SHEAR-FREE MODELS FOR RELATIVISTIC FLUIDS WITH HEAT FLOW AND PRESSURE ISOTROPY

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Shear-free models for relativistic fluids with heat flow and pressure isotropy

by

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As the candidate's supervisors, we have approved this thesis for submission.

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Abstract

We model the interior dynamics of a relativistic radiating fluid in a nonstatic spherically symmetric spacetime. The matter distribution takes the form of an imperfect fluid with a nonvanishing radially directed heat flux. The fluid pressure is isotropic and the spherically symmetric spacetime manifold is described by a shear-free line element. In our investigation, the isotropy of pressure is a consistency condition which realises a second order nonlinear ordinary differential equation with variable coefficients in the gravitational potentials. We examine this governing equation by imposing various forms for these potentials and review classes of physically acceptable models that are applicable in relativistic astrophysics. Several new classes of new exact solutions to the condition of pressure isotropy are also found. A comparison of our solutions with earlier well known results is undertaken. A physical analysis of two of the new models is performed where the spatial and temporal evolution of the matter and gravitational variables are probed. We demonstrate that the fluid pressure, energy density and heat flux are regular and well behaved for both models throughout the interior, and our results indicate that one of the models is consistent with the well established core-envelope framework for compact stellar scenarios. We also analyse the energy conditions for the radiating fluid and demonstrate consistent behaviour, with only the dominant condition being violated. Finally, an analysis of the relativistic thermodynamics of two solutions is undertaken in the Israel-Stewart theory and the temperature profiles for both the noncausal and causal cases are presented.

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DECLARATION 1 - PLAGIARISM

I, Byron Brassel, declare that

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Dedication

Dedicated to

the

Milvinae

May they continue forever dancing across the skies.

In Memoriam

Colin Sady Brassel

(1950 - 2004)

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Chapter 1

Introduction

The theory of general relativity has so far been the most successful in describing, for strong gravitational fields in particular, the phenomenon which is gravity. Preceding general relativity, gravitational interactions between bodies in the universe were described in the classical Newtonian framework, which seemed very fruitful for a time. However, there were some astronomical observations which could not be completely explained in the Newtonian structure, and as a consequence, justified the requirement for a new gravitational theory. General relativity describes the interactions between bodies in the universe as a result of their gravitational fields. It should be noted, also, that with this notion, gravity is now part of a much richer and more efficacious structure, the four-dimensional spacetime manifold, and cannot merely be described as an amorphous force.

The sempiternal notion that is evolution, applies also to the grand universe and in order to analyse the dynamics of celestial objects like stars and galaxies, it becomes necessary in many instances, to also understand the structure and behaviour of their gravitational fields. For further and more comprehensive information on the basic principles of general relativity, the reader is encouraged to read Eddington (1923), Foster and Nightingale (1994), Poisson (2004) and Straumann (2004).

In this study we aim to find exact solutions to Einstein's field equations that describe spherically symmetric manifolds which form the foundation of a relativistic model in astrophysics and cosmology. Despite the existence of many classes of exact solutions, very few of them are physically acceptable. Certain solutions that are found may be interesting, mathematically, but may not be appropriate for describing the physics that the problem entails. Despite this, any exact model can contribute to the understanding of the gravitational field, and can provide qualitative features useful for more complicated models. In order to study the physical features of the model, as well as make any kind of prediction, the exact solutions should be used in conjunction with other fundamental physical theories, such as electromagnetism and thermodynamics. Hence, for the success of that marriage, the process of generating exact solutions is a vital prelude point in the study of any self gravitating system. There exist various methods for solving the Einstein field equations exactly, ranging from the theory of Lie analysis to making assumptions, often ad hoc, for the matter and gravitational variables. Other techniques include the use of harmonic maps as well as numerical modus operandi. A comprehensive review of the methods and procedures utilized in the generation of exact solutions is provided by Stephani *et al* (2003).

Static spherically symmetric gravitational fields form the foundation for the description of highly dense objects in astrophysics where the matter distribution is normally considered to be a perfect neutral or charged fluid. The most well known solutions of Einstein's field equations are, not surprisingly, the first to have been found. These are the Schwarzschild exterior and interior solutions (Schwarzschild 1916a, 1916b), and the Reissner-Nordström solution (Nordström 1918, Reissner 1916). The exterior Schwarzschild solution is a vacuum solution which describes the gravitational field in the exterior spacetime of a spherical, uncharged and non-rotating body whereas the interior Schwarzschild solution describes the interior of an object in the limit when the mass density is constant. The Reissner-Nordström solution is a static solution to the Einstein-Maxwell field equations and represents the exterior gravitational field of a charged, non-rotating body. Another important solution is due to Vaidya (1951). The

Vaidya solution is important in situations where the exterior region contains null dust or pressureless radiation. Consequently, relativistic models for non-adiabatic gravitational collapse of massive stars, and dissipation in compact objects rely extensively on the Vaidya metric. Since the discovery of these pioneering solutions, others have been found and have been used to study physically reasonable stellar systems. Of these, the most recent solutions are for charged models obtained by Hansraj and Maharaj (2006), charged superdense stars by Komathiraj and Maharaj (2007a), and charged, compact spheres by Thirukkanesh and Maharaj (2006, 2009). These solutions have, embedded in them, the well known models obtained by Durgapal and Bannerji (1983), Finch and Skea (1989) and Tikekar (1990). There exist, also, exact analytical solutions of the Einstein field equations for shear-free spacetimes, the earliest model being attributed to Kustaanheimo and Qvist (1948). Shear-free models may also contain heat flow in the form of a non-vanishing radial heat flux through the interior and across the surface of a radiating star. Models with heat flux were obtained by Deng and Manheim (1990, 1991), in the field of cosmology, and more recently by Govender and Thirukkanesh (2009) and Maharaj *et al* (2011), in the field of astrophysics. The slightly more important classes of models related to tidal effects are the conformally flat radiating solutions obtained by Banerjee *et al* (1989), and they were applied to relativistic radiating stars by Herrera *et al* (2004, 2006), Maharaj and Govender (2005), and Misthry *et al* (2008). Stellar spacetimes in general, however, are continuums that have nonzero shear, acceleration and expansion, and in this context, the Einstein field equations are highly nonlinear, coupled partial differential equations. For further information and an erudition on the role of the theory of partial differential equations in general relativity, both from a mathematical and arguably philosophical point of view, the reader is requested to seek out Rendall (2008). Due to the difficulties that arise in solving the Einstein field equations for shearing spacetimes, very few solutions have been elucidated in the literature. The well established results are due to Maharaj *et al* (1993), Marklund and Bradley (1999) and Naidu *et al* (2006). These may be applied in cosmological processes where the heat flux is absent, and can be adapted without much difficulty to accommodate a heat flux for particular physical applications.

Upon generating exact solutions, another important notion to consider is the usefulness of the solutions with regard to the analysis of the physics they hope to describe. Are the solutions physically viable? To check this requires an analysis of the behaviour of the gravitational and matter variables, in order to establish whether they adhere to what is expected in physical reality. Over and above this, supposed stellar models need to obey the so-called energy conditions, namely the strong, weak and dominant conditions in general relativity, which were described in detail by Kolassis *et al* (1988), as well as causality, id est; the speed of sound must be positive and less than the speed of light everywhere. A solution which violates these conditions, or causality, cannot be considered realistic. Very few solutions exist in the literature which obey all of the conditions listed above, and this notion supports the fact that relativistic astrophysics, as a concept, is extraordinarily onerous, and very often, other theoretical approaches are required to study such models more successfully. The true usefulness of general relativity lies embedded within the compact objects formed by the gravitational collapse of supermassive stars. We have in mind objects such as neutron stars or their variants, pulsars and magnetars, quark stars or even more compact entities. Mitra (2002) described the final moments of the gravitational collapse of a supermassive star to an ultra compact object, and Germano (2014) showed that the original Schwarzschild metric permits compact objects having reasonable values for central pressure and density. An example of a model for quark stars was also studied in detail by Komathiraj and Maharaj (2007c). Neutron stars are extremely dense bodies and their gravitational and electromagnetic fields (resulting from magnetohydrodynamic dynamo processes in the very dense conducting fluid, before the star settles into its equilibrium configuration) are severely powerful, and this merits the use of general relativity to study their dynamical behaviour. However, the scenario does present problems. These compressed bodies, also called strange stars, often have an immense rotational period and there is no spacetime metric which describes the interior of a spherically symmetric object with the effects of rotation. Thus, any significant study is difficult, and a complete study of strange stars is absent in the literature.

This thesis is organised as follows:

- Chapter 1: Introduction

- Chapter 2: In this chapter a review of the fundamental concepts of differential geometry which are quintessential in the formulation of relativistic models, is presented for the reader's perusal. Key definitions and aphorisms as well as formalisms are underscored. The Einstein field equations for static and radiating models are generated. We make use of the pressure isotropy condition to construct a differential equation with variable coefficients. In the latter part of this chapter we consider the concept of gravitational collapse in detail, and discuss the physical and energy conditions, and causality.

- Chapter 3: A review of some known solutions for spherically symmetric radiating stellar models is presented. We review different exact solutions obtained for pressure isotropy. We also consider the boundary condition at the stellar surface and examine the different techniques used to generate models for a radiating star.

- Chapter 4: This chapter forms the most substantial part of this work. The governing pressure isotropy condition is solved in its natural state by choosing multitudinous forms for the gravitational potentials. Several new solutions are obtained in terms of elementary functions, one of which is an entire class. One of these solutions is similar to the conformally flat scenario. We then transform this master equation into a different form, and using the same methodological approach, obtain several new exact solutions.

- Chapter 5: In this chapter, we perform a physical analysis for two solutions that were generated in chapter 4. We plot graphs for the gravitational potentials as well as for the temporal and spatial evolutions of the matter variables. We then test for causality and perform an analysis for the energy conditions.

- Chapter 6: A brief description on the subject of causal thermodynamics in general relativity is provided in this chapter. We then generate the Maxwell-Cattaneo heat transport equation, and provide expressions for the causal and noncausal temperatures resulting from our model. Finally, we present explicit expressions for solutions obtained in chapter 4, after which we provide graphical profiles for both the noncausal and causal cases.

- Chapter 7: Conclusion

Chapter 2

Fundamental theory of relativistic astrophysics

2.1 Introduction

Einstein's general relativity is a successful theory in describing matter distributions in strong spherically symmetric gravitational fields. A review of the physics of compact objects, black holes and relativistic stellar processes such as core collapse, for example, is provided by Shapiro and Teukolsky (1983) and Glendenning (2000). For recent treatments of relativistic cosmological models see Gron and Hervik (2007). In this chapter, we provide the background theory that enables us to generate a spherical model for a relativistic radiating star or a cosmological system. A brief outline of the relevant differential geometry and the essential physical criteria for a reasonable stellar model are presented. For more extensive details on differential manifolds and tensor analysis, as well as related topics, the reader is referred to Bishop and Goldberg (1968), Misner *et al* (1973), Matzner and Shepley (1982), Wald (1984) and Foster and Nightingale (1994). In §2.2, the essential elements of differential geometry such as the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor are introduced. These constituents are the requirements for the generation of the

Einstein field equations which are the primary area of investigation in this project. We introduce, in §2.3, the energy momentum tensor and, for the purpose of modelling astrophysical and cosmological settings, the special case of an imperfect fluid. We then adduce a covariant formulation of the Einstein-Maxwell system of equations. In this system, the electromagnetic and matter fields are coupled nonlinearly to curvature. Descriptions of static and radiating spacetimes are presented respectively in §2.4 and §2.5. The field equations for both metrics are generated and a consistency condition is deduced for the radiating spacetime. In §2.6, we consider the physical conditions which are necessary for interior solutions of relativistic stellar systems and their matching to an exterior spacetime. We briefly describe the notion of the energy conditions which make a given stellar model realistic in §2.7. A description of gravitational collapse is elucidated upon in §2.8.

2.2 Spacetime geometry

In general relativity, we assume that spacetime, denoted by $\boldsymbol{M}$, is a four-dimensional, oriented, Hausdorff, pseudo-Riemannian manifold endowed with a symmetric, nondegenerate, smooth metric tensor field $\boldsymbol{g}$. In local regions, the manifold is homeomorphic (and diffeomorphic) to Euclidean space which implies that it may be covered by overlapping coordinate patches so that special relativity holds locally. The tensor field $\boldsymbol{g}$ has signature $(- + ++)$ and encodes the properties and dynamics of the gravitational field. Individual points in the manifold are labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$, where $x^0 = ct$ (c is the speed of light in vacuum) is the timelike coordinate and x^1, x^2, x^3 are spacelike coordinates. In this work, we take the speed of light c to be unity. For more comprehensive treatments of spacetime geometry, the reader is referred to the standard text books in differential geometry such as Hawking and Ellis (1973), de Felice and Clark (1990), Foster and Nightingale (1994) and Straumann (2004).

The invariant distance between neighbouring points in the manifold $\boldsymbol{M}$ is defined by the line element

$$ds^2 = g_{ab}dx^a dx^b, \tag{2.2.1}$$

which is a generalisation of the Cartesian formula

$$ds^2 = dx^2 + dy^2 + dz^2.$$

There exists a unique and symmetric connection Γ that preserves inner products under parallel transport (do Carmo 1992). The metric connection coefficient Γ is defined in terms of the metric tensor and its derivatives by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}), \tag{2.2.2}$$

where commas denote partial differentiation. The coefficients $\Gamma^a{}_{bc}$ are also known as the Christoffel symbols of the second kind and are components of the Levi-Civita connection (Christoffel 1869, Levi-Civita 1917). The Riemann-Christoffel (or Riemann curvature) tensor $\boldsymbol{R}$ is given by

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}. \tag{2.2.3}$$

Upon contracting (2.2.3) we acquire the Ricci tensor

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac}. \end{aligned} \tag{2.2.4}$$

Note that the Ricci tensor is symmetric and on contraction, we obtain

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab}R_{ab}, \end{aligned} \tag{2.2.5}$$

which is the Ricci (or curvature) scalar.

We can now use these definitions to construct the Einstein tensor $\boldsymbol{G}$, in terms of the Ricci tensor (2.2.4) and the Ricci scalar (2.2.5) as

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab}. \tag{2.2.6}$$

It is easy to see that the Einstein tensor $\boldsymbol{G}$ is symmetric and because it has zero divergence

$$G^{ab}{}_{;b} = 0, \tag{2.2.7}$$

which follows from the definition (2.2.6). A proof of the result (2.2.7) is a common exercise in most books on general relativity.

2.3 Relativistic fluids and electromagnetic fields

The matter distribution for a model in astrophysics and cosmology is usually described by a relativistic fluid. For an intricate erudition on relativistic fluid dynamics and magneto-fluid dynamics, the reader is referred to Anile (1989). The energy momentum tensor for uncharged matter is defined by the symmetric tensor $\boldsymbol{T}$ where

$$T^{ab} = (\rho + p)u^a u^b + p g^{ab} + q^a u^b + q^b u^a + \pi^{ab}. \tag{2.3.1}$$

In the above ρ is the energy density, p is the isotropic (kinetic) pressure, q^a is the heat flux vector $(q^a u_a) = 0$ and π^{ab} is the anisotropic pressure (stress) tensor $(\pi^{ab}u_a = 0 = \pi^a{}_a)$. These quantities are measured relative to a comoving fluid four-velocity $\boldsymbol{u}$ which is unit and timelike $(u^a u_a = -1)$. In perfect fluids there are no stress and heat conduction terms $(q^a = 0, \pi^{ab} = 0)$ and $\boldsymbol{T}$ is isentropic. Hence, for a perfect fluid, the energy momentum tensor (2.3.1) becomes

$$T^{ab} = (\rho + p)u^a u^b + p g^{ab}. \tag{2.3.2}$$

For the purposes of many applications we require that the matter distribution satisfies a barotropic equation of state on physical grounds

$$p = p(\rho). \tag{2.3.3}$$

Sometimes the particular equation of state (called the linear γ-law equation of state)

$$p = (\gamma - 1)\rho,$$

where $0 \leq \gamma \leq 1$, is assumed in cosmology, when probing the dynamics of matter on galactic and extragalactic length scales. The case $\gamma = 1$ corresponds to dust (vanishing pressure); $\gamma = 2$ gives a stiff equation of state in which the speed of sound and the speed of light are equal; $\gamma = 4/3$ corresponds to radiation. In the limit when $\gamma = 0$, the fluid pressure is negative, $p = -\rho$ (since $\rho > 0$). This is the characteristic property of the so-called dark energy or the existence of a possible scalar field that is responsible for the accelerated expansion of the universe. Often the particular equation of state

$$p = k\rho^{1+\frac{1}{n}},$$

where k and n are real constants, is assumed in relativistic astrophysics; this is called the polytropic equation of state. It is commonly used to model electron degenerate and neutron degenerate gases in white dwarfs and neutron stars, respectively.

The Einstein field equations in the absence of charge

$$G^{ab} = T^{ab}, \tag{2.3.4}$$

express the coupling of the matter content and curvature, and govern the dynamical interaction between the two. We have set the coupling constant to be unity in (2.3.4). From (2.2.7) and (2.3.4) we obtain

$$T^{ab}{}_{;b} = 0, \tag{2.3.5}$$

which is the conservation of matter.

We define the electromagnetic field tensor $\boldsymbol{F}$ in terms of the four-potential $\boldsymbol{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b},$$

which is a skew-symmetric tensor. The electromagnetic field tensor can be written in terms of the magnetic field $\boldsymbol{B} = (B^1, B^2, B^3)$ and the electric field $\boldsymbol{E} = (E^1, E^2, E^3)$, in a 4x4 skew-symmetric matrix in the following way

$$F^{ab} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}. \tag{2.3.6}$$

In the case when the fluid distribution contains electric charge, the energy momentum tensor becomes

$$T_{ab}^{(Total)} = T_{ab} + E_{ab},$$

where $\boldsymbol{E}$ is the electromagnetic contribution to the total energy momentum $\boldsymbol{T}^{(Total)}$ in the matter field. The electromagnetic matter tensor is given by

$$E_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \tag{2.3.7}$$

and the electromagnetic field is governed by Maxwell's equations, the fundamental equations of electromagnetism. In covariant form, the governing equations are given by

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{2.3.8a}$$

$$F^{ab}{}_{;b} = J^a, \tag{2.3.8b}$$

where $\boldsymbol{J}$ is the four-current density defined by

$$J^a = \sigma u^a, \tag{2.3.9}$$

and σ is the proper charge density. For more insight on Maxwell's equations (2.3.8) the reader is referred Misner *et al* (1973) and Narlikar (2002). It should be noted that Maxwell's equations (2.3.8) are the basic equations that govern the behaviour of the electromagnetic field in a manifold with curvature.

We alluded that the total energy momentum tensor is the sum of $\boldsymbol{T}$ and $\boldsymbol{E}$. With this notion we can now introduce the Einstein-Maxwell system of equations for a charged fluid in a gravitational field. The interaction between $\boldsymbol{T}$, $\boldsymbol{E}$ and $\boldsymbol{g}$ is described by the Einstein-Maxwell system of equations

$$\begin{aligned} G^{ab} &= T^{ab} + E^{ab}, && \text{(2.3.10a)} \\ F_{ab;c} + F_{bc;a} + F_{ca;b} &= 0, && \text{(2.3.10b)} \\ F^{ab}{}_{;b} &= J^a. && \text{(2.3.10c)} \end{aligned}$$

The reader should seek out Ivanov (2002), Thirukkanesh and Maharaj (2009) and Komathiraj and Maharaj (2011) for further insight into the Einstein-Maxwell equations and charged stellar models in general. The system (2.3.10) is a highly nonlinear system of coupled, partial differential equations governing the behaviour of gravitating systems in the presence of an electromagnetic field. In (2.3.10a), we use units in which the coupling constant is unity. We need to solve the system (2.3.10) to generate an exact solution. One approach is to stipulate particular forms for the matter distribution and electromagnetic field on physical grounds, and then attempt to integrate the partial differential equations to find the metric tensor field $\boldsymbol{g}$. For uncharged matter, the only equation that has to be satisfied is the Einstein field equation (2.3.10a) with $\boldsymbol{E} = 0$. Note that from (2.2.7) and (2.3.10a) we get

$$(T^{ab} + E^{ab})_{;b} = 0, \tag{2.3.11}$$

which is the total conservation of matter and charge that generalises (2.3.5).

2.4 Static spherical fields

The line element for a static, spherically symmetric spacetime can be written in the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.4.1}$$

The quantities $\nu(r)$ and $\lambda(r)$ are related to the gravitational potentials. The nonvanishing Christoffel symbols (2.2.2) are

$\Gamma^0{}_{01} = \nu'$ $\qquad$ $\Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)}$

$\Gamma^1{}_{11} = \lambda'$ $\qquad$ $\Gamma^1{}_{22} = -re^{-2\lambda}$

$\Gamma^1{}_{33} = -re^{-2\lambda}\sin^2\theta$ $\qquad$ $\Gamma^2{}_{12} = \frac{1}{r}$

$\Gamma^2{}_{33} = -\sin\theta\cos\theta$ $\qquad$ $\Gamma^3{}_{13} = \frac{1}{r}$

$\Gamma^3{}_{23} = \cot\theta$

for the metric element (2.4.1). The primes denote differentiation with respect to the radial coordinate r. Substituting the above connection coefficients into the definition (2.2.4), we acquire the nonvanishing Ricci tensor components

$$\begin{aligned}
R_{00} &= \left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r}\right] e^{2(\nu-\lambda)}, &(2.4.2a)\\
R_{11} &= -\left[\nu'' + \nu'^2 - \nu'\lambda' - \frac{2\lambda'}{r}\right], &(2.4.2b)\\
R_{22} &= 1 - \left[1 + r(\nu' - \lambda')\right] e^{-2\lambda}, &(2.4.2c)\\
R_{33} &= \sin^2\theta R_{22}. &(2.4.2d)
\end{aligned}$$

Using (2.4.2) and the definition for the Ricci scalar (2.2.5) we generate the following result

$$R = 2\left[\frac{1}{r^2} - \left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{1}{r^2}\right) e^{-2\lambda}\right]. \qquad (2.4.3)$$

The Ricci tensor components (2.4.2), along with the Ricci scalar (2.4.3), may be used to generate the nonvanishing Einstein tensor components (2.2.6). These are given by

$$\begin{aligned}
G^{00} &= \frac{1}{r^2} e^{-2\nu}\left[r\left(1 - e^{-2\lambda}\right)\right]', &(2.4.4a)\\
G^{11} &= e^{-2\lambda}\left[-\frac{1}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2\nu'}{r} e^{-2\lambda}\right], &(2.4.4b)\\
G^{22} &= \frac{1}{r^2} e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right), &(2.4.4c)\\
G^{33} &= \frac{1}{\sin^2\theta} G^{22}, &(2.4.4d)
\end{aligned}$$

for the metric (2.4.1).

As the fluid four-velocity is comoving we have $u^a = e^{-\nu}\delta^a_0$ for the line element (2.4.1). The perfect fluid energy momentum tensor (2.3.2) then has the nonvanishing components

$$
\begin{aligned}
T^{00} &= e^{-2\nu}\rho, && (2.4.5a)\\
T^{11} &= e^{-2\lambda}p, && (2.4.5b)\\
T^{22} &= \frac{1}{r^2}p, && (2.4.5c)\\
T^{33} &= \frac{1}{r^2\sin^2\theta}T^{22}. && (2.4.5d)
\end{aligned}
$$

On equating the components of the Einstein tensor (2.4.4) to the components of the energy momentum tensor (2.4.5), the Einstein field equations (2.3.4) are obtained in the form

$$
\begin{aligned}
\rho &= \frac{1}{r^2}\left[r\left(1-e^{-2\lambda}\right)\right]', && (2.4.6a)\\
p &= -\frac{1}{r^2}\left(1-e^{-2\lambda}\right)+\frac{2\nu'}{r}e^{-2\lambda}, && (2.4.6b)\\
p &= e^{-2\lambda}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right). && (2.4.6c)
\end{aligned}
$$

From the conservation law (2.3.5) we have the following

$$
\frac{dp}{dr} = -\left(\rho+p\right)\frac{d\nu}{dr}. \qquad (2.4.7)
$$

Note that (2.4.7) can also be acquired directly from the field equations (2.4.6); it may be used as a replacement for one of the field equations in the integration process. The evolution of the static spherically symmetric star, which we have modelled as a perfect fluid, is determined by the system of Einstein field equations (2.4.6). It is sometimes convenient to utilise the so-called Oppenheimer-Volkoff equations written as

$$
m(r) = \frac{1}{2}\int_0^r \rho(x)x^2dx, \qquad (2.4.8a)
$$

$$
\frac{dp}{dr} = -\frac{\left[\rho+p\right]\left[m+\frac{1}{2}r^3p\right]}{r\left[r-2m\right]}, \qquad (2.4.8b)
$$

where the quantity m is the gravitational mass of the star. They are important in describing relativistic stellar bodies and gravitational collapse processes.

2.5 Radiating spherical fields

Shear-free fluids are important in the modelling of inhomogeneous cosmological processes and radiating stars in relativistic astrophysics. Spherically symmetric spacetimes in the absence of shear can be written as

$$ds^2 = -A^2 dt^2 + B^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{2.5.1}$$

in comoving coordinates $(x^a) = (t, r, \theta, \phi)$. The metric functions A and B depend on both the timelike coordinate t and the radial coordinate r.

