of a segment in 2^X is also shown. A segment homotopy is then utilised to show that if one of 2^X or $\mathcal{C}(X)$ is contractible then so

is the other. This is presented in the Fundamental Theorem of Contractible Hyperspaces.

The relationship between local connectedness and connectedness im kleinen is examined in order to understand the properties of Peano continua. Property S, introduced by Sierpinski([10]), is considered and its connection to local connectedness is examined. Furthermore, a result of Wojdyslawski([15]), which shows that local connectedness is an extensional property of a continuum X to the hyperspaces 2^X and $\mathcal{C}(X)$, is given. Local connectedness is also reflective if either 2^X or $\mathcal{C}(X)$ is a locally connected metric continuum. Lastly, Property K, by Kelley([3]) is examined and is shown to be a sufficient condition for a continuum X to have its hyperspaces 2^X and $\mathcal{C}(X)$ to be contractible. Consequently, if X is a Peano continuum then 2^X and $\mathcal{C}(X)$ are contractible.

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Chapter 1

Introduction

Hyperspace theory had its early beginnings in the 1900's, due to the work of Hausdorff([1]) and Vietoris([11]). The foundation for understanding the structure of hyperspaces was later developed during the 1930's through the work of Mazurkiewicz([5]) and Wojdyslawski([15]). In 1942 J.L. Kelley([3]) introduced the construction of the hyperspace of a compact connected metric space X . Since this seminal paper appeared, the notion of a hyperspace has been defined for several classes of metric spaces which are not necessarily compact or connected. The role of continua in hyperspace theory only became apparent in the 1970's, but much earlier Kelley had determined basic facts which have since been of fundamental importance to the subject.

In this thesis, we will present some aspects of connectedness in the theory of hyperspaces. We give attention, specifically, to arcwise connectedness, local connectedness, connectedness in kleinen and Property S.

For the remainder of this chapter, we first present a brief synopsis of all necessary definitions and results that will be required. We include results for compactness, connectedness and continuity. This is followed by an introduction to the Hausdorff metric and the description of the hyperspaces that we will focus our study on. In Theorem 1.3.12 we show that if X is compact and connected, then so are the hyperspaces 2^X and $\mathcal{C}(X)$. Classical mappings on 2^X and 2^{2^X} are then presented, leading to a result by Kelley([3]) given in Theorem 1.4.12.

In Chapter 2, we use chain connectedness to prove the Cut Wire Theorem (Theorem 2.1.6). We then consider the conditions under which the component of a set intersects its boundary. This is presented in the Boundary Bumping Theorems.

In Chapter 3, we show the existence of Whitney maps for hyperspaces (defined by Whitney [12]), which Kelley ([3]) used in the study of hyperspaces. In Lemma 3.2.2 we present a proof of Kelley's result ([3]; Lemma 1.5.).

In Chapter 4 the arc structure of hyperspaces and Kelley's segments are investigated. We give the necessary conditions, in Theorem 4.1.13, for arc structures to exist in a hyperspace. As a consequence, we obtain Theorem 4.1.14 and Theorem 4.1.16, which show that 2^X and $\mathcal{C}(X)$ are arcwise connected when X is a continuum. In addition, attention is given to contractibility of hyperspaces. The Fundamental Theorem of Contractible Hyperspaces (Theorem 4.3.10), which provides an alternate way to determine if 2^X and $\mathcal{C}(X)$ is contractible, is presented.

In Chapter 5 we first discuss the properties of local connectedness and connectedness im kleinen. Property S, introduced by Sierpinski ([10]), is then examined and we obtain a description of Peano continua using Property S in Theorem 5.2.10. Theorem 5.2.11 shows that if X is a locally connected continuum then 2^X and $\mathcal{C}(X)$ are locally connected, and in Theorem 5.2.12 we show that for a continuum X , if 2^X or $\mathcal{C}(X)$ is locally connected then so is X . These are two important results of this thesis. Furthermore, Property K (by Kelley [3]) is investigated, and we obtain in Theorem 5.3.13 that Property K is a sufficient condition for 2^X and $\mathcal{C}(X)$ to be contractible. As a consequence of this, we finally show that 2^X and $\mathcal{C}(X)$ are contractible when X is a Peano continuum.

1.1 Preliminaries

In this thesis, we will be working within the theory of metric spaces. We begin by recalling the definition of a metric space and some important results. Proofs of these results will not be provided, but one can refer to ([6]) and ([9]) in each case.

Definition 1.1.1. Let X be a non-empty set and let $d : X \times X \rightarrow \mathbb{R}^+$ be a function, where $\mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$, satisfying the following :

1. $d(x, y) \geq 0$, for all $x, y \in X$.
2. $d(x, y) = 0$ iff $x = y$.
3. $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We say that (X, d) is a metric space.

Let (X, d) be a metric space. For $x \in X$ and $\varepsilon > 0$, define $S(x, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\}$. $S(x, \varepsilon)$ is called the sphere centered at x with radius ε .

Definition 1.1.2. Let (X, d) be a metric space.

1. For any $A \subseteq X$, A is called degenerate provided that $A = \{x\}$ for some $x \in X$.
Otherwise, A is called non-degenerate.
2. If A is bounded and non-empty, the diameter of A is defined to be the number
$$\text{diam}(A) = \sup\{d(a_1, a_2) | a_1, a_2 \in A\}.$$

Proposition 1.1.3. Let (X, d) be a metric space and $A \subseteq (X, d)$. Then $\text{diam}(\overline{A}) = \text{diam}(A)$.

Definition 1.1.4. Let $A \subseteq (X, d)$. A point $p \in X$ is called an accumulation point of A , if for all $\varepsilon > 0$, $(S(p, \varepsilon) \setminus \{p\}) \cap A \neq \emptyset$.

Proposition 1.1.5.

Let $A \subseteq (X, d)$. Then $p \in \overline{A}$ iff there exists a sequence $(x_n)_n$ in A such that $x_n \rightarrow p$.

Definition 1.1.6. Let $f : (X, d) \longrightarrow (Y, e)$ be a function between metric spaces (X, d) and (Y, e) . We say that f is continuous at $a \in X$ if given any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, a) < \delta$ then $e(f(x), f(a)) < \varepsilon$.

We say that f is continuous, if it is continuous at every point of X .

Example 1.1.7. Let (X, d) be a metric space. For a fixed $x \in X$, let $d(x, \cdot) : X \longrightarrow \mathbb{R}$ be given by $p \mapsto d(x, p)$. Then $d(x, \cdot)$ is continuous on X .

The next proposition give us some elementary but important properties of continuous maps.

Proposition 1.1.8.

Let (X, d) , (Y, e) and (Z, k) be metric spaces.

(a) $f : X \longrightarrow Y$ is continuous iff for every $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

(b) Let $f : X \longrightarrow Y$ be continuous and $A \subseteq X$. Then $f|_A : A \longrightarrow Y$ is continuous. (Where $f|_A$ is the map given by $f|_A(x) = f(x)$, for all $x \in A$.) $f|_A$ is called the restriction map of f to A .

(c) Let $f : X \longrightarrow Y$ be continuous and $g : Y \longrightarrow Z$ be continuous. Then the composition gf is continuous.

Definition 1.1.9. A function $f : X \longrightarrow Y$ between metric spaces X and Y is called a homeomorphism if f is a continuous function which is a bijection and f^{-1} is continuous.

When there exists a homeomorphism between X and Y , we say that X and Y are homeomorphic.

Definition 1.1.10. Let (X, d) and (Y, e) be metric spaces. A function $f : X \longrightarrow Y$ is called an *isometry*, if f is onto and provided that for every $x, y \in X$, $e(f(x), f(y)) = d(x, y)$.

If there exists an isometry f between metric space (X, d) and (Y, e) , then we say that X and Y are *isometric*.

Definition 1.1.11. Let (X, d) be a metric space. Then the map $f : X \longrightarrow X$ is a *contraction*, if for some real number $0 \leq c \leq 1$, we have $d(f(x), f(y)) \leq cd(x, y)$, for all $x, y \in X$.

Remark 1.1.12. It is easily seen that a contraction map is a continuous mapping.

Definition 1.1.13. Let $f : (X, d) \longrightarrow (Y, e)$ be a function between metric spaces (X, d) and (Y, e) . f is uniformly continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$d(x, x') < \delta \text{ implies that } e(f(x), f(x')) < \varepsilon.$$

Remark 1.1.14. In particular, uniformly continuous maps are always continuous, but not the other way around.

Definition 1.1.15. $A \subseteq (X, d)$, is said to be compact if given any open covering of X , there exists a finite subcovering of X .

That is, when $A = \bigcup_{\alpha \in I} G_\alpha$ and each G_α is open in X , then there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in I$ such that $A = \bigcup_{i=1}^n G_{\alpha_i}$.

One of the most important fact about compactness is that continuous images of compact spaces are compact.

Theorem 1.1.16.

Let (X, d) and (Y, e) be metric spaces, and let $f : X \longrightarrow Y$ be a continuous map.

- 1. If $A \subseteq X$ is compact, then $f(A)$ is compact.*
- 2. If X is compact, then f is uniformly continuous.*

Theorem 1.1.17.

- 1. A closed subset of a compact metric space must be compact.*
- 2. If A is a compact subset of (X, d) , then A is closed and bounded.*
- 3. If A is a compact subset of (X, d) , every sequence in A contains a convergent subsequence whose limit is in A .*

We will now recall an important criterion for a space to be compact. First we give a definition.

Definition 1.1.18. A collection \mathcal{A} of subsets of X , is said to have the finite intersection property if for every finite subcollection, $\{A_1, \dots, A_n\}$ of \mathcal{A} , the intersection $A_1 \cap \dots \cap A_n$ is non-empty.

A proof of the next result can be found in [13].

Theorem 1.1.19. *Let X be a metric space. Then X is compact iff for every collection \mathcal{A} of closed subsets of X having the finite intersection property, the intersection $\bigcap_{A \in \mathcal{A}} A$, of all the elements of \mathcal{A} is non-empty.*

Lemma 1.1.20. *(The Lebesgue Covering Lemma)*

Let \mathcal{A} be an open covering of the metric space (X, d) . If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ , there exists an element of \mathcal{A} containing it.

The number δ is called a Lebesgue number of the covering \mathcal{A} .

Definition 1.1.21. A metric space (X, d) is said to be disconnected if there exists non-empty open sets G, H of X such that $X = G \cup H$ and $G \cap H = \emptyset$.

X is connected, if it is not disconnected and $A \subseteq (X, d)$ is said to be connected if it is connected as a metric space in its own right.

Proposition 1.1.22.

The continuous image of a connected metric space is again connected.

Theorem 1.1.23.

Let (X, d) be a metric space.

- 1. The union of a collection of connected subspaces, of a metric space X , which have a point in common is connected.*
- 2. Let A be a connected subspace of metric space X , if $A \subseteq B \subseteq \bar{A}$, then B is also connected.*
- 3. If for all $x, y \in X$, there exists a connected set $A \subseteq X$ such that $x, y \in A$, then X is connected.*

Definition 1.1.24. Let X be a metric space and $x \in X$. The largest connected subset C_x of X containing x is called the component of x . It exists, being just the union of all connected subsets of X containing x .

If $x \neq y$ in X , then either $C_x = C_y$ or $C_x \cap C_y = \emptyset$; otherwise $C_x \cup C_y$ would be a connected set containing x and y and larger than C_x or C_y , which is impossible. Thus the components of points in X form a partition of X into maximal connected subsets. This justifies referring to them as components of X .

Definition 1.1.25. A metric space (X, d) is said to be arcwise connected, if for any two points x and y in X there exists a homeomorphism $h : [0, 1] \rightarrow h([0, 1]) \subseteq X$ such that $h(0) = x$ and $h(1) = y$.

We now state a few topological results and definitions that will be required for later use.

Definition 1.1.26. A topological space X is said to be a Hausdorff space, if given any pair of distinct points $p_1, p_2 \in X$, there exists neighbourhoods U_1 of p_1 and U_2 of p_2 with $U_1 \cap U_2 = \emptyset$.

Lemma 1.1.27. (*Closed map lemma*)

Let F is a continuous map from a compact space to a Hausdorff space. Then

- (a) *F is a closed map.*
- (b) *If F is bijective then it is a homeomorphism.*

Definition 1.1.28. Let X be a topological space. A collection \mathcal{B} of subsets of X is called a basis for the topology of X , if the following conditions hold :

- (1) Every element of \mathcal{B} is an open subset of X .
- (2) Every open subset of X is the union of some collection of elements of \mathcal{B} .

If the topology on X is understood, then we just say that \mathcal{B} is a basis for X .

Definition 1.1.29. A topological space X is said to be second countable if the topology on X has a countable basis.

Proposition 1.1.30. *Every compact metric space is second countable.*

Definition 1.1.31. A subset B of a topological space X is said to be dense in X if, $\overline{B} = X$.

Definition 1.1.32. A topological space X is said to be separable if X has a countable dense subset.

Proposition 1.1.33. *Every compact metric space is separable.*

1.2 Hyperspaces and the Hausdorff Metric

Named after *Felix Hausdorff*, we will now introduce the Hausdorff metric for an arbitrary metric space. In order to do this we begin by providing convenient notation.

Let (X, d) be a metric space and A be a closed bounded subset of X . We denote by $V_\varepsilon(A)$,

$$V_\varepsilon(A) = \bigcup_{x \in A} S(x, \varepsilon) = \bigcup_{x \in A} \{y \in X \mid d(x, y) < \varepsilon\}$$

Thus, for any $\{x\} \subseteq X$, we observe that $V_\varepsilon(\{x\}) = V_\varepsilon(x) = S(x, \varepsilon)$.

Furthermore, for $x \in X$ and any $A, B \subseteq X$, we let

$$d(x, B) = \inf\{d(x, y) \mid y \in B\} \quad \text{and} \quad d(A, B) = \inf\{d(x, y) \mid x \in A \text{ and } y \in B\}.$$

Definition 1.2.1. Let (X, d) be a metric space. The Hausdorff distance induced by d , which is denoted by ρ , is defined as follows: For any closed bounded subsets A, B of X ,

$$\rho(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq V_\varepsilon(B) \text{ and } B \subseteq V_\varepsilon(A)\}$$

We prove that ρ is, indeed, a metric.

Theorem 1.2.2 ([8]; (0.2)). *Let (X, d) be a metric space and 2^X denote the set of all non-empty closed bounded subsets of X . Then $(2^X, \rho)$ is a metric space.*

Proof.

It is clear that $\rho(A, B) > 0$, for all $A, B \in 2^X$. We also observe that for each $A, B \in 2^X$, $\rho(A, B) < \infty$, since A and B are bounded subsets.

Now for any $A, B \in 2^X$, $\rho(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq V_\varepsilon(B) \text{ and } B \subseteq V_\varepsilon(A)\} = \rho(B, A)$. Hence ρ is symmetric.

We shall now show that

$$\rho(A, B) = 0 \iff A = B. \tag{1.1}$$

(\implies) Suppose that $\rho(A, B) = 0$, for some $A, B \in 2^X$. We first show that $A \subseteq B$. Let $a \in A$

be arbitrary and $\varepsilon > 0$ be given. Since $\rho(A, B) = 0$, there exists $\delta > 0$ such that $A \subseteq V_\delta(B)$ and $B \subseteq V_\delta(A)$ and $\delta < \varepsilon$. Now there exists $b \in B$ such that $d(a, b) < \delta$, since $A \subseteq V_\delta(B)$. Hence $d(a, B) < \delta < \varepsilon$, and since ε is arbitrary, we have $d(a, B) = 0$. Thus since B is closed, $a \in B$ and so $A \subseteq B$. Similarly, $B \subseteq A$, and hence $A = B$.

(\Leftarrow) Assume $A = B$. Then for every $\varepsilon > 0$, we have $A \subseteq V_\varepsilon(B)$ and $B \subseteq V_\varepsilon(A)$. Hence $\rho(A, B) = 0$.

So Equation (1.1) holds.

It remains only to show the triangle inequality. That is, for $A, B, C \in 2^X$,

$$\rho(A, C) \leq \rho(A, B) + \rho(B, C).$$

Let $\varepsilon > 0$ and $A, B, C \in 2^X$. Now as a consequence of Definition 1.2.1, there exists $\delta > 0$ such that $\delta < \rho(A, B) + \frac{\varepsilon}{2}$ and $A \subseteq V_\delta(B)$ and $B \subseteq V_\delta(A)$. Also, there exists $\delta' > 0$ such that $\delta' < \rho(B, C) + \frac{\varepsilon}{2}$ and $B \subseteq V_{\delta'}(C)$ and $C \subseteq V_{\delta'}(B)$. Therefore,

$$A \subseteq V_\delta(B) \subseteq V_{\rho(A, B) + \frac{\varepsilon}{2}}(B) \quad (1.2)$$

and,

$$B \subseteq V_{\delta'}(C) \subseteq V_{\rho(B, C) + \frac{\varepsilon}{2}}(C) \quad (1.3)$$

Let $a \in A$. Then by Equation (1.2), there exists $b \in B$ such that $d(a, b) < \rho(A, B) + \frac{\varepsilon}{2}$, and by Equation (1.3), there exists $c \in C$ such that $d(b, c) < \rho(B, C) + \frac{\varepsilon}{2}$. Hence we have

$$\begin{aligned} d(a, c) &\leq d(a, b) + d(b, c) \\ &< \rho(A, B) + \rho(B, C) + \varepsilon \end{aligned}$$

Since $a \in A$ was arbitrary,

$$A \subseteq V_{[\rho(A, B) + \rho(B, C) + \varepsilon]}(C) \quad (1.4)$$

By a similar argument, interchanging A and C , we obtain

$$C \subseteq V_{[\rho(C, B) + \rho(B, A) + \varepsilon]}(A) = V_{[\rho(A, B) + \rho(B, C) + \varepsilon]}(A). \quad (1.5)$$

Therefore by Equation (1.4) and (1.5), we obtain by the definition that

$$\rho(A, C) \leq \rho(A, B) + \rho(B, C) + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we have $\rho(A, C) \leq \rho(A, B) + \rho(B, C)$. Therefore ρ is a metric and the set 2^X endowed with ρ is a metric space. \square

In Theorem 1.2.2, we let 2^X denote the set of all non-empty closed bounded subsets of X which ensures that $\rho < \infty$ on 2^X . We will now define the hyperspaces of X , and other hyperspaces closely related to that of X .

Definition 1.2.3. Let (X, d) be a metric space. We define the hyperspaces 2^X and $\mathcal{C}(X)$ as follows :

1. $2^X = \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is closed and bounded in } X\}$.
2. $\mathcal{C}(X) = \{A \in 2^X \mid A \text{ is connected}\}$.

Remark 1.2.4.

By Definition 1.2.3, we observe that $\mathcal{C}(X) \subseteq 2^X$, therefore $\mathcal{C}(X)$ inherits the ρ metric. We will later show in Theorem 1.3.11 that $\mathcal{C}(X)$ is a closed subspace of 2^X .

In the next theorem, we provide an alternate description of the metric ρ .

Theorem 1.2.5. Let (X, d) be a metric space and $A, B \in 2^X$, then

$$\rho(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}.$$

Proof.

Let $\rho^*(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$ for any $A, B \in 2^X$. We will show that for $A, B \in 2^X$,

$$\rho(A, B) = \rho^*(A, B) \tag{1.6}$$

Take any $A, B \in 2^X$. In order to prove Equation(1.6), we begin by showing that

$\rho^*(A, B) \leq \rho(A, B)$. Suppose that $\rho^*(A, B) = \sup_{a \in A} d(a, B)$. Fix any $a \in A$. For any $\varepsilon > 0$

such that $A \subseteq V_\varepsilon(B)$ and $B \subseteq V_\varepsilon(A)$, we have $a \in V_\varepsilon(B)$. So there exists $b \in B$ such that $d(a, b) < \varepsilon$. Therefore $d(a, B) \leq d(a, b) < \varepsilon$ and so, $\sup_{a \in A} d(a, B) \leq \varepsilon$. It follows that $\rho^*(A, B) \leq \varepsilon$ and that $\rho^*(A, B) \leq \rho(A, B)$, as required.

It now remains to show that $\rho(A, B) \leq \rho^*(A, B)$. That is, we have to show that,

$$\inf\{\varepsilon > 0 \mid A \subseteq V_\varepsilon(B) \text{ and } B \subseteq V_\varepsilon(A)\} \leq \rho^*(A, B).$$

Suppose that $\rho^*(A, B) = \sup_{a \in A} d(a, B)$ and note that we then have

$$\sup_{b \in B} d(b, A) \leq \sup_{a \in A} d(a, B) \tag{1.7}$$

Let $\delta > 0$ be arbitrary. We show that

$$\inf\{\varepsilon > 0 \mid A \subseteq V_\varepsilon(B) \text{ and } B \subseteq V_\varepsilon(A)\} \leq \sup_{a \in A} d(a, B) + \delta.$$

Now there exists $a' \in A$ such that

$$\sup_{a \in A} d(a, B) < d(a', B) + \delta \tag{1.8}$$

Also, there exists $b' \in B$ such that $d(a', b') < d(a', B) + \delta$. Let $\varepsilon = d(a', B) + \delta$.

Claim: $A \subseteq V_\varepsilon(B)$.

For $a \in A$, it follows from Equation (1.8) that $d(a, B) < d(a', B) + \delta$. Therefore there exists $b'' \in B$ such that $d(a, B) \leq d(a, b'') < d(a', B) + \delta = \varepsilon$. Thus $a \in V_\varepsilon(b'') \subseteq V_\varepsilon(B)$ and the claim holds.

Claim: $B \subseteq V_\varepsilon(A)$.

For any $b \in B$ we have that $d(b, A) \leq \sup_{a \in A} d(a, B) < d(a', B) + \delta$ (by Equations (1.7) and (1.8)). Therefore there exists $a'' \in A$ such that $d(b, A) \leq d(b, a'') < d(a', B) + \delta = \varepsilon$.

So $b \in V_\varepsilon(A)$ and the claim holds.

Thus $\rho(A, B) \leq \varepsilon = d(a', B) + \delta \leq \sup_{a \in A} d(a, B) + \delta$, and since $\delta > 0$ is arbitrary, we have $\rho(A, B) \leq \sup_{a \in A} d(a, B)$. Hence $\rho(A, B) \leq \rho^*(A, B)$ and Equation (1.2.5) holds, as required. \square

Remark 1.2.6. From now onwards, we will use the two descriptions of ρ interchangeably.

By Theorem 1.2.5, we note that for any $\{a\}, \{b\} \in 2^X$,

$$\rho(\{a\}, \{b\}) = d(a, b).$$

Definition 1.2.7. A *continuum* is a non-empty compact connected metric space. Analogously, a *subcontinuum* is a non-empty compact connected subspace of a continuum.

The theory of hyperspaces is intimately linked to the structure of continua. Thus once the ideas of compactness and connectedness are examined, hyperspace theory is an appropriate setting in which to expand ones understanding of these concepts.

Unless otherwise stated, when we refer to 2^X (resp. $\mathcal{C}(X)$) we mean the space 2^X (resp. $\mathcal{C}(X)$) obtained, when X is a compact metric space. Note that if X is compact, then all closed subsets of X are also bounded, thus for such X , $2^X = \{A \subseteq X \mid A \text{ is closed and non-empty}\}$.

Throughout this thesis, 2^X and $\mathcal{C}(X)$ will be endowed with the Hausdorff metric ρ .

1.3 Limits and Convergence Theorems

Most properties of continua and in particular, hyperspaces of continua, are best studied by means of sequences of sets. We will now lay down the foundation for using sequences of sets to examine hyperspaces. This is done by defining the notation for the convergence of sets and obtaining basic properties.

Definition 1.3.1. Let $(A_i)_{i=1}^{\infty}$ be a sequence in 2^X . Define

1. $\liminf A_i = \{x \in X \mid \text{whenever } U \text{ open with } x \in U, U \text{ meets all but finitely many } A_i\}$
2. $\limsup A_i = \{x \in X \mid \text{whenever } U \text{ open with } x \in U, U \text{ meets infinitely many } A_i\}$

Remark 1.3.2.

1. We see that $\liminf A_i \subseteq \limsup A_i$.

2. $\liminf A_i$ and $\limsup A_i$ are closed subsets of X :

Take $x \in \overline{\liminf A_i}$. We will show $x \in \liminf A_i$. Take any open U , such that $x \in U$, then $U \cap \liminf A_i \neq \emptyset$, and so there exists $p \in \liminf A_i$ with $p \in U$. Thus U meets all but finitely many A_i and hence $x \in \liminf A_i$. Therefore $\liminf A_i$ is a closed subset of X .

Similarly for $\limsup A_i$. □

3. If $(A_{i(k)})_{k=1}^{\infty}$ is a subsequence of $(A_i)_{i=1}^{\infty}$, then $\liminf A_i \subseteq \liminf A_{i(k)}$ and $\limsup A_{i(k)} \subseteq \limsup A_i$

Note : For notational purposes, we shall denote sequences $(A_i)_{i=1}^{\infty}$ by $(A_i)_i$.

Definition 1.3.3. Let $(A_i)_i$ be a sequence in 2^X . If $\liminf A_i = A = \limsup A_i$, then we say that $(A_i)_i$ converges to A (written $A_i \rightarrow A$).

Theorem 1.3.4 ([8]; (0.7)). Let $(A_i)_i$ be a sequence in 2^X . $A_i \rightarrow A$ iff $\rho(A_i, A) \rightarrow 0$.

Proof.

(\implies) Suppose $A_i \rightarrow A$. We first show that $\limsup A_i \neq \emptyset$. Suppose $\limsup A_i = \emptyset$. Then for each $x \in X$, there exists an open U_x , $x \in U_x$ such that U_x meets only finitely many A_i . Since X is compact, $X = \bigcup_{x \in F} U_x$, F is finite. So X only meets finitely many A_i . This gives a contradiction. Hence $\limsup A_i \neq \emptyset$.

We have that $\limsup A_i$ is non-empty and is a closed subset of X (by Remark 1.3.2). So $A = \limsup A_i \in 2^X$. It remains to be shown that $\rho(A_i, A) \rightarrow 0$.

Let $\varepsilon > 0$ be given. Now $\limsup A_i = A$ and $A \subseteq V_{\varepsilon}(A)$.

Now $V_{\varepsilon}(A)$ is open in X , therefore $(X \setminus V_{\varepsilon}(A))$ is closed. Hence $(X \setminus V_{\varepsilon}(A))$ is compact.

For each $x \in (X \setminus V_{\varepsilon}(A))$, there exists an open U_x , $x \in U_x$ such that U_x meets only finitely many A_i . Thus $(X \setminus V_{\varepsilon}(A)) \subseteq \bigcup_{x \in X \setminus V_{\varepsilon}(A)} U_x$. By compactness of $(X \setminus V_{\varepsilon}(A))$, there exists a finite subset F of $(X \setminus V_{\varepsilon}(A))$ such that

$$(X \setminus V_{\varepsilon}(A)) \subseteq \bigcup_{x \in F} U_x.$$

Since F is finite and U_x meets only finitely many A_i 's, $(X \setminus V_\varepsilon(A))$ can only meet finitely many A_i 's. Thus there exists $N_1 \in \mathbb{N}$ such that $i \geq N_1$ implies that $(X \setminus V_\varepsilon(A)) \cap A_i = \emptyset$. Therefore $A_i \subseteq V_\varepsilon(A)$ for each $i \geq N_1$.

Since A is compact, we can find a finite number of open subsets U_1, U_2, \dots, U_k of X such that $A \subseteq \bigcup_{j=1}^k U_j$, the diameter of each U_j for $1 \leq j \leq k$ is less than ε , and $U_j \cap A \neq \emptyset$ for each $j = 1, 2, \dots, k$. Now $A = \liminf A_i$. For each $j = 1, 2, \dots, k$, there exists $M_j \in \mathbb{N}$ such that $i \geq M_j$ implies that $U_j \cap A_i \neq \emptyset$.

Let $N_2 = \max\{M_j\}_{j=1}^k$. Then whenever $i \geq N_2$ we have $U_j \cap A_i \neq \emptyset$, for all j .

Claim: $A \subseteq V_\varepsilon(A_i)$, for $i \geq N_2$.

Take $x \in A$. Then $x \in U_j$ for some j , and $U_j \cap A_i \neq \emptyset$. Therefore there exists y such that $y \in A_i$ and $y \in U_j$. Now $d(x, y) \leq \text{diam}(U_j) < \varepsilon$. Hence $x \in V_\varepsilon(A_i)$ as required.

Let $N = \max\{N_1, N_2\}$. Then for $i \geq N$, we have $A \subseteq V_\varepsilon(A_i)$ and $A_i \subseteq V_\varepsilon(A)$. Thus $\rho(A_i, A) \leq \varepsilon$. Hence $\rho(A_i, A) \rightarrow 0$.

(\Leftarrow) Suppose $\rho(A_i, A) \rightarrow 0$. We will first show that $\limsup A_n \subseteq A$.

Let $\varepsilon > 0$ be arbitrary, then there exists $N \in \mathbb{N}$ such that $i \geq N$ implies that $\rho(A_i, A) < \varepsilon$. Therefore for $i \geq N$, $A_i \subseteq V_\varepsilon(A)$. Take $x \in \limsup A_n$. Then $V_\varepsilon(x) \cap A_i \neq \emptyset$ for some $i \geq N$. Thus for some $y \in A_i$ we have that $d(x, y) < \varepsilon$ and there exists $a \in A$ such that $d(y, a) < \varepsilon$. So $d(x, a) < 2\varepsilon$ and thus, $d(x, A) < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $d(x, A) = 0$ and hence $x \in A$, since A is closed. We have shown $\limsup A_n \subseteq A$, as required.

It remains to show that $A \subseteq \liminf A_n$.

Take $a \in A$ and open U such that $a \in U$. There exists $\varepsilon > 0$ such that $V_\varepsilon(a) \subseteq U$, and there exists $M \in \mathbb{N}$ such that $i \geq M$ implies that $\rho(A_i, A) < \varepsilon$.

Hence, we have that $A \subseteq V_\varepsilon(A_i)$ for each $i \geq M$. Thus $V_\varepsilon(a) \cap A_i \neq \emptyset$ and so $U \cap A_i \neq \emptyset$ for each $i \geq M$. So $a \in \liminf A_n$, as needed.

We have shown that $A \subseteq \liminf A_n \subseteq \limsup A_n \subseteq A$. Therefore $\liminf A_n = \limsup A_n$, ie. $A_i \rightarrow A$. □

Proposition 1.3.5. *If $(A_n)_n$ is a sequence in 2^X and $L = \limsup A_n$, then for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that if $n > m$ then $A_n \subseteq V_\varepsilon(L)$.*

Proof.

Suppose this is not the case. Then there exists $\varepsilon > 0$ such that for each m there exists $n > m$ with $A_n \not\subseteq V_\varepsilon(L)$. So there exists $n_1 < n_2 < \dots < \dots$ such that $A_{n_k} \not\subseteq V_\varepsilon(L)$ for every k . For each i , pick $x_i \in A_{n_i}$, where $x_i \notin V_\varepsilon(L)$. Then $x_i \notin V_\varepsilon(x)$, for all i .

If $\{x_i\}$ is finite, take any such i , then there exists x such that $x = x_i \in L$. So $x_i \in V_\varepsilon(L)$.

But we know that $x_i \notin V_\varepsilon(L)$, so we have a contradiction.

If $\{x_i\}$ is infinite, then it is an infinite set in a compact space. So $\{x_i\}$ must have an accumulation point, x . Thus any open set containing x must contain infinitely many x_i . So $x \in L$, contradicting the fact that x is an accumulation point of $\{x_i\}$.

So in each case, we can find an m such that $n > m$ implies that $A_n \subseteq V_\varepsilon(L)$. \square

Remark 1.3.6.

In the next proposition the following known property of the metric d is required.

For any compact and disjoint subsets A, B of metric space (X, d) ,

$$d(A, B) = \inf\{d(x, y) \mid x \in A \text{ and } y \in B\} > 0 :$$

Assume $d(A, B) = 0$. Now $d(x, B) = \inf\{d(x, y) \mid y \in B\}$ is a continuous mapping on the domain $X \times B$. Since A is compact, $d(x, B)$ must attain a minimum when defined on $A \times B$. Therefore there exists $a \in A$ such that $d(a, B) = \inf\{d(x, B) \mid x \in A\} = d(A, B) = 0$. Hence since B is closed in X , $a \in \overline{B} = B$. This is a contradiction since $A \cap B = \emptyset$. Thus $d(A, B) > 0$. \square

Proposition 1.3.7. *If $A_i \in \mathcal{C}(X)$ for each $i = 1, 2, 3, \dots$, and if $\liminf A_i \neq \emptyset$ then $\limsup A_i$ is connected.*

Proof.

Let $L = \limsup A_i$. By Remark 1.3.2, L must be closed. Suppose L is disconnected, then $L = A \cup B$, for some A, B where $A, B \neq \emptyset$ and closed in L (and hence closed in X), and $A \cap B = \emptyset$.

Now $\liminf A_i \subseteq L$, so we may assume that $A \cap \liminf A_i \neq \emptyset$. Let $x \in A \cap \liminf A_i$. Since A, B are closed in X and X is compact, it follows that A and B are compact. So by Remark 1.3.6, $d(A, B) > 0$. Let $d(A, B) = 3\varepsilon$ for $\varepsilon > 0$.

Claim: $V_\varepsilon(A) \cap V_\varepsilon(B) = \emptyset$.

Assume there exists $y \in V_\varepsilon(A) \cap V_\varepsilon(B)$. Then there exists $a \in A$, $b \in B$ such that $d(y, a) < \varepsilon$ and $d(y, b) < \varepsilon$. Thus $d(a, b) < 2\varepsilon$ and so $d(A, B) < 2\varepsilon$, which is a contradiction. Hence the claim holds.

Now $L = A \cup B$, therefore $V_\varepsilon(L) = V_\varepsilon(A) \cup V_\varepsilon(B)$. By Proposition 1.3.5, we can find $N \in \mathbb{N}$ such that $n > N$ implies that $A_n \subseteq V_\varepsilon(L) = V_\varepsilon(A) \cup V_\varepsilon(B)$ and $V_\varepsilon(x) \cap A_n \neq \emptyset$.

Hence $A_n \cap V_\varepsilon(A) \neq \emptyset$ for all $n > N$.

So for all $n > N$, $A_n = (A_n \cap V_\varepsilon(A)) \cup (A_n \cap V_\varepsilon(B))$, where $(A_n \cap V_\varepsilon(A))$ and $(A_n \cap V_\varepsilon(B))$ are each open disjoint subsets of A_n . Since A_n is connected and $A_n \cap V_\varepsilon(A) \neq \emptyset$, we must have $A_n \cap V_\varepsilon(B) = \emptyset$ for each $n > N$ (otherwise we would obtain a disconnection of A_n).

Therefore $A_n \subseteq V_\varepsilon(A)$.

Now since $B \neq \emptyset$, there exists $b \in B$, $b \in \limsup A_n$ and hence $V_\varepsilon(b)$ does not meet any of the A_n for $n > N$, a contradiction. Therefore $\limsup A_n$ is connected. \square

Lemma 1.3.8. *Let $(K_i)_i$ be a collection in 2^X , such that $K_i \supseteq K_{i+1}$. Then $\bigcap K_i = K$ iff $K_i \longrightarrow K$.*

Proof.

(\implies) Take $\varepsilon > 0$ and let $\bigcap K_i = K$. Note that $\{K_i\}_i$ has the finite intersection property, so by compactness of X we have that $\bigcap K_i \neq \emptyset$. We have to show that $\limsup K_i \subseteq \liminf K_i$. Let $x \in \limsup K_i$ and $x \in U$, where U is open in X . Now $U \cap K_i \neq \emptyset$ for infinitely many i . Thus there is an increasing sequence of natural numbers $n_1 < n_2 < n_3 < \dots$, such that $U \cap K_{n_i} \neq \emptyset$. Hence if $i \in \mathbb{N}$ is arbitrary, then there exists n_k such that $i \leq n_k$ implies that $K_i \supseteq K_{n_k}$. Therefore $U \cap K_i \neq \emptyset$, since $U \cap K_{n_k} \neq \emptyset$. Hence $x \in \liminf K_i$. Thus $\liminf K_i = \limsup K_i$ and so $K_i \longrightarrow K$.

(\impliedby) Suppose $K_i \longrightarrow K$, then $K = \liminf K_i = \limsup K_i$. We first show that $K \subseteq K_i$ for all i . Take $x \in K$. Assume $x \notin K_N$, for some N . Then $x \in (X \setminus K_N)$, which is open in X and meets infinitely many K_i . This is a contradiction, hence $K \subseteq \bigcap K_i$. Now take $x \in \bigcap K_i$, then $x \in K_i$ for all i . Thus $x \in \liminf K_i = K$, and so $\bigcap K_i \subseteq K$. We have shown that $K = \bigcap K_i$, as required. \square

The next result illustrates an important fact about the nested intersection of continua.

Proposition 1.3.9. *Let X be a metric continuum and $K_1 \supseteq K_2 \supseteq K_3 \dots$, where each K_n is a continuum. Then $\bigcap_{n=1}^{\infty} K_n$ is a continuum.*

Proof.

Let $K = \bigcap_{n=1}^{\infty} K_n$. Then K is closed and non-empty by the finite intersection property. Hence K must be compact. So it remains to show that K is connected. By Lemma 1.3.8, we have that $K_i \rightarrow K$. Therefore $K = \liminf K_i = \limsup K_i$, and so by Proposition 1.3.7, K must be connected. Hence $K = \bigcap_{n=1}^{\infty} K_n$ is a continuum. \square

We now prove a general convergence result which will be essential later.

Lemma 1.3.10. *Let $(A_n)_n$ and $(B_n)_n$ be sequences in 2^X .*

(a) *If $A_n \rightarrow A$, $B_n \rightarrow B$ in 2^X and $A_n \subseteq B_n$ for each n , then $A \subseteq B$.*

(b) *If $A_n \rightarrow A$ in 2^X and $A_n \cap B \neq \emptyset$ for each n , then $A \cap B \neq \emptyset$.*

Proof.

(a) Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for $n \geq N$, $\rho(A_n, A) < \frac{\varepsilon}{2}$ and $\rho(B_n, B) < \frac{\varepsilon}{2}$. Therefore there exists $\varepsilon' > 0$ such that $\varepsilon' < \frac{\varepsilon}{2}$ and the following hold : $A_N \subseteq V_{\varepsilon'}(A)$, $A \subseteq V_{\varepsilon'}(A_N)$ and $B \subseteq V_{\varepsilon'}(B_N)$, $B_N \subseteq V_{\varepsilon'}(B)$. Take any $a \in A$, then there exists $y \in A_N$ such that $d(y, a) < \varepsilon'$. Since $A_N \subseteq B_N$, there exists $b \in B$ such that $d(y, b) < \varepsilon'$. Hence we have,

$$d(a, b) \leq d(a, y) + d(y, b) < 2\varepsilon' \leq \varepsilon$$

Therefore $d(a, B) \leq \varepsilon$ and since ε is arbitrary, $d(a, B) = 0$. Hence $a \in B$, since B is closed. Thus $A \subseteq B$.

(b) Now for each n , there exists $a_n \in A_n \cap B$. But $A_n \cap B$ is compact, so there exists a subsequence of $(a_n)_n$ which converges to some $a \in A_n \cap B$. Without loss of generality, we may assume that $a_n \rightarrow a$.

Then $d(a_n, A) \leq \rho(A_n, A)$, and since $\rho(A_n, A) \rightarrow 0$, we must have that $d(a_n, A) \rightarrow 0$.

But $d(a_n, A) \rightarrow d(a, A)$. Hence $d(a, A) = 0$, proving that $a \in \bar{A} = A$. Hence $a \in A \cap B$, so $A \cap B \neq \emptyset$, as required. \square

It is important to know about the existence of convergent subsequences, as it is desirable to use sequences of subsets of a continuum X , to examine properties about X and its hyperspace. The next few results establish basic information about this. We begin by showing that $\mathcal{C}(X)$ is a closed subspace of 2^X .

Theorem 1.3.11. $\mathcal{C}(X)$ is a closed subspace of 2^X .

Proof.

Take $A \in \overline{\mathcal{C}(X)}$. Now there exists $A_n \in \mathcal{C}(X)$ such that $A_n \rightarrow A$ in 2^X , that is $\rho(A_n, A) \rightarrow 0$. Now $\liminf A_n = A \neq \emptyset$, so by Proposition 1.3.7, $A = \limsup A_n$ is connected. Thus $A \in \mathcal{C}(X)$. \square

Theorem 1.3.12 ([8]; (0.8)). *If (X, d) is a continuum, then the spaces 2^X and $\mathcal{C}(X)$ are each compact.*

Proof.

We will first show that 2^X is compact. It suffices to show that every sequence in 2^X has a convergent subsequence. Let $(A_i)_i$ be a sequence in 2^X .

We define sequences $(A_i^1)_{i=1}^\infty, (A_i^2)_{i=1}^\infty, \dots, (A_i^n)_{i=1}^\infty, \dots$ for $n = 1, 2, 3, \dots$ inductively as follows.

Since X is a compact metric space, it is a second countable topological space by Proposition 1.1.30, so let $\mathcal{U} = \{U_1, U_2, \dots, U_n, \dots\}$ be a countable base for the topology for X . Define $(A_i^1)_{i=1}^\infty$ by $A_i^1 = A_i$ for each $i = 1, 2, 3, \dots$

Assume inductively that we have defined the sequence $(A_i^n)_{i=1}^\infty$. We then define $(A_i^{n+1})_{i=1}^\infty$ in one of the following two ways :

- (1) If $(A_i^n)_{i=1}^\infty$ has a subsequence $(A_{i(j)}^n)_{j=1}^\infty$ such that $(\limsup A_{i(j)}^n) \cap U_n = \emptyset$, then let $(A_i^{n+1})_{i=1}^\infty$ be one such subsequence of $(A_i^n)_{i=1}^\infty$;
- (2) If every subsequence of $(A_i^n)_{i=1}^\infty$ has a point of its lim sup in U_n , then let $(A_i^{n+1})_{i=1}^\infty$ be given by $A_i^{n+1} = A_i^n$ for each $i = 1, 2, 3, \dots$

Given that $(A_i^n)_{i=1}^\infty$ has been defined for each $n = 1, 2, 3, \dots$, consider the diagonal sequence $(A_n^n)_{n=1}^\infty$. Then $(A_n^n)_{n=1}^\infty$ is a subsequence of $(A_i)_i$.

Claim: $(A_n^n)_{n=1}^\infty$ converges in 2^X .

Suppose that $(A_n^n)_{n=1}^\infty$ does not converge. Then there exists a point $p \in \limsup A_n^n$ such that $p \notin \liminf A_n^n$. Hence there exists $U_m \in \mathcal{U}$ such that $p \in U_m$ and such that $U_m \cap A_n^{n(i)} = \emptyset$ for some subsequence $(A_{n(i)}^{n(i)})_{i=1}^\infty$ of $(A_n^n)_{n=1}^\infty$. Clearly $(A_{n(i)}^{n(i)})_{i=m}^\infty$ is a subsequence of $(A_i^m)_{i=1}^\infty$ and furthermore, $\limsup A_{n(i)}^{n(i)} \cap U_m = \emptyset$. So $(A_i^m)_{i=1}^\infty$ satisfies (1) above (with m replacing n in (1)). Hence $(\limsup A_i^{m+1}) \cap U_m = \emptyset$ and since $(A_n^n)_{n=m+1}^\infty$ is a subsequence of $(A_i^{m+1})_{i=1}^\infty$, it follows that $(\limsup A_n^n) \cap U_m = \emptyset$. Now $p \in (\limsup A_n^n) \cap U_m$, thus we have a contradiction. Therefore $(A_n^n)_{n=1}^\infty$ converges and 2^X is compact as required.

We now show that $\mathcal{C}(X)$ is compact.

By Theorem 1.3.11 we know that $\mathcal{C}(X)$ is a closed subspace of 2^X . Since 2^X is compact, it follows that $\mathcal{C}(X)$ is compact. \square

Remark 1.3.13. It is true that each of 2^X and $\mathcal{C}(X)$ are continua. The proof of this result will be delayed, since we will prove a stronger result, which is that each of 2^X and $\mathcal{C}(X)$ is an arcwise connected continuum.

By Theorem 1.3.12, we know that 2^X is compact and therefore all subsets of 2^X are bounded. Hence the hyperspace 2^{2^X} , consists of all non-empty and closed subsets of 2^X . Analogous to our construction of ρ , we have $(2^{2^X}, \rho^1)$ where for $\mathcal{A}, \mathcal{B} \in 2^{2^X}$,

$$\rho^1(\mathcal{A}, \mathcal{B}) = \inf\{\varepsilon > 0 \mid \mathcal{A} \subseteq V_\varepsilon(\mathcal{B}) \text{ and } \mathcal{B} \subseteq V_\varepsilon(\mathcal{A})\}.$$

1.4 General Mappings in Hyperspaces

In the previous section, we provided a few basic results about the structure of hyperspaces. We now define and examine two of the classical mappings of the hyperspaces 2^X and 2^{2^X} , which will be used in subsequent results.

Definition 1.4.1. For $A \in 2^X$, we define $\phi(A) = \{\{a\} \mid a \in A\}$.

Thus $\phi(A) \subseteq 2^X$.

Theorem 1.4.2. $\phi(A) \in 2^{2^X}$.

Proof.

We need to show that $\phi(A)$ is closed in $(2^X, \rho)$.

Take $B \in \overline{\phi(A)}$. Then there exists a sequence $(\{a_n\})_n$ in 2^X such that $\rho(\{a_n\}, B) \rightarrow 0$, as $n \rightarrow \infty$.

So we have $(a_n)_n$ in A with A being compact. Therefore there exists a subsequence $(a_{n_k})_k$ such that $a_{n_k} \rightarrow a$, for some $a \in A$.

Claim: $B = \{a\}$.

Since $a_{n_k} \rightarrow a$, as $k \rightarrow \infty$, we have that $d(a_{n_k}, a) \rightarrow 0$ and hence $\rho(\{a_{n_k}\}, \{a\}) \rightarrow 0$.

But we have that $\{a_{n_k}\} \rightarrow B$ as $k \rightarrow \infty$. So by uniqueness of limits, $B = \{a\}$, as claimed.

Thus $\phi(A)$ is closed in $(2^X, \rho)$ and hence $\phi(A) \in 2^{2^X}$. \square

Remark 1.4.3. By Theorem 1.4.2, ϕ is a mapping from 2^X into 2^{2^X} .

Theorem 1.4.4. For $A \in 2^X$, $\phi(A)$ is isometric with A .

Proof.

For any $A \in 2^X$, since $A \subseteq X$, it follows that A inherits the metric d on X . Hence (A, d) is a metric space. Similarly for $\phi(A) \subseteq 2^X$, we have $(\phi(A), \rho)$ is a metric space in its own right. Define a map,

$$\begin{aligned} \alpha &: A \rightarrow \phi(A) \\ \text{by } a &\mapsto \{a\} \end{aligned}$$

Now for $a, b \in A$, we have, $\rho(\alpha(a), \alpha(b)) = \rho(\{a\}, \{b\}) = d(a, b)$, by Remark 1.2.6.

Hence α is a distance preserving map. So $\phi(A)$ is isometric with A . \square

Theorem 1.4.5. ϕ is an isometry onto $\phi(2^X)$.

Proof.

Recall $\phi : 2^X \rightarrow 2^{2^X}$.

Take $A, B \in 2^X$. We want to show that $\rho^1(\phi(A), \phi(B)) = \rho(A, B)$, for all $A, B \in 2^X$. By definition we have,

$$\rho^1(\phi(A), \phi(B)) = \max\left\{\sup_{a \in A} \rho(\{a\}, \phi(B)), \sup_{b \in B} \rho(\{b\}, \phi(A))\right\}$$

Now

$$\rho(\{a\}, \phi(B)) = \inf_{b \in B} \rho(\{a\}, \{b\}) = \inf_{b \in B} d(a, b) = d(a, B)$$

Substituting this, we have

$$\rho^1(\phi(A), \phi(B)) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} = \rho(A, B)$$

So ϕ is an isometry, as required. □

Corollary 1.4.6. $\phi : 2^X \longrightarrow 2^{2^X}$ is continuous.

Proof.

By Theorem 1.4.5, we have that ϕ is an isometry. Therefore ϕ is continuous. □

Definition 1.4.7. For $\mathcal{A} \in 2^{2^X}$, we define $\Gamma(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$.

Lemma 1.4.8 ([3]; Lemma 1.1.). Γ is a contraction.

Proof.

First it is necessary to show that $\Gamma(\mathcal{A})$ is closed in X for $\mathcal{A} \in 2^{2^X}$. So let $a \in \overline{\Gamma(\mathcal{A})}$. We shall show that $a \in \Gamma(\mathcal{A})$.

Now there exists a sequence $(a_n)_n$ in $\Gamma(\mathcal{A})$ such that $a_n \longrightarrow a$ as $n \longrightarrow \infty$, with $a_n \in A_n$ and $A_n \in \mathcal{A}$. Now \mathcal{A} is compact because \mathcal{A} is a closed subset of compact 2^X . So (A_n) has a subsequence $(A_{n_k})_k$ such that $A_{n_k} \longrightarrow A$ in 2^X and $A \in \mathcal{A}$. For simplicity of notation, we can assume that $\lim A_n = A$, since \mathcal{A} is closed and $A \in \mathcal{A}$.

Claim: $a \in A$.

We have $A_n \longrightarrow A$. Now $a_n \longrightarrow a$ in X , therefore $\{a_n\} \longrightarrow \{a\}$ in 2^X , since

$\rho(\{a_n\}, \{a\}) = d(a_n, a) \longrightarrow 0$. Thus since $\{a_n\} \subseteq A_n$ for each n , it follows by Lemma 1.3.10 that $\lim\{a_n\} \subseteq \lim A_n$. That is, $\{a\} \subseteq A$, so $a \in A$.

So $a \in \Gamma(\mathcal{A})$. Hence $\Gamma(\mathcal{A})$ is closed in X , that is, $\Gamma(\mathcal{A}) \in 2^X$.

Now, in order to show that Γ is a contraction, we will show that for $\mathcal{A}, \mathcal{B} \in 2^{2^X}$,

$$\rho(\Gamma(\mathcal{A}), \Gamma(\mathcal{B})) \leq \rho^1(\mathcal{A}, \mathcal{B})$$

Take $\mathcal{A}, \mathcal{B} \in 2^{2^X}$ and suppose $\rho(\Gamma(\mathcal{A}), \Gamma(\mathcal{B})) = h$. That is,

$$\max\left\{\sup_{b \in \Gamma(\mathcal{B})} d(b, \Gamma(\mathcal{A})), \sup_{a \in \Gamma(\mathcal{A})} d(a, \Gamma(\mathcal{B}))\right\} = h$$

Suppose $h = \sup\{d(b, \Gamma(\mathcal{A})) \mid b \in \Gamma(\mathcal{B})\}$. Then for each $n \in \mathbb{N}$, there exists $b_n \in \Gamma(\mathcal{B})$ such that, $h - \frac{1}{n} < d(b_n, \Gamma(\mathcal{A}))$. Since $\Gamma(\mathcal{B})$ is closed, it must be compact and hence a subsequence of $(b_n)_n$ converges to some $b \in \Gamma(\mathcal{B})$. Therefore for each $a \in \Gamma(\mathcal{A})$, by taking limits we obtain,

$$h \leq d(b, \Gamma(\mathcal{A})) \leq d(b, a)$$

So we have shown that there exists $b \in \Gamma(\mathcal{B})$ such that for each $a \in \Gamma(\mathcal{A})$, $h \leq d(b, a)$. Hence $h \leq d(b, \Gamma(\mathcal{A}))$. Now find $B \in \mathcal{B}$ such that $b \in B$.

Claim: $\rho^1(\mathcal{A}, \mathcal{B}) \geq h$.

For each $A \in \mathcal{A}$ we have, $\rho(B, A) \geq d(b, A) \geq d(b, \Gamma(\mathcal{A})) \geq h$. Therefore $\rho(B, \mathcal{A}) \geq h$.

Hence, we have

$$\rho^1(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{A \in \mathcal{A}} \rho(A, \mathcal{B}), \sup_{B \in \mathcal{B}} \rho(B, \mathcal{A})\right\} \geq h$$

Hence $\rho^1(\mathcal{A}, \mathcal{B}) \geq h$ and Γ is a contraction. □

Remark 1.4.9. Since Γ is a contraction by Lemma 1.4.8, it follows that Γ is, in fact, a continuous function.

Definition 1.4.10. Let $A \subseteq X$ and suppose that $r : X \rightarrow A$ is a continuous map such that $r(a) = a$ for each point $a \in A$. The map r is called a retraction of X onto A .

Lemma 1.4.11 ([3]; Lemma 1.1.). $\phi\Gamma$ is a retraction of 2^{2^X} onto $\phi(2^X)$.

Proof.

$\phi\Gamma$ is continuous, since both ϕ and Γ are each continuous.

We need to show that

$$\phi\Gamma|_{\phi(2^X)} = id_{\phi(2^X)} \tag{1.9}$$

For any $A \in 2^X$, we have that $\phi\Gamma(\phi(A)) = \phi\Gamma(\{\{a\} \mid a \in A\}) = \phi(A)$.

Thus Equation 1.9 holds and $\phi\Gamma$ is a retraction of 2^{2^X} onto $\phi(2^X)$. □

Theorem 1.4.12 ([3]; Lemma 1.2.). *If \mathcal{A} is a subcontinuum of 2^X and $\mathcal{A} \cap \mathcal{C}(X) \neq \emptyset$ then $\Gamma(\mathcal{A})$ is a continuum.*

Proof.

$\Gamma(\mathcal{A})$ is compact for each $\mathcal{A} \subseteq 2^X$, since $\Gamma(\mathcal{A})$ is a closed subset of X . So it remains to show that $\Gamma(\mathcal{A})$ is connected.