The nonvanishing connection coefficients (2.2.2) are given by

$\Gamma^0{}_{00} = \frac{\dot{A}}{A}$ $\qquad$ $\Gamma^0{}_{01} = \frac{A'}{A}$

$\Gamma^0{}_{11} = \frac{B\dot{B}}{A^2}$ $\qquad$ $\Gamma^0{}_{22} = r^2\frac{B\dot{B}}{A^2}$

$\Gamma^0{}_{33} = r^2\sin^2\theta\frac{B\dot{B}}{A^2}$ $\qquad$ $\Gamma^1{}_{00} = \frac{AA'}{B^2}$

$\Gamma^1{}_{11} = \frac{B'}{B}$ $\qquad$ $\Gamma^1{}_{22} = -r^2\left(\frac{B'}{B} + \frac{1}{r}\right)$

$\Gamma^1{}_{33} = -r^2\sin^2\theta\left(\frac{B'}{B} + \frac{1}{r}\right)$ $\qquad$ $\Gamma^1{}_{01} = \frac{\dot{B}}{B}$

$\Gamma^2{}_{02} = \frac{\dot{B}}{B}$ $\qquad$ $\Gamma^3{}_{03} = \frac{\dot{B}}{B}$

$\Gamma^2{}_{12} = \frac{B'}{B} + \frac{1}{r}$ $\qquad$ $\Gamma^3{}_{13} = \frac{B'}{B} + \frac{1}{r}$

$\Gamma^2{}_{33} = -\sin\theta\cos\theta$ $\qquad$ $\Gamma^3{}_{23} = \cot\theta$

for the metric (2.5.1). In the above, dots and primes denote differentiation with respect to t and r, respectively. Using the above Christoffel symbols and the definition for the Ricci tensor (2.2.4), we can write the nonvanishing Ricci tensor components as follows

$$R_{00} = \frac{AA''}{B^2} + AA'\frac{B'}{B^3} - 3\frac{\ddot{B}}{B} + 3\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{2}{r}\frac{AA'}{B^2}, \tag{2.5.2a}$$

$$R_{01} = 2\left(\frac{\dot{B}B'}{B^2} - \frac{\dot{B}'}{B} + \frac{A'\dot{B}}{AB}\right), \tag{2.5.2b}$$

$$\begin{aligned} R_{11} &= 2\frac{\dot{B}^2}{A^2} + \frac{A'}{A}\frac{B'}{B} - \frac{2}{r}\frac{B'}{B} - B\dot{B}\frac{\dot{A}}{A^3} - \frac{A''}{A} \\ &\quad + \frac{B\ddot{B}}{A^2} + 2\frac{B'^2}{B^2} - 2\frac{B''}{B}, \end{aligned} \tag{2.5.2c}$$

$$\begin{aligned} R_{22} &= r^2\frac{B\ddot{B}}{A^2} - r^2 B\dot{B}\frac{\dot{A}}{A^3} + 2r^2\frac{\dot{B}^2}{A^2} - r^2\frac{A'}{A}\frac{B'}{B} - r\frac{A'}{A} \\ &\quad - 3r\frac{B'}{B} - r^2\frac{B''}{B}, \end{aligned} \tag{2.5.2d}$$

$$R_{33} = \sin^2\theta R_{22}. \tag{2.5.2e}$$

Making use of (2.5.2), and the definition (2.2.5), we obtain the Ricci scalar

$$\begin{aligned} R &= -2\frac{1}{B^2}\frac{A''}{A} - \frac{4}{r}\frac{1}{B^2}\frac{A'}{A} + \frac{6}{A^2}\frac{\dot{B}^2}{B^2} - \frac{8}{r}\frac{B'}{B^3} + 2\frac{B'^2}{B^4} \\ &\quad - 2\frac{A'}{A}\frac{B'}{B^3} - 4\frac{B''}{B^3} - 6\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} + 6\frac{\ddot{B}}{BA^2}, \end{aligned} \tag{2.5.3}$$

for the metric (2.5.1). Now using (2.5.2), along with (2.5.3), we obtain the nonvanishing Einstein tensor components

$$G_{00} = 3\frac{\dot{B}^2}{B^2} - \frac{A^2}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right), \tag{2.5.4a}$$

$$G_{01} = -\frac{2}{B^2}\left(B\dot{B}' - B'\dot{B} - B\dot{B}\frac{A'}{A}\right), \tag{2.5.4b}$$

$$\begin{aligned} G_{11} &= \frac{1}{A^2}\left(-2B\ddot{B} - \dot{B}^2 + 2B\dot{B}\frac{\dot{A}}{A}\right) \\ &\quad + \frac{1}{B^2}\left(B'^2 + 2BB'\frac{A'}{A} + B^2\frac{2}{r}\frac{A'}{A} + BB'\frac{2}{r}\right), \end{aligned} \tag{2.5.4c}$$

$$\begin{aligned} G_{22} &= -2r^2\frac{B\ddot{B}}{A^2} + 2r^2 B\dot{B}\frac{\dot{A}}{A^3} - r^2\frac{\dot{B}^2}{A^2} \\ &\quad + r\frac{A'}{A} + r\frac{B'}{B} + r^2\frac{A''}{A} - r^2\frac{B'^2}{B^2} + r^2\frac{B''}{B}, \end{aligned} \tag{2.5.4d}$$

$$G_{33} = \sin^2\theta G_{22}. \tag{2.5.4e}$$

When shearing stresses are absent in the fluid ($\pi^{ab} = 0$), the nonvanishing components of the energy momentum tensor (2.3.1) are written in the following way:

$$T_{00} = \rho A^2, \tag{2.5.5a}$$

$$T_{01} = -AB^2 q, \tag{2.5.5b}$$

$$T_{11} = pB^2, \tag{2.5.5c}$$

$$T_{22} = pB^2 r^2, \tag{2.5.5d}$$

$$T_{33} = \sin^2\theta T_{22}, \tag{2.5.5e}$$

where $q = q^a q_a$. Furthermore, it is quite evident from (2.5.5b) that in the limit when the fluid is not conducting heat ($q = 0$), $\boldsymbol{T}$ has diagonal components. Using (2.5.4) and (2.5.5) we obtain the Einstein field equations

$$\rho = \frac{3\dot{B}^2}{A^2B^2} - \frac{1}{B^2}\left(\frac{2B''}{B} - \frac{B'^2}{B^2} + \frac{4B'}{rB}\right), \tag{2.5.6a}$$

$$\begin{aligned} p = {} & \frac{1}{A^2}\left(\frac{-2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{2\dot{A}\dot{B}}{AB}\right) \\ & + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + \frac{2A'B'}{AB} + \frac{2A'}{rA} + \frac{2B'}{rB}\right), \end{aligned} \tag{2.5.6b}$$

$$\begin{aligned} p = {} & -\frac{2\ddot{B}}{BA^2} + \frac{2\dot{A}\dot{B}}{BA^3} - \frac{\dot{B}^2}{A^2B^2} + \frac{A'}{rAB^2} \\ & + \frac{B'}{rB^3} + \frac{A''}{AB^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3}, \end{aligned} \tag{2.5.6c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right). \tag{2.5.6d}$$

The field equations (2.5.6) are a system of highly nonlinear, coupled partial differential equations that describe the dynamics of the matter field in the interior of the radiating sphere.

The fluid pressure is isotropic, and consequently equations (2.5.6b) and (2.5.6c) give rise to the consistency condition

$$\frac{A''}{A}\frac{1}{B^2} + \frac{B''}{B^3} - 2\frac{A'}{A}\frac{B'}{B^3} - 2\frac{B'^2}{B^4} - \frac{1}{B^2}\frac{1}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0. \tag{2.5.7}$$

This equation governs the gravitational behaviour of the radiating spacetime and can be solved to produce an exact solution of (2.5.6). In its present form, (2.5.7) is a nonlinear equation, and needs to be rewritten in a simpler way to make any progress. We make the observation that, after some algebraic manipulations, (2.5.7) can be rewritten as

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right), \tag{2.5.8}$$

which is an equivalent form of the condition of pressure isotropy (2.5.7). Introducing the new variable

$$x = r^2,$$

the pressure isotropy condition (2.5.8) becomes

$$\left(\frac{A}{B}\right)_{xx} = 2A\left(\frac{1}{B}\right)_{xx}, \tag{2.5.9}$$

where subscripts emblematize differentiation with respect to the new variable x.

2.6 Physical conditions

We will briefly consider the physical conditions which are applicable to a relativistic stellar model. For physical viability, any solution describing the interior of the stellar body should match smoothly to the appropriate exterior spacetime of that body.

We first consider a relativistic radiating star. The exterior spacetime is described by the metric

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right)dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.6.1}$$

where v is a timelike coordinate valid in the exterior spacetime, which was first found by Vaidya (1951), and where m represents the mass flow at the surface and is related to the gravitational energy within a given radius R. The matching of the interior (2.5.1) to the exterior (2.6.1) generates the conditions

$$(Adt)_\Sigma = \left[\left(1 - \frac{m}{R} + 2\frac{dR}{dv}\right)^{1/2} dv\right]_\Sigma, \tag{2.6.2a}$$

$$(rB)_\Sigma = R_\Sigma, \tag{2.6.2b}$$

$$[m(v)]_\Sigma = \left[\frac{r^3}{2}\left(\frac{\dot{B}^2B}{A^2} - \frac{B'^2}{B}\right) - r^2B'\right]_\Sigma, \tag{2.6.2c}$$

$$p_\Sigma = (qB)_\Sigma, \tag{2.6.2d}$$

across the boundary Σ. Conditions (2.6.2a) and (2.6.2b) connect the coordinates across the boundary. The equation (2.6.2c) defines the mass of the radiating star as measured by an observer at infinity. We observe that the pressure at the boundary Σ is nonzero and is proportional to the magnitude of the heat flux in (2.6.2d), a result first found by Santos (1985). This condition requires generating an exact solution to the resulting nonlinear differential equation. The boundary condition (2.6.2d) describes a dissipating star having an exterior atmosphere that consists of pure null radiation (photons) or null dust. Generating a reasonable exact solution is difficult, and consequently only a few models are presented in the literature. Some models found recently are given by Misthry *et al* (2008), Rajah and Maharaj (2008) and Thirukkanesh and Maharaj (2009).

We consider next a relativistic static star. The gravitational field outside a static spherically symmetric body, in the absence of charge, is given by

$$ds^2 = -\left(1 - \frac{2m}{R}\right) dt^2 + \left(1 - \frac{2m}{R}\right)^{-1} dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.6.3)$$

which is the famous exterior Schwarzschild solution. Here the quantity m is the mass of the stellar object as measured by an observer at infinity. The exterior gravitational field of a static spherically symmetric body, in the presence of charge, has the form,

$$ds^2 = -\left(1 - \frac{2m}{R} + \frac{\epsilon^2}{R^2}\right) dt^2 + \left(1 - \frac{2m}{R} + \frac{\epsilon^2}{R^2}\right)^{-1} dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)(2.6.4)$$

Here, the quantity ϵ is the constant related to the total charge of the gravitating sphere. The line element (2.6.4) is the exterior Reissner-Nordström solution. For a derivation of the Reissner-Nordström solution to the Einstein-Maxwell field equations, see Poisson (2004). The radial electric field is given by the equation $E = \frac{\epsilon}{R^2}$ and, consequently, the proper charge density is $\sigma = 0$. As a result of this, the four-current density $\boldsymbol{J} = 0$ is consistent with an exterior spacetime with no barotropic matter. When $\epsilon = 0$, the Reissner-Nordström line element, (2.6.4) reduces to the exterior Schwarzschild line element (2.6.3). The exterior gravitational field of a radiating spherically symmetric object, in the presence of charge, has the form

$$ds^2 = -\left(1 - \frac{2m(v)}{R} + \frac{\epsilon^2}{R^2}\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.6.5}$$

where $m(v)$ represents the mass of the body generating null radiation. This is the charged Vaidya (or Vaidya-Bonnor) solution, and it describes radially outgoing radiation flux from a charged spherically symmetric body.

It should be noted that appropriately defined physical conditions will restrict the solutions of the Einstein-Maxwell system of equations (2.3.10) for a realistic star. It is very often assumed by researchers that realistic stellar models for isotropic matter should satisfy the following conditions: The energy density ρ and the pressure p should be positive and finite throughout the interior of the star and the radial pressure should vanish at the boundary $R = b$. The energy density ρ and the pressure p should be monotonically decreasing functions from the centre to the boundary of the star. Causality should be satisfied: the speed of sound should always remain less than the speed of light throughout the interior of the star. The metric functions and the electric field intensity E should realistically be positive and nonsingular throughout the interior of the star. At the boundary of the star, the interior gravitational potentials should match smoothly to the exterior line elements (2.6.3) and (2.6.5) for neutral and charged matter, respectively. This matching generates the following conditions on the gravitational potentials:

$$e^{2\nu(b)} = e^{-2\lambda(b)} = 1 - \tfrac{2m}{b}, \quad (E = 0)$$

$$e^{2\nu(b)} = e^{-2\lambda(b)} = 1 - \tfrac{2m}{b} + \tfrac{\epsilon^2}{b^2}, \quad (E \neq 0)$$

at the boundary $R = b$. The electric field intensity E should be continuous across the boundary for the case of charged models $E(b) = \frac{\epsilon}{b^2}$. There should not exist any instability in the stellar model, with respect to radial perturbations.

An important observation that should be made is that not all relativistic stellar models satisfy all the conditions detailed above throughout the stellar interior. Particular solutions may only be valid in certain regions of the spacetime. An example with this feature arises in the Schwarzschild interior solution, which has a singularity at the centre. Solutions like this need to be treated as an envelope of the star, which should be matched smoothly to another solution valid for the core. An example of such a foliation was described by Thomas *et al* (2005). Any exact solution to the Einstein field equations that does not satisfy all the physical conditions can still be valuable, because it can assist in the qualitative analysis of relativistic stellar models.

2.7 Energy conditions and causality

In addition, for a stellar model to be deemed realistic, it must also adhere to the so-called energy conditions of general relativity, as well as not violating the law of causality.

Investigating the nature of the energy conditions is a mathematical problem, specifically an algebraic problem (Kolassis *et al* 1988) related to the eigenvalue problem of the energy momentum tensor $\boldsymbol{T}$. In a four-dimensional spacetime manifold, investigating these energy conditions involves solving a quartic polynomial. Solving such a polynomial, while not impossible, is usually very difficult and can lead to certain situations where we are faced with complicated analytical expressions of the eigenvalues which makes the problem difficult to solve, in a non-numerical sense, in general relativity. In order for a relativistic fluid to be deemed physically reasonable, it must obey the three energy conditions:

(i) *The weak energy condition*: For any future pointing timelike vector w^a, the total energy density $T_{ab}w^a w^b \geq 0$, at each event in the spacetime.

(ii) *The strong energy condition*: For any future pointing timelike unit vector w^a, the stresses of the matter, at each event in the spacetime are restricted by the condition $2T_{ab}w^a w^b + T \geq 0$ where T is the trace of the energy momentum tensor $\boldsymbol{T}$.

(iii) *The dominant energy condition*: For any future pointing timelike vector w^a, the four-momentum density vector $T_{ab}w^b$ must be future pointing and timelike, or null at each event in the spacetime.

A detailed discussion of the energy conditions is contained in Hawking and Ellis (1973) and Kolassis *et al* (1988).

To further elucidate on the above three conditions, we express them in terms of the matter variables, so that

(a) *The weak energy condition*: $\rho - p + \Delta \geq 0$.

(b) *The strong energy condition*: $2p + \Delta \geq 0$.

(c) *The dominant energy condition*: $\rho - 3p + \Delta \geq 0$.

We have defined

$$\Delta = \sqrt{(p+q)^2 - 4q^2},$$

in the above. In the absence of heat flux we obtain $\rho \geq 0$ and $p \geq 0$. The speed of sound for a relativistic fluid is given by

$$\frac{dp}{d\rho} = c_s^2. \tag{2.7.1}$$

A violation of causality would result if the above quantity is, at any point, negative or greater than the speed of light.

2.8 Gravitational collapse

When a supermassive star of mass greater than $8M_\odot$ reaches the end of the luminous phase of its life it experiences an inwardly directed gravitational collapse. This is an extremely violent and energetic process that occurs on timescales of the order of seconds and is observed as a type II supernova explosion. The entire collapse process is usually divided into an early, intermediate and late stage. It is understood that at the end of the late stage of the collapse, the remnant core constitutes either a white dwarf or a neutron star, depending on the initial mass of the collapsing star. At this point the effects of gravity become very strong and general relativity is needed to understand the collapse dynamics and the evolution of the compact stellar object. The compact star after some length of time reaches the end of its lifespan and may undergo a further collapse. This scenario, again, can only be modeled in the framework of relativistic gravity.

A supermassive star cannot simply explode, because it has no more thermonuclear energy to release and it is gravitationally bound. It also cannot reach a static equilibrium state since there does not exist such a state for a mass that large. The only alternative is collapse. It must collapse, perhaps several times, until it reaches an infinite density and negligible volume, or until the laws of general relativity break down, and new, yet-to-be-found quantum gravitational forces halt the collapse. The catastrophic notion of collapse was first brought to light by Oppenheimer and Snyder (1939), and they described in detail the free-fall collapse of a spherical body in which the pressure forces were completely overwhelmed by the gravitational forces. The exact equations of collapse, as analysed analytically by Misner (1965), Shapiro and Teukolsly (1983), Goswami and Joshi (2004), Misthry *et al* (2008) and Maharaj *et al* (2011), as well as by numerical methods elsewhere (May and White 1966, Bodenheimer *et al* 2007, Kuroda and Umeda 2010, Müller *et al* 2012) have produced significant neoteric insight into gravitational collapse. They have, moreover, confirmed quantitatively and qualitatively, the results first obtained by Oppenheimer and Snyder.

In the remarkable treatment of Thorne (1966), for example, collapse is described in terms of Schwarzschild coordinates. Several interesting physical features arise in the collapse process. For example, it was observed that as time increases, the radius of the body decreases asymptotically to twice its initial radius.

In the process of its very long life, a supermassive stellar object will live in a state of suspended collapse, converting hydrogen into helium, carbon, neon, oxygen, magnesium and silicon through nucleosynthesis, creating an internal pressure gradient resulting in the release of outward energy (radiation, conduction and convection). For a relativistic star, this pressure gradient forms part of a vital system of equations which govern its stability and contraction. These are called the Oppenheimer-Volkoff equations (2.4.8). Thermonuclear fusion ends at iron-56, which is the most bound nuclear species. Beyond iron, fusion is no longer exothermic. This outward energy just about balances the inward gravitational pull and this notion is called quasi-hydrodynamic stability. The above process continues for some time and is called the *Main Sequence* phase of a star's life. For further deep insight on gravitational contraction, the reader is referred to Glendenning (2000).

The process of gravitational collapse is usually complicated. Once the hydrogen has burned out in the core, the next phase of thermonuclear burning - helium - commences (hydrogen in some surrounding shell will continue to burn). The helium which builds up in the core undergoes an increasingly intense compression, until these helium atoms commence fusion into heavier elements like carbon, neon, oxygen and so on. Concentric burning shells are created as one element after the other is synthesized. Enormous amounts of gamma rays in the core produce electron-positron pairs which annihilate, producing neutrino pairs. At the exhaustion of each elemental fuel, the core contracts further until the ignition temperature for the next step in the next chain is attained. Each successive burning stage is quicker than the preceding one. Iron-56 is the end point of nucleosynthesis (the result of silicon burning). Burning in the outer shells of the star add to the now iron core's mass and since iron will not easily fuse to form a heavier element, it remains as such and a hydrodynamical instability sets in, where the inward pressure of gravity in the star will begin to overwhelm the outward energy being released. Gravity crushes the core to such an extent that electrons become relativistic and the pressure they provide increases less rapidly with increasing density.

The inward gravity crushes the iron core very quickly, and it becomes extremely hot ($\sim 10^{11}K$). The infalling material in the core, overshoots the equilibrium configuration and rebounds from the stiffened core, acting in a similar way to a piston. The catastrophic result of this induces what is called a post-bounce-pre-supernova shockwave which will propagate outward from some point within the collapsing core reaching relativistic speeds. The heat energy released in the process is transported away from the core through the interior and across the stellar surface by neutrinos and it is this shockwave that drives the outflow of the neutrinos. A physically reasonable relativistic model for gravitational collapse should include these features. In view of this, Glass (1990) modeled the emission of neutrinos in dissipative collapse and (Herrera and Núñez 1987, Barreto 1993) investigated the associated shock structure and propagation in the interior of a radiating star. As this shockwave travels outward, its energy is dissipated by neutrino losses and by photodisintegration of all the nuclei in its path, and will eventually stall. The plasma material that surrounded the pre-collapsed core will also, very quickly, fill the space now available, and this notion produces a decompression shockwave which travels outward at the speed of sound in the diffuse stellar material, and this material now begins to freefall. The freefalling material is seized as it meets the stalled shock front, and this turns the latter into an accretion shock, which is heated by this infalling matter. A rarefied bubble-like region develops between the highly dense core and the accreting shock front. Neutrino pairs diffusing from the extremely hot interior, annihilate, heat up and expand the bubble. Through a complex sequence of events, i.e. some interplay of convection and neutrino heating, a fraction of the deeply intense gravitational binding energy of the dense core remnant is transported to the accretion front. This small fraction provides the kinetic energy for the ejection of all but the core remnant of the progenitor star in what is called a type II supernova explosion. The resulting supernova remnants and this core remnant are pushed away from each other during this phase. The reader is encouraged to seek out Smoller and Temple (1997) and Temple and Smoller (1999) for further information on general relativistic shock wave theory.

What is left behind is a very dense, compact and hot core remnant which becomes, if the mass of the initial star was enormous, a proto-neutron star. Otherwise it becomes a white dwarf. The resulting compact object is usually supported by either electron degeneracy pressure in the case of a white dwarf or neutron degeneracy pressure in the case of a neutron star. Over an interval of a few seconds, this proto-neutron star loses its trapped neutrinos and cools, and at this point, the collapsed core remnant has reached its equilibrium configuration composition of neutrons, protons, hyperons, leptons and perhaps quarks. Thus, the neutron star is born. The radius of such a star is around $10km$ and it has a density in the region of 10^{14} times greater than that of the Earth. It may itself manifest into a pulsar or magnetar at a later point. The limit of neutron degeneracy pressure is known as the Tolman-Oppenheimer-Volkoff limit and if the neutron star is more massive than this limit, it must undergo a further collapse to some denser and more compact form, which is the hypothetical quark/ultra-compact star. The final state for a collapsed star is an astrophysical black hole (also referred to as a collapsar) which inherits the mass, angular momentum and electric charge (if any) of the initial object. This is characterised in the famous no-hair conjecture proposed by John Wheeler (Misner *et al* 1973).

During the contraction phases, an important notion to consider is the temporal evolution of the fluid from the internal core region through to the stellar surface in which the radiation is lost. To achieve this requires solving the Maxwell-Cattaneo equation, a causal heat transport equation; for particular values of the model parameters and integration constants we need to generate the corresponding temperature profiles. For a recent treatment of causal thermodynamics and solutions of the Maxwell-Cattaneo equation in shear-free spherical spacetimes see Nyonyi *et al* (2014). Another physical quantity which is crucial in modeling a dissipative collapsing stellar fluid is the effective adiabatic index

$$\Gamma_{eff} = \left[\frac{\partial(\ln p)}{\partial(\ln \rho)}\right]_{\Sigma}, \tag{2.8.1}$$

at the stellar surface Σ. The effective adiabatic index measures the ability of the stellar matter to resist compression under gravity, and is a measure of the dynamical stability of this matter at a given instant of time. It depends on the fluid pressure and energy density profiles which are obtained by generating an exact solution to the Einstein field equations.

Yet another important quantity which is used to describe the dynamics of a stellar object is the Weyl tensor (a deformation tensor) defined by

$$\begin{aligned} C_{abcd} &= R_{abcd} + \frac{1}{n-2}[g_{ad}R_{bc} - g_{ac}R_{bd} + g_{bc}R_{ad} - g_{bd}R_{ac}] \\ &\quad + \frac{1}{(n-1)(n-2)}[g_{ac}g_{bd} - g_{ad}g_{cb}]R, \end{aligned} \tag{2.8.2}$$

where n is a real constant and corresponds to the spacetime dimensions. The Weyl tensor is crucial in understanding the two types of gravitational effects on matter: compression and tidal deformation. The difference between this tensor quantity and the Riemann curvature tensor is the fact that the Weyl tensor conveys the distortion effects on a body due to the result of a tidal force. It gives no information on how the volume of the body changes. In essence, the Weyl tensor is a traceless component of the Riemann tensor and any metric contraction on any pair of indices yields zero. For a discussion on the algebraic classification of the Weyl tensor in higher dimensional Lorentzian manifolds, the reader is encouraged to peruse Coley *et al* (2004). The recent papers by Batista (2013) and Hofmann *et al* (2013) give an interesting classification and interpretation of the Weyl tensor in higher dimensional manifolds, including both static and dynamical spacetimes. Radiating relativistic stellar models have been found by Herrera *et al* (2004), Maharaj and Govender (2005) and Herrera *et al* (2006) by restricting the form of the Weyl tensor so that the spacetime is conformally flat.

Other important parameters that are used to describe the dynamics of a collapsing object in general relativity are the gravitational redshift and the velocity of the collapsing fluid. These have been investigated in detail by Bonnor *et al* (1989), Govender *et al* (2003) and Maharaj and Govender (2005).

Chapter 3

A review of some well known solutions

3.1 Introduction

There exist many exact solutions to the Einstein and Einstein-Maxwell field equations corresponding to spherically symmetric spacetimes for matter which is neutral or charged, respectively. However, very few of these are fully acceptable in view of the physical scenarios which they are meant to describe. The reader is encouraged to seek out Banerjee and Som (1981), Finch and Skea (1989), Delgaty and Lake (1998) and Ivanov (2002) for insightful reviews of many solutions to the field equations. In some modernistic treatments attempts have been made to find general classes of solutions which marry particular cases found in antiquity. Some examples are the papers by Komathiraj and Maharaj (2007a, 2007b, 2007c), Thirukkanesh and Maharaj (2006, 2009) and more recently, by Ivanov (2012), Msomi *et al* (2012) and Nyonyi *et al* (2013). In §3.2 we discuss solutions obtained by solving the condition of pressure isotropy. We present solutions for particles in geodesic motion as well as the conformally flat models obtained by Maiti (1982), Banerjee *et al* (1989) and others. We describe the Deng algorithm and the Ngubelanga-Maharaj algorithm and then elucidate on the Lie symmetry analysis of differential equations and present the solutions of Msomi *et al* (2011) and Nyonyi *et al* (2013). Finally a brief discussion on general metrics is provided. A description of the boundary condition for a radiating star is then presented in §3.3. Again, the geodesic and conformally flat scenarios are discussed as well as the initially static models and the models that can be found using Lie point symmetries.

3.2 Condition of pressure isotropy

As mentioned in the previous chapter, in order to solve the Einstein field equations for the interior radiating sphere, the pressure isotropy condition (2.5.8), given by the following

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right), \tag{3.2.1}$$

needs to be solved. Several solutions for pressure isotropy are known from the past, including those of Bergmann (1981), Maiti (1982), Deng (1989), Krasinski (1997) and John and Maharaj (2006).

3.2.1 Geodesic models

A simple class of exact solutions arises when particles are traveling on timelike geodesics. In this case $A = 1$, and (3.2.1) becomes

$$\frac{B_{rr}}{B} = 2\left(\frac{B_r}{B}\right)^2 + \frac{1}{r}\frac{B_r}{B}. \tag{3.2.2}$$

Equation (3.2.2) is nonlinear but note that it can be written in the compact form

$$\left(\frac{1}{B}\right)_{rr} - \frac{1}{r}\left(\frac{1}{B}\right)_r = 0. \tag{3.2.3}$$

We note that equation (3.2.3) is a linear Cauchy-Euler equation in the variable $\frac{1}{B}$. Integration of (3.2.3) gives

$$\left(\frac{1}{B}\right)_r = C_1(t)r, \tag{3.2.4}$$

where C_1 is a temporal function. A second integration gives the gravitational potential B. It is convenient to express the potential in the form

$$B(r,t) = \frac{d}{C_2(t) - C_1(t)r^2}, \tag{3.2.5}$$

where C_2 is another temporal function and d is a constant.