Since $\mathcal{A} \cap \mathcal{C}(X) \neq \emptyset$, there exists $E \in \mathcal{A} \cap \mathcal{C}(X)$. Assume that $\Gamma(\mathcal{A})$ is disconnected.

So $\Gamma(\mathcal{A}) = A_1 \cup A_2$, where A_1, A_2 are non-empty closed subsets of $\Gamma(\mathcal{A})$ (and hence also closed in X) satisfying $A_1 \cap A_2 = \emptyset$.

Now $\Gamma(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$ and $E \in \mathcal{A}$, therefore $E \subseteq A_1 \cup A_2$. Since E is connected, we may assume that $E \subseteq A_1$. Define

$$\mathcal{A}_1 = \{A \in \mathcal{A} \mid A \subseteq A_1\}$$

$$\mathcal{A}_2 = \{A \in \mathcal{A} \mid A \cap A_2 \neq \emptyset\}$$

Now $E \in \mathcal{A}_1$, and since $A_2 \neq \emptyset$ we must have $\mathcal{A}_2 \neq \emptyset$. Thus \mathcal{A}_1 and \mathcal{A}_2 are non-empty and we have that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. We will show that \mathcal{A}_1 and \mathcal{A}_2 are closed in 2^X .

Take $B \in \overline{\mathcal{A}_1}$, then there exists a sequence $(B_n)_n$ in \mathcal{A}_1 such that $B_n \rightarrow B$. By Lemma 1.3.10, $B \subseteq A_1$ and so $B \in \mathcal{A}_1$. Hence \mathcal{A}_1 is closed in 2^X .

Now to show that \mathcal{A}_2 is closed in 2^X . Take $B \in \overline{\mathcal{A}_2}$. Then there exists a sequence $(B_n)_n$ in \mathcal{A}_2 such that $B_n \rightarrow B$.

Since $B_n \in \mathcal{A}_2$, we have that $B_n \cap A_2 \neq \emptyset$ for each n . Thus by Lemma 1.3.10, $B \cap A_2 \neq \emptyset$. Hence $B \in \mathcal{A}_2$ and \mathcal{A}_2 is closed in 2^X .

So $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, with $\mathcal{A}_1, \mathcal{A}_2$ non-empty and closed in 2^X and $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. This is a disconnection of \mathcal{A} . Hence a contradiction. So $\Gamma(\mathcal{A})$ is a continuum. \square

Chapter 2

Boundary Bumping Theorems

The results that are to follow in Theorem 2.2.2, Theorem 2.2.3 and Theorem 2.2.4 are known as the Boundary Bumping Theorems, since under mild conditions, a component of a set must intersect (that is, bump) the boundary of the set. But first we begin with the Cut Wire Theorem.

2.1 The Cut Wire Theorem

Definition 2.1.1. Let (X, d) be any metric space and $\varepsilon > 0$. An ε -chain from a point a to b in X is a finite sequence of points $x_1, x_2, \dots, x_n \in X$, such that $a = x_1$, $b = x_n$ and $d(x_i, x_{i+1}) < \varepsilon$ for all $i = 1, 2, \dots, n - 1$.

Definition 2.1.2. X is *well-chained* or *chain connected*, if for any $\varepsilon > 0$, any two points in X can be joined by an ε -chain.

Proposition 2.1.3.

- (1) If X is connected, then X is well-chained.
- (2) If X is compact and well-chained, then X is connected.

Proof.

(1) Suppose X is connected. Take any $\varepsilon > 0$ and any points $a \in X$ and $b \in X$.

Let $X_a = \{x \in X \mid a \text{ can be joined to } x \text{ by an } \varepsilon\text{-chain}\}$. Note $X_a \neq \emptyset$, since $a \in X_a$.

Claim: X_a is open.

Let $x \in X_a$. Then we have an ε -chain, $a = x_1, x_2, \dots, x_n = x$. Take $y \in S(x, \varepsilon)$, then $d(x, y) < \varepsilon$. So $a = x_1, x_2, \dots, x_n = x, y$ is an ε -chain from a to y . Thus $y \in X_a$, that is $S(x, \varepsilon) \subseteq X_a$. So X_a is open.

Claim: X_a is closed.

For any $x \in \overline{X_a}$ we have that $S(x, \varepsilon) \cap X_a \neq \emptyset$. Take $y \in S(x, \varepsilon) \cap X_a$. Then $d(x, y) < \varepsilon$ and $a = x_1, x_2, \dots, x_n = y, x$ is an ε -chain from a to x . Thus $x \in X_a$, so X_a is closed.

So X_a is clopen, $X_a \neq \emptyset$ and we cannot have $(X \setminus X_a) \neq \emptyset$ (because X is connected). Thus $b \in X_a$, that is, a can be joined to b by an ε -chain. Hence X is well-chained.

(2) Suppose X is compact and well-chained. We show that X is connected. Suppose not, with a view to contradiction. Then $X = A \cup B$, where A, B are some non-empty closed subsets of X and $A \cap B = \emptyset$. Since A and B are closed in X , each must be compact and $d(A, B) > 0$ by Remark 1.3.6. Take $a \in A$ and $b \in B$ and let $\varepsilon = d(A, B)$. There exists an ε -chain from a to b , that is $a = x_1, x_2, \dots, x_n = b$. Let k be the least natural number such that $x_k \in B$. Now since $k \in \{2, 3, \dots, n\}$, it follows that $x_{k-1} \in A$ and $x_k \in B$. Therefore $d(x_{k-1}, x_k) < \varepsilon$ but $d(x_{k-1}, x_k) \geq d(A, B) = \varepsilon$, which is a contradiction. So X is connected. \square

Example 2.1.4. *By the above proposition, although connectedness implies well-chainedness, it is not the case that if X is well-chained then X is connected.*

Consider $X = \mathbb{Q}$ with the normal metric, then X is well chained but X is not connected.

Lemma 2.1.5. *Let X be a compact metric space, and suppose A_n is a $\frac{1}{n}$ -chain for each $n \in \mathbb{N}$. Suppose $A_n \rightarrow A_0$ in 2^X . Then A_0 is connected.*

Proof.

We first note that since A_n is finite, it is closed and hence $A_n \in 2^X$.

A_0 is compact, so it suffices to show that A_0 is well-chained, for then A_0 will be connected by Proposition 2.1.3. Let $a, b \in A_0$ and $\varepsilon > 0$ be arbitrary. Then we can find $N \in \mathbb{N}$ such that $\rho(A_N, A_0) < \frac{\varepsilon}{3}$ and $\frac{1}{N} < \frac{\varepsilon}{3}$.

Now there exists $\varepsilon' > 0$, $A_N \subseteq V_{\varepsilon'}(A_0)$, $A_0 \subseteq V_{\varepsilon'}(A_N)$ and $\varepsilon' < \frac{\varepsilon}{3}$. So we have that, for $a \in A_0$ there exists $q \in A_N$ such that $d(a, q) < \varepsilon'$ and similarly, for $b \in A_0$ there exists

$r \in A_N$ such that $d(b, r) < \varepsilon'$

Thus we can find, $q_1, q_2, \dots, q_m \in A_N$ with $q_1 = q$ and $q_m = r$, which is a $\frac{1}{N}$ -chain in A_N , since $q_i \in A_N$ and $d(q_i, q_{i+1}) < \frac{1}{N}$ for $i = 1, \dots, m$.

Now for each $i = 2, \dots, m - 1$, $q_i \in A_N$ so there exists $q'_i \in A_0$ such that $d(q_i, q'_i) < \varepsilon'$

So $a, q'_2, q'_3, \dots, q'_{m-1}, b$ are all in A_0 and

$$\begin{aligned} d(a, q'_2) &\leq d(a, q_1) + d(q_1, q_2) + d(q_2, q'_2) \\ &< \varepsilon' + \frac{1}{N} + \varepsilon' \\ &< \varepsilon \end{aligned}$$

Similarly, $d(q'_i, q'_{i+1}) < \varepsilon$ for $i = 2, 3, \dots, m - 2$ and $d(q'_{m-1}, b) < \varepsilon$.

This implies that A_0 is well chained. So A_0 is connected. □

Theorem 2.1.6 ([8]; Theorem 20.6). (*Cut Wire Theorem*)

Let X be a compact metric space and suppose that A, B are non-empty and $A, B \in 2^X$ such that no connected subset of X intersects both A and B .

Then there is a separation $X = X_0 \cup X_1$ of X (ie. $X_0, X_1 \neq \emptyset$, X_0, X_1 closed, $X_0 \cap X_1 = \emptyset$ such that $A \subseteq X_0$ and $B \subseteq X_1$).

Proof.

First note that, $A \cap B = \emptyset$, since no connected subset of X intersects both A and B .

Claim: There exists $\varepsilon > 0$ such that there is no ε -chain from any point of A to any point B . Suppose not (with a view to a contradiction).

Then for each $n \in \mathbb{N}$, there exists a $\frac{1}{n}$ -chain, Q_n , from some point $a_n \in A$ to some point $b_n \in B$.

So $(Q_n)_n$ is a sequence in 2^X and 2^X is compact by Theorem 1.3.12, therefore there exists a subsequence $(Q_{n_k})_k$ such that $Q_{n_k} \rightarrow Q_0$, say, and Q_0 is connected, the latter following from Lemma 2.1.5. Without loss of generality, we may assume that $Q_n \rightarrow Q_0$.

Now $Q_n \cap A \neq \emptyset$, $Q_n \cap B \neq \emptyset$ and $Q_n \rightarrow Q_0$, therefore by Lemma 1.3.10, $Q_0 \cap A \neq \emptyset$ (because A is closed) and $Q_0 \cap B \neq \emptyset$ (because B is closed). Hence a contradiction, so the claim holds true.

We may now define $X_0 = \{x \in X \mid \text{there is an } \varepsilon\text{-chain from some point of } A \text{ to } x\}$.

Then $A \subseteq X_0$, since for each $a \in A$, $\{a\}$ is an ε -chain.

Claim: X_0 open.

Take $x \in X_0$, we shall show that $S(x, \varepsilon) \subseteq X_0$. Now $x \in X_0$ implies that there is an ε -chain from some $a \in A$ to x . Take $y \in S(x, \varepsilon)$. Then there must be a ε -chain from a to y . Thus $y \in X_0$, and hence X_0 is open.

Claim: X_0 is closed.

Now $x \in \overline{X_0}$ implies $S(x, \varepsilon) \cap X_0 \neq \emptyset$. Take $y \in S(x, \varepsilon) \cap X_0$. Then there exists an ε -chain from some point $a \in A$ to y . Hence there is an ε -chain from a to x , that is, $x \in X_0$. Thus X_0 is closed.

Thus we have that X_0 is clopen with $A \subseteq X_0$. Furthermore, $B \subseteq X \setminus X_0$ and this follows from the first claim of the proof. This is the required separation. \square

2.2 The Boundary Bumping Theorems

There are three special cases for which the boundary bumping theorems apply :

- (1) open sets (2) closed sets and (3) arbitrary sets.

Before we proceed, we first recall a definition.

Definition 2.2.1. Let X be a metric space and $U \subseteq X$. Then the boundary of U (in X), denoted $Bd_X(U)$ or, more simply, by $Bd(U)$, is defined by $Bd(U) = \overline{U} \cap \overline{(X \setminus U)}$.

Note: if U is open, then $Bd(U) = \overline{U} \cap (X \setminus U) = \overline{U} \setminus U$.

Theorem 2.2.2. (open sets) [8]; Theorem 20.1] Let X be a metric continuum and let U be a non-empty proper open subset of X . If K is a component of \overline{U} , then $K \cap Bd(U) \neq \emptyset$.

Thus since U is open, $K \cap (\overline{U} \setminus U) \neq \emptyset$.

Proof.

Suppose $K \cap (\overline{U} \setminus U) = \emptyset$.

Note : $\overline{U} \setminus U \neq \emptyset$. For if $\overline{U} \setminus U = \emptyset$, then $U = \overline{U}$ and hence U is clopen (since U is also

closed). So $X = U \cup (X \setminus U)$, where U and $X \setminus U$ are each non-empty open disjoint sets. This contradicts the connectedness of X , so $\overline{U} \setminus U$ must be non-empty.

By the Cut Wire Theorem, there exists compact subsets M_1, M_2 of \overline{U} , such that $\overline{U} = M_1 \cup M_2$, $K \subseteq M_1$, $\overline{U} \setminus U \subseteq M_2$ and $M_1 \cap M_2 = \emptyset$. Let $M_3 = M_2 \cup (X \setminus \overline{U})$. Then

$$\begin{aligned} X &= \overline{U} \cup (X \setminus \overline{U}) \\ &= (M_1 \cup M_2) \cup (X \setminus \overline{U}) \\ &= M_1 \cup M_3 \\ &= M_1 \cup \overline{M_3} \end{aligned}$$

We now show that $M_1 \cap \overline{M_3} = \emptyset$. Now

$$\begin{aligned} \overline{M_3} &= \overline{M_2 \cup (X \setminus \overline{U})} \\ &= \overline{M_2} \cup \overline{(X \setminus \overline{U})} \\ &= M_2 \cup \overline{(X \setminus \overline{U})} \end{aligned} \tag{2.1}$$

and note that

$$\overline{(X \setminus \overline{U})} \subseteq (X \setminus \overline{U}) \cup (\overline{U} \setminus U) \tag{2.2}$$

To see this take $p \in \overline{(X \setminus \overline{U})}$. If $p \in (X \setminus \overline{U})$ then done. So assume $p \in \overline{U}$ and we show that $p \in (\overline{U} \setminus U)$. Assume $p \in U$. Then $U \cap (X \setminus \overline{U}) \neq \emptyset$ which is a contradiction. So $p \notin U$. Hence $p \in (\overline{U} \setminus U)$.

By Equation (2.1) and Equation (2.2), we have that $\overline{M_3} \subseteq M_2 \cup (X \setminus \overline{U}) \cup (\overline{U} \setminus U)$.

Since $M_1 \cap M_2 = \emptyset$, $M_1 \cap (X \setminus \overline{U}) = \emptyset$ and $M_1 \cap (\overline{U} \setminus U) = \emptyset$, it follows that $M_1 \cap \overline{M_3} = \emptyset$.

This contradicts the connectedness of X . Hence $K \cap (\overline{U} \setminus U) \neq \emptyset$. \square

Theorem 2.2.3. (closed sets) [8]; Theorem 20.2] *Let X be a metric continuum and let A be a non-empty proper closed subset of X . If K is a component of A , then $K \cap \text{Bd}(A) \neq \emptyset$.*

Proof.

Since A is a proper non-empty subset of X and A is closed, there exists $p \notin A$ such that $d(p, A) > 0$. Let $\varepsilon_1 = d(p, A)$.

Note: $p \notin V_{\varepsilon_1}(A)$, for if $p \in V_{\varepsilon_1}(A)$ then there exists $a \in A$ such that

$$d(a, p) < \varepsilon_1 = d(p, A) \leq d(p, a)$$

which is a contradiction.

For each n , let $\varepsilon_n = (\frac{1}{n})\varepsilon_1$. So $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$, and $\varepsilon_n \rightarrow 0$. Then

$$V_{\varepsilon_{n+1}}(A) \subseteq V_{\varepsilon_n}(A) \subseteq V_{\varepsilon_1}(A)$$

and so $p \notin V_{\varepsilon_n}(A)$ for each n .

We have that $V_{\varepsilon_n}(A) \neq \emptyset$ and is a proper open subset of X , with $K \subseteq A \subseteq V_{\varepsilon_n}(A) \subseteq \overline{V_{\varepsilon_n}(A)}$.

So let K_n be the component of $\overline{V_{\varepsilon_n}(A)}$ containing K . Then for each n , $K \subseteq K_n \subseteq \overline{V_{\varepsilon_n}(A)}$,

and therefore

$$K_n \cap Bd(V_{\varepsilon_n}(A)) \neq \emptyset \tag{2.3}$$

Note : For each n , K_n is a sub-continuum of X and $K \subseteq K_{n+1} \subseteq \overline{V_{\varepsilon_{n+1}}(A)} \subseteq \overline{V_{\varepsilon_n}(A)}$.

Since K_n is the component of $\overline{V_{\varepsilon_n}(A)}$ containing K , we have $K_{n+1} \subseteq K_n$ for each n . So $(K_n)_n$

is a decreasing collection of subcontinua. By Proposition 1.3.9, $\bigcap_{n=1}^{\infty} K_n$ is a subcontinuum

of X . Now $K \subseteq K_n$ for each n , therefore $K \subseteq \bigcap_{n=1}^{\infty} K_n$. We have $A \subseteq V_{\varepsilon_n}(A)$ for each n ,

and $V_{\varepsilon_n}(A)$ is open, hence $A \cap Bd(V_{\varepsilon_n}(A)) = \emptyset$.

By Equation (2.3), it follows that $K_n \cap (X \setminus A) \neq \emptyset$. Now $\overline{(X \setminus A)}$ is compact and

$\{K_n \cap \overline{(X \setminus A)}\}_{n=1}^{\infty}$ is a family of closed subsets of X with the finite intersection property.

Since X is compact, $\bigcap_n (K_n \cap \overline{(X \setminus A)}) \neq \emptyset$, that is $(\bigcap_n K_n) \cap \overline{(X \setminus A)} \neq \emptyset$.

Furthermore for each n , $K_n \subseteq \overline{V_{\varepsilon_n}(A)}$, hence $(\bigcap_n K_n) \subseteq \bigcap_n \overline{V_{\varepsilon_n}(A)}$.

Claim: $\bigcap_n \overline{V_{\varepsilon_n}(A)} = A$.

Take $y \in \bigcap_n \overline{V_{\varepsilon_n}(A)}$. Suppose $y \notin A$. Then $d(y, A) > 0$ and so there exists n such that

$2\varepsilon_n < d(y, A)$. For this particular n , $y \in \overline{V_{\varepsilon_n}(A)}$ implies that $d(y, V_{\varepsilon_n}(A)) = 0$ and therefore

there exists $x \in V_{\varepsilon_n}(A)$ such that $d(y, x) < \varepsilon_n$.

Thus there exists $a \in A$ such that $d(x, a) < \varepsilon_n$, so we have $d(y, A) < 2\varepsilon_n < d(y, A)$ which

is a contradiction. So $\bigcap_n \overline{V_{\varepsilon_n}(A)} = A$.

Now A is closed, therefore $Bd(A) = A \cap \overline{X \setminus A}$.

Thus $(\bigcap_n K_n) \cap Bd(A) = (\bigcap_n K_n) \cap A \cap \overline{X \setminus A} = (\bigcap_n K_n) \cap \overline{X \setminus A} \neq \emptyset$,

and $K \subseteq \bigcap_n K_n \subseteq A$. That is, K , which is a component of A , is a subset of $\bigcap_n K_n$ which is connected. By maximality, we must have that $K = \bigcap_n K_n$. Hence $K \cap Bd(A) \neq \emptyset$. \square

Theorem 2.2.4. (arbitrary sets) [8]; Theorem 20.3] *Let X be a metric continuum and E be a non-empty proper subset of X . If K is a component of E , then $\overline{K} \cap Bd(E) \neq \emptyset$.*

Proof.

We have $K \neq \emptyset$, so take $x_0 \in K$. If $x_0 \in \overline{X \setminus E}$, then $x_0 \in \overline{E} \cap \overline{X \setminus E} = Bd(E)$. Therefore $K \cap Bd(E) \neq \emptyset$, and $\overline{K} \cap Bd(E) \neq \emptyset$ as required.

So assume $x_0 \notin \overline{X \setminus E}$. Then similar to the proof in Theorem 2.2.3 where we showed $A = \bigcap_n \overline{V_{\varepsilon_n}(A)}$, we can find open sets W_n such that $W_1 \supseteq W_2 \supseteq W_3 \supseteq \dots$, and $\overline{X \setminus E} \subseteq W_n$ for each n , $x_0 \notin W_n$ for each n and $\overline{X \setminus E} = \bigcap_n \overline{W_n}$.

Let $A_n = (X \setminus W_n)$, then $x_0 \in A_n$ for each n , and $A_n = (X \setminus W_n) \subseteq E$ for each n . For each n , let K_n be the component of A_n containing x_0 . So $x_0 \in K_n \subseteq A_n$. Now A_n is a non-empty proper closed subset of X , for each n . By Theorem 2.2.3, $K_n \cap Bd(A_n) \neq \emptyset$, for each n .

Now for each n , $x_0 \in K_n \subseteq A_n \subseteq E$, hence $K_n \subseteq K$ for each n , because K is a component of E containing x_0 . This implies that $K \cap Bd(A_n) \neq \emptyset$.

Now $Bd(A_n) = \overline{A_n} \cap \overline{X \setminus A_n} = (A_n \cap \overline{W_n}) \subseteq \overline{W_n}$, for each n , therefore $K \cap \overline{W_n} \neq \emptyset$. Let $L_n = \overline{K} \cap \overline{W_n}$. Thus L_n is non-empty, closed and $L_{n+1} = (\overline{K} \cap \overline{W_{n+1}}) \subseteq (\overline{K} \cap \overline{W_n}) \subseteq L_n$, for each n . So L_n has the finite intersection property and therefore $\bigcap_n L_n \neq \emptyset$, by compactness of X .

Thus $\overline{K} \cap \overline{X \setminus E} = \overline{K} \cap (\bigcap_n \overline{W_n}) = \bigcap_n (\overline{K} \cap \overline{W_n}) = \bigcap_n L_n \neq \emptyset$.

Now $\overline{K} \cap Bd(E) = \overline{K} \cap \overline{E} \cap \overline{X \setminus E} = \overline{K} \cap \overline{X \setminus E}$ (since $K \subseteq E$ implies that $\overline{K} \subseteq \overline{E}$).

Hence $\overline{K} \cap Bd(E) \neq \emptyset$. \square

2.3 Application : Continuum of Convergence Theorem

One of the important concepts for the study of local connectedness is the concept of connectedness im kleinen. We will define this below and then use the Boundary Bumping Theorem for open sets and Theorem 1.3.12 to prove a general theorem on the existence of continua arising out of convergence.

Definition 2.3.1. Let X be a metric space. X is said to be connected im kleinen at x , if given any open U , $x \in U$, there exists open V with $x \in V$ and connected C such that $x \in V \subseteq C \subseteq U$.

Remark 2.3.2. We will revisit the concept of connected im kleinen later, where we will study it in more detail along with local connectedness.

Theorem 2.3.3 ([14]). *Let X be a metric continuum. If X is not connected im kleinen at p , then there exists $\varepsilon > 0$ such that there is a sequence $(K_n)_{n=1}^{\infty}$ of distinct components of $\overline{V_\varepsilon(p)}$ satisfying:*

- (1) $(K_n)_n$ converges to a subcontinuum K of X such that $p \in K$.
- (2) $K \cap (\bigcup_{n=1}^{\infty} K_n) = \emptyset$.
- (3) $\text{diam}(K) \geq \varepsilon$.
- (4) X is not connected im kleinen at any point of K .

Proof.

(1) Recall that if X is connected im kleinen at x , then for U open and $x \in U$, there exists an open V , connected C such that $x \in V \subseteq C \subseteq U$. Hence for $x \in U$, and U open, it is true that the component C_x of x in U is a neighbourhood of x .

By the premise, X is not connected im kleinen at p , therefore there exists an open U containing p , such that for all V open, $p \in V \subseteq U$ and $V \not\subseteq C$, where C is the component of p in U . Since every metric space is regular, we can find $\varepsilon > 0$ such that $\overline{V_\varepsilon(p)} \subseteq V_{2\varepsilon}(p) \subseteq \overline{V_{2\varepsilon}(p)} \subseteq U$. Then for each $\delta > 0$, $V_\delta(p) \not\subseteq C$, where C is the component of p in $\overline{V_{2\varepsilon}(p)}$.

Let $\delta = \frac{\varepsilon}{2}$ and choose $x_1 \in V_{\frac{\varepsilon}{2}}(p)$ such that $x_1 \notin C$. Let C_1 be the component of $\overline{V_\varepsilon(p)}$ that contains the point x_1 . Now if $C \cap C_1 \neq \emptyset$, then $C_1 \subseteq C$ and $x_1 \in C$, which is a contradiction. Hence $C \cap C_1 = \emptyset$.

So C_1 is closed and $p \notin C_1$. So there exists $x_2 \in V_{\frac{\varepsilon}{2^2}}(p)$, such that $x_2 \notin C \cup C_1$. Let C_2 be the component of $\overline{V_\varepsilon(p)}$ containing x_2 . Now if $C_2 \cap C \neq \emptyset$ then $C_2 \subseteq C$ and hence $x_2 \in C$ which is a contradiction, so $C \cap C_2 = \emptyset$. Also $C_1 \neq C_2$, since $x_2 \in C_2$ but $x_2 \notin C_1$. Hence C_1, C_2 and C are pairwise disjoint they are components of $V_\varepsilon(p)$ and hence $C_1 \cap C_2 = \emptyset$. Thus $p \notin C_1 \cup C_2$ and $C_1 \cup C_2$ is closed, therefore there exists $x_3 \in V_{\frac{\varepsilon}{2^3}}(p)$, such that $x_3 \notin C \cup C_1 \cup C_2$. If we continue in this way, then there exists $x_n \in V_{\frac{\varepsilon}{2^n}}(p)$ for each n , such that $x_n \notin C \cup C_1 \cup C_2 \cup \dots \cup C_{n-1}$, and with $(C_n)_n$ pairwise disjoint and also $C \cap C_n = \emptyset$ for all n .

So we have $p \notin C_n$ for each n , where C_n 's are distinct components of $\overline{V_\varepsilon(p)}$.

Claim : $p \in \liminf C_n$.

Take $p \in W$ such that W is open. There exists m such that $V_{\frac{\varepsilon}{2^m}}(p) \subseteq W$ and therefore $W \cap C_n \neq \emptyset$ for each $n \geq m$. Thus the claim is true. Now 2^X is compact by Theorem 1.3.12. Hence $(C_n)_n$ has a convergent subsequence $(C_{n_i})_i$. Let $K_i = C_{n_i}$, for each i .

Now $p \in \liminf C_n \subseteq \liminf C_{n_i} = \liminf K_i$. Since $(K_i)_i$ is closed connected, it follows from Proposition 1.3.7 that $\limsup K_i$ is connected. We know that $K_i \rightarrow K$ in $(2^X, \rho)$ and $K = \limsup K_i$, so K is connected. Hence K is a subcontinuum of X .

(2) It suffices to show that $K \cap K_n = \emptyset$ for each n .

Now $K_i \subseteq \overline{V_\varepsilon(p)}$ for each i , and $K_i \rightarrow K$ in 2^X , therefore by Lemma 1.3.10 we have $K \subseteq \overline{V_\varepsilon(p)}$. Now $p \in K$ and $p \in C$, therefore $p \in K \subseteq C \subseteq \overline{V_\varepsilon(p)}$ (since K is connected). Hence $K \cap K_n = \emptyset$, for all n .

(3) We show $\text{diam}(K) \geq \varepsilon$. For each i , K_i is a component of $\overline{V_\varepsilon(p)}$. Hence by Theorem 2.2.2 it follows that $K_i \cap \text{Bd}(V_\varepsilon(p)) \neq \emptyset$, and therefore $K_i \cap (\overline{V_\varepsilon(p)} \setminus V_\varepsilon(p)) \neq \emptyset$. So there exists $y_i \in K_i$ such that $d(y_i, p) \geq \varepsilon$ for each i . By compactness of X , $(y_i)_i$ has a subsequence converging to y (say). We can assume that $y_i \rightarrow y$, for simplicity of notation.