Bergmann (1981) obtained the solution in the form

$$\begin{aligned} A(r,t) &= 1, & (3.2.6a) \\ B(r,t) &= \frac{l(t)}{1 + \frac{1}{4}k(t)r^2}, & (3.2.6b) \end{aligned}$$

where k and l are temporal functions of integration, by integrating an alternate form of the pressure isotropy condition

$$A_{xx} + 2\left(\frac{F_x}{F}\right)A_x - \left(\frac{F_{xx}}{F}\right)A = 0, \tag{3.2.7}$$

where $F = B^{-1}$ and $x = r^2$. The technique of relating F to the multiplicative inverse of B was first employed by Glass (1979). For particular values of the temporal functions, we can see that this solution is identical to (3.2.5). It is interesting to note that the Bergmann solution is in a form that resembles the Friedmann solution for an isotropic universe; the only difference is that k is a temporal function and not constant. These models may be used to describe highly compact objects in astrophysics where the gravitational fields are very strong. Kolassis *et al* (1988) modeled a radiating star which has the Friedmann solution as a limiting case. There are also several applications for geodesic fluids in cosmology as pointed out by Stephani *et al* (2003).

3.2.2 Conformally flat metrics

The Weyl tensor C_{abcd} describes tidal effects in the spacetime manifold. The spacetime described by the line element (2.5.1) has the following nonzero Weyl tensor component

$$C_{2323} = \frac{r^4}{3}B^2\sin^2\theta\left[\left(\frac{A_r}{A} - \frac{B_r}{B}\right)\left(\frac{1}{r} + 2\frac{B_r}{B}\right) - \left(\frac{A_{rr}}{A} - \frac{B_{rr}}{B}\right)\right]. \tag{3.2.8}$$

All the nonzero Weyl tensor components are related by

$$C_{2323} = -r^4 \left(\frac{B}{A}\right)^2 \sin^2\theta C_{0101} = 2r^2 \left(\frac{B}{A}\right)^2 \sin^2\theta C_{0202}$$
$$= 2r^2 \left(\frac{B}{A}\right)^2 C_{0303} = -2r^2 \sin^2\theta C_{1212} = -2r^2 C_{1313}. \tag{3.2.9}$$

For conformal flatness, all the Weyl tensor components must vanish, so that $C_{2323} = 0$ from (3.2.8) and (3.2.9) above. With these notions, we acquire the relation

$$r\left(\frac{A_{rr}}{A} - \frac{B_{rr}}{B}\right) - \left(\frac{A_r}{A} - \frac{B_r}{B}\right)\left(1 + 2r\frac{B_r}{B}\right) = 0, \tag{3.2.10}$$

which is a nonlinear equation relating the potentials A and B.

We show that equation (3.2.10) can be integrated in general. We observe that equation (3.2.10) can be written in the form

$$\frac{\left(\frac{A_r}{A} - \frac{B_r}{B}\right)_r}{\frac{A_r}{A} - \frac{B_r}{B}} + \frac{A_r}{A} + \frac{B_r}{B} + 2\frac{B_r}{B} + \frac{1}{r} = 0. \tag{3.2.11}$$

After a reparametrisation of the coordinate t and integrating the above expression, we obtain

$$A(r,t) = [C_1(t)r^2 + 1]B(r,t), \tag{3.2.12}$$

where C_1 is an arbitrary integration function. This is a general solution.

The pressure isotropy condition (2.5.8) and the conformally flat condition (3.2.12), generates the differential equation

$$\frac{B_{rr}}{B_r} - 2\frac{B_r}{B} - \frac{1}{r} = 0. \tag{3.2.13}$$

Equation (3.2.13) may be immediately integrated yielding

$$B(r,t) = \frac{1}{C_2(t)r^2 + C_3(t)}, \tag{3.2.14}$$

where C_2 and C_3 are arbitrary temporal functions. Whence, the metric element (2.5.1) becomes

$$ds^2 = -\left[\frac{C_1(t)r^2 + 1}{C_2(t)r^2 + C_3(t)}\right]^2 dt^2 + \left[\frac{1}{C_2(t)r^2 + C_3(t)}\right]^2 [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{3.2.15}$$

This is the most general conformally flat metric for shear-free spherically symmetric spacetimes. It was first obtained by Banerjee *et al* (1989) in Cartesian coordinates.

Maiti (1982) sought solutions which are conformally flat with vanishing shear and vorticity. He obtained expressions for the potentials A and B which were of the following form

$$A(r,t) = b(t) + \frac{a(t)}{1+kr^2/4}, \qquad (3.2.16a)$$

$$B(r,t) = \left[\frac{c(t)}{1+kr^2/4}\right]^2, \qquad (3.2.16b)$$

where a, b and c are arbitrary functions of time and k is a constant. With a suitable temporal transformation, the function $b(t)$ can be reduced to unity. The form of the solution (3.2.16) is similar to the Friedmann-Lemaître-Robertson-Walker (FLRW) metrics and as the relativistic fluid thermalises, we regain homogeneous and isotropic spacetimes. The Maiti (1982) model is a special case of (3.2.15). Another paper generating a solution similar in form to (3.2.16) is that of Modak (1984). The conformally flat models (3.2.15) have proven to be useful in studying heat transport phenomena in inhomogeneous spherically symmetric universes which are close to FLRW metrics. For exact solutions to the Eckart temperature equation and the generalised heat transport equation of the Maxwell-Cattaneo type the reader is referred to the treatment of Triginer and Pavon (1995).

3.2.3 The Deng algorithm

A general method for generating solutions to the pressure isotropy condition (3.2.1) was provided by Deng (1989). It should be noted that the condition of the isotropy of pressure is a second order partial differential equation, however since there is no explicit time dependence in t, it can be treated as an ordinary differential equation in r. The form for (3.2.1) can be reduced to something simpler if either the potential A or B is a known function. Choosing a form for any particular potential will reduce the pressure isotropy condition to an easier equation in terms of the other potential.

The Deng algorithm uses this idea in an infinite loop. A form for A is chosen, for example, say $A = A_1$ and is substituted into the master equation (3.2.1) to find the most general solution for the other potential B, say $B = B_1$. (It should be noted that either potential may be chosen to start the procedure.) The pair $(A = A_1, B = B_1)$ provides the first class of solutions to (3.2.1). The next step is to take $B = B_1$ and substitute it into the isotropy equation. This will induce an equation in terms of A: from this, a second solution $A = A_2$ maybe obtained which is linearly independent of A_1. The linear combination $A_3 = \alpha A_2 + \beta A_1$ will give a solution which satisfies (3.2.1), so that the pair $(A = A_3, B = B_1)$ is a second set of solutions. Continuing the pattern, substituting $A = A_3$ into the master equation yields an equation in terms of B. From this, it is possible to obtain $B = B_2$ in exactly the same way A_2 was obtained. Hence the pair $(A = A_3, B = \gamma B_1 + \delta B_2)$ is the third class of solutions to (3.2.1). This process can be repeated such that an infinite class of solutions are obtained.

Using equation (3.2.7) with $F = B^{-1}$ in his approach, Deng (1989) considered the simple case

$$A = A_1 = 1, \tag{3.2.17}$$

as a starting point. With this form he obtained

$$F = F_1 = a(t)r^2 + b(t), \tag{3.2.18}$$

as a solution in terms of $F = 1/B$ where a and b are temporal functions. We then substitute the above equation (3.2.18) back into (3.2.7), and after integrating, obtain another solution for A:

$$A = A_2 = \frac{c(t)r^2 + d(t)}{a(t)r^2 + b(t)}, \tag{3.2.19}$$

where c and d are further integration functions. Hence, we see that the pair

$$A = A_1 = 1, \tag{3.2.20a}$$

$$F = F_1 = a(t)r^2 + b(t), \tag{3.2.20b}$$

is the first class of solutions, and the pair

$$A = A_2 = \frac{c(t)r^2 + d(t)}{a(t)r^2 + b(t)}, \tag{3.2.21a}$$

$$F = F_1 = a(t)r^2 + b(t), \tag{3.2.21b}$$

are the second class of solutions. The algorithm can be repeated a third time to yield

$$A(r,t) = A_2 = \frac{c(t)r^2 + d(t)}{a(t)r^2 + b(t)}, \tag{3.2.22a}$$

$$\begin{aligned} F(r,t) &= F_3 = g(a(t)r^2 + b(t)) \\ &\quad - \frac{h(t)}{3a(t)}\left[\left(\frac{c(t)r^2 + d(t)}{a(t)r^2 + b(t)}\right)^2 + \frac{c(t)}{a(t)}\frac{c(t)r^2 + d(t)}{a(t)r^2 + b(t)} + \frac{c(t)^2}{a(t)^2}\right], \end{aligned} \tag{3.2.22b}$$

where g and h are further temporal functions. Consequently the pair (A_2, F_3) is a third class of exact solutions to the condition of pressure isotropy. The Deng algorithm is a powerful tool for finding new solutions. Finding such solutions may become a complicated affair with each new implementation of the algorithm, since the resulting integrations become more and more difficult. It must be noted that all known solutions to the pressure isotropy condition can be generated from the Deng algorithm. The cases which are physically important are listed by Krasinski (1997).

3.2.4 The Ngubelanga-Maharaj algorithm

Another algorithm was developed by Ngubelanga and Maharaj (2013) for finding solutions to the pressure isotropy condition (3.2.1). Before proceeding, it is prudent to write (3.2.1) in a different form. Introduction of the new variables (first suggested by Kustaanheimo and Qvist (1948))

$$x \equiv r^2, \tag{3.2.23a}$$

$$L \equiv B^{-1}, \tag{3.2.23b}$$

$$G \equiv LA, \tag{3.2.23c}$$

transforms (3.2.1) into

$$LG_{xx} = 2GL_{xx}, \tag{3.2.24}$$

which is an equivalent form for the condition of the isotropy of the pressure. The algorithmic procedure begins in the following way.

A known solution of the form $(\bar{L}, \bar{G})$ must be assumed so that the equation

$$\bar{L}\bar{G}_{xx} = 2\bar{G}\bar{L}_{xx}, \tag{3.2.25}$$

holds. A new solution (L, G) with

$$L = \bar{L}e^{g(x)}, \tag{3.2.26a}$$

$$G = \bar{G}e^{f(x)}, \tag{3.2.26b}$$

is required where $f(x)$ and $g(x)$ are arbitrary functions. Substituting (3.2.26) into (3.2.24) yields the following

$$(\bar{L}\bar{G}_{xx} - 2\bar{G}\bar{L}_{xx}) + 2(\bar{L}\bar{G}_x f_x - 2\bar{G}\bar{L}_x g_x) + \bar{L}\bar{G}(f_{xx} - g_{xx}) + \bar{L}\bar{G}(f_x^2 - 2g_x^2) = 0, \tag{3.2.27}$$

where the derivatives of the arbitrary functions now appear explicitly. Since $(\bar{L}, \bar{G})$ is a known solution of (3.2.24) (so that (3.2.25) is satisfied), we obtain

$$(f_{xx} - 2g_{xx}) + 2\left(\frac{\bar{G}_x}{\bar{G}}f_x - 2\frac{\bar{L}_x}{\bar{L}}g_x\right) + \left(f_x^2 - g_x^2\right) = 0. \tag{3.2.28}$$

We must integrate (3.2.28) to find the unknown functions $f(x)$ and $g(x)$.

Equation (3.2.28) is difficult to integrate in general due to the existence of the two arbitrary functions and the nonlinearity. We can assume a simple form for f or g to find an exact solution. For example we can take

$$g(x) = 1, \tag{3.2.29}$$

in the hope of generating an integrable equation in f. With this assumption (3.2.28) reduces to

$$f_{xx} + 2\frac{\bar{G}_x}{\bar{G}}f_x + f_x^2 = 0, \tag{3.2.30}$$

which is nonlinear in f, and a Bernoulli equation in f_x. Integrating this equation yields

$$f_x = \bar{G}^{-2}\left(\int \bar{G}^{-2}dx + c_1\right)^{-1}. \tag{3.2.31}$$

Integrating this expression finally generates an expression for f:

$$f(x) = \int \left[\bar{G}^{-2} \left(\int \bar{G}^{-2} dx + c_1 \right)^{-1} \right] dx + c_2, \tag{3.2.32}$$

where c_1 and c_2 are arbitrary constants. With the forms of $f(x)$ and $g(x)$ given above, we generate a new solution to the condition (3.2.24)

$$\begin{aligned} L &= \bar{L}, && (3.2.33a) \\ G &= \bar{G} \exp \left(\int \left[\bar{G}^{-2} \left(\int \bar{G}^{-2} dx + c_1 \right)^{-1} \right] dx + c_2 \right), && (3.2.33b) \end{aligned}$$

which completes the algorithm.

Once a solution $(\bar{L}, \bar{G})$ is known to the field equations, a new solution (L, G) can be produced, given by the system (3.2.33).

Other choices of f and g will lead to new solutions. Note that we can always integrate (3.2.28) if there exists a relationship between the functions f and g. If we assume that

$$g(x) = af(x), \tag{3.2.34}$$

where a is an arbitrary constant, (3.2.28) becomes

$$f_{xx} + \frac{1}{1-2a} \left(\frac{\bar{G}_x}{\bar{G}} - 2a \frac{\bar{L}_x}{\bar{L}} \right) f_x + \left(\frac{1-2a^2}{1-2a} \right) f_x^2 = 0, \tag{3.2.35}$$

which is a Bernoulli equation in f_x. We obtain the general solution of (3.2.35) in the form

$$\begin{aligned} L &= \bar{L} \exp a \left[\int \left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^{\eta} \left[\Theta \int \left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^{\eta} dx + c_1 \right]^{-1} \right) dx + c_2 \right], && (3.2.36a) \\ G &= \bar{G} \exp \left[\int \left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^{\eta} \left[\Theta \int \left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^{\eta} dx + c_1 \right]^{-1} \right) dx + c_2 \right], && (3.2.36b) \end{aligned}$$

where we have the following

$$\begin{aligned} \Theta &= \left(\frac{1-2a^2}{1-2a} \right), && (3.2.37a) \\ \eta &= \frac{2}{1-2a}, && (3.2.37b) \end{aligned}$$

and $a \neq \frac{1}{2}$.

3.2.5 Lie symmetries

For a systematic approach to generalise known solutions or generate new solutions to the field equations, we use a group theoretic technique, namely the Lie analysis of differential equations. For further insight into the technique, the reader is referred to Cantwell (2002). The general approach is as follows: Given a system of ordinary differential equations with two dependent variables $A(r,t)$ and $B(r,t)$, say, a basic feature of the Lie analysis requires that the one-parameter (ε) Lie group of transformations

$$\begin{aligned} \bar{r} &= f(r,B,A,\varepsilon), && \text{(3.2.38a)} \\ \bar{B} &= g(r,B,A,\varepsilon), && \text{(3.2.38b)} \\ \bar{A} &= h(r,B,A,\varepsilon), && \text{(3.2.38c)} \end{aligned}$$

needs to be determined such that the solution set is left invariant. Due to the difficulty in calculating these transformations in a direct manner, seeking the infinitesimal form of the transformations becomes a necessary endeavour. These are given by

$$\begin{aligned} \bar{r} &= r + \varepsilon\xi(r,B,A) + O(\varepsilon^2), && \text{(3.2.39a)} \\ \bar{B} &= B + \varepsilon\eta(r,B,A) + O(\varepsilon^2), && \text{(3.2.39b)} \\ \bar{A} &= A + \varepsilon\zeta(r,B,A) + O(\varepsilon^2), && \text{(3.2.39c)} \end{aligned}$$

which are obtained once their so-called generator

$$G = \xi\frac{\partial}{\partial r} + \eta\frac{\partial}{\partial B} + \zeta\frac{\partial}{\partial A}, \tag{3.2.40}$$

is found. This generator, which is a set of vector fields, is called the symmetry of the differential equation. In solving

$$\begin{aligned} \frac{d\bar{r}}{d\varepsilon} &= \xi(\bar{r},\bar{B},\bar{A}), && \text{(3.2.41a)} \\ \frac{d\bar{B}}{d\varepsilon} &= \eta(\bar{r},\bar{B},\bar{A}), && \text{(3.2.41b)} \\ \frac{d\bar{A}}{d\varepsilon} &= \zeta(\bar{r},\bar{B},\bar{A}), && \text{(3.2.41c)} \end{aligned}$$

subject to the following

$$\bar{r} \mid_{\varepsilon=0} = r, \tag{3.2.42a}$$

$$\bar{B} \mid_{\varepsilon=0} = B, \tag{3.2.42b}$$

$$\bar{A} \mid_{\varepsilon=0} = A, \tag{3.2.42c}$$

the global (finite) form of the transformation (3.2.38) can be regained. More details are provided in the excellent texts by Bluman and Kumei (1989), Olver (1993), Ibragimov (1993, 1994, 1996) and Dorodnitsyn (2011). The latter text also gives good insight into the group theoretic approaches of difference equations.

Msomi *et al* (2011) used the Lie analysis to extend the Deng algorithm by using simple transformations which were based on the invariance properties of the pressure isotropy condition (2.5.8). Using this condition with $B = 1/V$ and $u = r^2$, they obtained the following Lie point symmetries

$$G_1 = \frac{\partial}{\partial u}, \tag{3.2.43a}$$

$$G_2 = u\frac{\partial}{\partial u}, \tag{3.2.43b}$$

$$G_3 = A\frac{\partial}{\partial A}, \tag{3.2.43c}$$

$$G_4 = V\frac{\partial}{\partial V}, \tag{3.2.43d}$$

$$G_5 = u^2\frac{\partial}{\partial u} + uV\frac{\partial}{\partial V}, \tag{3.2.43e}$$

for the consistency condition (3.2.1).

The combination of the transformation of symmetries leads to the following

$$\bar{u} = \frac{e^{a_2}(a_1 + u)}{1 - a_5 e^{a_2}(a_1 + u)}, \tag{3.2.44a}$$

$$\bar{A} = e^{a_3}, \tag{3.2.44b}$$

$$\bar{V} = \frac{e^{a_4}}{1 - a_5 e^{a_2}(a_1 + u)}, \tag{3.2.44c}$$

where the a_i ($i \in \{1, ..., 5\}$) are functions of time. Therefore any known solution of (3.2.1) can be transformed into a new solution of equation (3.2.1) via (3.2.44). All solutions in the Deng (1989) class, the conformally flat models of Sanyal and Ray (1984), Modak (1984) and Banerjee *et al* (1989), the result in Krasinski (1997) and Stephani *et al* (2003) can be extended by (3.2.44) to produce new solutions of (3.2.1).

We can reduce the complexity of the differential equation to be solved by choosing relationships between the potentials A and B and using the Lie generators (3.2.43). As an example we consider the ratio

$$W = \frac{V}{A}, \tag{3.2.45}$$

and take a combination of the generators G_3 and G_4 given by

$$G_3 + G_4 = A\frac{\partial}{\partial A} + V\frac{\partial}{\partial V}.$$

Then W given by (3.2.45) is an invariant. We can then find a solution in the form

$$A = \bar{C}(t)\exp\left[\pm\int\sqrt{\frac{W''}{2W}}du\right], \tag{3.2.46}$$

where $\bar{C}(t)$, $a(t)$ and $b(t)$ are integration functions. Given any function $W = W(u)$ it is possible to integrate (3.2.46) and find A explicitly. The simplest example is when W is a linear function. Msomi *et al* (2011) were able to, using this approach, obtain solutions to the Einstein field equations that the Deng algorithm would not have been able to produce.

Another group theoretic analysis was performed by Nyonyi *et al* (2013) for charged shear-free fluids. Their analysis initially led to a modified form of the pressure isotropy condition:

$$VA_{uu} + 2V_uA_u - AV_{uu} - V^2\frac{F(u)}{4u} = 0, \tag{3.2.47}$$

where $V = 1/B$, $u = r^2$ and $F(u)$ is an arbitrary function which depends on the electric charge. The Lie group theoretic approach leads to the invariant

$$W = \frac{V}{A}. \tag{3.2.48}$$

We can then generate a solution in the form

$$A = \exp\left[\pm C\int\sqrt{\frac{W''}{2W} + W\frac{F(u)}{8u}}du\right], \tag{3.2.49}$$

where C is a constant of integration. It can be shown that upon making appropriate choices for C, $F(u)$ and W, we get explicit expressions for the potentials A and $V = 1/B$. When $F(u) = 0$, for example, we regain the uncharged solution

$$A = \exp\left[\pm C \int \sqrt{\frac{W''}{2W}} du\right],$$

given by (3.2.46). It is interesting to note that the condition of pressure isotropy for charged shear-free fluids in higher dimensions has the form

$$4uVA_{uu} + 8uA_uV_u - 4u(n-1)AV_{uu} - V^nF(u) = 0, \tag{3.2.50}$$

where n is the dimensionality of the spacetime. The Lie group analysis gives the invariant

$$W = \frac{V}{A^{\frac{1}{n-1}}}, \tag{3.2.51}$$

and (3.2.50) has the solution

$$A = \exp\left(\pm C \int \sqrt{\frac{(n-1)^2}{n}\left[\frac{W''}{W} + \frac{W^{n-1}}{n-1}\frac{F(u)}{4u}\right]} du\right). \tag{3.2.52}$$

This result was generated by Nyonyi *et al* (2014). We regain the four-dimensional metric (3.2.49) when $n = 2$, and the uncharged metric (3.2.46) when $F(u) = 0$.

3.2.6 General metrics

It would be desirable to integrate (3.2.7) in general without making any assumptions on the forms of the gravitational potentials A and B. This seems unlikely because of the nonlinear nature of the differential equation. Remarkably it is possible to generate a general solution for a particular functional relation between the potentials. To generate this solution Sanyal and Ray (1984) assumed the functional relationship $A = A(F, t)$ and $F = 1/B$ where $A \neq 0$. We then have the derivatives

$$A_x = A_F F_x, \tag{3.2.53a}$$

$$A_{xx} = A_{FF} F_x^2 + A_F F_{xx}, \tag{3.2.53b}$$

and (3.2.7) becomes

$$A_{FF}F_x^2 + A_F F_{xx} + 2A_F\left(\frac{F_x^2}{F}\right) - A\left(\frac{F_{xx}}{F}\right) = 0, \tag{3.2.54}$$

and integrating the above yields

$$-\int \frac{\left[A_{FF} + 2\frac{A_F}{F}\right]}{\left[A_F - \frac{A}{F}\right]} F_x dx = \ln(F_x) + \alpha(t), \tag{3.2.55}$$

where $\alpha(t)$ is a function of integration. Integrating the above equation once more yields the remarkable formula

$$\int \left[\exp \int \left(\frac{A_{FF} + 2A_F/F}{A_F - A/F}\right) dF\right] dF = \alpha(t)x + \beta(t), \tag{3.2.56}$$

where $A = A(F, t)$, $x = r^2$ and $F = 1/B$. Therefore, for any particular choice of $A = A(F, t)$, a solution can be found by performing the above integration.

3.3 The boundary condition

In chapter 2, we considered the matching of the interior and exterior spacetimes valid at the boundary of the star. The boundary condition (2.6.2d) is a differential equation that arises because of the existence of a heat flux in the interior. This differential equation has to be solved to describe a radiating star in spherically symmetric spacetimes. Writing (2.6.2d) explicitly, we have

$$\begin{aligned} &\frac{A'' - \dot{B}'}{AB^2} + \frac{2A'\dot{B} - \dot{B}^2}{A^2B^2} + \frac{2\dot{A}B' - 2A\ddot{B}}{BA^3} \\ &+ \frac{A'B + AB' + 2rB'\dot{B}}{rAB^3} + \frac{BB'' + B'^2}{B^4} = 0, \end{aligned} \tag{3.3.1}$$

which is valid at the stellar surface Σ. Solutions to equation (3.3.1) have been obtained using various physical assumptions. We consider some of the interesting exact solutions that have been recently found. The general case of shearing fluids which are expanding and accelerating was considered by Thirukkanesh *et al* (2012).

3.3.1 Geodesic models

A simple class of solutions are generated when the particles are traveling in geodesic motion along timelike trajectories. The solution obtained by Thirukkanesh and Maharaj (2009) for matter in geodesic motion can be written in terms of elementary functions.

The first class of exact models is given by the line element

$$ds^2 = -dt^2 + \frac{d^2}{\alpha^2}\left[\frac{b^2 - \exp(\frac{\alpha(t+e)}{bd})}{r^2 - \exp(\frac{\alpha(t+e)}{bd})}\right]^2 [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{3.3.2}$$

where α, b, d and e are additive constants. An interesting feature of this model is that when $\alpha = d$ as $t \to \infty$, the associated spacetime becomes the flat Minkowski spacetime. The second class of exact models is of the form

$$ds^2 = -dt^2 + \frac{d^2}{\beta^2}\left[\frac{b^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)}{r^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)}\right]^2 (t+\gamma)^{4/3}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{3.3.3}$$

where, again, β, f, γ, b and d are constants. For certain choices of these constants, the above solution reduces to the model found by Kolassis *et al* (1988). In the limit of negligible heat flow the interior Friedmann dust solution is regained. Other solutions in terms of Kummer special functions are admitted by (3.3.1); these are given by Thirukkanesh and Maharaj (2009).

For a shearing fluid undergoing gravitational collapse in geodesic motion, the equivalent of (3.3.1) is a Riccati equation which admits exact solutions as shown by Rajah and Maharaj (2008).

3.3.2 Conformally flat models

Another class of exact models are generated when the spacetime is conformally flat so that tidal forces are absent. Herrera *et al* (2004) found a model by approximately solving the boundary condition. Exact models were found by Maharaj and Govender (2005), Herrera *et al* (2006) and Misthry *et al* (2008).