Claim: If $y_i \rightarrow y$ for $y_i \in K_i$ and $K_i \rightarrow K$, then $y \in K$.

Now $y_i \rightarrow y$, implies that $\{y_i\} \rightarrow \{y\}$ in 2^X , and we have that $\{y_i\} \subseteq K_i$. So by Lemma 1.3.10, $\{y\} \subseteq K$. Hence $y \in K$. Now, $d(y, p) = d(\lim y_i, p) = \lim d(y_i, p) \geq \varepsilon$, for each i . Hence $\text{diam}(K) \geq \varepsilon$, as required.

(4) We will now show that X is not connected im kleinen at any $q \in K$. Take any $q \in K$. Observe that since

$$p \in K \subseteq \overline{V_\varepsilon(p)} \subseteq V_{2\varepsilon}(p) \subseteq \overline{V_{2\varepsilon}(p)},$$

K is connected and C is a component of $\overline{V_{2\varepsilon}(p)}$ containing p , we must have $p \in K \subseteq C$. If X is connected im kleinen at $q \in K$, then since $q \in V_{2\varepsilon}(p)$, there must be a connected neighbourhood G of q such that $q \in G \subseteq V_{2\varepsilon}(p)$. But then $G \cap C \neq \emptyset$ (since q belongs to this intersection) implies that $G \cup C$ is connected with $p \in G \cup C \subseteq \overline{V_{2\varepsilon}(p)}$.

Since C is the component of p , we have $G \cup C = C$, so $G \subseteq C$. Hence $q \in G \subseteq C$. Now $q \in K$ implies that $q \in \limsup K_i$, so G meets infinitely many K_i . Therefore for some j , $G \cap K_j \neq \emptyset$. But then this implies $C \cap K_j \neq \emptyset$. This is impossible since $C \cap C_n = \emptyset$ for all n .

□

Chapter 3

Whitney Maps

Whitney maps were first described in the early 1930's, by Hassler Whitney (in [12]), for the purpose of studying families of curves. In 1942, J.L. Kelley (see [3]) used Whitney maps extensively to study hyperspaces. In particular, Whitney maps are closely connected to arcs and segments in hyperspaces (see Chapter 4). It is for this reason, that we now dedicate this chapter to constructing Whitney maps for hyperspaces and proving their existence.

Definition 3.0.4. (Whitney map) Let X be a compact metric space and $\mathcal{H} \subseteq 2^X$. A Whitney map for \mathcal{H} is a continuous function $\mu : \mathcal{H} \rightarrow [0, \infty)$ that satisfies the following two conditions:

- (1) whenever $A, B \in \mathcal{H}$ such that $A \subsetneq B$ then $\mu(A) < \mu(B)$
- (2) $\mu(A) = 0$ iff $A = \{a\} \in \mathcal{H}$ for some $a \in X$.

Furthermore, if $\mu(X) = 1$, then we call μ a *normalised* Whitney map.

3.1 Construction and Existence of Whitney Maps

We will prove that Whitney maps exist for any hyperspace of a compact metric space. But first we deal with its construction and a few preliminary results. We now revisit the definition of *diameter*, and reformulate the concept by means of a *diameter map*, for the purpose of constructing a Whitney map.

Definition 3.1.1. Let X be a compact metric space with metric d , the diameter map with respect to metric d , is the function $\text{diam}_d : 2^X \rightarrow [0, \infty)$, that is defined as follows : For

each $A \in 2^X$,

$$\text{diam}_d(A) = \sup \{d(x, y) | x, y \in A\}$$

Lemma 3.1.2. *Let X be compact metric space with metric d . Then the diameter map $\text{diam}_d : 2^X \rightarrow [0, \infty)$ is uniformly continuous.*

Proof.

Let $\varepsilon > 0$ and $A, B \in 2^X$ such that $\rho(A, B) < \frac{\varepsilon}{3}$. Since A is compact, there exists $a_1, a_2 \in A$ such that $\text{diam}_d(A) = d(a_1, a_2)$. Since $a_1, a_2 \in A$ and $A \subseteq V_{\frac{\varepsilon}{3}}(B)$, there exists $b_1, b_2 \in B$ such that $d(a_1, b_1) < \frac{\varepsilon}{3}$ and $d(a_2, b_2) < \frac{\varepsilon}{3}$. Hence,

$$\begin{aligned} d(a_1, a_2) &\leq d(a_1, b_1) + d(b_1, b_2) + d(b_2, a_2) \\ &< d(b_1, b_2) + \frac{2\varepsilon}{3} \end{aligned}$$

Therefore since $\text{diam}_d(A) = d(a_1, a_2)$, we have that

$$\text{diam}_d(A) \leq d(b_1, b_2) + \frac{2\varepsilon}{3} \leq \text{diam}_d(B) + \frac{2\varepsilon}{3} \quad (3.1)$$

Also, a similar symmetric argument shows

$$\text{diam}_d(B) \leq \text{diam}_d(A) + \frac{2\varepsilon}{3} \quad (3.2)$$

By Equation (3.1) and Equation (3.2), $|\text{diam}_d(A) - \text{diam}_d(B)| \leq \frac{2\varepsilon}{3} < \varepsilon$. Hence the diameter map is uniformly continuous. \square

Lemma 3.1.3. *Let X and Y be compact metric spaces, and let $f : X \rightarrow Y$ be continuous.*

Let $F : 2^X \rightarrow 2^Y$ be defined as follows : for each $A \in 2^X$, $F(A) = f(A)$.

Then F is uniformly continuous.

Proof.

If $A \in 2^X$ then A is a closed subset of X which is compact. Now f is continuous, so $f(A)$ is compact in Y . Therefore $F(A) \in 2^Y$.

Let $\varepsilon > 0$. Now $f : X \rightarrow Y$ is continuous and X is compact, thus f is uniformly continuous. Hence there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \text{ implies that } d_Y(f(x), f(y)) < \frac{\varepsilon}{2} \quad (3.3)$$

Let ρ_X and ρ_Y be the Hausdorff metrics in 2^X and 2^Y , respectively. Now, let $\rho_X(A, B) < \delta$ in 2^X . Then $A \subseteq V_\delta(B)$ and $B \subseteq V_\delta(A)$. We will show that $f(A) \subseteq V_{\frac{\varepsilon}{2}}(f(B))$ and $f(B) \subseteq V_{\frac{\varepsilon}{2}}(f(A))$. By symmetry, we need only show that $f(A) \subseteq V_{\frac{\varepsilon}{2}}(f(B))$. Let $y \in f(A)$. Then $y = f(a)$, where $a \in A$, and so $a \in V_\delta(B)$. Hence there exists $b \in B$ such that $d_X(a, b) < \delta$ and so $d_Y(f(a), f(b)) < \frac{\varepsilon}{2}$, by Equation (3.3). So $f(a) \in V_{\frac{\varepsilon}{2}}(f(b)) \subseteq V_{\frac{\varepsilon}{2}}(f(B))$. Thus we have shown that $f(A) \subseteq V_{\frac{\varepsilon}{2}}(f(B))$ and $f(B) \subseteq V_{\frac{\varepsilon}{2}}(f(A))$, and hence

$$F(A) \subseteq V_{\frac{\varepsilon}{2}}(F(B)) \text{ and } F(B) \subseteq V_{\frac{\varepsilon}{2}}(F(A)).$$

This proves that $\rho_Y(F(A), F(B)) \leq \frac{\varepsilon}{2} < \varepsilon$. Therefore F is uniformly continuous. □

We now begin the construction of our Whitney map.

Let X be a compact metric space. Then, as is well known, X is separable. Let $T = \{a_1, a_2, \dots, a_i, \dots\}$ be a countable dense subset in X . Define $f_i : X \rightarrow \mathbb{R}$ by,

$$f_i(p) = \frac{1}{1 + d(a_i, p)}, \quad (3.4)$$

for each $p \in X$.

Note the following :

- (1) For each i , $f_i : X \rightarrow [0, 1]$ and f_i is continuous (because $p \mapsto d(a_i, p)$ is continuous).
- (2) For any $p, q \in X$ we have

$$\begin{aligned}
|f_i(p) - f_i(q)| &= \left| \frac{1}{1 + d(a_i, p)} - \frac{1}{1 + d(a_i, q)} \right| \\
&= \frac{|d(a_i, p) - d(a_i, q)|}{(1 + d(a_i, p))(1 + d(a_i, q))} \\
&\leq \frac{d(p, q)}{(1 + d(a_i, p))(1 + d(a_i, q))} \\
&\leq d(p, q)
\end{aligned} \tag{3.5}$$

Next, for each i , consider the mapping $\mu_i : 2^X \rightarrow [0, 1]$, where for $A \in 2^X$

$$\mu_i(A) = \text{diam}(f_i(A)) \tag{3.6}$$

Now define $\mu : 2^X \rightarrow [0, 1]$, as follows : for any $A \in 2^X$

$$\mu(A) = \sum_i \frac{\mu_i(A)}{2^i} \tag{3.7}$$

Theorem 3.1.4 ([2]). $\mu : 2^X \rightarrow [0, 1]$ is a Whitney map.

Proof.

To see that μ is continuous, we simply note that μ_i is continuous for each i , by Lemma 3.1.2 and Lemma 3.1.3, since $\mu_i = \text{diam}_\rho \circ F_i$, where ρ is the usual metric for $[0, 1]$, and where $F_i(A) = f_i(A)$.

We now show that μ satisfies (1) of Definition 3.0.4.

Let $A, B \in 2^X$ such that $A \subseteq B$ and $A \neq B$. Since $A \subsetneq B$, we have that $f_n(A) \subseteq f_n(B)$ for each n , hence $\mu_n(A) \leq \mu_n(B)$ for each n . So it remains for us to show that $\mu(A) < \mu(B)$, thus it suffices to prove that : for some i

$$\mu_i(A) < \mu_i(B) \tag{3.8}$$

Let $p \in (B \setminus A)$ and let $r = \frac{1}{2}d(p, A)$.

Note that $r > 0$ (Since $p \notin A$ and A is closed in X) and recall that T is dense in X . So

there exists $a_i \in T$ such that $d(p, a_i) < r$. Therefore

$$f_i(p) = \frac{1}{1 + d(a_i, p)} > \frac{1}{1 + r} \quad (3.9)$$

Claim: $d(a_i, A) > r$.

For any $a \in A$, $d(p, A) \leq d(p, a) \leq d(p, a_i) + d(a_i, a)$. Therefore $2r - d(p, a_i) \leq d(a_i, a)$ and thus we obtain that $r + (r - d(p, a_i)) \leq d(a_i, a)$. This implies that $r < d(a_i, A)$, as required.

So since $d(a_i, A) > r$, it follows that for each $a \in A$,

$$f_i(a) = \frac{1}{1 + d(a_i, a)} < \frac{1}{1 + r} \quad (3.10)$$

By combining Equation (3.9) and Equation (3.10), we see that

$$\sup f_i(A) \leq \frac{1}{1 + r} < f_i(p) \quad (3.11)$$

Thus by Equation (3.11) and since $p \in B$, we obtain

$$\sup f_i(A) < \sup f_i(B) \quad (3.12)$$

and since $A \subseteq B$, we know that

$$\inf f_i(B) \leq \inf f_i(A) \quad (3.13)$$

Therefore by Equation (3.12) and Equation (3.13) it follows that $\mu_i(A) < \mu_i(B)$, as required.

Finally, we show that μ satisfies (2) of Definition 3.0.4.

(\implies) Let $A \in 2^X$. Assume $A = \{x\}$, for some $x \in X$, $\mu_n(A) = \text{diam}(f_n(\{x\})) = 0$ for each $n = 1, 2, 3, \dots$. Therefore $\mu(A) = 0$.

(\impliedby) Assume that A is not a singleton set in 2^X .

Claim: $f_k(A)$ is non-degenerate, for some k .

Since A is not a singleton, then there exists $x, y \in A$, $x \neq y$. Let $w = d(x, y) > 0$. Since T is

dense in X there exists a_k such that $a_k \in V_{\frac{w}{2}}(a_k)$. Now $y \notin V_{\frac{w}{2}}(a_k)$, for if $y \in V_{\frac{w}{2}}(a_k)$ then $d(x, y) \leq d(x, a_k) + d(a_k, y) < \frac{w}{2} + \frac{w}{2} = w$. This implies that $w < w$, which is a contradiction. Thus $d(a_k, y) \geq \frac{w}{2}$ and $d(a_k, x) < \frac{w}{2}$. Therefore $d(a_k, y) \neq d(a_k, x)$ and thus we have that

$$f_k(x) = \frac{1}{1 + d(a_k, x)} \neq \frac{1}{1 + d(a_k, y)} = f_k(y).$$

Thus $f_k(A)$ is non-degenerate. Hence $\mu_k(A) > 0$, and so $\mu(A) > 0$.

We have shown that μ satisfies both (1) and (2) for the definition of a Whitney map. Hence μ is a Whitney map for 2^X . \square

Corollary 3.1.5 ([2]). *If X is a compact metric space, then there exists a Whitney map for any hyperspace of X .*

Proof.

μ is a Whitney map for 2^X by Theorem 3.1.4. By the definition of a Whitney map, it is evident that for any $\mathcal{H} \subseteq 2^X$ containing all singletons, the restricted map $\mu|_{\mathcal{H}}$ is a Whitney map for \mathcal{H} . \square

Remark 3.1.6.

$\mathcal{C}(X) \subseteq 2^X$ and $\mathcal{C}(X)$ contains all singletons, therefore by Corollary 3.1.5, $\mu|_{\mathcal{C}(X)} : \mathcal{C}(X) \rightarrow [0, 1]$ is a Whitney map for $\mathcal{C}(X)$. This restriction map (along with μ itself) will be used later to lend elegance and clarity to proofs. From now on, we will denote $\mu|_{\mathcal{C}(X)}$ by ω .

3.2 Some Properties of Whitney maps

Proposition 3.2.1. *Let X be a compact metric space and $\mu : 2^X \rightarrow [0, 1]$ be a Whitney map. If $A \subseteq V_\varepsilon(A')$, then $\mu(A) \leq \mu(A') + 2\varepsilon$ for $A, A' \in 2^X$.*

Proof.

Take $p, q \in A$. Then for some $p' \in V_\varepsilon(A)$, we have $d(p, p') < \varepsilon$ for some $p' \in A'$, and for some

$q \in A'$, we have $d(q, q') < \varepsilon$. Now for arbitrary i , we have

$$\begin{aligned}
|f_i(p) - f_i(q)| &= |f_i(p) - f_i(p') + f_i(p') - f_i(q') + f_i(q') - f_i(q)| \\
&\leq |f_i(p) - f_i(p')| + |f_i(p') - f_i(q')| + |f_i(q') - f_i(q)| \\
&\leq |f_i(p) - f_i(p')| + \mu_i(A') + |f_i(q') - f_i(q)| \\
&\leq d(p, p') + \mu_i(A') + d(q, q') \quad \text{from (3.5)} \\
&< \mu_i(A') + 2\varepsilon
\end{aligned}$$

Hence $\mu_i(A) = \text{diam}(f_i(A)) \leq \mu_i(A') + 2\varepsilon$. Thus we now have

$$\begin{aligned}
\mu(A) &= \sum_i \frac{\mu_i(A)}{2^i} \leq \sum_i \frac{\mu_i(A') + 2\varepsilon}{2^i} \\
&= \sum_i \frac{\mu_i(A')}{2^i} + 2\varepsilon \sum_i \frac{1}{2^i} \\
&= \mu(A') + 2\varepsilon
\end{aligned}$$

Therefore $\mu(A) \leq \mu(A') + 2\varepsilon$. □

Lemma 3.2.2 ([3]; Lemma 1.5.). *For every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that if $A \subseteq B$, $A, B \in 2^X$ and $\mu(B) - \mu(A) < \eta(\varepsilon)$, then $\rho(A, B) < \varepsilon$.*

Proof.

Suppose to the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists $A_n \subseteq B_n$, $A_n, B_n \in 2^X$ and $\mu(B_n) - \mu(A_n) < \frac{1}{n}$ but $\rho(A_n, B_n) \geq \varepsilon$.

Now $(A_n)_n$ and $(B_n)_n$ are in 2^X and 2^X is compact by Theorem 1.3.12, therefore $(B_n)_n$ has a convergent subsequence, say $(B_{n_k})_k$ where $B_{n_k} \rightarrow B$ in 2^X . Similarly $(A_n)_n$ has a convergent subsequence $(A_{n_k})_k$. Now $(A_{n_k})_{k=1}^\infty$ is a sequence in 2^X which must itself have a convergent subsequence $(A_{n_{kl}})_{l=1}^\infty$. Thus there exists $A \in 2^X$ such that $A_{n_{kl}} \rightarrow A$, as $l \rightarrow \infty$. Now $(B_{n_{kl}})_{l=1}^\infty$ is a subsequence of $(B_{n_k})_k$, and since $(B_{n_k})_k$ converges, this subsequence $(B_{n_{kl}})_{l=1}^\infty$ must converge to the same limit, that is $B_{n_{kl}} \rightarrow B$. Thus $(A_n)_n$ and $(B_n)_n$ have subsequences with the same indexing such that whenever $l \rightarrow \infty$ then $A_{n_{kl}} \rightarrow A$ and $B_{n_{kl}} \rightarrow B$. Since the indexing is the same, we may assume for the sake of

notational simplicity that $A_n \rightarrow A$ and $B_n \rightarrow B$. Now we have that,

$$\mu(B_n) < \mu(A_n) + \frac{1}{n}.$$

By continuity of μ , we have that $\mu(B_n) \rightarrow \mu(B)$ and $\mu(A_n) \rightarrow \mu(A)$. Hence

$$\lim \mu(B_n) \leq \lim(\mu(A_n) + \frac{1}{n}) = \lim \mu(A_n) + \lim \frac{1}{n} \leq \lim \mu(A_n).$$

So $\mu(B) \leq \mu(A)$. For each n , $A_n \subseteq B_n$, therefore $A \subseteq B$ by Lemma 1.3.10. Hence we have shown, $\mu(B) \leq \mu(A) \leq \mu(B)$, and this implies that $A = B$, from the definition of μ . Thus $\rho(A, B) = \rho(\lim A_n, \lim B_n) = \lim \rho(A_n, B_n) \geq \varepsilon$, which is a contradiction. \square

Chapter 4

Structure of Hyperspaces

4.1 Arcs in Hyperspaces

The arc structure of hyperspaces is the most important ingredient in the general theory of hyperspaces. We now study the arc structure of 2^X and $\mathcal{C}(X)$, with a systematic treatment. The notion of an order arc will be introduced in Definition 4.1.2, which will be used to prove that 2^X and $\mathcal{C}(X)$ are arcwise connected, when X is a continuum.

Definition 4.1.1. Let X be a metric space. An arc, α in X , is a homeomorphism, $\alpha : [a, b] \longrightarrow \alpha[a, b] \subseteq X$, where $a, b \in \mathbb{R}$ with $a < b$ (That is, $[a, b]$ is non-degenerate).

Convention for the rest of this chapter:

1. In what follows, we shall assume that X is a metric continuum.
2. When we refer to a collection \mathcal{A} as a subcontinuum of 2^X , we mean that \mathcal{A} is a non-empty compact connected subset of the space 2^X .
3. By $\alpha \in 2^X$ is an arc, we mean that the image $\alpha[a, b]$ is a subset of 2^X for some non-degenerate interval $[a, b]$, and we shall denote $A \in \alpha[a, b]$ by simply $A \in \alpha$.

Definition 4.1.2. An order arc in 2^X is an arc α in 2^X such that, whenever $A, B \in \alpha$ then $A \subseteq B$ or $B \subseteq A$.

Lemma 4.1.3 ([8]; Lemma 1.3.). *Let $\mathcal{A} \subseteq 2^X$ such that if $A, B \in \mathcal{A}$ then either $A \subseteq B$ or $B \subseteq A$. Then μ is one-to-one on \mathcal{A} , where $\mu : 2^X \rightarrow [0, 1]$, is a Whitney map. Furthermore, if \mathcal{A} is compact, then $\mu|_{\mathcal{A}}$ is a homeomorphism onto $\mu(\mathcal{A})$.*

Proof.

$\mu|_{\mathcal{A}} : \mathcal{A} \rightarrow [0, 1]$ is a continuous, since it is a restriction of a continuous map.

Suppose $\mu(A) = \mu(B)$, for $A, B \in \mathcal{A}$. Assume $A \subseteq B$, then $A = B$, because if $A \subseteq B$ with $A \neq B$ then $\mu(A) < \mu(B)$, which is a contradiction. So μ is one-to-one.

Thus since a one-to-one, continuous function from a compact space onto a Hausdorff topological space is a homeomorphism (by Lemma 1.1.27; (b)), it must follow that $\mu|_{\mathcal{A}}$ is a homeomorphism, if \mathcal{A} is compact. \square

Theorem 4.1.4. *Let \mathcal{A} be a non-degenerate subcontinuum of 2^X . Then \mathcal{A} is an order arc iff whenever $A, B \in \mathcal{A}$, then either $A \subseteq B$ or $B \subseteq A$.*

Proof.

(\implies) Suppose \mathcal{A} is an order arc in 2^X , then there exists a homeomorphism $\alpha : [a, b] \rightarrow \mathcal{A}$, where $\mathcal{A} = \alpha[a, b]$. Hence the forward direction is clear.

(\impliedby) Suppose $A, B \in \mathcal{A}$ implies that $A \subseteq B$ or $B \subseteq A$. Then by Lemma 4.1.3, $\mu|_{\mathcal{A}} : \mathcal{A} \rightarrow \mu(\mathcal{A})$ is a homeomorphism. So since \mathcal{A} is compact, connected and non-degenerate, we have that $\mu(\mathcal{A})$ must be a closed bounded interval $[a, b]$ where $a < b$.

Hence we have that

$$\mu : \mathcal{A} \rightarrow \mu(\mathcal{A}) = [a, b] \text{ and, } [a, b] \xrightarrow{\mu^{-1}} \mathcal{A} \text{ is a homeomorphism.}$$

Now we assumed that whenever $A, B \in \mathcal{A}$ then either $A \subseteq B$ or $B \subseteq A$, so $\mu^{-1} = \mathcal{A}$ is an order arc. \square

Let $\alpha : [a, b] \rightarrow 2^X$ be an order arc in 2^X . We have the following useful notation for the intersection (respectively, union) of α :

$$\bigcap \alpha = \bigcap_{t \in [a, b]} \alpha(t)$$

$$\bigcup \alpha = \bigcup_{t \in [a, b]} \alpha(t)$$

Lemma 4.1.5 ([8]; Lemma 1.5.). *If α is an order arc in 2^X , then $\bigcap \alpha \in \alpha$ and $\bigcup \alpha \in \alpha$.*

Proof.

Define $t_0 = \inf\{\mu(A) \mid A \in \alpha\}$. Since α is compact in 2^X , we have that $\mu(\alpha)$ compact in \mathbb{R} and hence closed. So there exists $A_0 \in \alpha$ such that $t_0 = \mu(A_0)$.

Claim : $A_0 \subseteq A$ for all $A \in \alpha$.

Suppose there exists $A \in \alpha$ such that $A_0 \not\subseteq A$, then since α is an order arc, $A \subseteq A_0$ and $A \neq A_0$. Now $A \subsetneq A_0$ implies that $\mu(A) < \mu(A_0) = t_0$, and this is impossible.

Thus $A_0 \subseteq A$, for all $A \in \alpha$.

Therefore $A_0 \subseteq \bigcap \alpha \subseteq A_0$, so $\bigcap \alpha = A_0$.

It remains to show that $\bigcup \alpha \in \alpha$. Let $t_1 = \sup\{\mu(A) \mid A \in \alpha\}$. By a similar argument as earlier, there exists $A_1 \in \alpha$ such that $\mu(A_1) = t_1$.

Claim : $A \subseteq A_1$ for all $A \in \alpha$.

If not, then there exists $A \in \alpha$ such that $A \not\subseteq A_1$. But then $A_1 \subseteq A$ and $A_1 \neq A$.

So $A_1 \subsetneq A$ implies that $\mu(A_1) < \mu(A)$, and this is not possible. Thus $A \subseteq A_1$, for all $A \in \alpha$.

Therefore $\bigcup \alpha \subseteq A_1 \subseteq \bigcup \alpha$, and so we have $\bigcup \alpha \in \alpha$. \square

Theorem 4.1.6 ([8]; Theorem 1.6). *If α is an order arc in 2^X , then the two end points of α are $\bigcap \alpha$ and $\bigcup \alpha$.*

Proof.

Let $\alpha : [a, b] \rightarrow \alpha[a, b] \subseteq 2^X$ be an order arc. We show that $\{\alpha(a), \alpha(b)\} = \{\bigcup \alpha, \bigcap \alpha\}$.

Now, we have

$$[a, b] \xrightarrow{\alpha} \alpha[a, b] \xrightarrow{\mu} \mu(\alpha[a, b])$$

and $\mu(\alpha[a, b])$ is compact and connected in \mathbb{R} , so it is a closed bounded interval. By Lemma 4.1.5, we know that $\bigcap \alpha \in \alpha$ and $\bigcup \alpha \in \alpha$. Take any $A \in \alpha$, then $\bigcap \alpha \subseteq A \subseteq \bigcup \alpha$ and $\mu(\bigcap \alpha) \leq \mu(A) \leq \mu(\bigcup \alpha)$. Therefore $\mu(\alpha[a, b]) = [\mu(\bigcap \alpha), \mu(\bigcup \alpha)]$. So

$$[a, b] \xrightarrow[\text{homeomorphism}]{\alpha} \alpha[a, b] \xrightarrow[\text{homeomorphism}]{\mu} [\mu(\bigcap \alpha), \mu(\bigcup \alpha)]$$

We note that μ is a homeomorphism, since $\alpha[a, b]$ is compact.

Claim: $\mu\alpha(a) = \mu(\bigcap \alpha)$ or $\mu(\bigcup \alpha)$

Suppose $\mu\alpha(a) = t$ where $\mu(\bigcap \alpha) < t < \mu(\bigcup \alpha)$. Then the restriction

$\mu\alpha|_{(a,b)} : (a, b) \longrightarrow [\mu(\bigcap \alpha), t) \cup (t, \mu(\bigcup \alpha)]$, is a continuous map from the connected set (a, b) onto the disconnected set $[\mu(\bigcap \alpha), t) \cup (t, \mu(\bigcup \alpha)]$, which is not possible.

So $\mu\alpha(a) = \mu(\bigcap \alpha)$ or $\mu\alpha(a) = \mu(\bigcup \alpha)$.

Hence $\alpha(a) = \bigcap \alpha$ or $\alpha(a) = \bigcup \alpha$. Similarly, $\alpha(b)$ also has the same property. \square

In order to state Theorem 4.1.13 precisely, we will need to reformulate our description of an order arc. In view of Theorem 4.1.6, we now introduce the following terminology.