Substituting (3.2.12) and (3.2.14) into (3.3.1) yields the following relation

$$\left\{\ddot{C}_2 r^2 + \ddot{C}_3 - \frac{3}{2}\frac{(\dot{C}_2 r^2 + \dot{C}_3)}{C_2 r^2 + C_3} - \frac{\dot{C}_1 r^2 (\dot{C}_2 r^2 + \dot{C}_3)}{C_1 r^2 + 1} - 2(\dot{C}_3 C_1 - \dot{C}_2) r\right\}_\Sigma$$
$$+\left\{2\frac{(C_1 r^2 + 1)}{C_2 r^2 + C_3}\left[C_2(C_2 - 2C_1C_3) r^2 + C_3(C_1C_3 - 2C_2)\right]\right\}_\Sigma = 0, \qquad (3.3.4)$$

at the surface Σ. To integrate (3.3.4), we let

$$C_2 = \alpha C_3, \quad C_1 = \text{const.}, \quad \alpha = \text{const.}, \qquad (3.3.5)$$

so that (3.3.4) transforms to

$$C_3\ddot{C}_3 - \frac{3}{2}\dot{C}_3^2 - \frac{2(C_1 - \alpha) r_\Sigma^2}{\alpha r_\Sigma^2 + 1} C_3 \dot{C}_3$$
$$+\frac{2(C_1 r_\Sigma^2 + 1)}{\alpha r_\Sigma^2 + 1}\left[\alpha(\alpha - 2C_1) r_\Sigma^2 + (C_1 - 2\alpha)\right] C_3^2 = 0. \qquad (3.3.6)$$

This equation admits three solutions:

1. For $(C_1 - \alpha)^2 r_\Sigma^2 + (C_1 r_\Sigma^2 + 1)\left[\alpha(\alpha - 2C_1) r_\Sigma^2 + (C_1 - 2\alpha)\right] > 0$,

$$C_3(t) = \left\{\beta_1 \exp\left(\frac{(C_1 - \alpha) r_\Sigma + \sqrt{D}}{\alpha r_\Sigma^2 + 1}\right) t + \beta_2 \exp\left(\frac{(C_1 - \alpha) r_\Sigma + \sqrt{D}}{\alpha r_\Sigma^2 + 1}\right) t\right\}^{-2}, \qquad (3.3.7)$$

   where we have denoted $D = (C_1 - \alpha)^2 r_\Sigma^2 + (C_1 r_\Sigma^2 + 1)\left[\alpha(\alpha - 2C_1) r_\Sigma^2 + (C_1 - 2\alpha)\right]$ for convenience.

2. For $(C_1 - \alpha)^2 r_\Sigma^2 + (C_1 r_\Sigma^2 + 1)\left[\alpha(\alpha - 2C_1) r_\Sigma^2 + (C_1 - 2\alpha)\right] < 0$,

$$C_3(t) = \left\{e^{\frac{(C_1 - \alpha) r_\Sigma}{\alpha r_\Sigma^2 + 1} t} \beta_1 \cos\left(\frac{\sqrt{D}}{\alpha r_\Sigma^2 + 1}\right) t + \beta_2 \sin\left(\frac{\sqrt{D}}{\alpha r_\Sigma^2 + 1}\right) t\right\}^{-2}, \qquad (3.3.8)$$

   where D has the value given above.

3. For $(C_1 - \alpha)^2 r_\Sigma^2 + (C_1 r_\Sigma^2 + 1)\left[\alpha(\alpha - 2C_1) r_\Sigma^2 + (C_1 - 2\alpha)\right] = 0$,

$$C_3(t) = (\beta_1 + \beta_2 t)^{-2} \exp\left(\frac{2(C_1 - \alpha)}{\alpha r_\Sigma^2 + 1} t\right). \qquad (3.3.9)$$

   β_1 and β_2 are constants of integration in all three of the above solutions.

Using a different approach, Misthry *et al* (2008) found another class of conformally flat models. We can assume

$$U = C_1 b^2 + 1, \tag{3.3.10}$$

for $C_1(t)$ with $b = r_\Sigma$ so that equation (3.3.4) becomes

$$\begin{aligned} &\dot{U}(\dot{C}_2 b^2 + \dot{C}_3) + U\left[\frac{3}{2}\frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{C_2 b^2 + C_3} - \frac{2}{b}(\dot{C}_2 b^2 + \dot{C}_3) - (\ddot{C}_2 b^2 + \ddot{C}_3)\right] \\ &+2U^2\left[\frac{\dot{C}_3}{b} - \frac{1}{C_2 b^2 + C_3}\left(C_2^2 - \frac{C_3^2}{b^2}\right)\right] + 2U^3 \frac{2C_2 b^2 - C_3}{C_2 b^2 + C_3}\frac{C_3}{b^2} = 0. \end{aligned} \tag{3.3.11}$$

The above equation is difficult to analyse but note that it has a generic structure

$$\mathcal{A}\dot{U} + \mathcal{B}U + \mathcal{C}U^2 + \mathcal{D}U^3 = 0, \tag{3.3.12}$$

where

$$\mathcal{A} = \dot{C}_2 b^2 + \dot{C}_3, \tag{3.3.13a}$$

$$\mathcal{B} = \frac{3}{2}\frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{C_2 b^2 + C_3} - \frac{2}{b}(\dot{C}_2 b^2 + \dot{C}_3) - (\ddot{C}_2 b^2 + \ddot{C}_3), \tag{3.3.13b}$$

$$\mathcal{C} = 2\left[\frac{\dot{C}_3}{b} - \frac{1}{C_2 b^2 + C_3}\left(C_2^2 - \frac{C_3^2}{b^2}\right)\right], \tag{3.3.13c}$$

$$\mathcal{D} = 2\left(\frac{2C_2 b^2 - C_3}{C_2 b^2 + C_3}\frac{C_3}{b^2}\right). \tag{3.3.13d}$$

We note that (3.3.12) is an Abel equation of the first kind in U. Such equations are very difficult to solve in general. It is possible to integrate (3.3.12) to generate solutions in terms of elementary functions by restricting the forms of $\mathcal{A}$, $\mathcal{B}$, C and $\mathcal{D}$; these are given by Misthry *et al* (2008).

3.3.3 Initially static models

In this approach, the fluid distribution is in an initially static configuration, the star begins to radiate before eventually succumbing to radiative gravitational collapse. Such models have been studied extensively by de Oliveira *et al* (1985, 1988), de Oliveira and Santos (1987) and Bonnor *et al* (1989). Govender *et al* (1999) used an initially static model to show that relaxational effects in stellar transport can be significant in causal thermodynamics.

A recent model in this setting was found by Tewari (2013). Separable forms for the gravitational variables in (2.5.1) are assumed, namely

$$
\begin{aligned}
A(r,t) &= A_0(r)g(t), && (3.3.14a)\\
B(r,t) &= B_0(r)f(t), && (3.3.14b)
\end{aligned}
$$

where $f(t)$ and $g(t)$ are temporal functions. The isotropy of the pressure yields the following relation

$$
\frac{A_0''}{A_0} + \frac{B_0''}{B_0} = \left(2\frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right), \qquad (3.3.15)
$$

where the subscript, 0, represents the static configuration of the model with components of the metric $A_0(r)$ and $B_0(r)$. The boundary condition $p_\Sigma = (qB)_\Sigma$, namely (3.3.1), then yields at $r = r_\Sigma$:

$$
2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2} - \frac{2\dot{g}\dot{f}}{gf} = \frac{2\alpha g\dot{f}}{f^2}, \qquad (3.3.16)
$$

where

$$
\alpha = \left(\frac{A_0'}{B_0}\right)_\Sigma .
$$

Assuming $g(t) = f(t)$, we obtain the solution

$$
\begin{aligned}
\dot{f} &= 2\alpha f + \beta\sqrt{f}, && (3.3.17a)\\
t &= \frac{1}{\alpha}\ln\left(1 + \frac{2\alpha}{\beta}\sqrt{f}\right), && (3.3.17b)
\end{aligned}
$$

where β is an arbitrary constant of integration. This solution is similar to the models obtained by de Oliveira *et al* (1985) and Bonnor *et al* (1989). When the static model is undergoing gravitational collapse, $\dot{f}(t) \leq 0$. If we choose $\beta = -2\alpha$, we have $\dot{f} \to 0$ as $f \to 1$ and then the solution (3.3.14) and (3.3.17b) represents a static isentropic fluid as $t \to -\infty$. Gradually the fluid falls into a non-adiabatic gravitational collapse.

Initially static models in the presence of shear have been extensively studied by Chan (2003), Nogueira and Chan (2004) and Pinheiro and Chan (2008, 2010).

3.3.4 Lie symmetries

The application of the symmetry analysis to the boundary condition (3.3.1) for shear-free models with isotropic pressures has not been accomplished. However some work has been done with anisotropic pressures by Abebe *et al* (2013, 2014) and Abebe *et al* (2014), particularly in the conformally flat scenario.

Abebe *et al* (2013) acquired solutions to the field equations for a particular conformally flat radiating star. They considered the line element

$$ds^2 = B^2(-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \tag{3.3.18}$$

where $B = B(r,t)$ which has vanishing Weyl stresses. The resulting master equation (arising from the junction condition $p_\Sigma = (qB)_\Sigma$) is given by

$$2rBB_{rt} + 2rBB_{tt} - 4rB_rB_t - rB_t^2 - 3rB_r^2 - 4BB_r = 0. \tag{3.3.19}$$

The group theoretic approach yields the following three Lie point symmetries

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t}, && (3.3.20a)\\ G_2 &= t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, && (3.3.20b)\\ G_3 &= B\frac{\partial}{\partial B}, && (3.3.20c) \end{aligned}$$

with the non-zero Lie bracket $[G_1, G_2] = G_1$. Using each symmetry systematically, in any particular order, we can generate group invariant solutions to the differential equation (3.3.19). However these solutions are often not helpful, or useful, and therefore an optimal system is required. For the above symmetries (3.3.20), an optimal system of one-dimensional subgroups can be created and is given by

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t}, && (3.3.21a)\\ G_2 &= t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, && (3.3.21b)\\ aG_2 + G_3 &= a\left(t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}\right) + B\frac{\partial}{\partial B}, && (3.3.21c) \end{aligned}$$

which are subalgebras of (3.3.20). All group invariant solutions can be transformed to those obtained via this optimal system. For further insight into the theory of groups and algebras, the reader is referred to Dixon (1971) and Smirnov (2011).

Each G_i is a generator for optimal system (3.3.20) and using the first generator $G_1 = \frac{\partial}{\partial t}$, the resulting invariant particular solution obtained for the line element (3.3.18) is given by

$$B = \left(\sqrt{7} + 5 - 6\frac{t}{r}\right)^{\frac{14-2\sqrt{7}}{21}} \left(\sqrt{7} - 5 + 6\frac{t}{r}\right)^{\frac{14+2\sqrt{7}}{21}}, \tag{3.3.22}$$

which is expressed in terms of elementary functions. It should be emphasised that the presence of a self-similar variable $x = t/r$ in the above solution implies the existence of a homothetic Killing vector. Previously, a homothetic vector was found, for shearing spherically symmetric spacetimes, by Wagh and Govinder (2006). The full conformal geometry for shear-free spacetimes is known and is given by Moopanar and Maharaj (2013). Other solutions can be obtained by considering the combination $bG_1 + G_3$ which was not originally in the optimal system. This results in the following solutions

$$B(r,t) = \exp\left(\frac{3t - r + 2f(r)}{3b}\right)\left(\frac{b + 2r + 2f(r)}{[2b + r + 2f(r)]^2}\right)^{1/3}, \tag{3.3.23a}$$

$$B(r,t) = \exp\left(\frac{3t - r - 2f(r)}{3b}\right)\left(\frac{[2b + r + 2f(r)]^2}{r^4[b + 2r + 2f(r)]}\right)^{1/3}, \tag{3.3.23b}$$

being obtained where $f(r) = \sqrt{b^2 + br + r}$.

The Lie analysis of differential equations is a helpful method in producing exact solutions to the boundary condition. It should be applied to the general case of a spherically symmetric, expanding, accelerating spacetime in the presence of shear. This is an area of research we will pursue in future.

Chapter 4

Neoteric solutions with heat conduction in the shear-free regime

4.1 Introduction

Shear-free spacetimes are extensively used to model the interior of relativistic stars which, in the form of radial heat flow, dissipate null radiation. The heat flows outward from the much hotter centre toward the stellar surface. There exist many models on radiative gravitational collapse that have been studied in antiquity. These include the models by Santos (1985), Glass (1990), Deng and Mannheim (1990, 1991), Stephani *et al* (2003), Ivanov (2012) and Sharif and Yousaf (2012). An imperative requirement for all these models is that the interior spacetime must match at the stellar boundary to the exterior Vaidya radiating spacetime. The reader is encouraged to read Herrera *et al* (2004), Maharaj and Govender (2005), Misthry *et al* (2008) and Maharaj *et al* (2011) for further information on dissipative models and the usefulness of radiating relativistic stars in understanding the cosmic censorship hypothesis. Wagh *et al* (2001) generated models for a spherically symmetric shear-free spacetime with non-vanishing heat flux, by imposing a barotropic equation of state, Herrera *et al* (2006) discovered analytical solutions to the Einstein field equations for fluid spheres undergoing collapse in the diffusion approximation, and Maharaj *et al* (2011) investigated the gravitational collapse of a star which eventually evolves into a final static configuration described by the interior Schwarzschild solution. They demonstrated the remarkable application of causal thermodynamics in modelling the thermal evolution of the compact object. In this chapter we construct models for a shear-free relativistic fluid exhibiting heat flow. In §4.2 we first generate new solutions to the untransformed governing equation. We then produce other new solutions to the transformed fundamental equation, by choosing particular forms for the gravitational potentials in §4.3. We briefly discuss the mathematics behind the generation of solutions as well as locate comparisons (if any) between the solutions we obtained and those found previously. Some general comments about the nature of the solutions found are made in §4.4.

4.2 Solutions to the untransformed equation

We present new exact solutions to the untransformed condition of isotropic pressure

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right), \tag{4.2.1}$$

that we derived in chapter 2. As mentioned in chapter 3, particular solutions exist for this equation. We mention the two classes which are most studied. For particles in geodesic motion we have the metric

$$ds^2 = -dt^2 + \left(\frac{d}{C_2(t) - C_1(t)r^2}\right)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{4.2.2}$$

For vanishing Weyl stresses we have the famous solution

$$ds^2 = -\left[\frac{C_1(t)r^2 + 1}{C_2(t)r^2 + C_3(t)}\right]^2 dt^2 + \left[\frac{1}{C_2(t)r^2 + C_3(t)}\right]^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{4.2.3}$$

Our solutions are generated by making appropriate choices for one of the gravitational potentials and attempting to determine an integrable equation in terms of the other potential. In this way we generate several new classes of exact solutions to the Einstein field equations with heat flow that have not been found previously.

If we take

$$A = \alpha(t), \tag{4.2.4}$$

where $\alpha(t)$ is an arbitrary function, then the untransformed equation (4.2.1) reduces to

$$\frac{B_{rr}}{B} = 2\left(\frac{B_r}{B}\right)^2 + \frac{1}{r}\frac{B_r}{B}. \tag{4.2.5}$$

This equation is identical to equation (3.2.2) of the last chapter. Hence this case will eventually produce a solution

$$ds^2 = -\alpha(t)^2 dt^2 + \left(\frac{e^{\kappa(t)}}{2\zeta(t) - r^2}\right)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{4.2.6}$$

where $\kappa(t)$ and $\zeta(t)$ are integration functions. The above solution is equivalent to (4.2.2) which is geodesic.

If we take

$$B = \alpha(t)A, \tag{4.2.7}$$

where $\alpha(t)$ is arbitrary. Then equation (4.2.1) simplifies to

$$\frac{A_{rr}}{A} = 2\left(\frac{A_r}{A}\right)^2 + \frac{1}{r}\frac{A_r}{A}. \tag{4.2.8}$$

The form of the above equation is identical to the form (4.2.5). Hence integration will produce the solution

$$ds^2 = -\left(\frac{e^{\gamma(t)}}{2\eta(t) - r^2}\right)^2 dt^2 + \alpha(t)^2\left(\frac{e^{\gamma(t)}}{2\eta(t) - r^2}\right)^2 [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{4.2.9}$$

where $\gamma(t)$ and $\eta(t)$ are temporal functions. This solution is contained in (4.2.3) which is conformally flat.

4.2.1 Solution I: $B = \alpha(t)r^n$

In an attempt to generate a new solution, we set

$$B(r,t) = \alpha r^n, \tag{4.2.10}$$

where $\alpha = \alpha(t)$ and $n \in \mathbb{R}$. After some simplification, the pressure isotropy condition (4.2.1) becomes

$$A_{rr} - \frac{1}{r}\left(2n+1\right)A_r - \frac{1}{r^2}\left(n^2+2n\right)A = 0, \tag{4.2.11}$$

which is a second order Cauchy-Euler differential equation in A. Utilising the standard transformation $A = r^m$ generates the corresponding characteristic equation

$$m^2 - (2n+2)m - (n^2+2n) = 0,$$

with roots

$$m = (n+1) \pm \sqrt{2n^2+4n+1}.$$

Then the general solution to (4.2.11) may be written as

$$A(r,t) = \psi(t)r^{n+1+\sqrt{2n^2+4n+1}} + \xi(t)r^{n+1-\sqrt{2n^2+4n+1}}, \tag{4.2.12}$$

where $\psi(t)$ and $\xi(t)$ are functions of integration. Thus, the line element has the form

$$\begin{aligned} ds^2 &= -\left[\psi(t)r^{n+1+\sqrt{2n^2+4n+1}} + \xi(t)r^{n+1-\sqrt{2n^2+4n+1}}\right]^2 dt^2 \\ &\quad +\alpha(t)r^{2n}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \end{aligned} \tag{4.2.13}$$

This represents an entire class of new solutions that has real parts in the range $n < -1 - 1/\sqrt{2}$ and $n > -1 + 1/\sqrt{2}$. The simple form of the line element (4.2.13) will be helpful in cosmological and astrophysical applications.

4.2.2 Solution II: $B = \alpha(t)r^{\beta(t)n+\gamma(t)}$

It is interesting to observe that the solution in §4.2.1 may be extended to a more general class. We make the choice

$$B(r,t) = \alpha r^{\beta n+\gamma}, \tag{4.2.14}$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$ and $n \in \mathbb{R}$. This reduces the pressure isotropy condition (4.2.1) to

$$A_{rr} - \frac{1}{r}\left(2\beta n + 2\gamma + 1\right)A_r - \frac{1}{r^2}\left(\beta^2 n^2 + 2\beta\gamma n + 2\beta n + \gamma^2 + \gamma\right)A = 0, \tag{4.2.15}$$

which is again a second order Cauchy-Euler differential equation in A. Utilising the standard transformation $A = r^m$ generates the corresponding characteristic equation

$$m^2 - (2\beta n + 2\gamma + 2)m - (\beta^2 n^2 + 2\beta\gamma n + \gamma^2 + \gamma) = 0,$$

with roots

$$m = (\beta n + \gamma + 1) \pm \sqrt{2\beta^2 n^2 + 4\beta\gamma n + 4\beta n + 2\gamma^2 + 6\gamma + 1}.$$

Finally, the general solution to (4.2.15) may be written as

$$\begin{aligned} A(r,t) &= \tau(t) r^{\beta n + \gamma + 1 + \sqrt{2\beta^2 n^2 + 4\beta\gamma n + 4\beta n + 2\gamma^2 + 6\gamma + 1}} \\ &\quad + \chi(t) r^{\beta n + \gamma + 1 - \sqrt{2\beta^2 n^2 + 4\beta\gamma n + 4\beta n + 2\gamma^2 + 6\gamma + 1}}, \end{aligned} \tag{4.2.16}$$

where $\tau(t)$ and $\chi(t)$ are functions of integration. Thus, the metric has the form

$$\begin{aligned} ds^2 &= -\Big[\tau(t) r^{\beta n + \gamma + 1 + \sqrt{2\beta^2 n^2 + 4\beta\gamma n + 4\beta n + 2\gamma^2 + 6\gamma + 1}} \\ &\quad + \chi(t) r^{\beta n + \gamma + 1 - \sqrt{2\beta^2 n^2 + 4\beta\gamma n + 4\beta n + 2\gamma^2 + 6\gamma + 1}}\Big]^2 dt^2 \\ &\quad + (\alpha r^{\beta n + \gamma})^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \end{aligned} \tag{4.2.17}$$

This is a new category of exact solutions for a shear-free fluid exhibiting heat conduction where α, β, γ, τ and χ are all free temporal functions, and $n \in \mathbb{R}$. This solution is valid in the range $n < -\frac{\gamma+1}{\beta} - \frac{\sqrt{2\beta^2 - 4\beta^2\gamma}}{\beta^2}$ and $n > -\frac{\gamma+1}{\beta} + \frac{\sqrt{2\beta^2 - 4\beta^2\gamma}}{\beta^2}$. If we set $\gamma(t) = 0$ and $\beta(t) = 1$ then we regain the metric (4.2.13) of §4.2.1.

4.3 Solutions to the transformed equation

We now consider solutions to the transformed condition of pressure isotropy (2.5.9). These solutions are generated by making particular choices for the respective gravitational variables. To proceed we first express (2.5.9) in the form

$$\left(\frac{1}{B}\right) A_{xx} + 2\left(\frac{1}{B}\right)_x A_x - \left(\frac{1}{B}\right)_{xx} A = 0. \tag{4.3.1}$$

It is easy to see that (4.3.1) is a partial differential equation in general. However, since the variable t makes no explicit appearance it can be treated as an ordinary differential equation. It becomes linear in the function $\frac{1}{B}$, if A is specified, or a linear equation in the function A, if $\frac{1}{B}$ is specified. This feature of (4.3.1) will be used in generating classes of new solutions.

4.3.1 Solution I: $B^{-1} = \alpha(t)k^{\beta(t)x+\gamma(t)}$

We make an exponential choice for $\frac{1}{B}$, so that

$$\frac{1}{B} = \alpha k^{\beta x+\gamma}, \tag{4.3.2}$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$ and $k \in \mathbb{R}$. Then equation (4.3.1) reduces to

$$A_{xx} + 2\beta(\ln k)A_x - \beta^2(\ln k)^2 A = 0, \tag{4.3.3}$$

which is a second order ordinary differential equation with constant coefficients. The characteristic equation of (4.3.3) is

$$m^2 + 2(\ln k)\beta m - (\ln k)^2\beta^2 = 0,$$

and its roots are

$$m = -\beta(\ln k) \pm \sqrt{2}\beta(\ln k),$$

which are real and distinct. Hence the general solution to (4.3.3) is given by

$$A(x,t) = \nu(t)k^{\beta(t)[-1-\sqrt{2}]x} + \kappa(t)k^{\beta(t)[-1+\sqrt{2}]x}, \tag{4.3.4}$$

where $\nu(t)$ and $\kappa(t)$ are integration functions. The line element for this class of exact models is given by

$$\begin{aligned} ds^2 &= -\left[\nu(t)k^{\beta(t)[-1-\sqrt{2}]r^2} + \kappa(t)k^{\beta(t)[-1+\sqrt{2}]r^2}\right]^2 dt^2 \\ &\quad +\alpha(t)^{-2}k^{-2[\beta(t)r^2+\gamma(t)]}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \end{aligned} \tag{4.3.5}$$

We point out that the metric (4.3.5) is a generalisation of the exponential solution of Govender (2007).

4.3.2 Solution II: $B^{-1} = \alpha(t) \sinh x$

We now choose a form for $\frac{1}{B}$ which is of hyperbolic trigonometric form, namely we set

$$\frac{1}{B} = \alpha \sinh x,$$

where $\alpha = \alpha(t)$. This reduces (4.3.1) to

$$\sinh x A_{xx} + 2 \cosh x A_x - \sinh x A = 0, \tag{4.3.6}$$

This second order differential equation has variable coefficients. We now let

$$w = \sinh x A, \tag{4.3.7}$$

so that (4.3.6) has the simpler form

$$\frac{d^2 w}{dx^2} - 2w = 0, \tag{4.3.8}$$

which is the fundamental second order linear equation. Transformations like (4.3.7) can be found in Zwillinger (1989). The general solution to (4.3.8) is given by

$$w = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}.$$

Then the gravitational potential A is written as

$$A(x,t) = \varrho(t) e^{\sqrt{2}x} \operatorname{csch} x + \varsigma(t) e^{-\sqrt{2}x} \operatorname{csch} x, \tag{4.3.9}$$

where $\varrho(t)$ and $\varsigma(t)$ are functions of integration. The line element (2.5.1) is then

$$\begin{aligned} ds^2 &= -\left[\varrho(t) e^{\sqrt{2}r^2} \operatorname{csch} r^2 + \varsigma(t) e^{-\sqrt{2}r^2} \operatorname{csch} r^2\right]^2 dt^2 \\ &\quad + \alpha(t)^{-2} \operatorname{csch}^2 r^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \end{aligned} \tag{4.3.10}$$

The model is singular at $r = 0$ because of the presence of the hyperbolic function $\operatorname{csch} r^2$. It is regular for all other regions in the manifold.

4.3.3 Solution III: $B^{-1} = \alpha(t)\cosh x$

Making another hyperbolic trigonometric choice for $\frac{1}{B}$, viz, setting

$$\frac{1}{B} = \alpha \cosh x,$$

where $\alpha = \alpha(t)$. This reduces (4.3.1) to

$$\cosh x A_{xx} + 2\sinh x A_x - \cosh x A = 0, \tag{4.3.11}$$

which is a second order linear ordinary differential equation with variable coefficients. Again it is convenient to let

$$w = \cosh x A, \tag{4.3.12}$$

which transforms equation (4.3.11) into the following

$$\frac{d^2 w}{dx^2} - 2w = 0, \tag{4.3.13}$$

with constant coefficients. This equation has the general solution

$$w = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}.$$

Then the metric function for A is written as

$$A(x,t) = \phi(t) e^{\sqrt{2}x} \operatorname{sech} x + \vartheta(t) e^{-\sqrt{2}x} \operatorname{sech} x, \tag{4.3.14}$$

where $\phi(t)$ and $\vartheta(t)$ are integration functions. The spacetime metric (2.5.1) thus becomes

$$\begin{aligned} ds^2 &= -\left[\phi(t) e^{\sqrt{2}r^2} \operatorname{sech} r^2 + \vartheta(t) e^{-\sqrt{2}r^2} \operatorname{sech} r^2\right]^2 dt^2 \\ &\quad + \alpha(t)^{-2} \operatorname{sech}^2 r^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \end{aligned} \tag{4.3.15}$$

This is another interesting new exact solution to the condition of pressure isotropy. This class of solutions is possible because we can write the fundamental equation in the simple form (4.3.13) with constant coefficients. This model is regular at the centre $r = 0$ since the hyperbolic $\operatorname{sech} r^2$ function is finite at the origin. Consequently this model is regular for all points in the spacetime unlike the solution in §4.3.2.

4.3.4 Solution IV: $B^{-1} = \alpha(t)x^2/(\beta(t)x+1)$

A rational functional choice can be made for $\frac{1}{B}$, namely we choose

$$\frac{1}{B} = \frac{\alpha x^2}{\beta x + 1},$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$. Equation (4.3.1) then reduces to

$$x^2(\beta x+1)^2 A_{xx} + 2x(\beta x+1)(\beta x+2)A_x - 2A = 0, \qquad (4.3.16)$$

which is a second order linear ordinary differential equation with variable coefficients. Differential equations of the form of equation (4.3.16) can be found in the excellent book by Polyanin and Zaitsev (2003). Unfortunately, with such equations, solution techniques are not at all unique or obvious, so a trial and error approach has to be employed to reduce (4.3.16) to a simpler form. In our case, we can proceed by setting

$$\Omega = i\sqrt{2}\left[\ln(x) - \ln(\beta x+1)\right], \qquad (4.3.17)$$

where $i = \sqrt{-1}$. The appearance of the complex quantity i should not be troublesome as we eventually will obtain a real solution. It is in fact due to certain special qualities of the complex quantity, i.e. the fact that its square is real and that its multiplicative inverse is its additive inverse, that allows for this. Then, equation (4.3.16) becomes

$$\frac{\left(\frac{1}{\beta e^{i\Omega/\sqrt{2}}-1}+1\right)^2}{(\beta e^{i\Omega/\sqrt{2}}-1)^2}A_{xx} + 2\frac{\left(\frac{1}{\beta e^{i\Omega/\sqrt{2}}-1}+1\right)\left(\frac{1}{\beta e^{i\Omega/\sqrt{2}}-1}+2\right)}{\beta e^{i\Omega/\sqrt{2}}-1}A_x - 2A = 0. \qquad (4.3.18)$$

Noting that with the following

$$A_x = \frac{d\Omega}{dx}A_\Omega, \qquad (4.3.19a)$$

$$A_{xx} = \left(\frac{d\Omega}{dx}\right)^2 A_{\Omega\Omega} + \frac{d^2\Omega}{dx^2}A_\Omega, \qquad (4.3.19b)$$

equation (4.3.18) reduces, after some calculation, to

$$-2A_{\Omega\Omega} + 3i\sqrt{2}A_\Omega - 2A = 0, \qquad (4.3.20)$$

which is a second order linear differential equation with constant coefficients. Its characteristic equation is given by

$$2m^2 - 3i\sqrt{2}m + 2 = 0.$$

Multiplying through by i yields

$$2im^2 + 3\sqrt{2}m + 2i = 0,$$

which has roots

$$m = \frac{3i}{2\sqrt{2}} \pm \frac{1}{2}i\sqrt{\frac{17}{2}}.$$

Hence, the general solution to (4.3.20) is given by

$$A = \gamma e^{\left(\frac{3i}{2\sqrt{2}} + \frac{1}{2}i\sqrt{\frac{17}{2}}\right)\Omega} + \varphi e^{\left(\frac{3i}{2\sqrt{2}} - \frac{1}{2}i\sqrt{\frac{17}{2}}\right)\Omega}.$$

When we substitute for Ω from (4.3.17) we find that

$$A(x,t) = \gamma(t)\left(\frac{x}{\beta(t)x+1}\right)^{-\frac{1}{2}\left(3+\sqrt{17}\right)} + \varphi(t)\left(\frac{x}{\beta(t)x+1}\right)^{\frac{1}{2}\left(-3+\sqrt{17}\right)}, \quad (4.3.21)$$

where $\gamma(t)$ and $\varphi(t)$ are integration functions and the solution is given in real functions only. In terms of the original variables we can write

$$\begin{aligned} A(r,t) &= \gamma(t)\left(\frac{r^2}{\beta(t)r^2+1}\right)^{-\frac{1}{2}\left(3+\sqrt{17}\right)} \\ &\quad + \varphi(t)\left(\frac{r^2}{\beta(t)r^2+1}\right)^{\frac{1}{2}\left(-3+\sqrt{17}\right)}, && (4.3.22a) \\ B(r,t) &= \frac{\beta(t)r^2+1}{\alpha(t)r^4}. && (4.3.22b) \end{aligned}$$

We have verified that (4.3.22) satisfies the differential equation (4.3.16) with the help of MATHEMATICA. Thus the metric line element (2.5.1) becomes

$$\begin{aligned} ds^2 &= -\left[\gamma(t)\left(\frac{r^2}{\beta(t)r^2+1}\right)^{-\frac{1}{2}\left(3+\sqrt{17}\right)} + \varphi(t)\left(\frac{r^2}{\beta(t)r^2+1}\right)^{\frac{1}{2}\left(-3+\sqrt{17}\right)}\right]^2 dt^2 \\ &\quad + \left(\frac{\beta(t)r^2+1}{\alpha(t)r^4}\right)^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. && (4.3.23) \end{aligned}$$

We have generated another class of exact solutions to the Einstein field equations which is new and has not been found before. This class is made possible because (4.3.16) can be transformed via a complex transformation to (4.3.20) with constant coefficients. The metric functions turn out to be expressible in terms of *real functions* only. This is an unusual feature, and it is interesting to see this application arising in spherically symmetric gravitational fields.

4.3.5 Solution V: $B^{-1} = \alpha(t)A^{\beta(t)}$

We now consider a coupling of the metric potentials (as was first done by Govender (2007) in his M.Sc. thesis) by allowing one potential to be expressed as the power of the second. Namely, we set

$$\frac{1}{B} = \alpha A^{\beta}, \tag{4.3.24}$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$. Equation (4.3.1) then reduces to

$$(1-\beta)AA_{xx} + (3\beta - \beta^2)A_x^2 = 0, \tag{4.3.25}$$

which is in terms of A. The above equation is a highly nonlinear equation and two cases arise corresponding to the choice of β: $\beta = 1$ and $\beta \neq 1$. When $\beta = 1$ in (4.3.25), we get

$$A_x = 0,$$

which gives the trivial solutions $A = A(t)$ and hence $B = B(t)$ and the radial dependence is vanquished. We therefore consider the more interesting case when $\beta \neq 1$. In this case, equation (4.3.25) can be written in the following way

$$\frac{dA_x{}^2}{A_x{}^2} = \frac{2\beta(\beta-3)}{1-\beta}\frac{dA}{A}. \tag{4.3.26}$$

The above expression is integrable since it is a separable equation. Upon integrating this expression, we obtain the first order ordinary differential equation

$$\frac{dA}{dx} = \psi A^{\frac{\beta(\beta-3)}{1-\beta}}. \tag{4.3.27}$$

The above equation (4.3.27) is also separable and therefore, integration yields the general solution

$$A(x,t) = \left[\frac{\beta(t)^2 - 2\beta(t) - 1}{\beta(t) - 1}\left[\zeta(t)x + \tau(t)\right]\right]^{\frac{\beta(t)-1}{\beta(t)^2-2\beta(t)-1}}, \tag{4.3.28}$$

where $\zeta(t)$ and $\tau(t)$ are functions resulting from the integration process above. This expression for A, along with the assumption $B = (1/\alpha)A^{-\beta}$ constitutes another solution to the the master equation (4.3.1). Expressing these in terms of the (r,t) coordinates, we get

$$A(r,t) = \left[\frac{\beta(t)^2 - 2\beta(t) - 1}{\beta(t) - 1}\left[\zeta(t)r^2 + \tau(t)\right]\right]^{\frac{\beta(t)-1}{\beta(t)^2-2\beta(t)-1}}, \tag{4.3.29a}$$

$$B(r,t) = \alpha(t)^{-1}A(r,t)^{-\beta(t)}. \tag{4.3.29b}$$

From these, the line element (2.5.1) becomes

$$\begin{aligned} ds^2 \;=\; & -\left[\left(\frac{\beta(t)^2 - 2\beta(t) - 1}{\beta(t) - 1}\right)\left[\zeta(t)r^2 + \tau(t)\right]\right]^{\frac{2\beta(t)-2}{\beta(t)^2-2\beta(t)-1}} dt^2 \\ & +\alpha(t)^{-2}\left[\left[\frac{\beta(t)^2 - 2\beta(t) - 1}{\beta(t) - 1}\left[\zeta(t)r^2 + \tau(t)\right]\right]^{\frac{\beta(t)-1}{\beta(t)^2-2\beta(t)-1}}\right]^{-2\beta(t)} \\ & \times\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \end{aligned} \tag{4.3.30}$$

which is another new solution with heat conduction.

4.3.6 Solution VI: $B^{-1} = \alpha(t)(\beta(t)x + A)$

In an attempt to generate another class of exact solutions we attempt another coupling of the gravitational potentials. In particular we set

$$\frac{1}{B} = \alpha(\beta x + A), \tag{4.3.31}$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$. This assumption reduces the master equation (4.3.1) to

$$\beta x A_{xx} + 2\beta A_x + 2A_x^2 = 0, \tag{4.3.32}$$

which is nonlinear. In order to solve this equation we must reduce the order, thus, we set

$$A_x = v,$$

where $v = v(x)$. Equation (4.3.32) then becomes

$$\beta x \frac{dv}{dx} + 2\beta v + 2v^2 = 0, \tag{4.3.33}$$

which is a separable equation. Integrating this equation yields

$$v = \frac{dA}{dx} = -\frac{D}{D - x^2}, \tag{4.3.34}$$

where $D = D(t)$. This equation is also separable and so can be integrated to yield the general solution

$$A(x,t) = -\chi(t) \tanh^{-1}\left(\frac{x}{\chi(t)}\right) + \eta(t), \tag{4.3.35}$$

where $\chi(t)$ and $\eta(t)$ are integration functions. This expression, along with the assumption $B = (1/\alpha)(\beta x + A)^{-1}$ constitutes another solution to the consistency condition (4.3.1). Expressing these in terms of the original variables r and t, we get

$$\begin{aligned} A(r,t) &= \eta(t) - \chi(t) \tanh^{-1}\left(\frac{r^2}{\chi(t)}\right), && \text{(4.3.36a)} \\ B(r,t) &= \alpha(t)^{-1}[\beta(t) r^2 + A(r,t)]^{-1}, && \text{(4.3.36b)} \end{aligned}$$

which is a further new solution in exact form to the field equations. The line element (2.5.1) is then

$$\begin{aligned} ds^2 &= -\left[\eta(t) - \chi(t) \tanh^{-1}\left(\frac{r^2}{\chi(t)}\right)\right]^2 dt^2 \\ &\quad + \alpha(t)^{-2}\left[\beta(t) r^2 + \eta(t) - \chi(t) \tanh^{-1}\left(\frac{r^2}{\chi(t)}\right)\right]^{-2} \\ &\quad \times [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \end{aligned} \tag{4.3.37}$$

This new exact solution of the Einstein field equations is also expressible in terms of elementary functions.

4.3.7 Solution VII: A general solution

It should be noted that equation (4.3.1) can be integrated directly. Following the approach of Sanyal and Ray (1984), we assume that $A \neq 0$ and consider a function of the form $A = A(B,t)$ so that we have

$$\begin{aligned} A_x &= A_B B_x, && \text{(4.3.38a)} \\ A_{xx} &= A_{BB} B_x^2 + A_B B_{xx}. && \text{(4.3.38b)} \end{aligned}$$

Hence, (4.3.1) becomes

$$\frac{1}{B} A_{BB} B_x^2 + \frac{1}{B} A_B B_{xx} + 2\left(\frac{1}{B}\right)_x A_B B_x - \left(\frac{1}{B}\right)_{xx} A = 0, \tag{4.3.39}$$

and integrating the above yields

$$-\int \frac{[BA_{BB} - 2A_B - 2A/B]}{[BA_B + A]} B_x dx = \ln(B_x) + \alpha(t), \tag{4.3.40}$$

where $\alpha(t)$ is some temporal function of integration. Integrating (4.3.40) once more yields

$$\int \exp\left[\int \frac{[BA_{BB} - 2A_B - 2A/B]}{[BA_B + A]} dB\right] dB = \alpha(t)x + \beta(t), \tag{4.3.41}$$

where $\beta(t)$ is another function of integration. The above expression can be written in an even more compact way as follows

$$\int \left[\exp \int \left(\frac{\left[B^2 \left(\frac{A}{B}\right)_{BB} - 4A/B\right]}{(AB)_B}\right) dB\right] dB = \alpha(t)x + \beta(t). \tag{4.3.42}$$

As far as we are aware, this expression has not been found previously. For any particular choice for the function $A = A(B,t)$, a solution can finally be found by performing the above integrations. In this new solution we have a direct link between the potentials A and B unlike in §3.2.6.

4.4 The mathematics of solution generation

It should be noted that infinitely many choices for the potentials A and B, that we may choose to use, exist. Examples include exponential, logarithmic, trigonometric, rational functions and combinations of these. However, most choices produce differential equations that have no known solutions or require very ad hoc or geriatric methods to make progress in solving them. Such methods include harmonic analysis (such as Fourier transforms), transformation of variables (perhaps several times over), group theoretic approaches (Lie symmetry analysis), and numerical analysis (numerical integration, finite difference and volume methods). Often, these approaches lead to very complicated and nonstandard expressions, such as solutions in terms of hypergeometric series, Legendrè polynomials, Meijer G-functions (Meijer 1936), Bessel functions and differential roots. While perhaps holding some mathematical value, these kinds of solutions are not reasonable physically, so therefore such solutions have been omitted from this thesis. For further insight into special functions the reader is referred to the standard texts on the subject such as Wimp (1964), Slater (1966), Lebedev (1972), Tolstov (1976) and Bell (2004). In our cases we were able to express the fundamental differential equation in forms which are familiar and hence, integrable. These included the second order Cauchy-Euler equations and constant coefficient equations (via real and complex transformations). Linear first order and separable equations were also obtained. Also an equation with the functional dependence $A = A(B, t)$ produced a general solution.

Chapter 5

An analysis of the physics of the heat conducting fluid

5.1 Introduction

In order for relativistic radiating solutions to be used as a basis in constructing physical models in astrophysics and cosmology, we need to first test the physics that is realised by a given exact solution. A comprehensive physical analysis is necessary to demonstrate the physical viability and applicability of the results of the previous chapter to realistic scenarios. Ideally, every exact solution generated should be tested to ensure that the fundamental physics of the model is not violated. For example, the energy density ρ, pressure p and heat flux q must all be finite, non-negative and monotonically decreasing through the fluid interior. This must be satisfied along with the energy conditions that have been discussed earlier. Another important test is that of the causality of the model. A well constructed model is one in which the general evolution of the matter and other dynamical variables are probed. This would allow for a more detailed study of the qualitative features of the gravitating system. However, due to the often complicated structure and nature of the solutions that are generated, a complete analysis, as mentioned above, is not always possible. A particular exact solution usually contains free parameters or functions that have to be fine tuned in order to generate good behaviour. Since almost all solutions that are generated analytically, are extremely sensitive to these parameters or functions, detailed investigations are difficult. We carry out a physical analysis for two solutions.

Extensive study of the physics and in particular, the strong, weak and dominant energy conditions have been implemented in various astrophysical and cosmological scenarios. Kolassis *et al* (1988) have provided a rigorous framework for probing the energy conditions of a heat conducting relativistic fluid in the context of a radiating star with strong gravitational fields. More recently, Chan (2003) modeled the dissipative gravitational collapse of a $6M_{\odot}$ star under the influence of shearing stresses. The radial and temporal evolution of the density, pressure, total mass and luminosity were probed in detail. The general framework due to Kolassis *et al* (1988), for the energy conditions was generalised for the anisotropic radiating fluid by Chan (2003). Govender *et al* (2012) investigated the energy conditions and dynamical stability of radiating stellar systems where the collapse is influenced by the pressure. Shear-free collapse models which include the effects of electric charge and heat conduction were studied by Pinheiro and Chan (2013). In cosmology, Schuecker *et al* (2003) tested the strong and null energy conditions on macroscopic scales. They used X-ray and type-Ia supernova data to show that for a universe with flat spatial sections, the strong energy condition might be violated, whereas the null energy condition may be satisfied. It is suggested that this provides a good observational argument for the current accelerated expansion of the universe. Santos *et al* (2007) have shown that the energy conditions can be used to provide limiting values on the distance modulus of cosmic sources for varying spatial curvature. It was demonstrated that the null, weak and dominant energy conditions associated with the so-called phantom fields appear to have been violated in the current epoch ($z \leq 2$) of the evolution of the universe. Their results also suggest that the strong energy condition was first violated billions of years ago at redshift $z \geq 1$.

5.2 Two exact models

For the purpose of this analysis, it is prudent to introduce some nomenclature. In this chapter we consider two exact solutions found in chapter 4. The purpose is to analyse the physical features of the solutions and to graphically plot quantities of interest. The first model we consider corresponds to the metric

$$\begin{aligned} ds^2 &= -\Big[\tau(t)r^{\beta n+\gamma+1+\sqrt{2\beta^2n^2+4\beta\gamma n+4\beta n+2\gamma^2+6\gamma+1}} \\ &\quad +\chi(t)r^{\beta n+\gamma+1-\sqrt{2\beta^2n^2+4\beta\gamma n+4\beta n+2\gamma^2+6\gamma+1}}\Big]^2 dt^2 \\ &\quad +(\alpha r^{\beta n+\gamma})^2\left[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right], \end{aligned} \tag{5.2.1}$$

which arises as a solution to the untransformed pressure isotropy condition. This solution is given by

$$\rho_{ut} = \frac{3\dot{B}_{ut}^2}{A_{ut}^2B_{ut}^2} - \frac{1}{B_{ut}^2}\left(\frac{2B''_{ut}}{B_{ut}} - \frac{B'^2_{ut}}{B^2_{ut}} + \frac{4B'_{ut}}{rB_{ut}}\right), \tag{5.2.2a}$$

$$\begin{aligned} p_{ut} &= \frac{1}{A_{ut}^2}\left(\frac{-2\ddot{B}_{ut}}{B_{ut}} - \frac{\dot{B}^2_{ut}}{B^2_{ut}} + \frac{2\dot{A}_{ut}\dot{B}_{ut}}{A_{ut}B_{ut}}\right) \\ &\quad + \frac{1}{B_{ut}^2}\left(\frac{B'^2_{ut}}{B^2_{ut}} + \frac{2A'_{ut}B'_{ut}}{A_{ut}B_{ut}} + \frac{2A'_{ut}}{rA_{ut}} + \frac{2B'_{ut}}{rB_{ut}}\right), \end{aligned} \tag{5.2.2b}$$

$$q_{ut} = -\frac{2}{A_{ut}B_{ut}^2}\left(-\frac{\dot{B}'_{ut}}{B_{ut}} + \frac{B'_{ut}\dot{B}_{ut}}{B^2_{ut}} + \frac{A'_{ut}}{A_{ut}}\frac{\dot{B}_{ut}}{B_{ut}}\right), \tag{5.2.2c}$$

$$\begin{aligned} A_{ut} &= \tau(t)r^{\beta n+\gamma+1+\sqrt{2\beta^2n^2+4\beta\gamma n+4\beta n+2\gamma^2+6\gamma+1}} \\ &\quad +\chi(t)r^{\beta n+\gamma+1-\sqrt{2\beta^2n^2+4\beta\gamma n+4\beta n+2\gamma^2+6\gamma+1}}, \end{aligned} \tag{5.2.2d}$$

$$B_{ut} = \alpha(t)r^{\beta(t)n+\gamma(t)}, \tag{5.2.2e}$$

where the subscript, ut, refers to the fact that (5.2.2) corresponds to the untransformed pressure isotropy condition. The second model that we analyse is related to the metric

$$\begin{aligned} ds^2 &= -\left[\left(\frac{\beta(t)^2-2\beta(t)-1}{\beta(t)-1}\right)\left[\zeta(t)r^2+\tau(t)\right]\right]^{\frac{2\beta(t)-2}{\beta(t)^2-2\beta(t)-1}} dt^2 \\ &\quad +\alpha(t)^{-2}\left[\left[\frac{\beta(t)^2-2\beta(t)-1}{\beta(t)-1}\left[\zeta(t)r^2+\tau(t)\right]\right]^{\frac{\beta(t)-1}{\beta(t)^2-2\beta(t)-1}}\right]^{-2\beta(t)} \\ &\quad \times\left[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right], \end{aligned} \tag{5.2.3}$$

which was generated as a solution to the transformed pressure isotropy condition. This has the form

$$\rho_{trans} = \frac{3\dot{B}^2_{trans}}{A^2_{trans}B^2_{trans}} - \frac{1}{B^2_{trans}}\left(\frac{2B''_{trans}}{B_{trans}} - \frac{B'^2_{trans}}{B^2_{trans}} + \frac{4B'_{trans}}{rB_{trans}}\right), \tag{5.2.4a}$$

$$\begin{aligned} p_{trans} &= \frac{1}{A^2_{trans}}\left(\frac{-2\ddot{B}_{trans}}{B_{trans}} - \frac{\dot{B}^2_{trans}}{B^2_{trans}} + \frac{2\dot{A}_{trans}\dot{B}_{trans}}{A_{trans}B_{trans}}\right) \\ &\quad + \frac{1}{B^2_{trans}}\left(\frac{B'^2_{trans}}{B^2_{trans}} + \frac{2A'_{trans}B'_{trans}}{A_{trans}B_{trans}} + \frac{2A'_{trans}}{rA_{trans}} + \frac{2B'_{trans}}{rB_{trans}}\right), \end{aligned} \tag{5.2.4b}$$

$$q_{trans} = -\frac{2}{A_{trans}B^2_{trans}}\left(-\frac{\dot{B}'_{trans}}{B_{trans}} + \frac{B'_{trans}\dot{B}_{trans}}{B^2_{trans}} + \frac{A'_{trans}}{A_{trans}}\frac{\dot{B}_{trans}}{B_{trans}}\right), \tag{5.2.4c}$$

$$A_{trans} = \left[\frac{\beta(t)^2 - 2\beta(t) - 1}{\beta(t) - 1}\left[\zeta(t)r^2 + \tau(t)\right]\right]^{\frac{\beta(t)-1}{\beta(t)^2-2\beta(t)-1}}, \tag{5.2.4d}$$

$$B_{trans} = \alpha(t)^{-1}A^{-\beta(t)}_{trans}, \tag{5.2.4e}$$

where we have used the subscript, *trans*, to denote the solution as being from the transformed condition of pressure isotropy.

We now focus our attention on the physical behaviour of the matter and gravitational variables realised by solutions (5.2.2) and (5.2.4). The temporal and spatial profiles for all quantities (A, B, ρ, p, q) are constructed for a particular choice of the free functions α (appearing in both solutions), β, ψ, ξ, ζ, τ (appearing in both solutions) and χ. Since the model is expressed in natural units, the above mentioned parameters are dimensionless. Where possible, the results generated from both solutions are weighed against each other. In addition, we also investigate the weak, dominant and strong energy conditions for the transformed solution (5.2.4), and finally the sound speed for the radiating fluid corresponding to the untransformed solution (5.2.2) is investigated. We have also imposed restrictions upon the temporal functions contained within our solutions. For the untransformed solution (5.2.2) we have

(a) $n = 0$,

(b) $\alpha(t) = 0.00001t$,

(c) $\beta(t) = 1$,

(d) $\gamma(t) = 0$,

(e) $\tau(t) = -0.00001t$,

(f) $\chi(t) = 0$.

For the transformed solution (5.2.4), we have

(a) $\alpha(t) = 10000$,

(b) $\beta(t) = 0.01t$,

(c) $\zeta(t) = 1/t$,

(d) $\tau(t) = \ln(0.6t)$.

5.3 Gravitational potentials

Figures 5.1 and 5.2 feature spatial plots for the potentials of the untransformed solution (5.2.2) at differing time slices. As time varies we see that the behaviour of the potential A_{ut} is shown over the range ($\Delta A_{ut} = 0.002$). The behaviour of potential B_{ut} is depicted over the range ($\Delta B_{ut} = 0.00001$). Both plots indicate smooth and monotonically increasing behaviour and in the case of B_{ut}, the profile is almost constant. Despite the variations in the gravitational field seeming small, the field is still strong enough to have a marked influence on the matter. In figures 5.3 and 5.4 the potentials A_{trans} and B_{trans} from the transformed solution (5.2.4) are plotted against the radial coordinate r at various time slices in the relativistic fluid. As we move temporally outward: from $t = 3$ through to a time $t = 10$, A_{trans} increases through positive values. B_{trans} decreases through positive values at all time slices. Both plots indicate smooth, monotonic behaviour for the gravitational variables, with the behaviour of A_{trans} being shown in the range $\Delta A_{trans} = 5$ and for B_{trans}, the range is $\Delta B_{trans} = 0.000005$. This suggests that the fluctuations in the gravitational potential B_{trans} are small compared to A_{trans}.

5.4 The evolution of the matter variables

We observe from figure 5.5 that the energy density ρ resulting from the untransformed solution (5.2.2) is smooth and regular throughout the interior. This is in agreement with the required physical conditions for a reasonable stellar model. In the region close to the centre $r = 0$ it is apparent that ρ diverges. This solution is therefore valid in the outer regions of core-envelope models that are regarded as being more realistic for stellar interiors. In such models the local interior that surrounds the central core is divided into two distinct regions, namely the inner core and the outer envelope (Paul and Tikekar 2005). For relativistic models each region is described by a different spacetime and consequently is governed by different exact solutions. These solutions have to be matched at the interface in order to provide a more complete description of the behaviour of the matter variables from the centre outwards. Our exact solution is only valid for the envelope region. We also see that the variation between time slices is of the order of $\Delta\rho \sim 2 \times 10^6$ which is reflected in the steepness of the profile. In figure 5.6 we observe that the pressure p from the untransformed solution is smooth and regular throughout the interior and diverges at points closer to the centre of the sphere, as was the case with the energy density. We can also observe that pressure variations between different time slices is of the order of $\Delta p \sim 0.5 \times 10^8$ which, again, is reflected in the steepness of the profile. It is interesting to note from figures 5.5 and 5.6, that the energy density profile is noticeably steeper than the pressure profile, i.e., $d\rho/dt > dp/dt$ despite the fact that individual pressure changes are greater than the density changes (by two orders of magnitude), i.e., $\Delta p = (1.0 \times 10^2)\Delta\rho$. Figure 5.7 indicates that the heat flux q is smooth and monotonically decreasing from the centre outwards with divergence at the centre. It is evident from figure 5.7 that the variations of the heat flux between different time slices is of the order of $\Delta q \sim 2 \times 10^{12}$ and the profile is similar to that of the density. We also make the observation that $dq/dt > dp/dt$. Furthermore, figures 5.5, 5.6 and 5.7 reveal that $\Delta q > \Delta p > \Delta\rho$ and suggests that closer to the centre, $q > p > \rho$.