Definition 4.1.7. If α is an order arc in 2^X , then α is said to be an order arc from $\bigcap \alpha$ to $\bigcup \alpha$.

Remark 4.1.8.

By the above definition, for any order arc $\alpha : [a, b] \longrightarrow 2^X$, we may assume $\alpha(a) = \bigcap \alpha$ and $\alpha(b) = \bigcup \alpha$.

For suppose not, say, $\alpha(a) = \bigcup \alpha$ and $\alpha(b) = \bigcap \alpha$, then we may define a homeomorphism $\sigma : [a, b] \longrightarrow [a, b]$ by $\sigma(x) = -x + (a + b)$. Then the composite map $\alpha\sigma$

$$\begin{array}{ccc} & \alpha\sigma & \\ & \curvearrowright & \\ [a, b] & \longrightarrow & [a, b] \longrightarrow 2^X \end{array}$$

satisfies $\alpha\sigma(a) = \alpha(b) = \bigcap \alpha$ and $\alpha\sigma(b) = \alpha(a) = \bigcup \alpha$.

Theorem 4.1.13 ([8]), is one of the main results in this section. It provides an equivalent condition for the existence of an order arc in a hyperspace. We first recall important definitions and state Zorn's Lemma.

Definition 4.1.9. A partially ordered set is a set E equipped with an order relation \leq , satisfying :

1. $x \leq x$ for all $x \in E$. (Reflexive)
2. $x \leq y$ and $y \leq x$ implies that $x = y$ for $x, y \in E$. (Anti-symmetric)
3. $x \leq y$ and $y \leq z$ implies that $x \leq z$ for $x, y, z \in E$. (Transitive)

Definition 4.1.10. If E is a partially ordered set, an element $y \in E$ is called maximal, provided whenever $y \leq z$ and $z \in E$ then $y = z$.

Definition 4.1.11. A partially ordered set E is called a chain if for any $x, y \in E$, either $x \leq y$ or $y \leq x$.

Lemma 4.1.12. (*[14]; Zorn's Lemma*)

If each chain in a non-empty partially ordered set E has an upper bound, then E has a maximal element.

Theorem 4.1.13.

Let $A_0, A_1 \in 2^X$ such that $A_0 \neq A_1$. Then the following conditions are equivalent ;

- (1) there exists an order arc from A_0 to A_1 .*
- (2) $A_0 \subseteq A_1$ and each component of A_1 intersects A_0 .*

Proof.

(1) \implies (2) : Let α be an order arc from A_0 to A_1 , then $A_0 = \bigcap \alpha$ and $A_1 = \bigcup \alpha$. So $A_0 \subseteq A_1$. Suppose there exists a component L of A_1 such that $L \cap A_0 = \emptyset$.

Since L is a component of A_1 , L is closed in A_1 (and hence in X). Therefore L is compact and A_0 is compact.

Claim : No connected subset of A_1 intersects both A_0 and L .

Suppose there exists a connected C such that $C \cap L \neq \emptyset$ and $C \cap A_0 \neq \emptyset$, then $C \cup L$ is a connected subset of A_1 . By maximality, $C \cup L = L$, thus $C \subseteq L$. This is a contradiction, so the claim is true.

By the Cut Wire Theorem (Theorem 2.1.6), there exists non-empty disjoint compact sets B_0 and B_1 , such that $A_1 = B_0 \cup B_1$, $A_0 \subseteq B_0$ and $L \subseteq B_1$. Define

$$\beta_0 = \{A \in \alpha \mid A \subseteq B_0\}$$

$$\beta_1 = \{A \in \alpha \mid A \cap B_1 \neq \emptyset\}.$$

Then $\beta_0 \neq \emptyset$ since $A_0 \in \beta_0$ and $\beta_1 \neq \emptyset$ since $A_1 \in \beta_1$.

Claim: β_0 is closed.

Take $A \in \overline{\beta_0}$, then there exists $(A_n)_n$ in β_0 such that $A_n \longrightarrow A$. We know that $A_n \subseteq B_0$ for

each n , therefore by Lemma 1.3.10, $A \subseteq B_0$. Thus $A \in \beta_0$, and hence β_0 is closed.

Claim: β_1 is closed.

Take $A \in \overline{\beta_1}$, then there exists $(A_n)_n \in \beta_1$ such that $A_n \rightarrow A$, $A \in \alpha$. Now $A_n \cap B_1 \neq \emptyset$ for each n , therefore by Lemma 1.3.10 it follows that $A \cap B_1 \neq \emptyset$. Hence β_1 is closed as required.

Now $\beta_0 \cap \beta_1 = \emptyset$, and $\alpha[a, b] = \beta_0 \cup \beta_1$. Thus $\{\beta_0, \beta_1\}$ forms a disconnection of $\alpha[a, b]$. This is impossible, since $\alpha[a, b]$ is connected.

(2) \implies (1) : Assume $A_0 \subsetneq A_1$, and each component of A_1 intersects A_0 . Let

$$\mathcal{F} = \{ \mathcal{A} \subseteq 2^X \mid \mathcal{A} \text{ satisfies (1), (2) and (3) } \}$$

where (1) : If $B \in \mathcal{A}$ then $A_0 \subseteq B \subseteq A_1$

(2) : If $B \in \mathcal{A}$ then every component of B intersects A_0

(3) : If $B_0, B_1 \in \mathcal{A}$ then $B_0 \subseteq B_1$ or $B_1 \subseteq B_0$

Note: $\mathcal{F} \neq \emptyset$ because $\mathcal{A}_1 = \{A_0, A_1\} \in \mathcal{F}$. Now \mathcal{F} is a partially ordered set (i.e. \mathcal{F} is reflexive, anti-symmetric and transitive) with respect to \subseteq . By a chain in \mathcal{F} , we mean a collection $\{\mathcal{A}_\alpha \mid \alpha \in I\}$ such that whenever $\alpha, \beta \in I$, either $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ or $\mathcal{A}_\beta \subseteq \mathcal{A}_\alpha$.

Let $\{\mathcal{A}_\beta\}_{\beta \in I}$ be an arbitrary chain in \mathcal{F} . We show that $\bigcup_{\beta \in I} \mathcal{A}_\beta \in \mathcal{F}$. Take $B \in \bigcup_{\beta \in I} \mathcal{A}_\beta$, then $B \in \mathcal{A}_\beta$ for some β . Thus $\bigcup_{\beta \in I} \mathcal{A}_\beta$ satisfies conditions (1) and (2), since \mathcal{A}_β does. Now suppose $B_0, B_1 \in \bigcup_{\beta \in I} \mathcal{A}_\beta$. Then there exists $\beta_0 \in I$ and $\beta_1 \in I$ such that $B_0 \in \mathcal{A}_{\beta_0}$ and $B_1 \in \mathcal{A}_{\beta_1}$. By the chain condition, either $\mathcal{A}_{\beta_0} \subseteq \mathcal{A}_{\beta_1}$ or $\mathcal{A}_{\beta_1} \subseteq \mathcal{A}_{\beta_0}$, say $\mathcal{A}_{\beta_0} \subseteq \mathcal{A}_{\beta_1}$. Thus $B_0, B_1 \in \mathcal{A}_{\beta_1}$ and hence condition (3) is satisfied. Therefore $\bigcup_{\beta \in I} \mathcal{A}_\beta \in \mathcal{F}$. Thus by Zorn's Lemma, there exists a maximal element of \mathcal{F} . Call it \mathcal{A}_0 .

We will show that \mathcal{A}_0 is an order arc from A_0 and A_1 . Now since $\mathcal{A}_0 \in \mathcal{F}$, \mathcal{A}_0 satisfies (1), (2) and (3).

Claim : $\overline{\mathcal{A}_0} \in \mathcal{F}$. It suffices to show that $\overline{\mathcal{A}_0}$ satisfies (1), (2), and (3).

(1) : Take $B \in \overline{\mathcal{A}_0}$, then there exists $(B_n)_n$ in \mathcal{A}_0 such that $B_n \rightarrow B$ in 2^X . Now $B_n \in \mathcal{A}_0$ for each n , therefore $A_0 \subseteq B_n \subseteq A_1$ and hence $A_0 \subseteq B \subseteq A_1$ (by Lemma 1.3.10). Hence $\overline{\mathcal{A}_0}$

satisfies (1).

(2) : Suppose $B \in \overline{\mathcal{A}_0}$. Then there exists $(B_n)_n$ in \mathcal{A}_0 such that $B_n \rightarrow B$. We have shown that $A_0 \subseteq B \subseteq A_1$. Assume there was a component C of B such that $C \cap A_0 = \emptyset$. Now B is a compact metric space and no connected subset of B can meet both A_0 and C , so by the Cut-Wire Theorem (Theorem 2.1.6) there exists closed subsets E_1 and E_2 of B (and hence also of X) such that $B = E_1 \cup E_2$, $A_0 \subseteq E_1$, $C \subseteq E_2$, and $E_1 \cap E_2 = \emptyset$.

Now E_1 and E_2 are compact disjoint sets, therefore by Remark 1.3.6, $d(E_1, E_2) > 0$. Let $\varepsilon = d(E_1, E_2)$.

Since $B_n \rightarrow B$, there exists $N \in \mathbb{N}$ such that $\rho(B_N, B) < \frac{\varepsilon}{3}$. Hence $B_N \subseteq V_{\frac{\varepsilon}{3}}(B)$ and $B \subseteq V_{\frac{\varepsilon}{3}}(B_N)$. Now $B = E_1 \cup E_2$, therefore $V_{\frac{\varepsilon}{3}}(B) = V_{\frac{\varepsilon}{3}}(E_1) \cup V_{\frac{\varepsilon}{3}}(E_2)$.

Claim: $V_{\frac{\varepsilon}{3}}(E_1) \cap V_{\frac{\varepsilon}{3}}(E_2) = \emptyset$.

If not, then there exists $x \in X, y \in E_1$ and $z \in E_2$ such that $d(x, y) < \frac{\varepsilon}{3}$ and $d(x, z) < \frac{\varepsilon}{3}$.

Now $d(y, z) \leq d(x, y) + d(x, z) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$.

But then $\varepsilon = d(E_1, E_2) \leq \frac{2\varepsilon}{3}$, a contradiction. Hence $V_{\frac{\varepsilon}{3}}(E_1) \cap V_{\frac{\varepsilon}{3}}(E_2) = \emptyset$.

Now $C \subseteq B \subseteq V_{\frac{\varepsilon}{3}}(B_N)$, so $C \cap V_{\frac{\varepsilon}{3}}(W) \neq \emptyset$, for some component W of B_N . Thus we can find $x \in C, w \in W$ such that $d(x, w) < \frac{\varepsilon}{3}$. Now $W \subseteq B_N \subseteq V_{\frac{\varepsilon}{3}}(B) = V_{\frac{\varepsilon}{3}}(E_1) \cup V_{\frac{\varepsilon}{3}}(E_2)$.

Also $W \cap A_0 \neq \emptyset$, so $W \cap V_{\frac{\varepsilon}{3}}(E_1) \neq \emptyset$ (Since $A_0 \subseteq E_1$). Since W is connected, we must have $W \subseteq V_{\frac{\varepsilon}{3}}(E_1)$, for if $W \cap V_{\frac{\varepsilon}{3}}(E_2) \neq \emptyset$ then $W = (W \cap V_{\frac{\varepsilon}{3}}(E_1)) \cup (W \cap V_{\frac{\varepsilon}{3}}(E_2))$, which would form a disconnection of W . This is impossible, since W is connected.

Now $w \in V_{\frac{\varepsilon}{3}}(E_1)$ implies that there exists $z \in E_1$ such that $d(w, z) < \frac{\varepsilon}{3}$. Now $x \in C \subseteq E_2, z \in E_1$ and

$$d(x, z) \leq d(x, w) + d(w, z) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$$

This implies $\varepsilon = d(E_1, E_2) \leq d(x, z) < \frac{2\varepsilon}{3}$, which is a contradiction. Thus $C \cap A_0 \neq \emptyset$. So $\overline{\mathcal{A}_0}$ satisfies (2).

(3) : Take $B_0, B_1 \in \overline{\mathcal{A}_0}$. Then there exists $(C_n)_n$ and $(D_n)_n$ in \mathcal{A}_0 such that $C_n \rightarrow B_0$ and $D_n \rightarrow B_1$. Since $C_n, D_n \in \mathcal{A}_0$ and satisfy (3) for each n , then either $C_n \subseteq D_n$ or $D_n \subseteq C_n$. If there are infinitely many n such that $C_n \subseteq D_n$, then there is an increasing sequence $n_1 < n_2 < n_3 < \dots$ of natural numbers such that $C_{n_k} \subseteq D_{n_k}$ for all $k = 1, 2, 3, \dots$. Thus since $C_{n_k} \rightarrow B_0$ and $D_{n_k} \rightarrow B_1$, we have that $B_0 \subseteq B_1$ by Lemma 1.3.10. On the

other hand, if $C_n \subseteq D_n$ for at most finitely many n , then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $D_n \subseteq C_n$. In this case we have $B_1 \subseteq B_0$. Hence $\overline{\mathcal{A}_0}$ satisfies (3).

By maximality, we then have $\mathcal{A}_0 = \overline{\mathcal{A}_0}$. Hence \mathcal{A}_0 is closed. By Theorem 1.3.12, 2^X is compact, so it follows that \mathcal{A}_0 is compact.

Let $\mu_0 = \mu|_{\mathcal{A}_0}$, that is $\mu_0 : \mathcal{A}_0 \rightarrow [0, 1]$. Since \mathcal{A}_0 is compact, then by Lemma 4.1.3, μ_0 is a homeomorphism, onto $\mu_0(\mathcal{A}_0)$.

Note : Since \mathcal{A}_0 is a maximal element of \mathcal{F} , we must have $A_0 \in \mathcal{A}_0$ and $A_1 \in \mathcal{A}_0$.

By (1), $A \in \mathcal{A}_0$ implies $A_0 \subseteq A \subseteq A_1$, which implies that $\mu_0(\mathcal{A}_0) \subseteq [\mu_0(A_0), \mu_0(A_1)]$, where $\mu_0(A_0) < \mu_0(A_1)$ because $A_0 \subsetneq A_1$. Thus in order to show that \mathcal{A}_0 is an order arc from A_0 to A_1 , it suffices to show that

$$[\mu_0(A_0), \mu_0(A_1)] \subseteq \mu_0(\mathcal{A}_0) \quad (4.1)$$

Suppose Equation (4.1) is false. Then there exists $t \in [\mu_0(A_0), \mu_0(A_1)]$ such that $t \notin \mu_0(\mathcal{A}_0)$. Now t cannot be the endpoints $\mu_0(A_0)$ and $\mu_0(A_1)$, since the endpoints are in $\mu_0(\mathcal{A}_0)$. Since $\mu_0(\mathcal{A}_0)$ is closed, $t \in I \setminus \mu_0(\mathcal{A}_0)$ which is open. Thus there exists $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \subseteq I \setminus \mu_0(\mathcal{A}_0)$. Now let

$$t_2 = \sup\{x \mid (t, x) \subseteq I \setminus \mu_0(\mathcal{A}_0)\} \text{ and } t_1 = \inf\{x \mid (x, t) \subseteq I \setminus \mu_0(\mathcal{A}_0)\}.$$

Then $t_1, t_2 \in \mu_0(\mathcal{A}_0)$. Furthermore, $s \in (t_1, t_2)$ implies that $s \notin \mu_0(\mathcal{A}_0)$, as can be easily verified. Thus we have shown that there exists a non-degenerate closed sub-interval $[t_1, t_2]$ of $[\mu_0(A_0), \mu_0(A_1)]$ such that

(a) $t_1, t_2 \in \mu_0(\mathcal{A}_0)$

(b) $s \notin \mu_0(\mathcal{A}_0)$ whenever $t_1 < s < t_2$

By (a), Let $T_1, T_2 \in \mathcal{A}_0$ such that $\mu_0(T_1) = t_1$ and $\mu_0(T_2) = t_2$.

Claim : If $G \in \mathcal{A}_0$ then $G \subseteq T_1$ or $G \supseteq T_2$.

Take $G \in \mathcal{A}_0$, then $\mu_0(G) \in \mu_0(\mathcal{A}_0)$. Hence by (b), either $\mu_0(G) \leq \mu_0(T_1)$ or $\mu_0(G) \geq \mu_0(T_2)$.

Now μ_0 is a homeomorphism, so it follows that $G \subseteq T_1$ or $G \supseteq T_2$, as required.

Also since $t_1 < t_2$, we have $T_1 \subseteq T_2$ and $T_1 \neq T_2$. So there exists $\varepsilon > 0$ such that $T_2 \not\subseteq \overline{V_\varepsilon(T_1)}$.

Let $p \in (T_2 \setminus \overline{V_\varepsilon(T_1)})$ and let K denote the component of T_2 such that $p \in K$. So by (2), $K \cap A_0 \neq \emptyset$. Hence $A_0 \subseteq T_1$ (by (1)) and therefore $K \cap T_1 \neq \emptyset$.

Now take $q \in K \cap T_1$ and let $U = V_\varepsilon(T_1) \cap K$. Then $q \in U$, so $U \neq \emptyset$ and since $p \notin U$, U is a proper open subset of continuum K (that is, $\emptyset \neq U \subsetneq K$). Thus letting M denote the component of \overline{U} such that $q \in M$, we have by the Boundary Bumping Theorem (open sets) that

$$M \cap Bd_K(U) \neq \emptyset \quad (4.2)$$

where $Bd_K(U)$ refers to the boundary of U in the space K . Let $J = T_1 \cup M$ and $\mathcal{A}_1 = \mathcal{A}_0 \cup \{J\}$. We will now show that \mathcal{A}_1 satisfies (1), (2) and (3), but first note the following facts :

- Clearly, $T_1 \subseteq J$ and by Equation (4.2) $T_1 \neq J$.
- $M \subseteq K \subseteq T_2$ and $T_1 \subseteq T_2 \implies J \subseteq T_2$.
- Since $p \in T_2 \setminus J$ we have $J \neq T_2$.

\mathcal{A}_1 satisfies (1) : $T_1, T_2 \in \mathcal{A}_0$ and $T_1 \subset T_2$, so by (1), $A_0 \subseteq T_1 \subseteq T_2 \subseteq A_1$. So $T_1 \subseteq J$ and $J \subseteq T_2$ therefore $A_0 \subseteq J \subseteq A_1$. Now $\mathcal{A}_1 = \mathcal{A}_0 \cup \{J\}$, and J satisfies (1), hence \mathcal{A}_1 must satisfy (1).

\mathcal{A}_1 satisfies (2) : $J = T_1 \cup M$, where M is connected and $M \cap T_1 \neq \emptyset$ (because $q \in M \cap T_1$). Thus each component of J contains a component of T_1 and since each component of T_1 intersects A_0 (by (2)), we have that each component of J intersects A_0 . Hence \mathcal{A}_1 satisfies (2).

\mathcal{A}_1 satisfies (3) : Recall (earlier we showed the following claim) if $G \in \mathcal{A}_0$ then $G \subseteq T_1$ or $G \supseteq T_2$. Now $T_1 \subseteq J$ and $J \subseteq T_2$ therefore $G \subseteq J$ or $J \subseteq G$. Since $\mathcal{A}_1 = \mathcal{A}_0 \cup \{J\}$, it follows that \mathcal{A}_1 satisfies (3).

So \mathcal{A}_1 satisfies (1), (2) and (3), hence $\mathcal{A}_1 \in \mathcal{F}$.

Now $\mathcal{A}_0 \subseteq \mathcal{A}_1$ but \mathcal{A}_0 is a maximal element of \mathcal{F} , so $\mathcal{A}_0 = \mathcal{A}_1$. Thus $J \in \mathcal{A}_0$ (that is, $\mu_0(J) \in \mu_0(\mathcal{A}_0)$) and $t_1 < \mu_0(J) < t_2$. This contradicts (b). Hence Equation (4.1) holds, and thus $[\mu_0(A_0), \mu_0(A_1)] = \mu_0(\mathcal{A}_0)$. So $\mu_0^{-1}[\mu_0(A_0), \mu_0(A_1)] = \mathcal{A}_0$ is an order arc. This completes the proof. \square

The next result shows that 2^X is arcwise connected even when X is a continuum which is not necessarily arcwise connected. We will then present an analogous result for $\mathcal{C}(X)$ in Theorem 4.1.16.

Theorem 4.1.14. 2^X is arcwise connected.

Proof.

Let $A_0 \in 2^X$ such that $A_0 \neq X$ and let $A_1 = X$. Then $A_0 \subsetneq A_1$ and every component of A_1 intersects A_0 (since X is continuum and hence connected). By Theorem 4.1.13, there exists an order arc, say α , in 2^X from A_0 to A_1 . So each member of $2^X \setminus \{X\}$ can be joined to X by an order arc in 2^X . Thus for some order arc α' from $A'_0 \in 2^X \setminus \{X\}$ to A_1 , by taking the composition of α with $(\alpha')^{-1}$, we have that any two members from 2^X can be joined by an order arc (and hence arc) in 2^X . Hence 2^X is arcwise connected. \square

Lemma 4.1.15 ([8]; Lemma 1.11.). *If α is an order arc in 2^X beginning with $A_0 \in \mathcal{C}(X)$, then $\alpha \subseteq \mathcal{C}(X)$.*

Proof.

Let α be an order arc in 2^X beginning with $A_0 \in \mathcal{C}(X)$. Let $B \in \alpha$ such that $B \neq A_0$ and let β be the subarc of α with endpoints A_0 and B . Since α begins at A_0 , we have that $A_0 = \bigcap \alpha$, and clearly $\bigcap \alpha \subseteq B$. Therefore β is an order arc from A_0 to B . Now by Theorem 4.1.13, $A_0 \subseteq B$ and every component of B intersects A_0 . Since $A_0 \subseteq B$, we have the following representation for B :

$$B = \bigcup \{A_0 \cup K \mid K \text{ is a component of } B\}$$

Now A_0 is connected and $A_0 \cap K \neq \emptyset$ therefore $A_0 \cup K$ is connected for every component K of B . Hence B is connected. Furthermore, B is a closed subset of compact 2^X (and 2^X is compact by Theorem 1.3.12), hence B must be compact. So $B \in \mathcal{C}(X)$ and therefore $\alpha \subseteq \mathcal{C}(X)$. \square

Theorem 4.1.16. $\mathcal{C}(X)$ is arcwise connected.

Proof.

Let $A_0 \in \mathcal{C}(X)$ such that $A_0 \neq X$ and let $A_1 = X$. By Theorem 4.1.13, there exists an order

arc, say α , in 2^X from A_0 to A_1 . Now $A_0 \in \mathcal{C}(X)$, hence $\alpha \subseteq \mathcal{C}(X)$ (by Lemma 4.1.15).

So each member of $\mathcal{C}(X) \setminus \{X\}$ can be joined to X by an order arc in $\mathcal{C}(X)$. Let α' be an order arc in $\mathcal{C}(X)$ from $A'_0 \in \mathcal{C}(X) \setminus \{X\}$ to A_1 , then taking the composition of α with $(\alpha')^{-1}$, we obtain an order arc (and hence arc) in $\mathcal{C}(X)$. Therefore $\mathcal{C}(X)$ is arcwise connected. \square

We are now in the position to give one of the most fundamental facts about 2^X and $\mathcal{C}(X)$, that is, that each are continua.

Corollary 4.1.17 ([8]; Theorem 1.13.). *2^X and $\mathcal{C}(X)$ are each arcwise connected continua.*

Proof.

Each of 2^X and $\mathcal{C}(X)$ are compact by Theorem 1.3.12, and we have shown that 2^X and $\mathcal{C}(X)$ are arcwise connected by Theorem 4.1.14 and respectively, Theorem 4.1.16. Hence since arcwise connectedness implies connectedness, the result follows. \square

4.2 Kelley's Segments

Using Whitney maps, J.L. Kelley defined and made use of special mappings which he called segments, to study hyperspaces. We now introduce Kelley's notion of a segment and will look at the relationships between segments and order arc's.

Definition 4.2.1. Let μ be a Whitney map for 2^X and $A_0, A_1 \in 2^X$. A function $\sigma : [0, 1] \longrightarrow 2^X$ is said to be a segment with respect to μ , from A_0 to A_1 provided that :

- (1) σ is continuous on $[0, 1]$;
- (2) $\sigma(0) = A_0$ and $\sigma(1) = A_1$;
- (3) $\mu(\sigma(t)) = (1 - t) \cdot \mu(\sigma(0)) + t \cdot \mu(\sigma(1))$, for $t \in [0, 1]$;
- (4) if $0 \leq t_1 \leq t_2 \leq 1$, then $\sigma(t_1) \subseteq \sigma(t_2)$.

Remark 4.2.2.

1. A segment with respect to one Whitney map may not be a segment with respect to another. It is often not important to emphasise the dependence of segments on Whitney maps. For this reason, we now refer to the following terminology :

$\sigma : [0, 1] \longrightarrow 2^X$ is a segment from A_0 to A_1 provided there exists a Whitney map μ for 2^X such that σ is a segment with respect to μ from A_0 to A_1 .

2. We may sometimes say “ σ is a segment”, meaning σ is a segment from some member of 2^X to some member of 2^X .

3. We will require that μ be a normalised Whitney map (i.e. $\mu(X) = 1$).

Any segment as defined in Definition 4.2.1, is a segment with respect to some normalised Whitney map. For if σ is a segment with respect to a Whitney map μ , then σ is also a segment with respect to the normalised Whitney map $\mu' = \left[\frac{1}{\mu(X)}\right] \cdot \mu$.

4. Any constant function $\sigma : [0, 1] \longrightarrow 2^X$ is a segment with respect to any Whitney map for 2^X

The next result shows that any non-constant segment is an arc.

Lemma 4.2.3 ([8]; Lemma 1.18.). *If $\sigma : [0, 1] \longrightarrow 2^X$ is a non-constant segment (onto its image), then σ is a homeomorphism.*

Proof.

Assume $\sigma : [0, 1] \longrightarrow 2^X$ is a non-constant segment. We first show that $\sigma(0) \neq \sigma(1)$. Suppose $\sigma(1) = \sigma(0) = B$, for some $B \in 2^X$.

Then for $0 \leq t \leq 1$, $B = \sigma(0) \subseteq \sigma(t) \subseteq \sigma(1) = B$. Hence $\sigma(t) = B$ for all $t \in [0, 1]$ implying that σ is a constant, which is a contradiction. So, $\sigma(0)$ and $\sigma(1)$ are distinct.

We now show that σ is 1-1 : Take $s, t \in [0, 1]$ such that $\sigma(s) = \sigma(t)$, then

$$\mu(\sigma(s)) = \mu(\sigma(t))$$

$$\text{which implies } (1 - s) \cdot \mu(\sigma(0)) + s \cdot \mu(\sigma(1)) = (1 - t) \cdot \mu(\sigma(0)) + t \cdot \mu(\sigma(1))$$

$$(t - s) \cdot \mu(\sigma(0)) = (t - s) \cdot \mu(\sigma(1)) \tag{4.3}$$

Now since $0 < 1$, it follows that $\sigma(0) \subsetneq \sigma(1)$ and therefore $\mu(\sigma(0)) < \mu(\sigma(1))$. So in order for Equation (4.3) to hold, we must have $t - s = 0$. Hence $t = s$ and σ is 1-1.

So $\sigma : [0, 1] \xrightarrow{\text{onto}} \sigma[0, 1] \subseteq 2^X$ is a continuous, 1-1 map where $[0, 1]$ is compact and 2^X is Hausdorff, hence σ is a homeomorphism. □

Theorem 4.2.4 ([8]; Theorem 1.19.). *Let μ be a given Whitney map for 2^X . A non-degenerate subset of 2^X , \mathcal{A} , is the range of a segment with respect to μ iff it is an order arc.*

Proof.

(\implies) Let $\mathcal{A} \subseteq 2^X$ be non-degenerate. Assume there exists $\sigma : [0, 1] \longrightarrow 2^X$ with respect to μ such that $\sigma([0, 1]) = \mathcal{A}$. Since \mathcal{A} is non-degenerate, σ is a non-constant segment. Therefore σ is a homeomorphism by Lemma 4.2.3, and so σ is an arc. It remains to show that if $A, B \in \mathcal{A}$ then either $A \subseteq B$ or $B \subseteq A$.

Take $A, B \in \mathcal{A}$, then there exists $t_1, t_2 \in [0, 1]$ such that $A = \sigma(t_1)$ and $B = \sigma(t_2)$.

Now either $t_1 \leq t_2$ or $t_2 \leq t_1$. Hence either $\sigma(t_1) \subseteq \sigma(t_2)$ or $\sigma(t_2) \subseteq \sigma(t_1)$. Thus \mathcal{A} is an order arc.

(\impliedby) Assume that \mathcal{A} is an order arc. Let $\mu_0 = \mu|_{\mathcal{A}}$ (that is, $\mu_0 : \mathcal{A} \longrightarrow [0, 1]$). Since \mathcal{A} is an order arc (and hence an arc), \mathcal{A} is the continuous image of a compact set and so is compact. By Lemma 4.1.3, μ_0 is a homeomorphism onto its image. Let $a, b \in [0, \infty)$ such that $\mu_0(\mathcal{A}) = [a, b]$. Define $\rho : [0, 1] \xrightarrow{\text{onto}} [a, b]$ by $\rho(t) = (1 - t)a + tb$, for $t \in [0, 1]$, and define $\sigma : [0, 1] \longrightarrow 2^X$ by $\sigma = (\mu_0)^{-1}\rho$.

$$\begin{array}{ccc} & \sigma & \\ & \curvearrowright & \\ [0, 1] & \xrightarrow[\rho]{\text{onto}} & [a, b] \xrightarrow[\mu_0^{-1}]{\text{onto}} \mathcal{A} \end{array}$$

Hence by the above mappings, we have that $\sigma([0, 1]) = \mathcal{A}$. We will now show that σ is a segment with respect to μ .

μ_0^{-1} is continuous since μ_0 is a homeomorphism, so it follows that σ which is the composition of continuous functions, is itself continuous. So (1) of Definition 4.2.1 is satisfied.

Now for $t \in [0, 1]$, $\mu\sigma(t) = \rho(t) = (1 - t)a + tb$ with $\mu\sigma(0) = \rho(0) = a$ and $\mu\sigma(1) = \rho(1) = b$.

Thus we have, $\mu\sigma(t) = (1 - t) \cdot \mu\sigma(0) + t \cdot \mu\sigma(1)$, for $t \in [0, 1]$, which satisfies (3) of Definition 4.2.1.

We now show that σ satisfies (4) of Definition 4.2.1. Let $0 \leq t_1 \leq t_2 \leq 1$. Since $a \leq b$, we

have that $(1 - t_1)a + t_1b \leq (1 - t_2)a + t_2b$.

Therefore $\mu\sigma(t_1) \leq \mu\sigma(t_2)$. Suppose $\sigma(t_1) \not\subseteq \sigma(t_2)$, then since $\sigma(t_1), \sigma(t_2) \in \mathcal{A}$, and \mathcal{A} is an order arc, we obtain $\sigma(t_2) \subseteq \sigma(t_1)$ and $\sigma(t_1) \neq \sigma(t_2)$. Hence $\mu\sigma(t_2) < \mu\sigma(t_1)$, which is a contradiction, therefore $\sigma(t_1) \subseteq \sigma(t_2)$ and (4) holds.

So, σ is a segment with respect to μ from $\sigma(0)$ to $\sigma(1)$, and $\sigma([0, 1]) = \mathcal{A}$ implies that \mathcal{A} is range of a segment as required. \square

Remark 4.2.5. In the proof of Theorem 4.2.4, we had

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mu_0} & [a, b] \\ \sigma \uparrow & \nearrow \rho & \\ [0, 1] & & \end{array}$$

where $\sigma = \mu_0^{-1}\rho$.

Now recall that $\bigcap \mathcal{A}$ and $\bigcup \mathcal{A}$ are the two endpoints of \mathcal{A} . So since μ_0 is a homeomorphism, we have that

$$\mu_0(\bigcap \mathcal{A}) = a \text{ and } \mu_0(\bigcup \mathcal{A}) = b$$

Furthermore, by the definition of ρ and σ , it follows that

$$\sigma(0) = \bigcap \mathcal{A} \text{ and } \sigma(1) = \bigcup \mathcal{A}$$

This idea is generalised in the next theorem.

Theorem 4.2.6 ([8]; Theorem 1.21.). *If $\sigma : [0, 1] \rightarrow 2^X$ is a segment and $\alpha = \sigma([0, 1])$, then $\sigma(0) = \bigcap \alpha$ and $\sigma(1) = \bigcup \alpha$.*

Proof.

If σ is constant, then done. So suppose that σ is non-constant, that is, α is non-degenerate. Then by Theorem 4.2.4, α is an order arc. So $\bigcup \alpha$ and $\bigcap \alpha$ are the endpoints of α . By Lemma 4.2.3, σ is a homeomorphism onto α (because σ is non-constant), thus $\sigma(0)$ and $\sigma(1)$ are two endpoints of α .

Hence $\{\sigma(0), \sigma(1)\} = \{\bigcap \alpha, \bigcup \alpha\}$.

By the properties of a segment, we know that $\sigma(0) \subseteq \sigma(1)$. Therefore $\sigma(0) = \bigcap \alpha$ and $\sigma(1) = \bigcup \alpha$, as required. \square

Theorem 4.2.7 ([8]; Theorem 1.22.). *Let μ be a Whitney map for 2^X . A subset, \mathcal{A} of 2^X is the range of a segment with respect to μ , iff \mathcal{A} is an order arc or $\mathcal{A} = A$ for some $A \in 2^X$.*

Proof.

(\implies) If $\mathcal{A} \subseteq 2^X$ is the range of a segment with respect to μ and \mathcal{A} is non-degenerate, the \mathcal{A} is an order arc by Theorem 4.2.4. Otherwise $\mathcal{A} = A$ for some $A \in 2^X$.

(\impliedby) Suppose $\mathcal{A} = A$, for some $A \in 2^X$, then any constant function from $[0, 1]$ taking the constant value A is a segment with respect to μ .

Now suppose that \mathcal{A} is an order arc and that \mathcal{A} is non-degenerate subset of 2^X . By Theorem 4.2.4, \mathcal{A} is the range of a segment with respect to μ . \square

The next theorem (see [8]; Theorem 1.25.) is used frequently and includes the use of Theorem 4.1.13.

Theorem 4.2.8. *Let $A_0, A_1 \in 2^X$. The following two statements are equivalent :*

- (1) *There exists a segment (with respect to any given Whitney map) in 2^X from A_0 to A_1 .*
- (2) *$A_0 \subseteq A_1$ and each component of A_1 intersects A_0 .*

Furthermore, if $A_0 \neq A_1$, then (1) and (2) are each equivalent to the following :

- (3) *There exists an order arc in 2^X from A_0 to A_1 .*

Proof.

If $A_0 = A_1$, then trivial, since (2) holds and the constant map from $[0, 1]$ into 2^X with value A_0 is a segment with respect to μ , from A_0 to A_1 .

So suppose $A_0 \neq A_1$. It suffices to show that (2) \iff (3) and (1) \iff (3). By Theorem 4.1.13, we know that (2) and (3) are equivalent.

(1) \implies (3) : Let $\sigma : [0, 1] \longrightarrow 2^X$ be a segment with respect to μ from A_0 to A_1 . Let $\alpha = \sigma([0, 1])$, then by (1), α is a non-degenerate subset of 2^X . By Theorem 4.2.4, α is an order arc, and by Theorem 4.2.6, $\sigma(0) = \bigcap \alpha$ and $\sigma(1) = \bigcup \alpha$.

But σ is a segment from A_0 to A_1 , therefore $\sigma(0) = A_0$ and $\sigma(1) = A_1$. Thus we have

$$\bigcap \alpha = A_0 \text{ and } \bigcup \alpha = A_1.$$

Hence α is an order arc from A_0 to A_1 , as required.

(3) \implies (1) : Now assume (3) holds and let α be an order arc from A_0 to A_1 . Therefore $\bigcap \alpha = A_0$ and $\bigcup \alpha = A_1$. Now by Theorem 4.2.4, there exists a segment $\sigma : [0, 1] \longrightarrow 2^X$ with respect to μ such that $\sigma([0, 1]) = \alpha$. Thus by Theorem 4.2.6, $\sigma(0) = \bigcap \alpha$ and $\sigma(1) = \bigcup \alpha$, and by (2), $\sigma(0) = A_0$ and $\sigma(1) = A_1$.

Therefore σ is a segment from A_0 to A_1 , and so (1) holds. \square

The next theorem is an analogue of Lemma 4.1.15, for segments.

Theorem 4.2.9 ([8]; Theorem 1.26.). *If $\sigma : [0, 1] \longrightarrow 2^X$ is a segment such that $\sigma(t_0) \in \mathcal{C}(X)$ for some $t_0 \in [0, 1]$, then $\sigma(t) \in \mathcal{C}(X)$ for each $t \in [t_0, 1]$. Hence if $\sigma(0) \in \mathcal{C}(X)$ then $\sigma([0, 1]) \subseteq \mathcal{C}(X)$.*

Proof.

Let σ be a non-constant segment with $t_0 < 1$. Hence by Lemma 4.2.3, σ is a homeomorphism.

Let $\alpha = \sigma([0, 1])$ and $\beta = \sigma([t_0, 1])$, then by Theorem 4.2.4 α is an order arc.

Now $\beta \subseteq \alpha$ and since any subarc of an order arc is an order arc, β must be an order arc.

Claim : $\bigcap \beta = \sigma(t_0)$.

$\sigma(t_0) \subseteq \bigcap \beta$, since $\sigma(t_0) \subseteq \sigma(t)$ for each $t \in [t_0, 1]$, and $\sigma(t_0) \in \beta$, therefore $\bigcap \beta \subseteq \sigma(t_0)$.

Hence $\bigcap \beta = \sigma(t_0)$.

Thus β is an order arc beginning with $\sigma(t_0)$, and by our premise we have that $\sigma(t_0) \in \mathcal{C}(X)$.

The result then follows by Lemma 4.1.15. \square

4.3 Contractibility

Wojdyslawski ([15]) was the first person to give attention to the contractibility of hyperspaces. Later Kelley generalised some of Wojdyslawski's work, thereby providing simpler proofs. We will now work towards what is considered the most fundamental theorem concerning the contractibility of hyperspaces. The theorem shows that if one of 2^X and $\mathcal{C}(X)$ is

contractible, then so is the other. It also provides, an additional way to determine whether 2^X and $\mathcal{C}(X)$ is contractible, namely iff $\phi(X)$ (the space of singletons of X) is contractible in 2^X or $\mathcal{C}(X)$

For the purposes of studying contractibility, Kelley also defined what is now called Property K. However, we will delay this and present this in detail in Chapter 5. We begin with the definition of homotopy.

Definition 4.3.1. Let Z_1 and Z_2 be two topological spaces. A homotopy is a continuous mapping $h : Z_1 \times [0, 1] \rightarrow Z_2$. If $f : Z_1 \rightarrow Z_2$ and $g : Z_1 \rightarrow Z_2$ are two continuous mappings and if $h : Z_1 \times [0, 1] \rightarrow Z_2$ is a homotopy such that for each $x \in Z_1$,

$$h(x, 0) = f(x) \text{ and } h(x, 1) = g(x)$$

then h is said to be a homotopy from f to g .

Also when there is a homotopy from f to g , then f and g are said to be homotopic, written $f \simeq g$.

Definition 4.3.2. A topological space Z_1 is said to be *contractible in itself* or *contractible*, provided that the identity map id_{Z_1} is homotopic to a constant map from Z_1 into Z_1 .

Proposition 4.3.3. Z_1 is contractible iff every continuous map $f : Z_1 \rightarrow Z_1$ is homotopic to a constant map.

Proof.

(\implies) Suppose Z_1 is contractible. Let $f : Z_1 \rightarrow Z_1$ be continuous. Now there exists a homotopy $h : Z_1 \times [0, 1] \rightarrow Z_1$ such that $h(x, 0) = x$ for all $x \in Z_1$, and $h(x, 1) = y$ for all $x \in Z_1$ and some $y \in Z_1$. Then $Z_1 \times [0, 1] \xrightarrow{h} Z_1 \xrightarrow{f} Z_1$ is continuous, and

$$fh(x, 0) = f(x) \text{ for all } x \in Z_1$$

$$fh(x, 1) = f(y) \text{ for all } x \in Z_1.$$

So f is homotopic to a constant map via the homotopy fh .

(\impliedby) Clear from the Definition 4.3.2. □

Definition 4.3.4. If $Z_1 \subseteq Z_2$, Z_1 is said to be contractible in Z_2 if $id : Z_1 \longrightarrow Z_2$ is homotopic to a constant map from Z_1 into Z_2 .

Lemma 4.3.5. *If Y is a contractible space, and $Z \subseteq Y$, then Z is contractible in Y .*

Proof.

Assume that Y is contractible, then we have a homotopy $h : Y \times [0, 1] \longrightarrow Y$ such that id_Y is homotopic to a constant mapping from Y to Y . Then the restriction map,

$h|_{Z \times [0,1]} : Z \times [0, 1] \longrightarrow Y$ is continuous and is the required homotopy between $id_Z : Z \longrightarrow Y$ and a constant map from Z to Y . □

Lemma 4.3.6. *If 2^X is contractible, then id_{2^X} is homotopic to any constant map on 2^X .*

Proof.

Suppose that 2^X is contractible, then there exists a homotopy

$h : 2^X \times [0, 1] \longrightarrow 2^X$ such that id_{2^X} is homotopic to a constant map, say, $g : 2^X \longrightarrow \{A_0\}$ for some fixed $A_0 \in 2^X$.

Let $f : 2^X \longrightarrow \{A_1\}$, for some $A_1 \in 2^X$ ($A_0 \neq A_1$). Since 2^X is arcwise connected, it is also path connected, therefore there exists a path, $p : [0, 1] \longrightarrow 2^X$ from A_0 to A_1 .

Define $h_1 : 2^X \times [0, 1] \longrightarrow 2^X$, by $h_1(A, t) = p(t)$ for $t \in [0, 1]$ and each $A \in 2^X$. Thus h_1 is continuous since p is continuous, and for each $A \in 2^X$,

$$h_1(A, 0) = A_0 = g(A) \text{ and } h_1(A, 1) = A_1 = f(A)$$

Therefore g is homotopic to f . Now homotopy is an equivalence relationship on continuous maps (and in this case with respect to continuous maps on 2^X) therefore $id_{2^X} \simeq g \simeq f$.

Hence $id_{2^X} \simeq f$, as required. □

We now construct a homotopy h' from an arbitrary homotopy h . h' will be required for some of the proofs to follow.

Lemma 4.3.7 ([8]; lemma 16.3.). *Let $\mathcal{A} \subseteq 2^X$, and $h : \mathcal{A} \times [0, 1] \longrightarrow 2^X$ be a homotopy. Then for each $(A, t) \in \mathcal{A} \times [0, 1]$,*

$$\bigcup \{h(A, s) | 0 \leq s \leq t\} \in 2^X$$

Furthermore, the function $h' : \mathcal{A} \times [0, 1] \longrightarrow 2^X$ defined by $h'(A, t) = \bigcup\{h(A, s) \mid 0 \leq s \leq t\}$ is continuous.

Proof.

For each $(A, t) \in \mathcal{A} \times [0, 1]$, let $g(A, t) = \{h(A, s) \mid 0 \leq s \leq t\}$. Now h is a continuous function into 2^X and for each $(A, t) \in \mathcal{A} \times [0, 1]$,

$$g(A, t) = h(\{A\} \times [0, t])$$

Note: $\{A\} \times [0, t]$ is compact, thus $h(\{A\} \times [0, t])$ is a closed subset of 2^X (since h is continuous and 2^X is Hausdorff). So we have $g(A, t) \in 2^{2^X}$, for each $(A, t) \in \mathcal{A} \times [0, 1]$. Hence $g : \mathcal{A} \times [0, 1] \longrightarrow 2^{2^X}$ is a well-defined map.

Recall the union function $\Gamma : 2^{2^X} \xrightarrow{\text{conts}} 2^X$, given in Definition 1.4.7.

$$\begin{array}{ccccc} & & \Gamma g & & \\ & \text{---} & \text{---} & \text{---} & \\ & & \text{---} & & \\ \mathcal{A} \times [0, 1] & \xrightarrow{g} & 2^{2^X} & \xrightarrow{\Gamma} & 2^X \end{array}$$

Thus for each $(A, t) \in \mathcal{A} \times [0, 1]$, $(\Gamma g)(A, t) \in 2^X$. Hence $\bigcup\{h(A, s) \mid 0 \leq s \leq t\} \in 2^X$, as required. It remains for us to show that h' is continuous.

Now $h'(A, t) = (\Gamma g)(A, t)$, for $(A, t) \in \mathcal{A} \times [0, 1]$, and Γ is continuous. Thus it suffices for us to show that g is continuous, for then h' will be continuous since it is the composition of two continuous functions.

Claim: $g : \mathcal{A} \times [0, 1] \longrightarrow 2^{2^X}$ is continuous.

We will show continuity at (A, t) . Let $\varepsilon > 0$. We will find $\delta > 0$ such that whenever $\rho(A, A') + |t - t'| < \delta$ for some $(A, t), (A', t') \in \mathcal{A} \times [0, 1]$, then $\rho^1(g(A, t), g(A', t')) < \varepsilon$. That is

$$\rho^1(h(\{A\}, [0, 1]), h(\{A'\}, [0, t'])) < \varepsilon. \quad (4.4)$$

Since $h : \mathcal{A} \times [0, 1] \longrightarrow 2^X$ is a homotopy, there exists $\delta > 0$ such that whenever $\rho(A, A') + |t - t'| < \delta$, then

$$\rho(h(A, t), h(A', t')) < \varepsilon. \quad (4.5)$$

Choose this δ . In order to show Equation (4.4), it suffices to show

$$h(\{A\} \times [0, t]) \subseteq V_\varepsilon^\rho(h(\{A'\} \times [0, t'])), \quad (4.6)$$

(and similarly $h(\{A'\} \times [0, t']) \subseteq V_\varepsilon^\rho(h(\{A\} \times [0, t])$) where $V_\varepsilon^\rho(h(\{A'\} \times [0, t'])$ denotes $V_\varepsilon(h(\{A'\} \times [0, t'])$ obtained with the specified metric ρ .

To show Equation (4.6), take any $h(A, s)$, where $0 \leq s \leq t$. Choose s' , $0 \leq s' \leq t'$ such that $|s - s'| = |t - t'|$. Then $\rho(A, A') + |t - t'| < \delta$, implies that $\rho(A, A') + |s - s'| < \delta$. By Equation (4.5), it follows that $\rho(h(A, s), h(A', s')) < \varepsilon$. Thus Equation (4.6) is satisfied, and this implies that Equation (4.4) holds, proving that g is indeed continuous.

Hence it follows that h' is continuous, as required \square

Remark 4.3.8.

The homotopy h' which is defined in terms of h , is called the segment homotopy associated with h . In the next lemma we look at important properties of h'

Lemma 4.3.9 ([8]; Lemma 16.5.).

Let $\mathcal{A} \subseteq 2^X$ and $h : \mathcal{A} \times [0, 1] \rightarrow 2^X$ be a homotopy. Then the segment homotopy h' associated with h has the following properties :

(1) For each $A \in \mathcal{A}$, $h'(\{A\} \times [0, 1])$ is the range of a segment in 2^X from $h'(A, 0)$ to $h'(A, 1)$.

Equivalently, $h'(A, t_1) \subseteq h'(A, t_2)$ whenever $0 \leq t_1 \leq t_2 \leq 1$

(2) For each $A \in \mathcal{A}$, $h'(A, 0) = h(A, 0)$.

(3) For each $(A, t) \in \mathcal{A} \times [0, 1]$, $h'(A, t) \supseteq h(A, s)$ for each $s \in [0, t]$.

(4) If $h'(A, t) \in \mathcal{C}(X)$ for some $(A, t) \in \mathcal{A} \times [0, 1]$, then $h'(\{A\} \times [t, 1]) \subseteq \mathcal{C}(X)$.

Proof.

(1) Let $A \in \mathcal{A}$. By Lemma 4.3.7, we know that h' is a continuous function into 2^X , hence $h'(\{A\} \times [0, 1])$ is a subcontinuum of 2^X (and possibly degenerate).

If $h'(\{A\} \times [0, 1])$ is degenerate then $h'(\{A\} \times [0, 1]) = B$ for some $B \in \mathcal{A}$.

So suppose that $h'(\{A\} \times [0, 1])$ is non-degenerate. Take $B_1, B_2 \in h'(\{A\} \times [0, 1])$, then there exists $t_1, t_2 \in [0, 1]$ such that $B_1 = h'(A, t_1)$ and $B_2 = h'(A, t_2)$. Now either $t_1 \leq t_2$ or $t_2 \leq t_1$.

By the definition of h' , this implies that either $h'(A, t_1) \subseteq h'(A, t_2)$ or $h'(A, t_2) \subseteq h'(A, t_1)$.

Thus by Theorem 4.1.4, $h'(\{A\} \times [0, 1])$ is an order arc in 2^X , and it follows by Theorem 4.2.4 that $h'(\{A\} \times [0, 1])$ is the range of a segment.

(2) Follows directly from the definition of h' .

(3) Follows directly from the definition of h' .

(4) Let $h'(A, t) \in \mathcal{C}(X)$ for some $(A, t) \in \mathcal{A} \times [0, 1]$. Now $h'(\{A\} \times [0, 1])$ is the range of a segment by (1). Hence by Theorem 4.2.9, since $h'(A, t) \in \mathcal{C}(X)$, we have that $h'(A, t_0) \in \mathcal{C}(X)$ for each $t_0 \in [t, 1]$. Therefore $h'(\{A\} \times [t, 1]) \subseteq \mathcal{C}(X)$, as required. \square

Recall the map $\phi : 2^X \longrightarrow 2^{2^X}$ as defined in Definition 1.4.1. This continuous mapping is required in the proof of the next theorem.

Theorem 4.3.10. *The Fundamental Theorem of Contractible Hyperspaces*

For any continuum X , the following are equivalent :

- (1) $\phi(X)$ is contractible in 2^X ;
- (2) 2^X is contractible (in itself);
- (3) $\mathcal{C}(X)$ is contractible (in itself);
- (4) $\phi(X)$ is contractible in $\mathcal{C}(X)$.

Proof.

(1) \implies (2) : Assume that $\phi(X)$ is contractible in 2^X , then there is a homotopy $f : \phi(X) \times [0, 1] \longrightarrow 2^X$ such that for each $\{x\} \in \phi(X)$ and fixed $A_0 \in 2^X$,

$$f(\{x\}, 0) = \{x\} \text{ and } f(\{x\}, 1) = A_0.$$

For each $(A, t) \in 2^X \times [0, 1]$, let

$$g(A, t) = f(\phi(A) \times t).$$

Now for each A , $\phi(A) \subseteq \phi(X)$ and $\phi(A)$ is non-empty compact, thus since f is continuous $f(\phi(A) \times t)$ is compact, and hence we have that $g(A, t) \in 2^{2^X}$ for each $(A, t) \in 2^X \times [0, 1]$.

That is,

$$g : 2^X \times [0, 1] \longrightarrow 2^{2^X}$$

Now f is continuous and ϕ is continuous (by Corollary 1.4.6) hence g is continuous. Also recall the union function $\Gamma : 2^{2^X} \xrightarrow{\text{conts}} 2^X$. Define $K : 2^X \times [0, 1] \longrightarrow 2^X$ by,
 $K(A, t) = (\Gamma \circ g)(A, t)$, for each $(A, t) \in 2^X \times [0, 1]$.

$$\begin{array}{ccccc}
 & & K & & \\
 & & \curvearrowright & & \\
 2^X \times [0, 1] & \xrightarrow{g} & 2^{2^X} & \xrightarrow{\Gamma} & 2^X
 \end{array}$$

K is the composition of continuous functions, so K itself is continuous, and hence a homotopy. We now check that K is the required homotopy for 2^X to be contractible in itself. Let $A \in 2^X$ be arbitrary, then we have the following

$$\begin{aligned}
 K(A, 0) &= \Gamma \circ g(A, 0) \\
 &= \Gamma \circ f(\phi(A) \times 0) \\
 &= \Gamma(\{\{a_1\}, \{a_2\}, \dots\}) \text{ each } a_i \in A \\
 &= A \quad (\text{that is, the identity map})
 \end{aligned}$$

$$\begin{aligned}
 K(A, 1) &= \Gamma \circ g(A, 1) \\
 &= \Gamma \circ f(\phi(A) \times 1) \\
 &= \Gamma(\{A_0, A_0, \dots\}) \\
 &= A_0 \quad (\text{that is, a constant map})
 \end{aligned}$$

Therefore we have that id_{2^X} is homotopic to a constant mapping from 2^X into 2^X . Hence 2^X is contractible in itself.

(2) \implies (3) : Assume 2^X is contractible. By Theorem 4.1.14, 2^X is arcwise connected. Since 2^X is contractible and arcwise connected, by Lemma 4.3.6 there is a homotopy $h : 2^X \times [0, 1] \longrightarrow 2^X$ such that for each $A \in 2^X$

$$h(A, 0) = A \text{ and } h(A, 1) = X.$$

Let h' be the segment homotopy associated with h . Now $h'(A, 0) = h(A, 0) = A$, for each

$A \in 2^X$. So for $A \in \mathcal{C}(X)$, we have that $h'(A, 0) \in \mathcal{C}(X)$ and hence $h'(\mathcal{C}(X) \times [0, 1]) \subseteq \mathcal{C}(X)$ (by (4) of Lemma 4.3.9). Now we also have that, for each $A \in 2^X$,

$$h'(A, 1) \supseteq h(A, s) \text{ for } s \in [0, 1], \text{ thus } h'(A, 1) = \bigcup\{h(A, s) | s \in [0, 1]\} = X.$$

By restricting h' to $\mathcal{C}(X) \times [0, 1]$, we observe that $h'|_{\mathcal{C}(X) \times [0, 1]} : \mathcal{C}(X) \times [0, 1] \rightarrow \mathcal{C}(X)$ is continuous, hence is the required homotopy such that $\mathcal{C}(X)$ is contractible in itself.