In figure 5.8 we observe that the energy density ρ from the transformed solution (5.2.4) is smooth, finite and regular throughout the interior and is thus in agreement with the required physical conditions. It is evident also, that the variations between time slices are of the order of $\Delta\rho \sim 2 \times 10^6$ which is reflected in the steepness of the profile. We also notice that at small instances of time, the energy density decreases at a faster rate than at longer intervals, and it appears that all three curves intersect each other at certain points in time for some spatial region. We can also observe that the maximum energy density $(\rho_{trans})_{max}$ appears to be at a value much larger than 1×10^7. Furthermore, it is evident that $(\rho_{trans})_{max}$ occurs at a smaller instant of time, $t = 3$; as time progresses, i.e. when $t = 6$ and $t = 10$, the maximum energy density $(\rho_{trans})_{max}$ occurs at 8×10^6 and 6×10^6 respectively at the centre of the fluid distribution. Figure 5.9 depicts the fluid pressure p resulting from the transformed solution (5.2.4). It can be seen that the curves are smooth, finite and regular throughout the interior of the fluid with variations at instances of time being of the order of $\Delta p \sim 1 \times 10^7$ which are larger than those for the energy density, despite the apparent steepness of the profile being much less. We notice, also, that the maximum pressure $(p_{trans})_{max}$, which again occurs at the centre, appears to be a value much larger than 6×10^7 which is larger than that of the energy density, i.e. $(p_{trans})_{max} > (\rho_{trans})_{max}$. When $t = 6$ and $t = 10$, the maximum pressure peaks around 4×10^7 and close to 2×10^7 at the centre, respectively. We observe from figure 5.10 that the heat flux q is finite and regular throughout the interior with time slice variations which are much smaller, i.e. $\Delta q \sim 1 \times 10^5$. Here, again, at different instances of time, it is apparent that all the curves intersect each other at varying points in the interior. We can also see that the maximum heat flux $(q_{trans})_{max}$ for small instances of time, appears to be at a value much greater than 500000, hinted by the break in the curve at $t = 3$ which is due to the scale of the plot. $(q_{trans})_{max}$ also appears to be slightly away from the origin. As time progresses, i.e. when $t = 6$ and $t = 10$, the maximum heat flux appears to be in between 400000 and 500000, and close to 200000 respectively. However, at the origin, the heat flux vanishes. This could be in fact, due to the nature of the exact solution obtained, or if the gradient of the temperature is zero at the centre, i.e. $\nabla T = 0$, which is a common feature in many realistic stellar configurations.

We now consider the behaviour of the energy density, pressure and heat flux in the same plots for both the transformed and untransformed solutions. Figure 5.11 depicts the energy density from both the solutions. Again, the curve resulting from the transformed solution is finite, smooth and regular throughout the interior of the fluid sphere, while the curve from the untransformed solution is regular and smooth, but appears to diverge closer to the centre. It is interesting to note that the two curves appear to intersect at a radial slice close to $r = 2$. We can also see that the steepness of the profile of the untransformed energy density is more profound than that of the transformed variant with variations between the radial slices being of the order of $\Delta\rho \sim 2 \times 10^6$. In figure 5.12, we again see behaviour similar to that of the energy densities. Both curves are smooth, and regular but whereas the transformed pressure is finite, the untransformed variant diverges closer to the centre. The two curves appear to intersect at a radial slice close to $r = 1$, and, again, the untransformed pressure appears to induce a much steeper profile than the transformed variant. The variation of p between radial slices is of the order of $\Delta p \sim 1 \times 10^9$ which is larger than that of the energy densities. It is evident from figure 5.13 that the untransformed solution admits a curve which is smooth and monotonic, but divergent close to the centre of the fluid distribution, whereas the transformed heat flux is finite and continuous throughout the interior. It is important to note that both solutions admit realistic behaviour. This could be due to the nature of the solutions. To further understand these notions, it would be prudent to use a different theoretical approach, such as a gas-kinetic model, where the micro-physics can be studied in detail.

5.5 Energy conditions and sound speed

The weak energy condition is given by $W = \rho - p + \sqrt{(p+q)^2 - 4q^2}$. Figure 5.14 depicts the weak energy condition, denoted, W_t for the transformed solution at different instances of time. We observe that this energy condition is continuous, positive and finite throughout the interior of the heat conducting fluid. The strong energy condition is defined by $S = 2p + \sqrt{(p+q)^2 - 4q^2}$. In figure 5.15 we present the strong energy condition S_t for the transformed solution, and, again we notice the curves are continuous, smooth and positive throughout the interior. The dominant energy condition is given by $D = \rho - 3p + \sqrt{(p+q)^2 - 4q^2}$. We observe from figure 5.16 that the dominant energy condition D_t for the transformed solution is negative throughout the interior of the fluid: this energy condition is violated. This is probably due to the fact that our results are highly model-dependent. The numerical values for the dominant energy condition parameters D_t overlap identically for $t = 6$ and $t = 10$. Other choices for the parameters may produce a positive value for D. Also, a different choice for the shear-free gravitating fluid or the model parameters may produce a different result with positive dominant energy.

Finally, the speed of sound $c_s = (dp/d\rho)^{1/2}$ is considered. Figures 5.17 and 5.18 depict the numerical profiles for the sound speed at the different radial and time slices for the untransformed solution and different instances of time and different radial slices, respectively. We observe that, for both cases, the speed of sound is positive, finite and regular throughout the fluid interior. This implies that c_s is a regular function.

5.6 Discussion

In this chapter we have performed a physical analysis for the matter and gravitational variables of two exact solutions: (5.2.2) from the untransformed case and (5.2.4) from the transformed case. It was shown that the gravitational variables A and B for the transformed solution admit behaviour which is regular and smooth throughout the interior of the heat conducting fluid. In the case of the untransformed solution, a similar trend was evident. For the matter variables ρ, p and q, it was shown that in the transformed case, these quantities admit physically reasonable behaviour at different instances of time. Also, in the untransformed case the matter variables admit reasonable behaviour physically at different time intervals. All the curves appear to diverge closer to the centre which is consistent with the so called core-envelope scenario. Finally, an analysis was orchestrated for the energy conditions (resulting from the transformed solution) and the following notable behaviour was observed.

$$S_t \geq 0, \;\; W_t \geq 0, \;\; D_t \leq 0, \;\; (c_s)_{ut} \geq 0. \tag{5.6.1}$$

In summary, we found that only the dominant energy condition was violated. An attempt was made to analyse the behaviour of the energy conditions for the untransformed solution as well; the software packages used could not produce plots with the required behaviour. Further attempts in this direction will be made in future with more specialised computing packages.

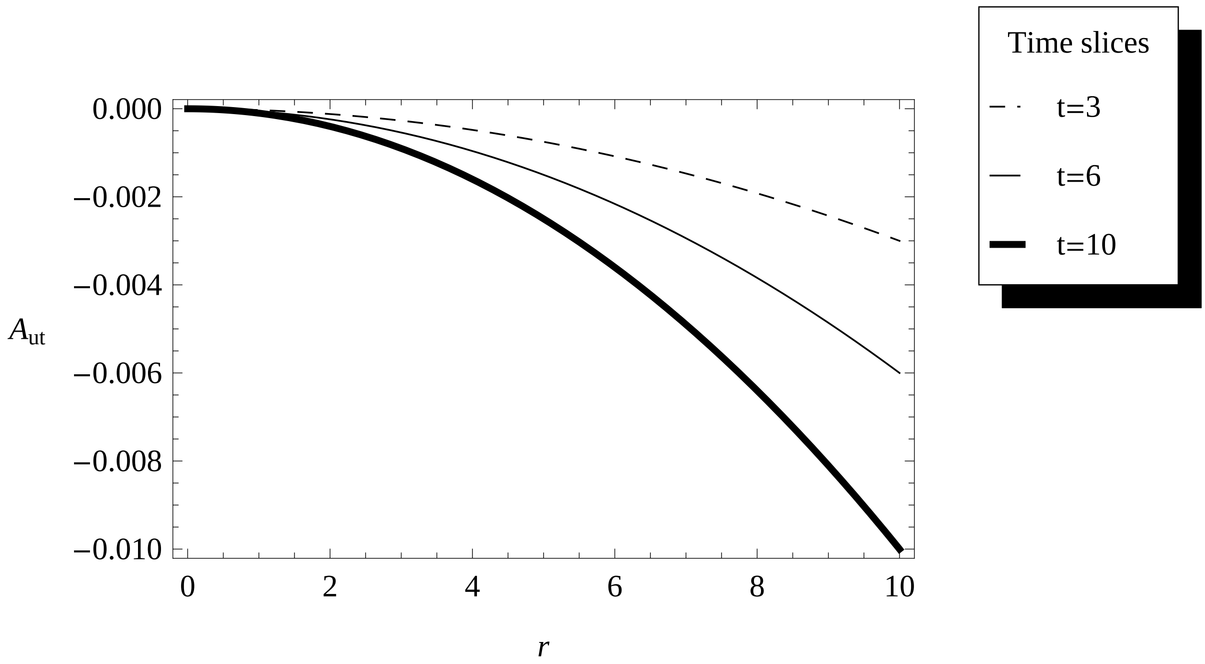

Figure 5.1: The potential A versus r for the untransformed solution (5.2.2) at varying units of time.

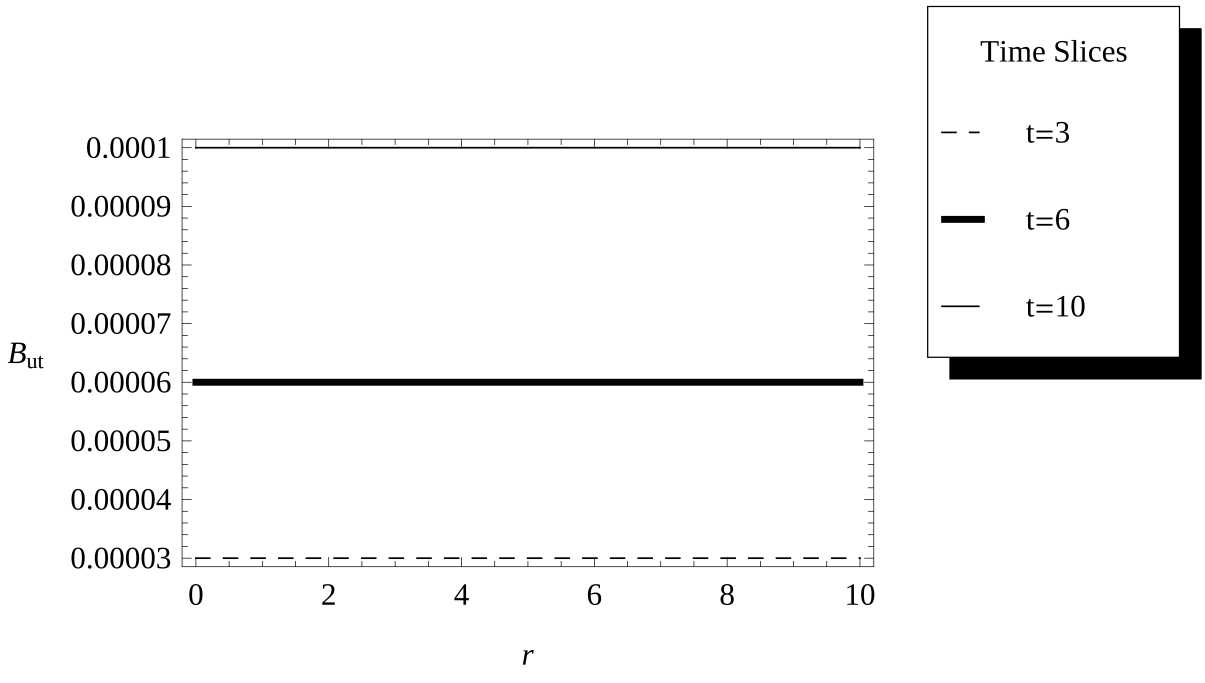

Figure 5.2: The potential B versus r for the untransformed solution (5.2.2) at varying units of time.

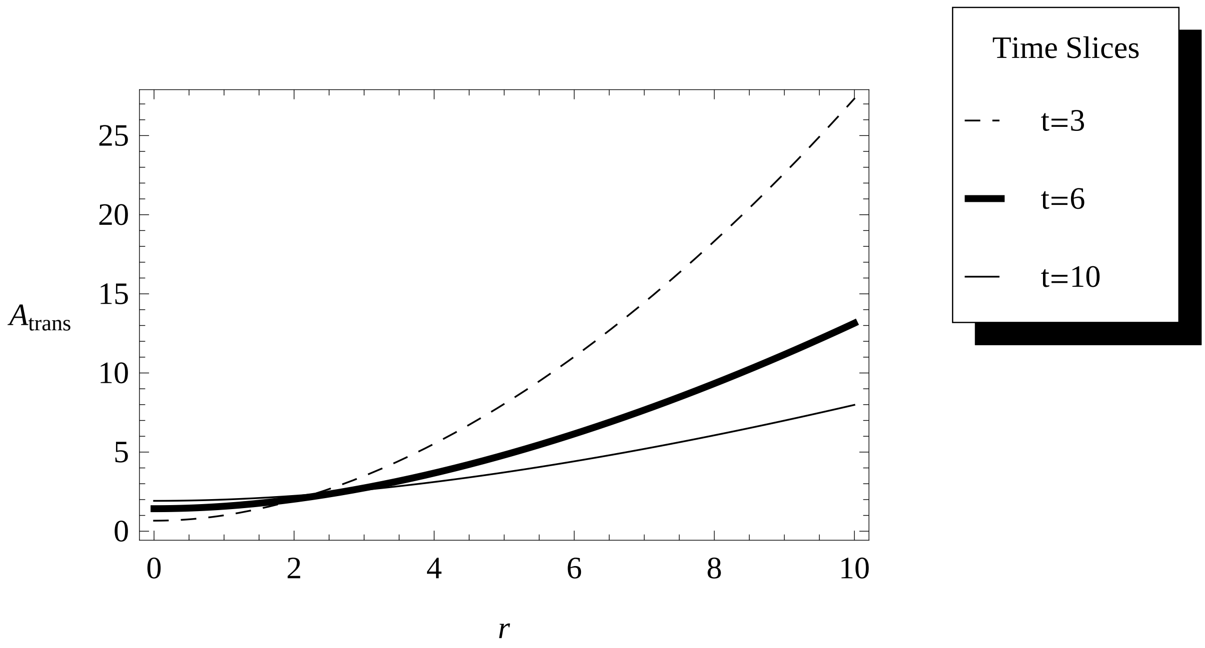

Figure 5.3: The potential A versus r for the transformed solution (5.2.4) at varying units of time.

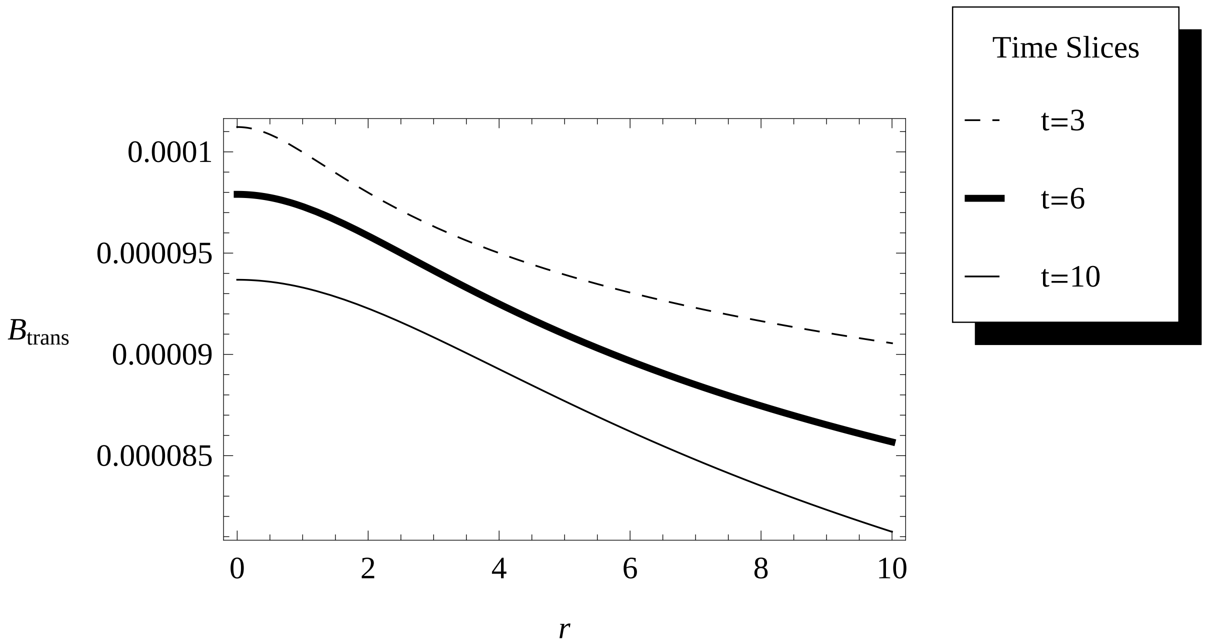

Figure 5.4: The potential B versus r for the transformed solution (5.2.4) at varying units of time.

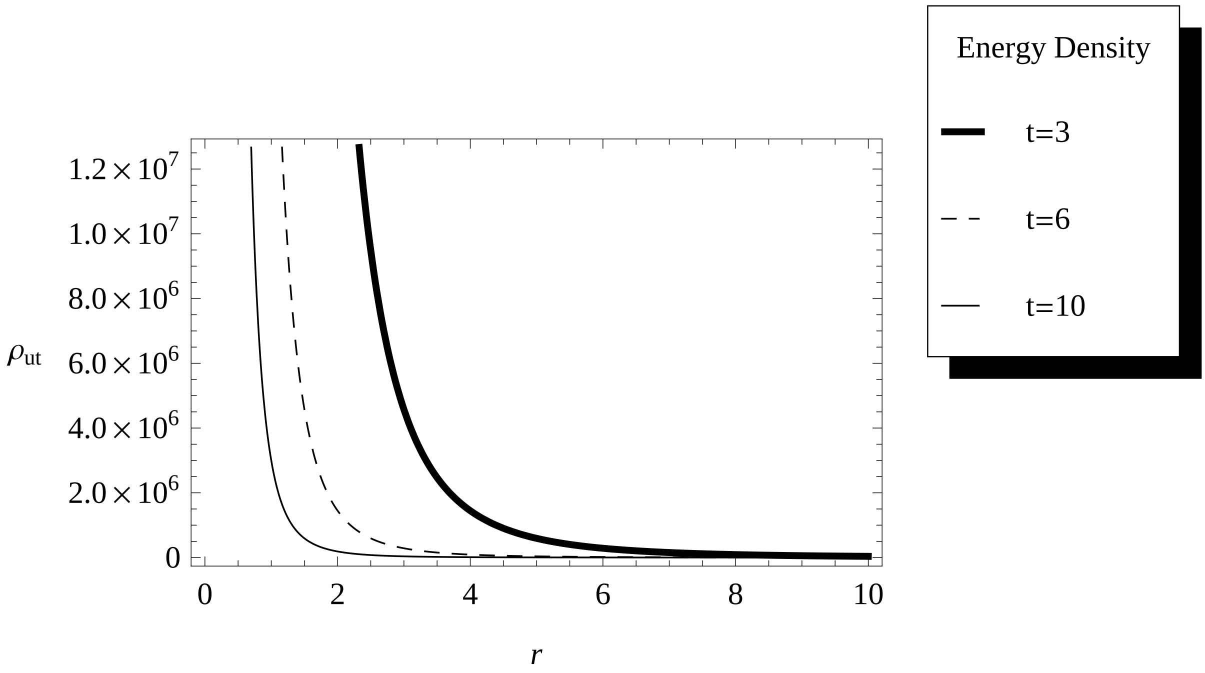

Figure 5.5: The radial profile for the energy density ρ resulting from the untransformed solution (5.2.2) at various units of time.

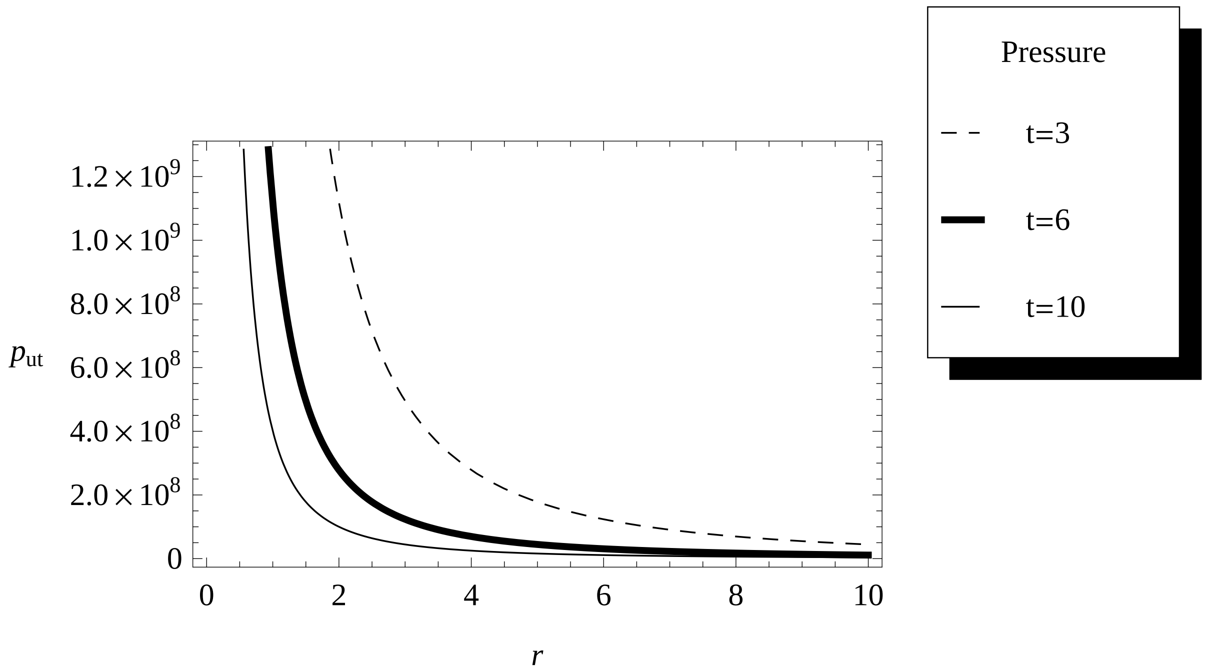

Figure 5.6: The radial profile for the pressure p resulting from the untransformed solution (5.2.2) at various units of time.

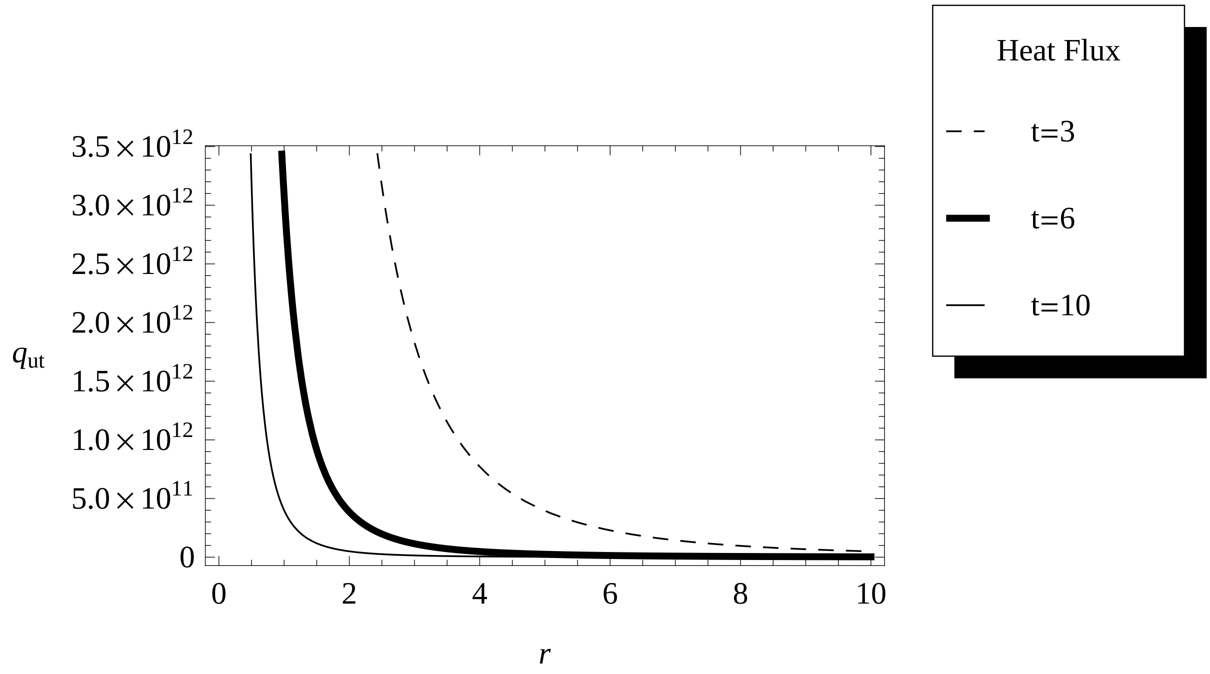

Figure 5.7: The radial profile for the heat flux q resulting from the untransformed solution (5.2.2) at various units of time.

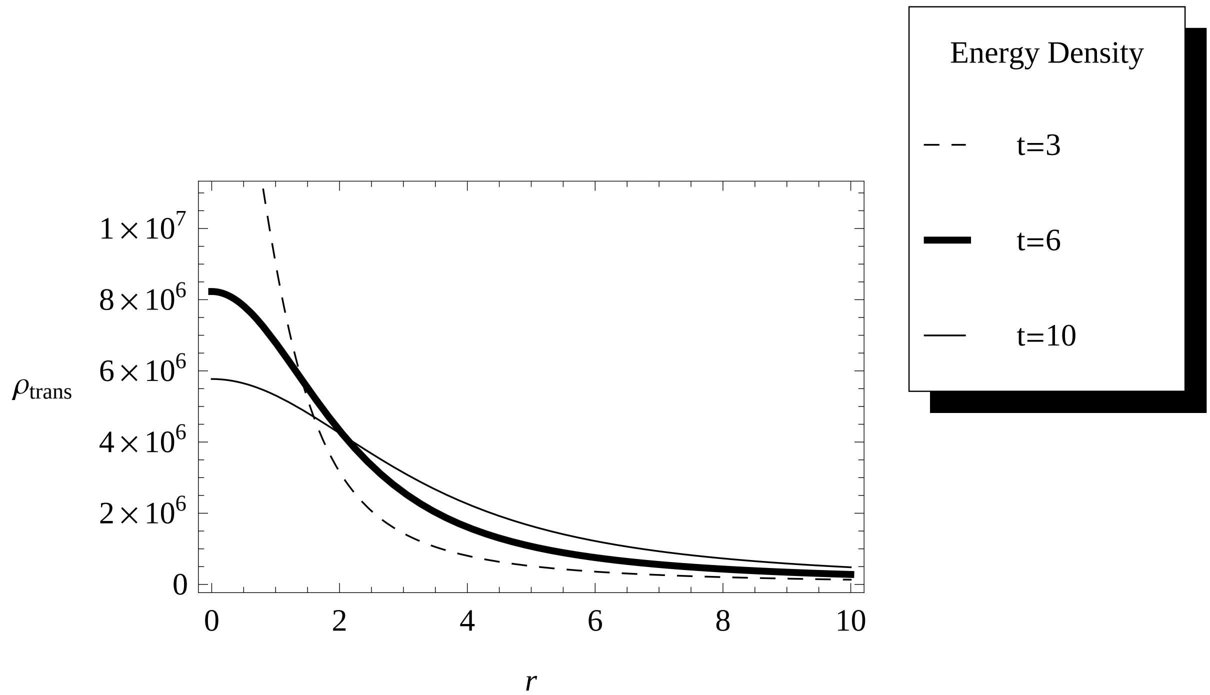

Figure 5.8: The radial profile for the energy density ρ resulting from the transformed solution (5.2.4) at various units of time.