(3) \implies (4) : Assume that $\mathcal{C}(X)$ is contractible in itself. $\phi(X)$ is a closed subset of 2^X (as shown in Theorem 1.4.2) and $\phi : 2^X \rightarrow 2^{2^X}$ is continuous, therefore $\phi(X)$ is a subcontinuum of 2^X . Hence $\phi(X) \subseteq \mathcal{C}(X)$. By Lemma 4.3.5, it follows immediately that $\phi(X)$ is contractible in $\mathcal{C}(X)$.

(4) \implies (1) : Suppose that $\phi(X)$ is contractible in $\mathcal{C}(X)$, that is there is a homotopy $h : \phi(X) \times [0, 1] \rightarrow \mathcal{C}(X)$ between the identity map $\phi(X) \rightarrow \mathcal{C}(X)$ and a constant map. But $\mathcal{C}(X) \subseteq 2^X$, hence $\phi(X)$ is then contractible in 2^X . \square

The above result is a general theorem by J. L. Kelley (see [2]). As a consequence of the Fundamental theorem of contractible hyperspaces, we have the following.

Corollary 4.3.11 ([8]; Corollary 16.8). *If X is a contractible continuum, then 2^X and $\mathcal{C}(X)$ are each contractible.*

Proof.

Let X be a contractible continuum. By Theorem 1.4.5, $\phi(X)$ is isometric to X , therefore $\phi(X)$ is contractible in itself. Also $\phi(X) \subseteq \mathcal{C}(X)$, thus $\phi(X)$ is contractible in $\mathcal{C}(X)$. It follows from the Fundamental theorem of contractible hyperspaces that 2^X and $\mathcal{C}(X)$ are contractible. \square

Chapter 5

Hyperspaces of Peano Continua

5.1 Local Connectedness and Connected im Kleinen

We now turn to the study of locally connected spaces. The first results about hyperspaces of locally connected spaces (see Theorem 5.2.11), are due to Vietoris ([11]), and Wojdyslawski ([15]). We dedicate this section to developing our theory around local connectedness and its relation to connectedness im kleinen. These concepts will be vital for our attention to *Peano continua*.

For this section, X will be a metric space.

Definition 5.1.1. Let X be a metric space. X is said to be locally connected at $x \in X$, if given any open U and $x \in U$ then there exists open connected V with $x \in V$ such that $x \in V \subseteq U$.

Further, X is said to be locally connected iff it is locally connected at every $x \in X$.

Remark 5.1.2.

Recall Definition 2.3.1. A metric space X is said to be connected im kleinen at x , if given any open U , $x \in U$, there exists open V with $x \in V$ and connected C such that $x \in V \subseteq C \subseteq U$. It follows that local connectedness at x implies connected im kleinen at x . But a space may be connected im kleinen at a point x and yet not be locally connected at x .

Proposition 5.1.3. X is connected im kleinen at x iff for any open U with $x \in U$, there exists open V with $x \in V$ such that any two points in V lie together in a connected set

contained in U .

Proof.

(\implies) Suppose X is connected im kleinen at x . Take open U , $x \in U$. Then there exists an open set V with $x \in V$ and connected C such that $x \in V \subseteq C \subseteq U$. Hence true.

(\impliedby) Take any open U with U containing x . Then there exists an open set V with $x \in V$ such that $x \in V \subseteq U$ and any two points in V lie together in a connected set contained in U . For each $y \in V$, we have that $x, y \in C_y \subseteq U$, where C_y is connected. Take $C = \bigcup_{y \in V} C_y$, then C_y is connected since $x \in C_y$ for all $y \in V$. Hence $x \in V \subseteq C \subseteq U$ and X is connected im kleinen at x . \square

Proposition 5.1.4 ([13]; Theorem 27.16). *X is locally connected iff X is connected im kleinen at every $x \in X$.*

Proof.

(\implies) Follows immediately from the definitions.

(\impliedby) Suppose X is connected im kleinen at every $x \in X$. Take any $x \in X$ and an open set U with $x \in U$. Then we can find an open V and connected C_x such that $x \in V \subseteq C_x \subseteq U$. Let C be the component of U containing x . C is connected because it is a component, so it suffices to show that C is open. Take any $y \in C$. We have that X is connected im kleinen at y , so there exists an open set V_y and a connected set C_y such that $y \in V_y \subseteq C_y \subseteq U$. Now $y \in C_y \cap C$, therefore $C_y \cup C$ is connected, hence $C_y \cup C \subseteq U$.

Hence $C \subseteq C_y \cup C \subseteq U$. By maximality of C , we must have $C = C_y \cup C$, so it follows that $C_y \subseteq C$. Thus $y \in V_y \subseteq C_y \subseteq C$ and C is open as required. \square

The next Theorem will be required for the proof of two of this chapters main results, namely, Theorem 5.2.11 and Theorem 5.2.12.

Theorem 5.1.5. *X is connected im kleinen at a point $p \in X$, if given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, p) < \delta$, then x, p lie together in a connected set C such that $\text{diam}(C) < \varepsilon$.*

Proof.

(\implies) Suppose X is connected im kleinen at $p \in X$. Let $\varepsilon > 0$. By im kleinen connectedness there exists an open connected set C and an open set U such that $p \in U \subseteq C \subseteq V_{\frac{\varepsilon}{3}}(p)$. Since U is open, there exists $\delta > 0$ such that $p \in V_{\delta}(p) \subseteq U$. Then $d(x, p) < \delta$ implies that $x \in V_{\delta}(p)$ and therefore $p \in U \subseteq C$ and $\text{diam}(C) < \varepsilon$.

(\impliedby) Let $p \in U$, where U is an open subset of X . Then there exists $\varepsilon > 0$ such that $p \in V_{\varepsilon}(p) \subseteq U$. Now there exists $\delta > 0$ such that if $d(x, p) < \delta$, then $x, p \in C_x$ and C_x is connected with $\text{diam}(C_x) < \varepsilon$.

Let $C = \bigcup_{x \in V_{\delta}(p)} C_x$. Then C is connected. Also $p \in V_{\delta}(p) \subseteq C$. Furthermore, $C_x \subseteq V_{\varepsilon}(p)$ since $x \in C_x$ implies that $d(z, p) \leq \text{diam}(C_x) < \varepsilon$, so $C_x \subseteq U$. Hence $p \in V_{\delta}(p) \subseteq C \subseteq U$, and thus X is connected im kleinen at p . \square

Proposition 5.1.6 ([13]; Theorem 27.9). *X is locally connected iff each component of each open set is open.*

Proof.

(\implies) Let X be locally connected. Let U be an open subset of X . Let $p \in C$, where C is a component of U . Since X is locally connected, there exists an open connected set, say, V such that $p \in V \subseteq U$. Since C is a component of U containing p , we must have that $V \subseteq C$. Hence C is open.

(\impliedby) Suppose that each component of each open set is open. Let $x \in X$ and U be an open set such that $x \in U$. Let C be the component of U which contains p . Then C is open connected and so X is locally connected. \square

We have given attention to local connectedness for the purpose of providing a background to a special class of continua called *Peano continua*. Named in honor of Giuseppe Peano, we now present its definition.

Definition 5.1.7. A locally connected continuum a Peano continuum.

Remark 5.1.8.

Hence a non-empty metric space which is compact, connected and locally connected is a Peano continuum. We shall see that Peano continua are well behaved since they can be

written as the union of finitely many arbitrarily small Peano subcontinua. We will do this next by making use of a special property.

5.2 Local connectedness of 2^X and $\mathcal{C}(X)$

Property S is a notion due to Sierpinski ([10]). This will play a crucial role in determining the structure of *locally connected spaces* and *Peano continua*. We begin with a definition.

Definition 5.2.1. Let X be a metric space. A non-empty subset Y of X is said to have Property S provided that for each $\varepsilon > 0$, there are finitely many connected subsets A_1, \dots, A_n of Y such that

$$Y = \bigcup_{i=1}^n A_i \text{ and } \text{diam}(A_i) < \varepsilon \text{ for each } i = 1, \dots, n.$$

We aim to show that for compact metric spaces, being a Peano space is equivalent to having Property S.

Theorem 5.2.2 ([7]; Theorem 8.3.). *If a metric space X has Property S, then X is locally connected.*

Proof.

By Proposition 5.1.4, it suffices to show that X is connected im kleinen at every point in X . Let $p \in X$ be arbitrary and $\varepsilon > 0$, then since X has Property S, there are finitely many connected sets A_1, \dots, A_n such that

$$X = \bigcup_{i=1}^n A_i \text{ and } \text{diam}(A_i) < \frac{\varepsilon}{2} \text{ for each } i = 1, \dots, n.$$

Let $G = \bigcup\{A_i \mid p \in \overline{A_i}\}$. Clearly, G is connected, $\text{diam}(G) < \varepsilon$ and since $p \notin \overline{X \setminus G}$, G is a neighbourhood of p . Hence $p \in X \setminus \overline{(X \setminus G)} \subseteq G \subseteq V_\varepsilon(p)$. Thus X is connected im kleinen at p , and hence X is locally connected. \square

Theorem 5.2.3 ([7]; Theorem 8.4.). *A compact metric space X is locally connected iff X has Property S. Thus, in particular, a continuum X is a Peano continuum iff for each $\varepsilon > 0$, X is the union of finitely many subcontinua each having diameter $< \varepsilon$.*

Proof.

(\implies) Let X be locally connected and let $\varepsilon > 0$. Since each point of X has a neighbourhood base of open connected sets, then for each $x_j \in X$ let us denote by U_j , the open connected set such that $x_j \in U_j$ and $\text{diam}(U_j) < \varepsilon$. This family of sets $\{U_j\}_{j \in J}$ is a covering for X , hence by the compactness of X , $\{U_j\}_{j \in J}$ must have a finite subcovering. That is, $X = \bigcup_{j=1}^n U_j$. Thus X has Property S.

Furthermore, if X is a continuum then by replacing U_j with $\overline{U_j}$ in the above argument, we have that $\overline{U_j}$ is connected (since it is the closure of a connected set) and compact (since it is a closed subset of a compact space). We also have $\text{diam}(\overline{U_j}) = \text{diam}(U_j) < \varepsilon$, therefore X will be the union of finitely many subcontinua with diameter $< \varepsilon$.

(\impliedby) This is immediate by Theorem 5.2.2. \square

Remark 5.2.4. Notice that the subcontinua which are guaranteed in the second part of Theorem 5.2.3, may not be Peano continua. Although it is true that it can be chosen such that it is Peano, this fact is not trivial to prove. So our next goal is to obtain this result (see Theorem 5.2.10), which is one of the most important results for the structure of Peano continua.

Proposition 5.2.5. *let X be a metric space and let $Y \subseteq X$ such that Y has Property S. Then for any Z such that $Y \subseteq Z \subseteq \overline{Y}$, Z has Property S and hence is locally connected.*

Proof.

Let $\varepsilon > 0$ and $Y = \bigcup_{i=1}^n A_i$ where A_i is connected and has diameter $< \varepsilon$, for $1 \leq i \leq n$. Then since $\overline{Y} = \bigcup_{i=1}^n \overline{A_i}$,

$$Z = \bigcup_{i=1}^n (\overline{A_i} \cap Z).$$

Now $A_i \subseteq (\overline{A_i} \cap Z) \subseteq \overline{A_i}$ for each $i = 1, \dots, n$. Thus $\overline{A_i} \cap Z$ is connected and $\text{diam}(\overline{A_i} \cap Z) < \varepsilon$. Hence by Definition 5.2.1, Z has Property S and therefore is locally connected by Theorem 5.2.2. \square

The next technical definition, will be required for Theorem 5.2.10.

Definition 5.2.6. Let X be a metric space and let $\varepsilon > 0$. An $\mathcal{S}(\varepsilon)$ – chain is a non-empty, finite, indexed collection $\mathfrak{L} = \{L_1, \dots, L_n\}$ of subsets of X satisfying :

1. $L_i \cap L_{i+1} \neq \emptyset$ for each $i = 1, \dots, n - 1$;
2. L_i is connected for each $i = 1, \dots, n$;
3. $\text{diam}(L_i) < \varepsilon \cdot 2^{-i}$ for each $i = 1, \dots, n$.

If $\mathfrak{L} = \{L_1, \dots, L_n\}$ is an $\mathcal{S}(\varepsilon)$ -chain, then each $L_i \in \mathfrak{L}$ is called a *link* of \mathfrak{L} . If $x \in L_1$ and $y \in L_n$, then we say that \mathfrak{L} is an $\mathcal{S}(\varepsilon)$ -chain from x to y . If $A \subseteq X$, then we define $\mathcal{S}(A, \varepsilon)$ as follows :

$$\mathcal{S}(A, \varepsilon) = \{y \in X \mid \text{there is an } \mathcal{S}(\varepsilon)\text{-chain from some point of } A \text{ to } y\}$$

We now obtain a few properties of the sets $\mathcal{S}(A, \varepsilon)$.

Proposition 5.2.7 ([7]; Proposition 8.7.). *If a metric space X has Property S, then for any subset A of X and any $\varepsilon > 0$, $\mathcal{S}(A, \varepsilon)$ has Property S.*

Proof.

Fix $\delta > 0$. We want to show that $\mathcal{S}(A, \varepsilon)$ has Property S. To do this choose a $k \in \mathbb{N}$ such that

$$\sum_{i=k}^{\infty} \varepsilon \cdot 2^{-i} < \frac{\delta}{4} \tag{5.1}$$

Let $K = \{y \in \mathcal{S}(A, \varepsilon) \mid \text{there is an } \mathcal{S}(\varepsilon)\text{-chain with at most } k \text{ links from some point of } A \text{ to } y\}$.

Since X has Property S, there is a finite cover of X by connected sets each having diameter $< \varepsilon \cdot 2^{-k-1}$. Let E_1, \dots, E_n denote the members of this cover which intersect K (note that if none of them intersects K , then $K = \emptyset$ but $A \subseteq K$, therefore $A = \emptyset$, which is a contradiction).

Claim : $E_i \subseteq \mathcal{S}(A, \varepsilon)$ for each $i = 1, \dots, n$.

We first take note of the following facts:

$$K \subseteq \bigcup_{i=1}^n E_i \quad (5.2)$$

$$E_i \cap K \neq \emptyset \text{ for each } i \quad (5.3)$$

$$E_i \text{ is connected for each } i \quad (5.4)$$

$$\text{diam}(E_i) < \varepsilon \cdot 2^{-k-1} \text{ for each } i \quad (5.5)$$

To prove the claim, fix $i \in \{1, \dots, n\}$, then by (5.3) there is an $\mathcal{S}(\varepsilon)$ -chain $\{L_1, \dots, L_t\}$ with $t \leq k$ from a point of A to a point of $E_i \cap K$. Then by (5.4), (5.5) and Definition 5.2.6, we see that $\{L_1, \dots, L_t, L_{t+1} = E_i\}$ is an $\mathcal{S}(\varepsilon)$ -chain from a point of A to any point of E_i . Hence the claim is true.

For each $i = 1, \dots, n$, let \mathfrak{P}_i denote the collection of sets M satisfying the following conditions:

$$M \subseteq \mathcal{S}(A, \varepsilon) \quad (5.6)$$

$$M \cap E_i \neq \emptyset \quad (5.7)$$

$$M \text{ is connected} \quad (5.8)$$

$$\text{diam}(M) < \frac{\delta}{4} \quad (5.9)$$

Now let $B_i = \bigcup \mathfrak{P}_i$ for each $i = 1, \dots, n$. Note that each E_i satisfies (5.6) (since $E_i \subseteq \mathcal{S}(A, \varepsilon)$ for each $i = 1, \dots, n$), (5.7) since $E_i \neq \emptyset$, (5.8) by (5.4), and (5.9) by (5.1) and (5.5). Hence for each i ,

$$E_i \subseteq B_i \quad (5.10)$$

We will show that $\mathcal{S}(A, \varepsilon) = \bigcup_{i=1}^n B_i$, some $n < \infty$, where each B_i is connected and has diameter $< \delta$, for then $\mathcal{S}(A, \varepsilon)$ will have Property S.

By (5.4), (5.7) and (5.8), each B_i is connected. It follows from (5.1), (5.5), (5.7) and (5.9) that each B_i has diameter $< \delta$. Now by (5.6), we have that $B_i \subseteq \mathcal{S}(A, \varepsilon)$ for each $i = 1, \dots, n$.

So it only remains for us to show that

$$\mathcal{S}(A, \varepsilon) \subseteq \bigcup_{i=1}^n B_i \quad (5.11)$$

Let $y \in \mathcal{S}(A, \varepsilon)$. We have that $K \subseteq \bigcup_{i=1}^n B_i$ (by (5.2) and (5.10)). Thus to prove (5.11), assume that $y \notin K$. Since $y \in \mathcal{S}(A, \varepsilon)$, there is an $\mathcal{S}(\varepsilon)$ -chain $\mathfrak{L} = \{L_1, \dots, L_m\}$ from a point of A to y . Since $y \notin K$, $m > k$. Let

$$H = \bigcup_{i=k}^m L_i, \text{ written as } \bigcup \mathfrak{L}.$$

By the definition of K we have that $L_k \subseteq K$. Hence by (5.2), $L_k \cap E_i \neq \emptyset$ for some $i \in \{1, \dots, n\}$. We show that $H \subseteq B_i$ by showing that H satisfies (5.6) - (5.9). By the definition of $\mathcal{S}(A, \varepsilon)$, $\bigcup \mathfrak{L} \subseteq \mathcal{S}(A, \varepsilon)$, so H satisfies (5.6). Since $L_k \cap E_i \neq \emptyset$, H satisfies (5.7). By the first two properties of Definition 5.2.6, H satisfies (5.8). Furthermore $L_i \cap L_{i+1} \neq \emptyset$ for each $i = 1, \dots, m-1$ (by Definition 5.2.6). Therefore $\text{diam}(H) \leq \sum_{i=k}^m \text{diam}(L_i)$, and since $\text{diam}(L_i) < \varepsilon \cdot 2^{-i}$ for each $i = 1, \dots, m$ (by Definition 5.2.6), we have that $\text{diam}(H) \leq \sum_{i=k}^m \varepsilon \cdot 2^{-i}$. By (5.1), H satisfies (5.9). So we have shown that H satisfies (5.6)-(5.9), and hence $H \subseteq B_i$. Since $y \in L_m \subseteq H$, this implies that $y \in B_i$. Therefore (5.11) holds and $\mathcal{S}(A, \varepsilon)$ has Property S. \square

Lemma 5.2.8 ([7]; Lemma 8.8.). *Let X be a metric space, A be a non-empty subset of X and let $\varepsilon > 0$. Then the following conditions hold :*

- (1) $\text{diam}(\mathcal{S}(A, \varepsilon)) \leq \text{diam}(A) + 2\varepsilon$;
- (2) *If A is connected, then $\mathcal{S}(A, \varepsilon)$ is connected;*
- (3) *If X has Property S, then $\mathcal{S}(A, \varepsilon)$ is an open subset of X .*

Proof.

- (1) Let $x, y \in \mathcal{S}(A, \varepsilon)$, such that there is an $\mathcal{S}(\varepsilon)$ -chain \mathfrak{L} from a point in A to x , and an

$\mathcal{S}(\varepsilon)$ – chain \mathfrak{K} from a point in A to y .

$$\begin{aligned} d(x, y) &\leq \text{diam}(\bigcup \mathfrak{L}) + \text{diam}(A) + \text{diam}(\bigcup \mathfrak{K}) \\ &\leq \sum (\varepsilon \cdot 2^{-i}) + \text{diam}(A) + \sum (\varepsilon \cdot 2^{-i}) \\ &\leq \text{diam}(A) + 2\varepsilon \end{aligned}$$

as required.

(2) Take any $x, y \in \mathcal{S}(A, \varepsilon)$. There exists an $\mathcal{S}(\varepsilon)$ – chain \mathfrak{L} from some point of A to x and there exists an $\mathcal{S}(\varepsilon)$ – chain \mathfrak{D} from some point of A to y . Now $\bigcup \mathfrak{L}$ and $\bigcup \mathfrak{D}$ are connected (using Definition 5.2.6, (1) and (2)), and $\bigcup \mathfrak{L} \subseteq \mathcal{S}(A, \varepsilon)$, $\bigcup \mathfrak{D} \subseteq \mathcal{S}(A, \varepsilon)$ and $A \subseteq \mathcal{S}(A, \varepsilon)$.

Furthermore, $A \cap (\bigcup \mathfrak{L}) \neq \emptyset$, $A \cap (\bigcup \mathfrak{D}) \neq \emptyset$, and since A is connected, we have $A \cup (\bigcup \mathfrak{L}) \cup (\bigcup \mathfrak{D})$ is a connected subset of $\mathcal{S}(A, \varepsilon)$. Thus x and y lie together in a connected subset of $\mathcal{S}(A, \varepsilon)$. This shows that $\mathcal{S}(A, \varepsilon)$ is connected by Theorem 1.1.23.

(3) Let $y \in \mathcal{S}(A, \varepsilon)$. Then there is an $\mathcal{S}(\varepsilon)$ – chain $\{L_1, \dots, L_n\}$ from some point of A to y . By Theorem 5.2.2, X is a locally connected so there is a connected open subset U of X such that $y \in U$ and

$$\text{diam}(U) < \varepsilon \cdot 2^{-n-1}.$$

Then, $\{L_1, \dots, L_n, L_{n+1} = U\}$ is an $\mathcal{S}(\varepsilon)$ – chain from a point of A to any point of U . Hence, $U \subseteq \mathcal{S}(A, \varepsilon)$ and so $\mathcal{S}(A, \varepsilon)$ is an open subset of X . \square

Theorem 5.2.9. *If X is a metric space and X has Property S, then for any $\varepsilon > 0$, X is the union of finitely many connected sets each of which has Property S and is of diameter $< \varepsilon$. Furthermore, these sets may be chosen to be open in X or closed in X .*

Proof.

Since X has Property S, $X = \bigcup_{i=1}^n A_i$, some $n < \infty$, where each A_i is connected and of diameter $< \frac{\varepsilon}{3}$. By Proposition 5.2.7 and Lemma 5.2.8, $\mathcal{S}(A_i, \frac{\varepsilon}{3})$ has Property S, is open and connected in X for each $i = 1, 2, \dots, n$, and $A_i \subseteq \mathcal{S}(A_i, \frac{\varepsilon}{3})$ for each i . Thus $X = \bigcup_{i=1}^n \mathcal{S}(A_i, \frac{\varepsilon}{3})$.

Also, $\text{diam}(\mathcal{S}(A_i, \frac{\varepsilon}{3})) \leq \text{diam}(A_i) + 2(\frac{\varepsilon}{3}) < \frac{\varepsilon}{3} + 2(\frac{\varepsilon}{3}) = \varepsilon$, for each $i = 1, 2, \dots, n$. Hence $\mathcal{S}(A_i, \frac{\varepsilon}{3})$ satisfies the desired properties.

Taking the closed sets $\overline{\mathcal{S}(A_i, \frac{\varepsilon}{3})}$ for each $i = 1, \dots, n$, we see that this also satisfies the required properties. This follows because $\text{diam}(\overline{\mathcal{S}(A_i, \frac{\varepsilon}{3})}) = \text{diam}(\mathcal{S}(A_i, \frac{\varepsilon}{3}))$, and Property S is preserved under taking closures as can be seen from the proof of Proposition 5.2.5. Hence the proof is complete. \square

We now come to the result that we have been working towards obtaining.

Theorem 5.2.10 ([7]; Theorem 8.10.). *If X is a Peano continuum, then for any $\varepsilon > 0$, X is the union of finitely many Peano continua each of which has diameter $< \varepsilon$.*

Proof.

Since X is a Peano space, X must have Property S (by Theorem 5.2.3). Hence by Theorem 5.2.9, X is the union of finitely many connected sets A_1, \dots, A_n each of which has Property S, is of diameter $< \varepsilon$ and is closed. Each A_i is compact since each is a closed subset of compact X , and hence we have each A_i is a continuum. Furthermore, A_i has Property S for each $i = 1, \dots, n$, hence each A_i is a Peano continuum, as required. \square

We now come to an important result by Wojdyslawski (see [15]) presented in Theorem 5.2.11 and Theorem 5.2.12, below.

Theorem 5.2.11. *If a continuum X is locally connected then 2^X and $\mathcal{C}(X)$ are locally connected.*

Proof.

Let X be locally connected, then X has Property S by Theorem 5.2.3. For a given $\varepsilon > 0$, take $X = \bigcup_{i=1}^n C_i$, such that C_i is connected and open and $\text{diam}(C_i) < \varepsilon$. Then since X is compact, there exists a Lebesgue number $\delta > 0$ (as in Lemma 1.1.20) such that if $A \subseteq X$ and $\text{diam}(A) < \delta$ then $A \subseteq C_i$ for some i .

Let $\nu(\varepsilon) = \min\{\delta, \frac{\varepsilon}{2}\}$. If $d(x, y) < \nu(\varepsilon)$ for $x, y \in X$, then $\{x, y\} \subseteq C_i \subseteq \overline{C_i}$, for some i and $\text{diam}(\overline{C_i}) = \text{diam}(C_i) < \varepsilon$. Hence for any $x, y \in X$ and $d(x, y) < \nu(\varepsilon)$, there exists a continua C such that $x, y \in C$ and $\text{diam}(C) < \varepsilon$.

Take $A, B \in 2^X$, with $\rho(A, B) < \nu(\varepsilon)$ with a view to showing that 2^X is connected im kleinen at $B \in 2^X$. Let

$$C = \{x \in X \mid \text{there exists continua } C_x, \text{ such that } \text{diam}(C_x) < \varepsilon, x \in C_x \text{ and } C_x \cap A \neq \emptyset\}.$$

Note that $A \subseteq C$. We now show that $\rho(A, C) \leq \varepsilon$.

We have that $A \subseteq V_\varepsilon(C)$, so it remains to show that $C \subseteq V_\varepsilon(A)$. Take $y \in C$, then $y \in C_y$, $C_y \cap A \neq \emptyset$ and $\text{diam}(C_y) < \varepsilon$. For $a \in C_y \cap A$, we have that $d(y, a) < \varepsilon$, therefore $y \in V_\varepsilon(A)$. Hence $C \subseteq V_\varepsilon(A)$ and $\rho(A, C) \leq \varepsilon$, as required.

Furthermore by the triangle inequality, we have that $\rho(B, C) < 2\varepsilon$.

Claim: $A \cup B \subseteq C$.

We know that $A \subseteq C$, so it suffices to show that $B \subseteq C$. Take $b \in B$, then there exists $a \in A$ such that $d(a, b) < \nu(\varepsilon)$. Thus there exists a continuum C' such that $a, b \in C'$, $\text{diam}(C') < \varepsilon$ and $C' \cap A \neq \emptyset$. Thus $b \in C$, as required.

Next we show that every component of C intersects A and B .