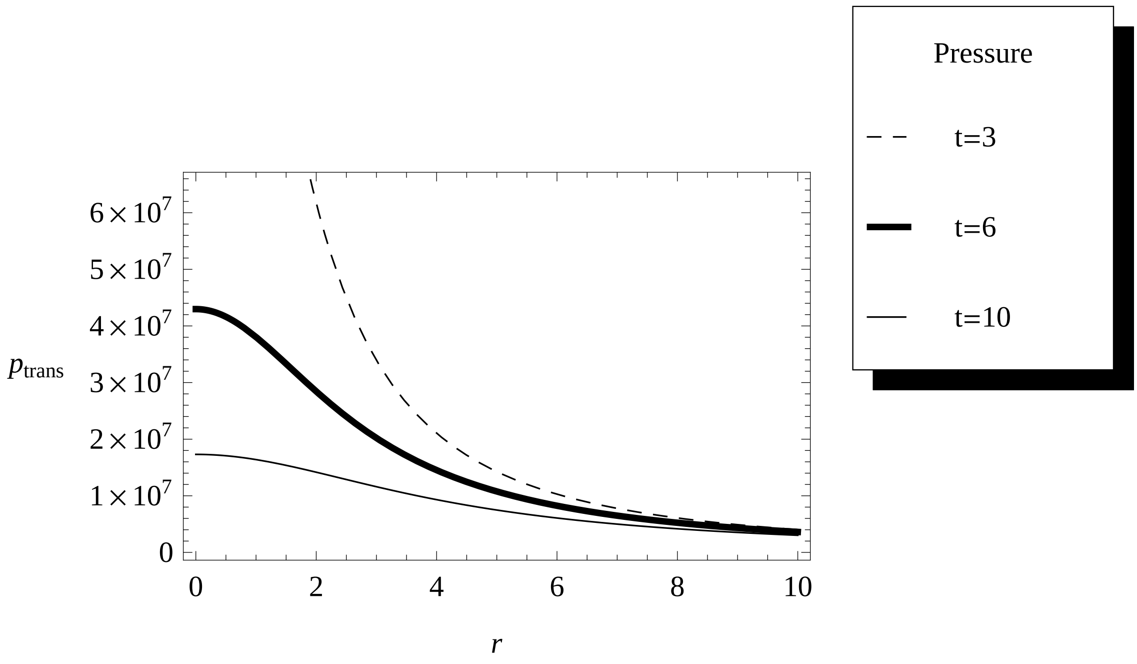

Figure 5.9: The radial profile for the fluid pressure p resulting from the transformed solution (5.2.4) at various units of time.

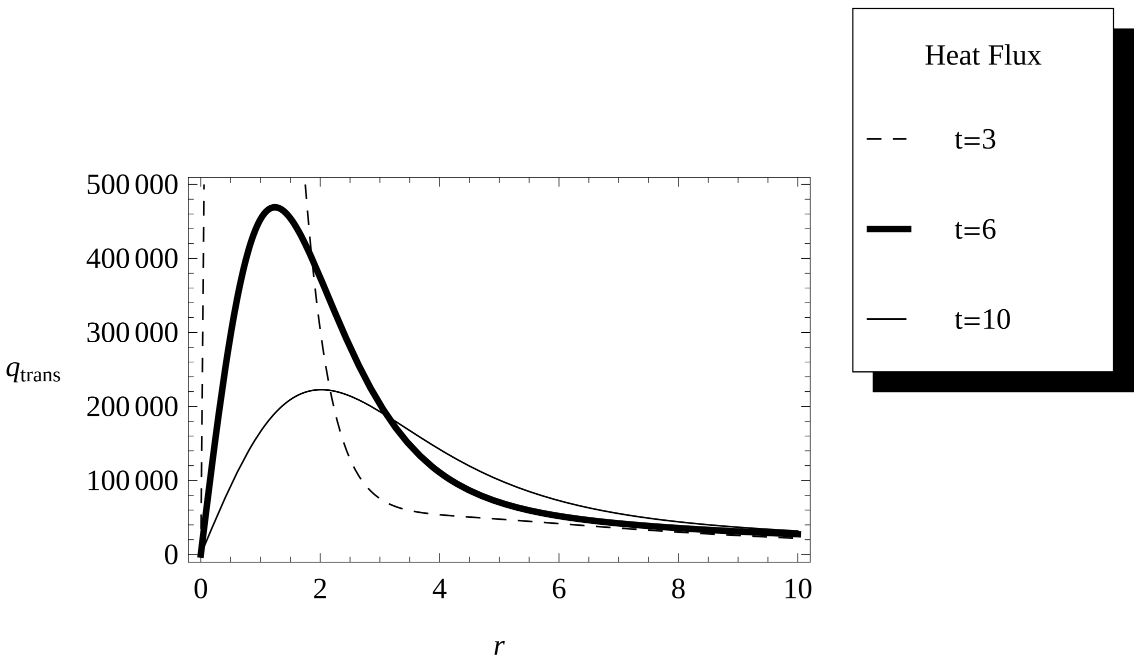

Figure 5.10: The radial profile for the heat flux q resulting from the transformed solution (5.2.4) at various units of time.

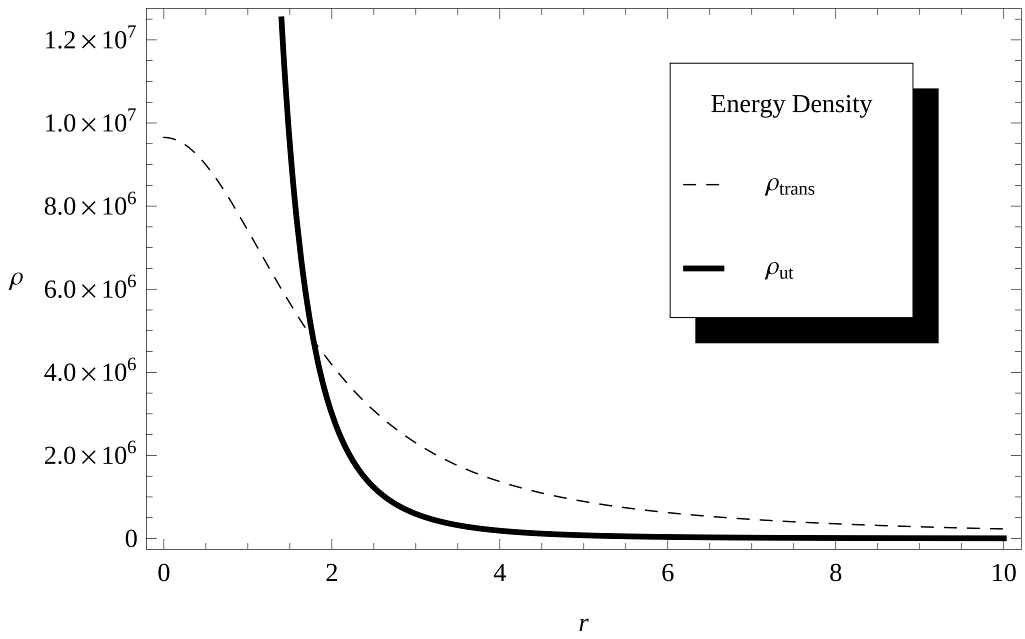

Figure 5.11: The radial profile of the energy densities for both solutions at various time slices.

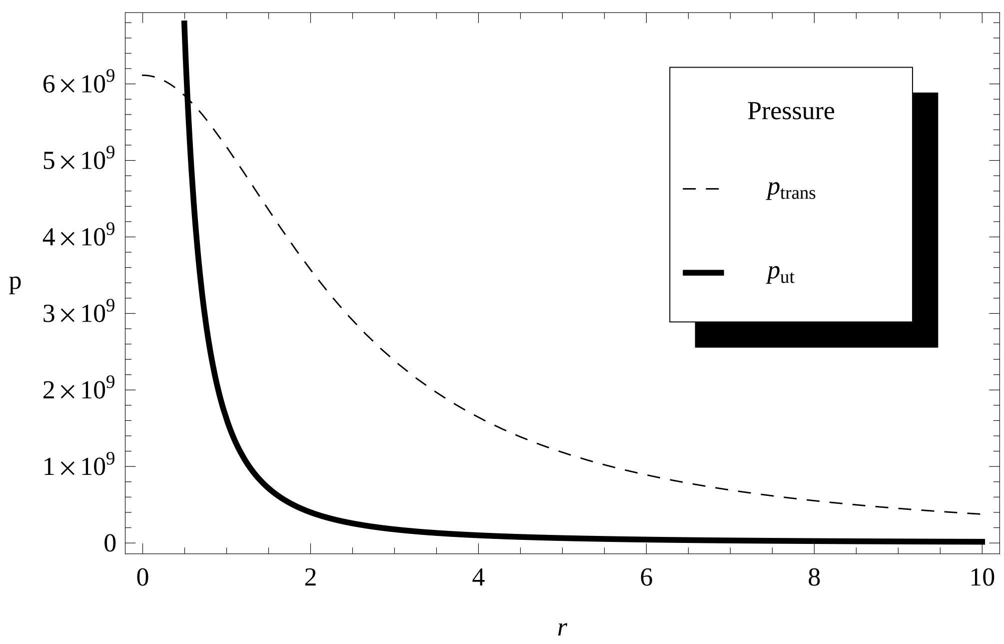

Figure 5.12: The radial profile of the fluid pressures for both solutions at various timeslices.

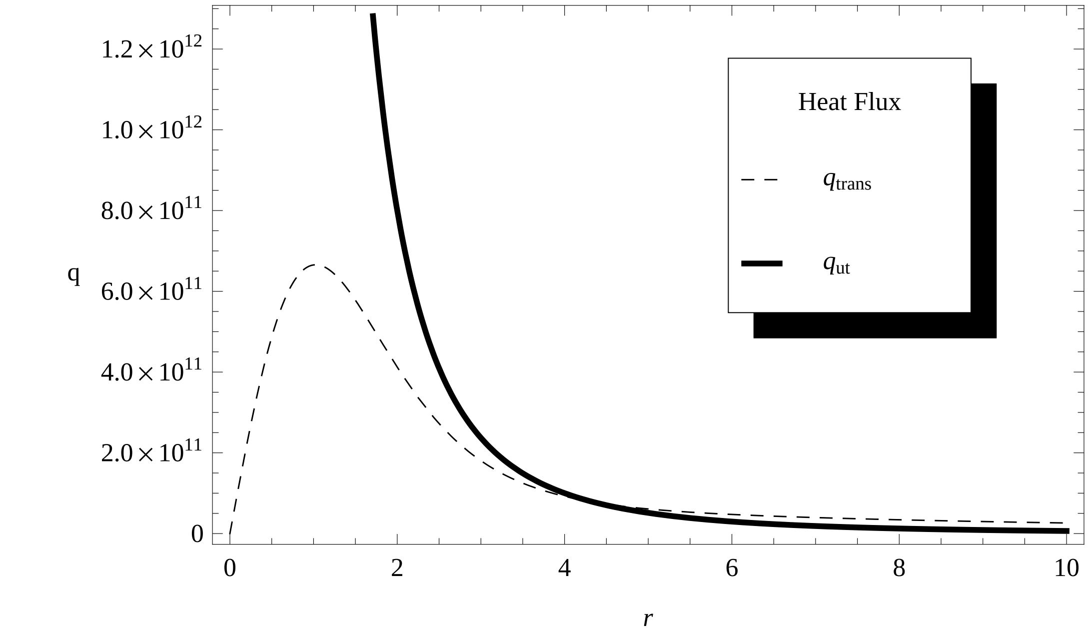

Figure 5.13: The radial profile of the heat flux for both solutions at various time slices.

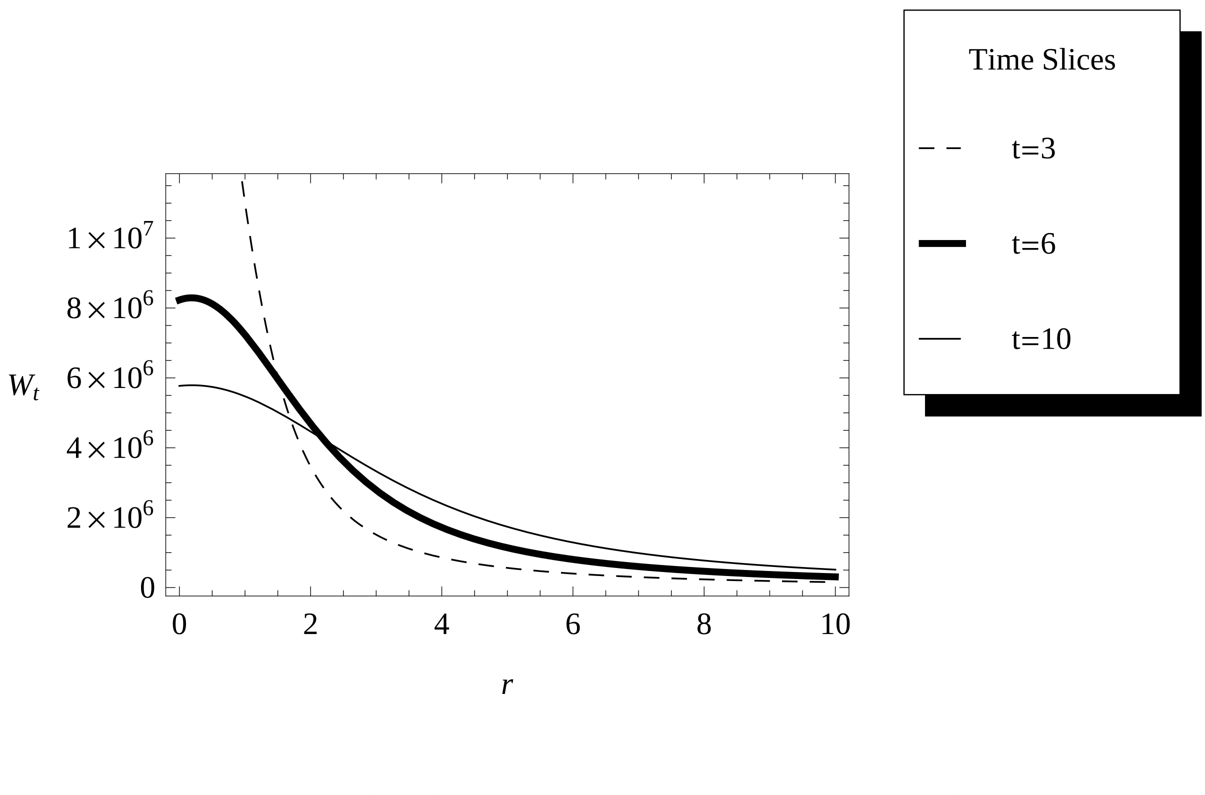

Figure 5.14: The radial profile of the weak energy condition resulting from the transformed solution (5.2.4) at various units of time.

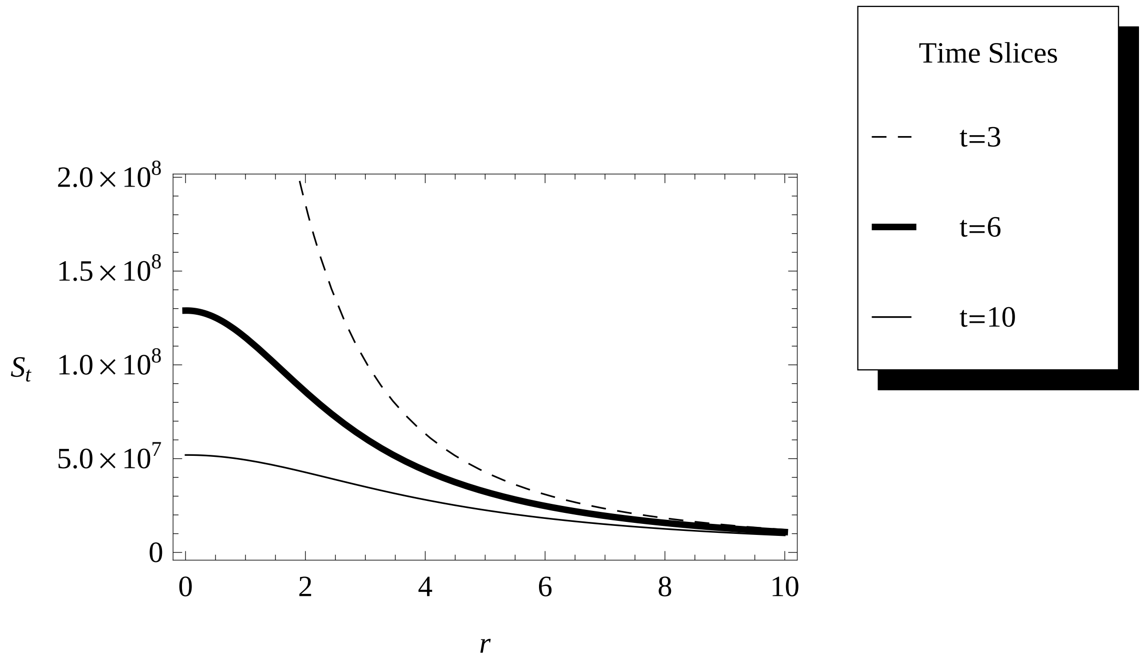

Figure 5.15: The radial profile of the strong energy condition for the transformed solution (5.2.4) at various units of time.

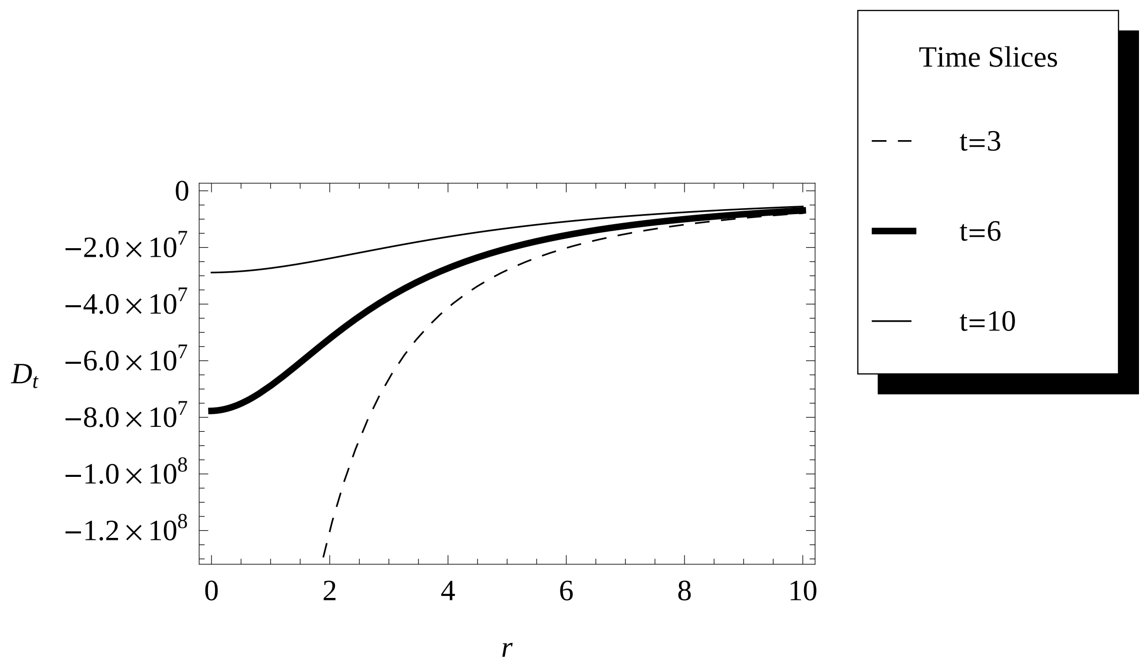

Figure 5.16: The radial profile of the dominant energy condition for the transformed solution (5.2.4) at various units of time.

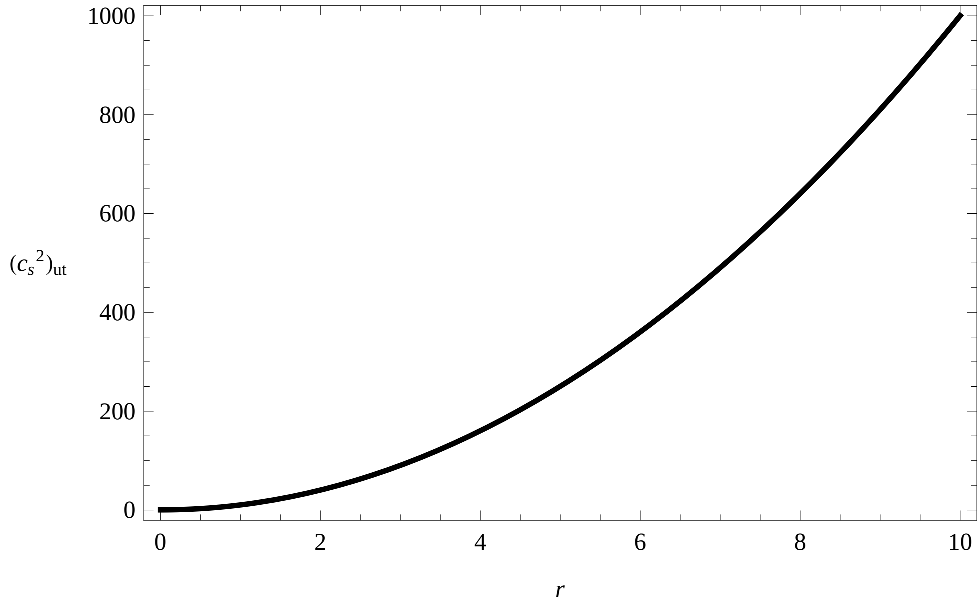

Figure 5.17: Radial profile of the sound speed for the untransformed solution (5.2.2).

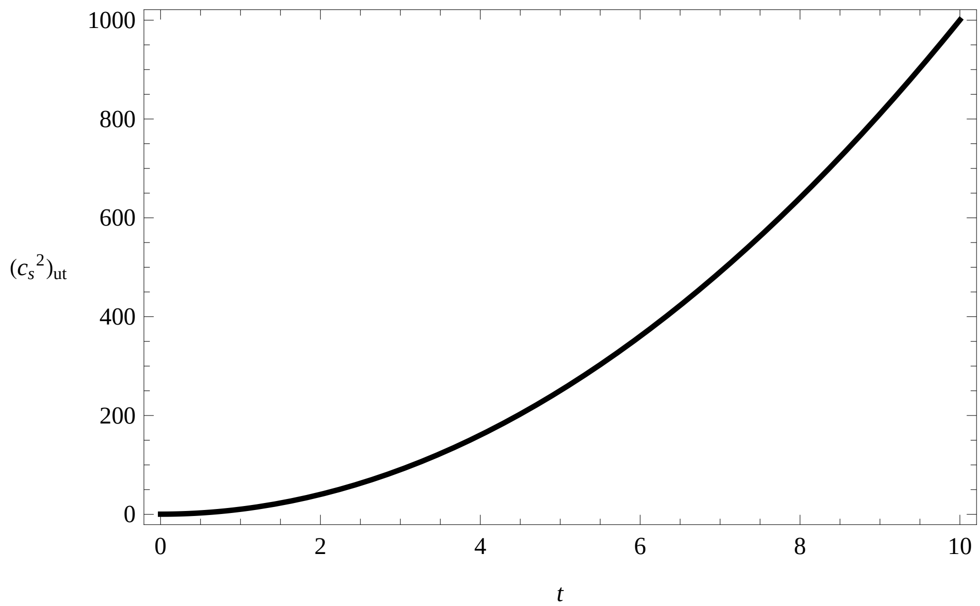

Figure 5.18: Temporal profile of the sound speed for the untransformed solution (5.2.2).

Chapter 6

Relativistic thermodynamics of heat conducting fluids

6.1 Introduction

The study of relativistic thermodynamics in radiating stars has been extensively pursued by Herrera and Santos (1997) and Herrera *et al* (2004, 2006, 2009). In many of these prior models, the shear-free dissipative gravitational collapse of stars was described in the form of radial heat flow in the so-called free streaming approximation. It was shown that the relaxational effects are important during late stages of collapse and that these effects lead to much higher temperatures within the interior of the star. Closer to the surface the temperature can be lower. The earlier Eckart (Eckart 1940) framework for heat transport has deficiencies. This is due mainly to the noncausal nature of the theory. Eckart first extended irreversible thermodynamics from Newtonian to relativistic fluids, however its main deficiency is the fact that dissipative perturbations propagated at infinite speeds which was unreasonable in a relativistic theory. This has led to the use of causal heat transport equations that have the Maxwell-Cattaneo form. As mentioned by Maartens (1997), extended irreversible thermodynamics arises from the fact that an extended set including the dissipative variables was required to study the non-equilibrium states. This notion leads to stable and causal behaviour under a wide variety of conditions. Israel and Stewart (Israel 1976, Israel and Stewart 1979) studied the relativistic version of this theory, and this is called causal/second-order thermodynamics (due to the dissipative variables being of second order in entropy) and transient thermodynamics (transient phenomena were included in the theory outside the quasi-stationary system of the classical theory). The main difference between the Eckart and Israel-Stewart equations is that the former are mere algebraic equations while the latter are differential evolution equations which are much more difficult to solve in general. In §6.2 we present the heat transport equation of Maxwell and Cattaneo. We proceed to generate the temperature profiles with potentials from the metric (2.5.1) in general. The causal and noncausal temperature profiles are obtained for various cases in §6.3. Numerical plots are generated.