Let D be a component of C containing x (that is, $C_x \subseteq D$). Then $x \in C_x$, so $C_x \cap A \neq \emptyset$. Take $a \in C_x \cap A$. We know that $A \subseteq V_{\nu(\varepsilon)}(B)$, because $\rho(A, B) < \nu(\varepsilon)$. Hence there exists $b \in B$ such that $d(a, b) < \nu(\varepsilon)$, and this implies that there exists a continuum E such that $a, b \in E$ and $\text{diam}(E) < \varepsilon$. So $b \in C$, since $E \cap A \neq \emptyset$. Now $a \in C_x \cap E \neq \emptyset$ and $x \in C_x \cup E \subseteq D$ since D is a component, therefore it follows that $D \cap B \neq \emptyset$. Now $A, B \in 2^X$ and every component of C intersects A and B . Thus, by Theorem 4.2.8, there exists segments $\sigma_1 : [0, 1] \rightarrow 2^X$ from A to C , and $\sigma_2 : [0, 1] \rightarrow 2^X$ from B to C .

Let $\sigma_1([0, 1]) \cup \sigma_2([0, 1]) = \mathcal{A}$. Each of $\sigma_1([0, 1])$ and $\sigma_2([0, 1])$ is compact and connected and $A, B \in \mathcal{A}$. Since $C \in \sigma_1([0, 1]) \cap \sigma_2([0, 1])$, we have that \mathcal{A} is a continuum. Now

$$\begin{aligned} \text{diam}(\mathcal{A}) &\leq \text{diam}(\sigma_1([0, 1])) + \text{diam}(\sigma_2([0, 1])) \\ &\leq \rho(A, C) + \rho(B, C) \\ &\leq 3\varepsilon. \end{aligned}$$

Thus it follows from Theorem 5.1.5 that 2^X is connected im kleinen at $B \in 2^X$. Since B is

arbitrary, 2^X is locally connected by Proposition 5.1.4.

Now, if $A, B \in \mathcal{C}(X)$, then $\mathcal{A} \subseteq \mathcal{C}(X)$ by Theorem 4.2.9, where \mathcal{A} is defined as above. Hence similarly $\mathcal{C}(X)$ is locally connected. \square

Theorem 5.2.12. *For a metric continuum X , if 2^X or $\mathcal{C}(X)$ is locally connected then X is locally connected.*

Proof.

Let $\varepsilon > 0$. Similar to the proof in the first part of Theorem 5.2.11, there exists $\nu(\varepsilon) > 0$ which we may assume is $< \frac{\varepsilon}{4}$, such that whenever $\rho(A, B) < \nu(\varepsilon)$, then there exists continuum \mathcal{A} containing A and B with $\text{diam}(\mathcal{A}) < \frac{\varepsilon}{4}$. Take $a, b \in X$ such that $d(a, b) < \nu(\varepsilon)$, then $\phi(a) = \{a\}$, $\phi(b) = \{b\}$ and $\rho(\{a\}, \{b\}) < \nu(\varepsilon)$. Hence there exists a continuum \mathcal{A} of 2^X such that $\{a\}, \{b\} \in \mathcal{A}$ and $\text{diam}(\mathcal{A}) < \frac{\varepsilon}{4}$. Since $\{a\}, \{b\} \in \mathcal{C}(X)$, $\mathcal{A} \cap \mathcal{C}(X) \neq \emptyset$ and by Theorem 1.4.12, it follows that $\Gamma(\mathcal{A})$ is a continuum in X with $a, b \in \Gamma(\mathcal{A})$.

Claim: $\text{diam}(\Gamma(\mathcal{A})) < \varepsilon$.

Take any $x, y \in \Gamma(\mathcal{A})$. Then there exists $A \in \mathcal{A}$, $B \in \mathcal{A}$ such that $x \in A$ and $y \in B$. Now $\rho(\{a\}, A) < \frac{\varepsilon}{4}$, so $A \subseteq V_{\frac{\varepsilon}{4}}(\{a\})$. Hence since $x \in A$, we have $d(x, a) < \frac{\varepsilon}{4}$.

Similarly, $d(y, b) < \frac{\varepsilon}{4}$. Hence

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, b) + d(b, y) \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}. \end{aligned}$$

Thus $\text{diam}(\Gamma(\mathcal{A})) = \sup\{d(x, y) \mid x, y \in \Gamma(\mathcal{A})\} \leq \frac{3\varepsilon}{4} < \varepsilon$. Thus by Theorem 5.1.5 and Proposition 5.1.4, X is locally connected. \square

5.3 Property K

Earlier, in Chapter 4, we considered the contractibility of 2^X and $\mathcal{C}(X)$, and showed that if one of 2^X and $\mathcal{C}(X)$ is contractible then so is the other. We now give attention to a useful sufficient condition for 2^X and $\mathcal{C}(X)$ to be contractible. The condition is due to Kelley and is called *Property K*. Let us define the property and consider relevant results as a consequence of it.

Definition 5.3.1. A continuum X has Property K provided that given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\text{if } a, b \in X, \quad d(a, b) < \delta(\varepsilon) \quad \text{and } a \in A \in \mathcal{C}(X)$$

then there exists $B \in \mathcal{C}(X)$ such that $b \in B$ and $\rho(A, B) < \varepsilon$.

Lemma 5.3.2. *Any Peano continuum has Property K.*

Proof.

Let X be a Peano continuum with metric d and let $\varepsilon > 0$. By Theorem 5.2.3 and Theorem 5.2.9, there is a finite open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of X such that each U_i is connected and has diameter $< \varepsilon$.

Now X compact, thus every open cover of X has a Lebesgue number by Lemma 1.1.20.

Let δ denote a Lebesgue number of \mathcal{U} . Now let $p, q \in X$ such that $d(p, q) < \delta$, and let $A \in \mathcal{C}(X)$ such that $p \in A$. Then there exists j such that $p, q \in U_j$. Let $B = \overline{U_j} \cup A$. B is compact since it is the union of two compact sets and connected (since $p \in U_j \cap A$). Hence $B \in \mathcal{C}(X)$. Furthermore, $q \in B$, so it remains for us to check that $\rho(A, B) < \varepsilon$.

Now we have that $A \subseteq B \subseteq V_\varepsilon(B)$. To show that $B \subseteq V_\varepsilon(A)$, we need only show that $\overline{U_j} \subseteq V_\varepsilon(A)$, since $A \subseteq V_\varepsilon(A)$. Take any $z \in \overline{U_j}$. Now $p \in \overline{U_j}$ and $p \in A$, so $d(z, p) \leq \text{diam}(\overline{U_j}) = \text{diam}(U_j) < \varepsilon$. Therefore $z \in V_\varepsilon(p)$, that is, $\overline{U_j} \subseteq V_\varepsilon(A)$. Hence $B \subseteq V_\varepsilon(A)$, which implies that $\rho(A, B) < \varepsilon$. Thus Property K is satisfied. \square

Throughout the rest of this section, X will be a continuum. We now aim to show that if X has Property K then 2^X and $\mathcal{C}(X)$ are contractible. To do this, we shall required the concept of a containment hyperspace and we will define a new function F_ω .

Definition 5.3.3. Let X be a metric continuum and $K \subseteq X$ be a subcontinuum. Define 2_K^X and $\mathcal{C}_K(X)$ as follows :

$$2_K^X = \{A \in 2^X \mid A \supseteq K\} \quad \text{and} \quad \mathcal{C}_K(X) = \{A \in \mathcal{C}(X) \mid A \supseteq K\}$$

We refer to 2_K^X as the containment hyperspace of 2^X and $\mathcal{C}_K(X)$ as the containment hyper-

space of $\mathcal{C}(X)$. Furthermore, for a point $p \in X$, we write 2_p^X and $\mathcal{C}_p(X)$ for the containment hyperspaces of the singleton set $\{p\}$.

Remark 5.3.4.

1. $\mathcal{C}_K(X)$ is closed in $\mathcal{C}(X)$:

Take $B \in \overline{\mathcal{C}_K(X)}$ with $B \in \mathcal{C}(X)$. Then there exists $(B_n)_n$ in $\mathcal{C}_K(X)$ such that $B_n \rightarrow B$. Now $B_n \supseteq K$ for each n , implies that $K \subseteq B$ (by Lemma 1.3.10) and thus we have $B \in \mathcal{C}_K(X)$.

Hence $\mathcal{C}_K(X)$ is closed in $\mathcal{C}(X)$. □

2. Similarly, we have that 2_K^X is closed in 2^X .

Definition 5.3.5. Let X be a continuum and $\omega : \mathcal{C}(X) \rightarrow [0, 1]$, where ω is the Whitney map on $\mathcal{C}(X)$ (as constructed in Chapter 3). Define $F_\omega : X \times [0, \omega(X)] \rightarrow 2^{2^X}$ by

$$F_\omega(x, t) = \{A \in \mathcal{C}(X) | x \in A \in \omega^{-1}(t)\} \quad \text{for each } (x, t) \in X \times [0, \omega(X)]$$

Remark 5.3.6.

Taking Definition 5.3.5 into consideration, we have for each $(x, t) \in X \times [0, \omega(X)]$,

$$\begin{aligned} F_\omega(x, t) &= \{A \in \mathcal{C}(X) | x \in A \in \omega^{-1}(t)\} \\ &= \mathcal{C}_x(X) \cap \omega^{-1}(t) \end{aligned}$$

Proposition 5.3.7. $F_\omega(x, t) \in 2^{2^X}$ for each $(x, t) \in X \times [0, \omega(X)]$.

Proof.

Fix $(x, t) \in X \times [0, \omega(X)]$. $\mathcal{C}(X)$ is compact (by Theorem 1.3.12) and $\mathcal{C}_x(X)$ is closed in $\mathcal{C}(X)$, hence $\mathcal{C}_x(X)$ must be compact. Also, $\omega^{-1}(t)$ is closed, since the inverse of a singleton is closed by the continuity of ω . Thus $F_\omega(x, t) = \mathcal{C}_x(X) \cap \omega^{-1}(t)$ is closed in 2^X

Claim: $F_\omega(x, t) \neq \emptyset$.

For $\{x\}, X \in \mathcal{C}(X)$, we have that every component of X intersects $\{x\}$ because X is connected. By Theorem 4.1.13, there exists an order arc α in 2^X from $\{x\}$ to X .

By Lemma 4.1.15, since $\{x\} \in \mathcal{C}(X)$ it follows that $\alpha \subseteq \mathcal{C}(X)$ and so $\omega : \alpha[0, 1] \longrightarrow [0, \omega(X)]$. Now $\omega(\alpha(0)) = \omega(\{x\}) = 0$ and $\omega(\alpha(1)) = \omega(X)$. By the continuity of ω and the connectedness of $\alpha[0, 1]$, $\omega(\alpha[0, 1])$ is connected, and so it must be all of $[0, \omega(X)]$. Thus there exists $A \in \alpha$ such that $\omega(A) = t$, where

$$[0, 1] \xrightarrow{\alpha} \alpha[0, 1] \xrightarrow[\text{onto}]{\omega} [0, \omega(X)] \quad \text{with } \alpha(0) = \{x\}, \quad \alpha(1) = X.$$

Now $A \in F_\omega(x, t)$, so $F_\omega(x, t) \neq \emptyset$. Hence $F_\omega(x, t) \in 2^{2^X}$, since it is a closed and non-empty subset of 2^X . \square

Definition 5.3.8. Let (Y, d_Y) , (Z, d_Z) and (M, d_M) be metric spaces and $f : Y \times Z \longrightarrow M$. We say that f is equi-continuous in the first variable provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $z \in Z$

$$d_M(f(y_1, z), f(y_2, z)) < \varepsilon \quad \text{whenever } d_Y(y_1, y_2) < \delta$$

Let $f_z(y) = f(y, z)$ for each $y \in Y$ and $z \in Z$, then we say that the family $\{f_z | z \in Z\}$ is equi-continuous. Similarly, f is equi-continuous in the second variable provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in Y$

$$d_M(f(y, z_1), f(y, z_2)) < \varepsilon \quad \text{whenever } d_Z(z_1, z_2) < \delta.$$

The next lemma is a general metric space result which says that if a function is equi-continuous in each of its variables (of its domain), then it is in fact continuous.

Lemma 5.3.9. Let (Y, d_Y) , (Z, d_Z) and (M, d_M) be metric spaces and let D denote the usual metric for $Y \times Z$, that is, $D = \sqrt{d_Y^2 + d_Z^2}$.

If $f : Y \times Z \longrightarrow M$ is equi-continuous in each variable then f is uniformly continuous.

Proof.

Let $\varepsilon > 0$. f is equi-continuous in the first variable, therefore there exists $\delta_1 > 0$, such that if $d_Y(y_1, y_2) < \delta_1$ for $y_1, y_2 \in Y$ then $d_M(f(y_1, z), f(y_2, z)) < \frac{\varepsilon}{2}$ for each $z \in Z$. Similarly, since f is equi-continuous in the second variable there exists $\delta_2 > 0$, such that if $d_Z(z_1, z_2) < \delta_2$

for $z_1, z_2 \in Z$ then $d_M(f(y, z_1), f(y, z_2)) < \frac{\varepsilon}{2}$ for each $y \in Y$.

Take $\delta = \min\{\delta_1, \delta_2\} > 0$ and let $D[(y_1, z_1), (y_2, z_2)] < \delta$ for $(y_1, z_1), (y_2, z_2) \in Y \times Z$.

Now, $d_Y \leq D$ and $d_Z \leq D$, therefore $d_Y(y_1, y_2) < \delta$ for $y_1, y_2 \in Y$ and $d_Z(z_1, z_2) < \delta$ for $z_1, z_2 \in Z$. So we have that,

$$\begin{aligned} d_M(f(y_1, z_1), f(y_2, z_2)) &\leq d_M(f(y_1, z_1), f(y_2, z_1)) + d_M(f(y_2, z_1), f(y_2, z_2)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Hence f is uniformly continuous. □

We aim to show that F_ω is continuous. The next two lemmas focus on this.

Lemma 5.3.10 ([8]; Lemma 16.14.). *If X is a continuum, then $F_\omega : X \times [0, \omega(X)] \rightarrow 2^{2^X}$ is equi-continuous in the second variable.*

Proof.

Let ρ^1 denote the Hausdorff metric for 2^{2^X} (as induced by ρ). Let D denote the metric on $X \times [0, \omega(X)]$ given by $D[(x_1, t_1), (x_2, t_2)] = \sqrt{d(x_1, x_2)^2 + |t_1 - t_2|^2}$, for each $(x_1, t_1), (x_2, t_2) \in X \times [0, \omega(X)]$.

Let $\varepsilon > 0$. We want to show that there exists a $\delta > 0$ such that for each $x \in X$, $\rho^1(F_\omega(x, t_1), F_\omega(x, t_2)) < \varepsilon$ whenever $D[(x, t_1), (x, t_2)] = |t_1 - t_2| < \delta$ for some $t_1, t_2 \in [0, \omega(X)]$.

Choose $\delta = \eta(\varepsilon)$ as in Lemma 3.2.2 and assume that $|t_1 - t_2| < \delta = \eta(\varepsilon)$ for $t_1, t_2 \in [0, \omega(X)]$.

For convenience, assume that $t_1 < t_2$. We will show that $F_\omega(x, t_1) \subseteq V_\varepsilon(F_\omega(x, t_2))$.

Let $A_1 \in F_\omega(x, t_1)$. It suffices to show that there exists $A_2 \in F_\omega(x, t_2)$ such that $\rho(A_1, A_2) < \varepsilon$. Since every component of X intersects A_1 , then by Theorem 4.1.13, there exists an order arc α from A_1 to X , and since $A_1 \in \mathcal{C}(X)$ we have that $\alpha \subseteq \mathcal{C}(X)$ (by Lemma 4.1.15).

Now since $\omega : \alpha[0, 1] \rightarrow [t_1, \omega(X)]$ is continuous and onto, there exists $A_2 \in \alpha$ such that $\omega(A_2) = t_2$.

Now $x \in A_1$ and $A_1 \subseteq A_2$, hence $x \in A_2$. Furthermore, $A_2 \in \alpha \subseteq \mathcal{C}(X)$, so $A_2 \in \mathcal{C}(X)$. Thus

$$A_2 \in F_\omega(x, t_2) \tag{5.12}$$

Note : $\omega(A_1) = t_1 < t_2 = \omega(A_2)$, so we have that $|\omega(A_2) - \omega(A_1)| < \eta(\varepsilon) = \delta$.

Since $A_1 \subseteq A_2$, it follows immediately by Lemma 3.2.2 that

$$\rho(A_1, A_2) < \varepsilon \tag{5.13}$$

By Equation (5.12) and Equation (5.13), we have shown that $F_\omega(x, t_1) \subseteq V_\varepsilon(F_\omega(x, t_2))$.

Next, we show $F_\omega(x, t_2) \subseteq V_\varepsilon(F_\omega(x, t_1))$. Let $B_2 \in F_\omega(x, t_2)$, then by Theorem 4.1.13 and Lemma 4.1.15 there exists an order arc $\beta \subseteq \mathcal{C}(X)$ from $\{x\}$ to B_2 . Since ω is continuous, $\omega(\{x\}) = 0 \leq t_1 < t_2 = \omega(B_2)$, and $\omega : \beta[0, 1] \rightarrow [0, t_2]$ is onto, there exists $B_1 \in \beta$ such that $\omega(B_1) = t_1$. Now $\{x\} = \bigcap \beta \subseteq B_1$, therefore $x \in B_1$. Also $B_1 \subseteq B_2$ and since $\beta \subseteq \mathcal{C}(X)$, we have $B_1 \in \mathcal{C}(X)$. Hence $B_1 \in F_\omega(x, t_1)$. Now $|\omega(B_2) - \omega(B_1)| < \eta(\varepsilon)$, thus by Lemma 3.2.2, $\rho(B_1, B_2) < \varepsilon$. Hence $F_\omega(x, t_2) \subseteq V_\varepsilon(F_\omega(x, t_1))$.

So we have shown that $\rho^1(F_\omega(x, t_1), F_\omega(x, t_2)) < \varepsilon$. □

Lemma 5.3.11 ([8]; Lemma 16.14.). *Let X be a continuum with Property K , then $F_\omega : X \times [0, \omega(X)] \rightarrow 2^{2^X}$ is equi-continuous in the first variable.*

Proof.

Let $\varepsilon > 0$ and let D denote the metric on $X \times [0, \omega(X)]$ given by

$$D[(x_1, t_1), (x_2, t_2)] = \sqrt{d(x_1, x_2)^2 + |t_1 - t_2|^2}, \text{ for each } (x_1, t_1), (x_2, t_2) \in X \times [0, \omega(X)].$$

We show there exists $\delta > 0$ such that if for $t \in [0, \omega(X)]$, $D[(x_1, t), (x_2, t)] = d(x_1, x_2) < \delta$, then $\rho^1(F_\omega(x_1, t), F_\omega(x_2, t)) < \varepsilon$. That is,

$$F_\omega(x_1, t) \subseteq V_\varepsilon(F_\omega(x_2, t)) \text{ and } F_\omega(x_2, t) \subseteq V_\varepsilon(F_\omega(x_1, t)).$$

Let $\eta(\frac{\varepsilon}{2})$ be as in Lemma 3.2.2 with $\frac{\varepsilon}{2}$ replacing ε .

Now ω is continuous and $\mathcal{C}(X)$ compact, thus ω is uniformly continuous. Therefore there

exists $\gamma > 0$ such that $\gamma < \frac{\varepsilon}{2}$ and if $K, L \in \mathcal{C}(X)$ such that $\rho(K, L) < \gamma$ then

$$|\omega(K) - \omega(L)| < \eta\left(\frac{\varepsilon}{2}\right) \quad (5.14)$$

Since X has Property K, there exists $\delta(\gamma) > 0$ (with ε replaced by γ) such that if $a, b \in X$, $d(a, b) < \delta(\gamma)$ and $a \in A \in \mathcal{C}(X)$, then there exists $B \in \mathcal{C}(X)$, $b \in B$ and $\rho(A, B) < \gamma$. Choose $\delta = \delta(\gamma)$ and assume

$$D[(x_1, t), (x_2, t)] = d(x_1, x_2) < \delta = \delta(\gamma) \quad (5.15)$$

Take $A \in F_\omega(x_1, t)$. We show that $A \in V_\varepsilon(F_\omega(x_2, t))$. Now $x_1 \in A \in \mathcal{C}(X)$, thus by Equation (5.15) and Property K, there exists $A_1 \in \mathcal{C}(X)$ such that $x_2 \in A_1$ and $\rho(A, A_1) < \gamma$. Therefore by Equation (5.14) since $\omega(A) = t$, we have $|t - \omega(A_1)| < \eta\left(\frac{\varepsilon}{2}\right)$. We will define a set B , after considering 3 cases.

Case 1: if $\omega(A_1) = t$ then $A_1 \in F_\omega(x_2, t)$, and in this case $A \in V_\varepsilon(F_\omega(x_2, t))$.

Case 2: if $\omega(A_1) < t$ then by Theorem 4.1.13 and Lemma 4.1.15, there exists an order arc $\alpha \subseteq \mathcal{C}(X)$ from A_1 to X . Since ω is continuous and $\omega(A_1) < t \leq \omega(X)$, there exists $A_2 \in \alpha$ such that $\omega(A_2) = t$. Now $A_1 = \bigcap \alpha \subseteq A_2$ (that is, $A_1 \subseteq A_2$) and $\omega(A_2) = t$, hence $|\omega(A_2) - \omega(A_1)| < \eta\left(\frac{\varepsilon}{2}\right)$. Hence by Lemma 3.2.2, $\rho(A_1, A_2) < \frac{\varepsilon}{2}$, and therefore

$$\begin{aligned} \rho(A, A_2) &\leq \rho(A, A_1) + \rho(A_1, A_2) \\ &< \gamma + \frac{\varepsilon}{2} \\ &< \varepsilon \quad \text{since } \gamma < \frac{\varepsilon}{2} \end{aligned}$$

$x_2 \in A_1 \subseteq A_2$, $\omega(A_2) = t$ and $A_2 \in \mathcal{C}(X)$, therefore $A_2 \in F_\omega(x_2, t)$. Thus $A \in V_\varepsilon(F_\omega(x_2, t))$.

Case 3 : if $\omega(A_1) > t$, then once again by Theorem 4.1.13 and Lemma 4.1.15, there exists an order arc $\beta \subseteq \mathcal{C}(X)$ from $\{x_2\}$ to A_1 . Using the ideas in *Case 2*, we similarly find $A_3 \in \mathcal{C}(X)$ such that $\rho(A, A_3) < \varepsilon$ and $A_3 \in F_\omega(x_2, t)$.

By the 3 cases, let

$$B = \begin{cases} A_1 & \text{if } \omega(A_1) = t \\ A_2 & \text{if } \omega(A_1) < t \\ A_3 & \text{if } \omega(A_3) > t \end{cases}$$

Then in any case, $B \in F_\omega(x_2, t)$ and $\rho(A, B) < \varepsilon$. Hence $F_\omega(x_1, t) \subseteq V_\varepsilon(F_\omega(x_2, t))$.

With a similar argument, one can show that $F_\omega(x_2, t) \subseteq V_\varepsilon(F_\omega(x_1, t))$.

Therefore $\rho^1(F_\omega(x_1, t), F_\omega(x_2, t)) < \varepsilon$. □

Proposition 5.3.12 ([2]; Proposition 20.11.). *Let X be a continuum and let ω be the Whitney map for $\mathcal{C}(X)$. Then X has Property K iff F_ω is continuous.*

Proof.

(\implies) Assume X has Property K. By Lemma 5.3.10 and Lemma 5.3.11, F_ω is equi-continuous in its first and second variable. Thus by Lemma 5.3.9, it follows that F_ω is uniformly continuous and hence continuous as required.

(\impliedby) Assume $F_\omega : X \times [0, \omega(X)] \longrightarrow 2^{2^X}$ is continuous. Let ρ^1 be the metric on 2^{2^X} induced by ρ . Since $X \times [0, \omega(X)]$ is compact, F_ω must be uniformly continuous. Therefore for $\varepsilon > 0$, there exists $\delta > 0$ such that for $(x_1, t_1), (x_2, t_2) \in X \times [0, \omega(X)]$, if

$$D[(x_1, t_1), (x_2, t_2)] = \sqrt{d(x_1, x_2)^2 + |t_1 - t_2|^2} < \delta \text{ then } \rho^1(F_\omega(x_1, t_1), F_\omega(x_2, t_2)) < \varepsilon.$$

So for $(x_1, t), (x_2, t) \in X \times [0, \omega(X)]$ such that $D[(x_1, t), (x_2, t)] = d(x_1, x_2) < \delta$ and

$A \in F_\omega(x_1, t)$ (that is, A such that $x_1 \in A \in \mathcal{C}(X)$ and $\omega(A) = t$), there exists $B \in F_\omega(x_2, t)$ such that $\rho(A, B) < \varepsilon$ and $x_2 \in B \in \mathcal{C}(X)$. This shows that X has Property K. □

We have established that F_ω is continuous if and only if X has Property K, and are now in a position to show that Property K is a sufficient condition for 2^X and $\mathcal{C}(X)$ to be contractible.

Theorem 5.3.13 ([3]; Theorem 3.3.). *If X is a continuum with Property K, then 2^X and $\mathcal{C}(X)$ are each contractible.*

Proof.

Assume X has Property K, and let F_ω be defined as in Definition 5.3.5. By Proposition

5.3.12, F_ω is continuous. Let $g : \phi(X) \rightarrow X$ defined by

$$g(\{x\}) = x \text{ for each } \{x\} \in \phi(X).$$

Clearly g is continuous. Now define $h : \phi(X) \times [0, \omega(X)] \rightarrow 2^X$ by

$$h(\{x\}, t) = \Gamma F_\omega(g(\{x\}), t) \text{ for each } (\{x\}, t) \in \phi(X) \times [0, \omega(X)],$$

where $\Gamma : 2^{2^X} \xrightarrow{\text{conts}} 2^X$ is the union function. So we have the following mappings,

$$\phi(X) \times [0, \omega(X)] \xrightarrow{g \times id} X \times [0, \omega(X)] \xrightarrow{F_\omega} 2^{2^X} \xrightarrow{\Gamma} 2^X.$$

Since h is the composition of continuous functions, h itself must be continuous. It is true that $\omega^{-1}(0) = \phi(X)$, therefore by the definition of F_ω , $F_\omega(x, 0) = \{\{x\}\}$ for each $x \in X$ and so $h(\{x\}, 0) = \{x\}$ for each $\{x\} \in \phi(X)$.

Also, $\omega^{-1}(\omega(X)) = \{X\}$, thus we have $h(\{x\}, \omega(X)) = X$ for each $\{x\} \in \phi(X)$ (by the definition of F_ω and h). Hence we have shown that $\phi(X)$ is contractible in 2^X . By the Fundamental Theorem of Contractible Hyperspaces, it follows that 2^X and $\mathcal{C}(X)$ are each contractible. \square

Corollary 5.3.14 ([2]; Corollary 20.14.). *If X is a Peano continuum then 2^X and $\mathcal{C}(X)$ are each contractible.*

Proof.

Since X is a Peano continuum, then by Lemma 5.3.2, X must have Property K. Hence 2^X and $\mathcal{C}(X)$ must be contractible by Theorem 5.3.13. \square

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