6.2 The Maxwell-Cattaneo heat transport equations

The Eckart temperature equation is known to produce unrealistic behaviour for highly relativistic matter. This issue may be remedied with the application of the causal thermodynamics of Israel and Stewart (1979). The relativistic causal transport equation for the temperature, or the Maxwell-Cattaneo heat transport equation, has been developed in general by Maartens and Triginer (1997). In the absence of rotation and viscous stresses the equation has the form

$$\tau h_{\varrho}^{\varsigma}\dot{q}_{\varsigma} + q_{\varrho} = -\kappa(h_{\varrho}^{\varsigma}T_{;\varsigma} + T\dot{u}_{\varrho}), \tag{6.2.1}$$

where $h_{\varrho\varsigma} = g_{\varrho\varsigma} + u_{\varrho}u_{\varsigma}$ is the projection tensor, T is the temperature, κ (≥ 0) is the coefficient of thermal conductivity and τ is the relaxational time scale. It is the presence of τ that gives rise to causal and stable behaviour in the Israel-Stewart theory. When $\tau = 0$ we regain the Fourier transport equation.

The equation (6.2.1) is truncated and insight into the values of τ and κ are required in order to solve it. Following the approach of Martínez (1996), we assume

$$\kappa = \gamma T^3 \tau_c, \tag{6.2.2}$$

where γ is a constant and $\tau_c = \frac{1}{n\sigma v}$ is the mean collision time (with σ being the collision cross section, v the mean particle speed and n is the total number of particles). We can make the following assumption (as was done by Nyonyi *et al* 2013)

$$\tau = \left(\frac{\beta\gamma}{\phi}\right)\tau_c = \beta T^{-\sigma}, \tag{6.2.3}$$

on physical grounds where ϕ, β and σ are non-negative constants. Therefore for the spherically symmetric metric (2.5.1), the causal evolution equation (6.2.1) becomes

$$\beta T^{-\sigma} h_\varrho^\varsigma \dot{q}_\varsigma + q_\varrho = -\phi T^{3-\sigma}\left(h_\varrho^\varsigma T_{;\varsigma} + T\dot{u}_\varrho\right).$$

The Maxwell-Cattaneo equation reduces to the following

$$\beta T^{-\sigma}(qB)_{,t} + q(AB) = -\phi\left(\frac{T^{3-\sigma}(AT)_{,r}}{B}\right), \tag{6.2.4}$$

which is a nonlinear partial differential equation.

6.3 Causal and noncausal temperatures

To integrate the transport equation (6.2.4), certain assumptions need to be made for the parameters β and σ. Following Govinder and Govender (2001), we consider two cases: $\beta = 0$ which is the noncausal Eckart theory, and the causal case $\beta \neq 0$. We shall present profiles for the temperature resulting from the exact solutions (4.2.6) and (4.2.9). For the purposes of the next sections, we introduce the following nomenclature:

(a) $T_{NC1}(r,t)$ - The noncausal temperature resulting from solution (4.2.6).

(b) $T_{C1}(r,t)$ - The causal temperature resulting from solution (4.2.6).

(c) $T_{NC2}(r,t)$ - The noncausal temperature resulting from solution (4.2.9).

(d) $T_{C2}(r,t)$ - The causal temperature resulting from solution (4.2.9).

6.3.1 Noncausal temperatures

For the case $\beta = 0$, the noncausal solutions obtained by the integration of (6.2.4) are

$$(AT)^{4-\sigma} = \frac{\sigma - 4}{\phi} \int \left(A^{4-\sigma} q B^2\right) dr + F(t), \tag{6.3.1}$$

for $\sigma \neq 0$. We obtain

$$\ln(AT) = -\frac{1}{\phi} \int qB^2 dr + F(t), \tag{6.3.2}$$

for $\sigma = 0$. In both cases, $F(t)$ is an integration function and for the purpose of this analysis, we set it to zero.

For solution (4.2.6), when $\alpha(t) = t$, $\kappa(t) = -t$ and $\zeta(t) = -t$, the analytical expressions for equations (6.3.1) and (6.3.2), respectively, are

$$\begin{aligned} T_{NC1}(r,t) &= \frac{1}{t}\left[\frac{12t^2}{r^2+2t}\right]^{1/3}, \quad 0 \neq \sigma = 1, \ \phi = 1, && \text{(6.3.3a)} \\ T_{NC1}(r,t) &= \frac{1}{t}\exp\left[\frac{4}{t(r^2+2t)}\right], \quad \sigma = 0, \ \phi = 1. && \text{(6.3.3b)} \end{aligned}$$

For solution (4.2.9), when $\alpha(t) = t$, $\gamma(t) = t$ and $\eta(t) = t$, equations (6.3.1) and (6.3.2) respectively become

$$\begin{aligned} T_{NC2}(r,t) &= \frac{2t-r^2}{e^t}\left[\frac{3e^{2t}(1+t)}{t(2t-r^2)^2}\right]^{1/3}, \quad 0 \neq \sigma = 1, \ \phi = 1, && \text{(6.3.4a)} \\ T_{NC2}(r,t) &= \frac{2t-r^2}{e^t}\exp\left[\frac{4e^{-t}r^2(1+t)}{t}\right], \quad \sigma = 0, \ \phi = 1. && \text{(6.3.4b)} \end{aligned}$$

Figures 6.1 and 6.2 depict the numerical noncausal temperature profiles for the untransformed solution (4.2.6) for the cases $\sigma \neq 0$ and $\sigma = 0$ respectively. It is easy to see that the thermal behaviour in both situations is smooth and finite throughout the interior fluid distribution. It is also important to notice that both of the noncausal temperatures resulting from this solution are positive and decreasing as we move radially outward. In figures 6.3 and 6.4, the numerical noncausal temperatures emanating from solution (4.2.9) are displayed. We note that the behaviour of the curves is smooth and monotonic; however at a certain radial value it appears that both become negative. The probable reason for this behaviour could be mathematical and not necessarily physical due to the form of the solution (4.2.9) itself. We note the smooth, finite behaviour and general decrease with r. Open models (for which the fluid distribution has no finite boundary) leave many physical questions that need to be addressed. We believe that this physical behaviour arises because we have noncausality inherent in the model since $\beta = 0$.

6.3.2 Causal temperature: $\sigma = 0$

When the mean collision time vanishes, equation (6.2.4) can be integrated to give the following expression

$$(AT)^4 = -\frac{4}{\phi}\left[\beta \int A^3 B (qB)_{,t} dr + \int A^4 q B^2 dr\right] + F(t), \qquad (6.3.5)$$

where $F(t)$ is a function of integration.

For solution (4.2.6), when $\alpha(t) = t$, $\kappa(t) = -t$ and $\zeta(t) = -t$, equation (6.3.5) becomes

$$T_{C1}(r,t) = \frac{1}{t}\left[\frac{16t^2(t-2) - 32t(t-1) + 16t^3}{r^2 + 2t} - 4(t-1)\ln(r^2 + 2t)\right]^{1/4}, \qquad (6.3.6)$$

when $\phi = 1 = \beta$. For solution (4.2.9), when $\alpha(t) = t$, $\gamma(t) = t$ and $\eta(t) = t$, the expression for (6.3.5) becomes

$$T_{C2}(r,t) = \frac{2t - r^2}{e^t}\left[\frac{f(t)}{3t^2(r^2 - 2t)^3} - \frac{g(t)}{3t(2t - r^2)^3} + \frac{h(t)}{3(2t - r^2)^3}\right]^{1/4}, \qquad (6.3.7)$$

for $\phi = 1 = \beta$. In the above expression, we have set $f(t) = 4e^{2t}(2 + 3t + 2t^2)(2t - 3r^2)$, $g(t) = 32e^{2t}(1 + 2t + 2t^2)$ and $h(t) = 16e^{3t}(1 + t)$.

6.3.3 Causal temperature: $\sigma \neq 0$

In the case when $\sigma \neq 0$ the evolution equation (6.2.4) can be integrated by treating it as a Bernoulli equation in the variable (AT). We obtain the temperature

$$\begin{aligned}(AT)^4 &= \frac{-4\beta}{\phi}\exp\left(-\int \frac{4qB^2}{\phi} dr\right)\int A^3 B (qB)_{,t} \exp\left(\int \frac{4qB^2}{\phi} dr\right) \\ &\quad + F(t)\exp\left(-\int \frac{4qB^2}{\phi} dr\right). \qquad (6.3.8)\end{aligned}$$

For simplicity the integration function $F(t)$ is taken to be zero. In the first solution, (4.2.6), with $\alpha(t) = t$, $\kappa(t) = -t$ and $\zeta(t) = -t$, the expression for the temperature (6.3.8) is

$$\begin{aligned} T_{C1}(r,t) &= \frac{1}{t}\left[-4\exp\left(\frac{6}{t(r^2+2t)}\right)\right. \\ &\quad \left.\times\int\left(\frac{j(t)}{(r^2+2t)^2}+\frac{k(t)}{(r^2+2t)^2}\right)\exp\left(\frac{-6}{t(r^2+2t)}\right)dr\right], \end{aligned} \tag{6.3.9}$$

where $j(t) = 8t(t-1)r^3$ and $k(t) = 8t^2(t-2)r$. The second solution (4.2.9) admits an expression for the temperature in the form

$$\begin{aligned} T_{C2}(r,t) &= \frac{2t-r^2}{e^t}\left[-4\exp\left(\frac{2e^{-t}(1+t)r^2}{t}\right)\right. \\ &\quad \left.\times\int\left(\frac{m(t)r}{t^2(2t-r^2)^4}-\frac{n(t)r^3}{t^2(2t-r^2)^4}\right)\exp\left(-\frac{w(t)r^2}{t}\right)dr\right], \end{aligned} \tag{6.3.10}$$

with parameter values of $\alpha(t) = t$, $\gamma(t) = t$ and $\eta(t) = t$. In the above we have set $m(t) = 8te^{2t}(1+2t+2t^2)$, $n(t) = 4e^{2t}(2+3t+2t^2)$ and $w(t) = 2e^{-t}(1+t)$.

Figure 6.5 depicts the numerical causal temperature profile for solution (4.2.6) when $\sigma = 0$. It is apparent that the behaviour of the curve is smooth and monotonically decreasing as we move radially outward in the fluid. The causal temperature is also always positive throughout the interior which is physically acceptable. In figure 6.6 the numerical causal temperature for the solution (4.2.9) is shown in the case when $\sigma = 0$. The smooth and positive behaviour depicted by the curve is clearly noted. We compare these favourable profiles and positive temperatures in this section arising from causal thermodynamics with the acausal behaviour is §6.3.2. An additional point that requires mentioning is the fact that the causal temperatures, in the cases when $\sigma \neq 0$ could not be produced for either of these two solutions due to the very complicated nature of the analytical temperature profiles.

6.3.4 Discussion

Making use of the solutions (4.2.6) and (4.2.9), we were able to generate curves which describe the temperature profiles for two particular models. For each solution, the numerical noncausal temperatures were plotted against r. Solution (4.2.6) depicted smooth and finite behaviour and for both cases (noncausal and causal), the resulting curves were monotonically decreasing and positive. The latter solution (4.2.9) realised profiles with smooth and finite behaviour, which, however, at certain values of r, become negative in the noncausal case. This notion highlights the deficiencies in the noncausal Eckart theory. When the collision time was nonvanishing, causal temperature profiles could not be produced. Furthermore, it is worthwhile to note that the temperatures generated by solutions (4.2.6) and (4.2.9) are finite and maximum at the centre ($r = 0$) of the fluid distribution, which is the physical requirement in a stellar model.

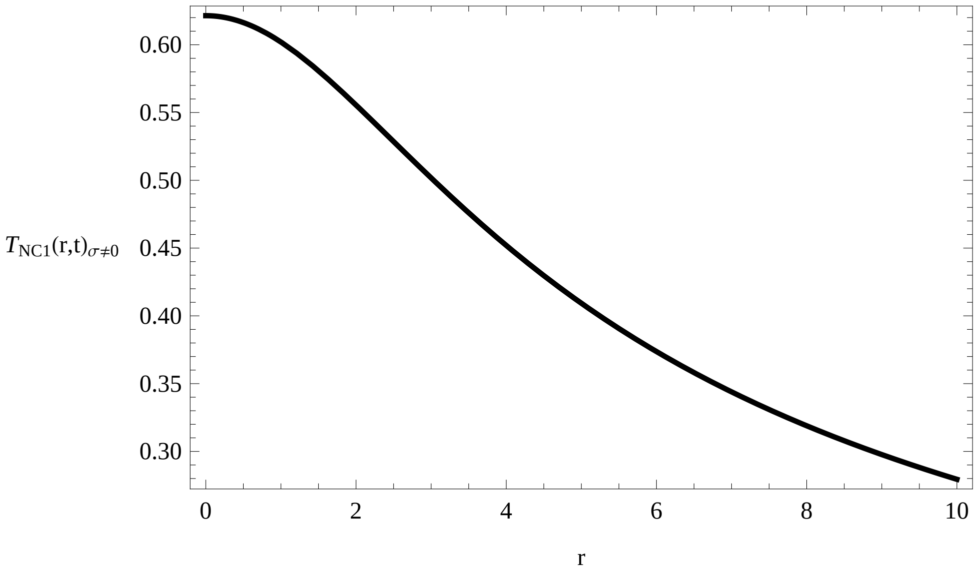

Figure 6.1: Radial profile of the noncausal temperature for the untransformed solution (4.2.6) when $\sigma \neq 0$.

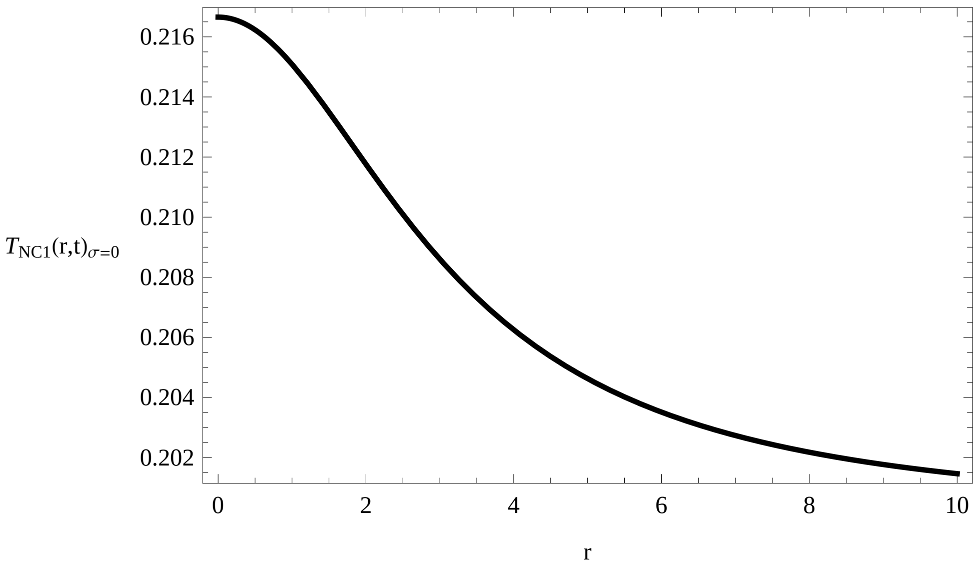

Figure 6.2: Radial profile of the noncausal temperature for the untransformed solution (4.2.6) when $\sigma = 0$.

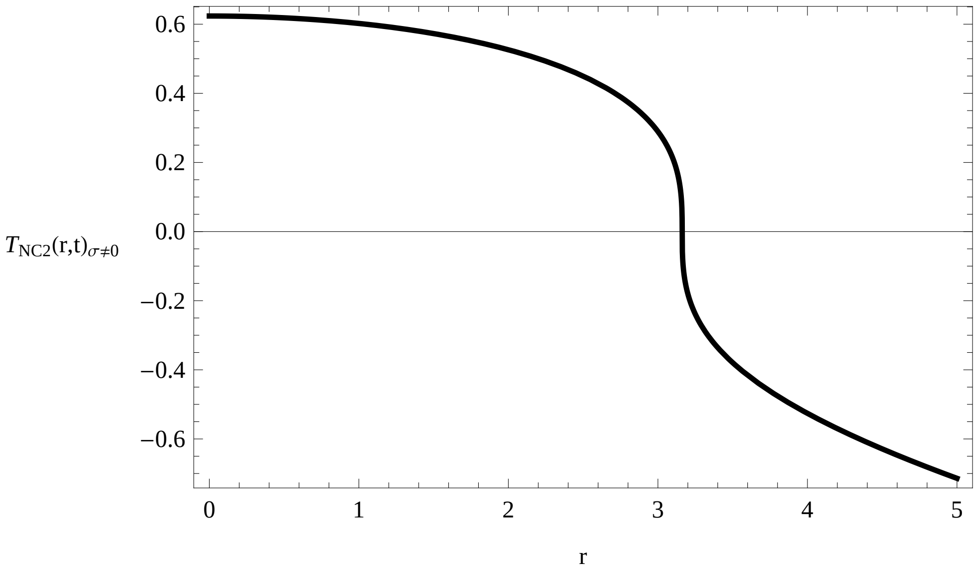

Figure 6.3: Radial profile of the noncausal temperature for the untransformed solution (4.2.9) when $\sigma \neq 0$.

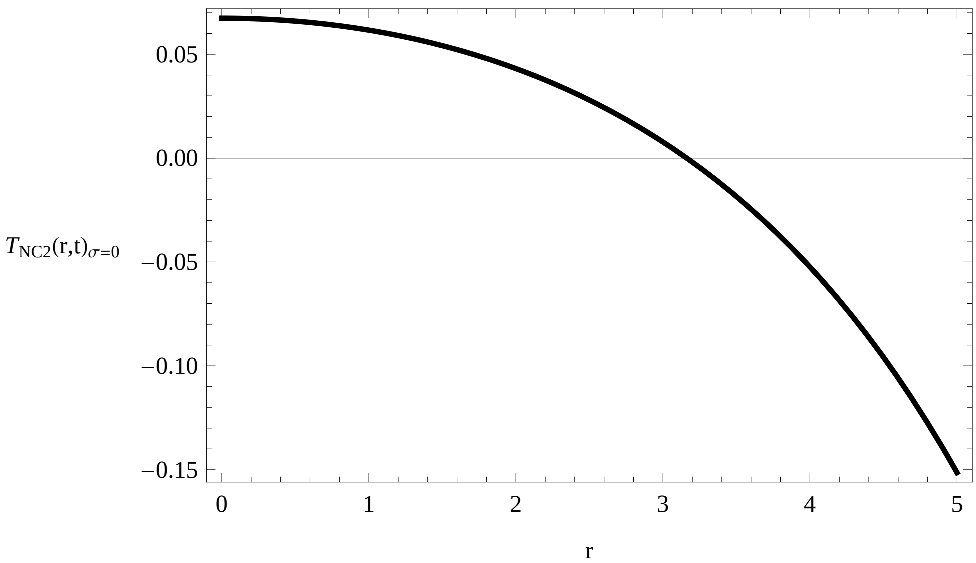

Figure 6.4: Radial profile of the noncausal temperature for the untransformed solution (4.2.9) when $\sigma = 0$.

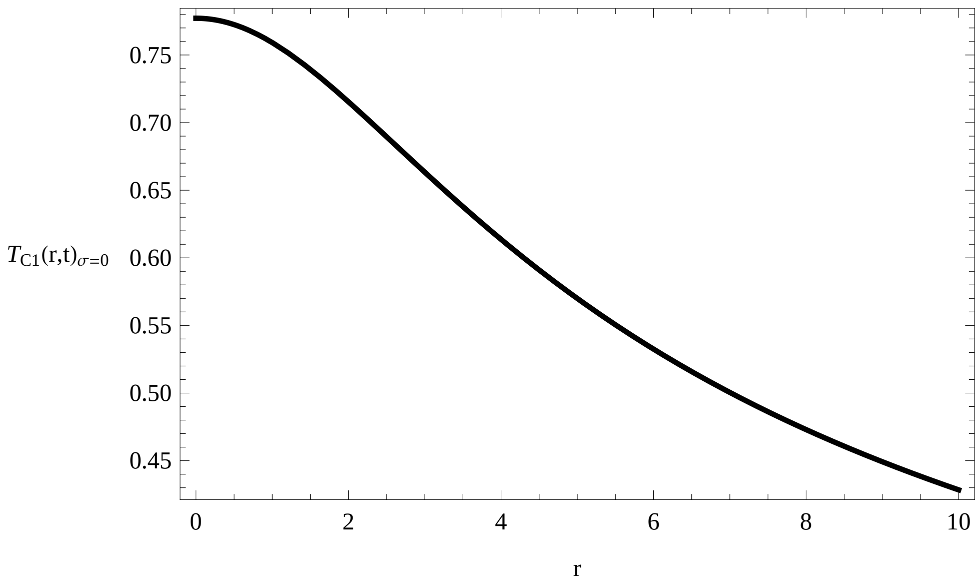

Figure 6.5: Radial profile of the causal temperature for the untransformed solution (4.2.6) when $\sigma = 0$.

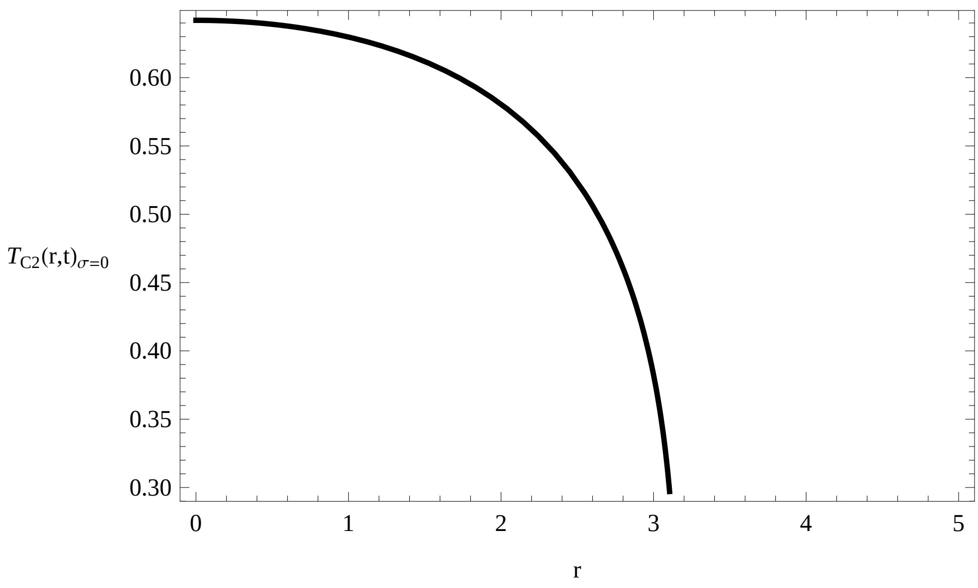

Figure 6.6: Radial profile of the causal temperature for the untransformed solution (4.2.9) when $\sigma = 0$.

Chapter 7

Conclusion

In this thesis we have studied spherically symmetric shear-free spacetimes and the associated models used to describe fluid distributions. We have generated new exact solutions which are applicable to stellar objects in relativistic astrophysics. Our models have vanishing shear and describe heat flow in the interior of the fluid. The consistency condition arising from the isotropy of the fluid pressure was analysed. It was shown that the resulting master equation was a second order ordinary differential equation with variable coefficients in the gravitational variables A and B. In general this is not easy to solve. However, in this investigation, particular forms for the gravitational potentials were chosen and several new solutions were obtained in terms of elementary functions, one of which is an entire class. In the same vane, a simple transformation was imposed upon the consistency condition, and further new solutions to the resulting equation were found, again in terms of elementary functions.

Below is an overview of our study:

- In chapter 2, we introduced the relevant formalisms and aphorisms for differential geometry and general relativity. A brief description of electromagnetism was provided along with a note on the physical and energy conditions for a stellar model. The general framework for the dynamics of the relativistic fluid distribution was constructed in both the static and radiating cases. We demonstrated that, for the latter, the pressure isotropy condition could be written as an ordinary differential equation

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right).$$

It was also realised that this equation could be transformed and presented in the more compact form

$$\left(\frac{A}{B}\right)_{xx} = 2A\left(\frac{1}{B}\right)_{xx}.$$

The chapter closes with a portraiture of the gravitational collapse of stellar objects.

- Chapter 3 consisted of a review of some known solutions for shear-free heat conducting models. We discussed solutions obtained for pressure isotropy and these included the early geodesic model of Bergmann (1981), and the conformally flat models of Maiti (1982), Modak (1984) and Banerjee *et al* (1989). The algorithms of Deng (1989) and Ngubelanga and Maharaj (2013), in which they used methods which inspired our approach, were also investigated in detail. Finally, solutions resulting from a group theoretic approach were discussed. These included the results of Msomi *et al* (2011) and Nyonyi *et al* (2013, 2014). Solutions to the boundary condition for relativistic radiating stars were also described. The geodesic model of Thirukkanesh and Maharaj (2009) as well as the conformally flat models of Herrera *et al* (2006) and Misthry *et al* (2008) were studied. A brief discussion on initially static models was provided, and the solution of Tewari (2013) was described. Finally, the models of Abebe *et al* (2013) and Nyonyi *et al* (2014), in which they used the Lie analysis technique to find new solutions to the consistency condition, were probed.

- The quintessential focus of this thesis resides in chapter 4. New solutions for the untransformed pressure isotropy equation were generated when certain restrictions were imposed on the gravitational variables. For the untransformed equation, the choices $A = \alpha(t)$ and $B = \alpha(t)A$ yielded solutions equivalent to the geodesic and conformally flat scenarios respectively. For the assumption $B = \alpha(t)r^n$, we obtained the metric

$$\begin{aligned} ds^2 &= -\left[\psi(t)r^{n+1+\sqrt{2n^2+4n+1}} + \xi(t)r^{n+1-\sqrt{2n^2+4n+1}}\right]^2 dt^2 \\ &\quad +\alpha(t)r^{2n}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \end{aligned}$$

The choice $B = \alpha(t)r^{\beta(t)n+\gamma(t)}$ yielded

$$\begin{aligned} ds^2 &= -\left[\tau(t)r^{\beta n+\gamma+1+\sqrt{2\beta^2n^2+4\beta\gamma n+4\beta n+2\gamma^2+6\gamma+1}}\right. \\ &\quad \left. +\chi(t)r^{\beta n+\gamma+1-\sqrt{2\beta^2n^2+4\beta\gamma n+4\beta n+2\gamma^2+6\gamma+1}}\right]^2 dt^2 \\ &\quad +(\alpha r^{\beta n+\gamma})^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \end{aligned}$$

as a solution which generalises the preceding expression. In a similar manner, we generated new solutions for the transformed pressure isotropy equation. For the choice $B^{-1} = \alpha(t)k^{\beta(t)x+\gamma(t)}$, the resulting metric was

$$\begin{aligned} ds^2 &= -\left[\nu(t)k^{\beta(t)[-1-\sqrt{2}]r^2} + \kappa(t)k^{\beta(t)[-1+\sqrt{2}]r^2}\right]^2 dt^2 \\ &\quad +\alpha(t)^{-2}k^{-2[\beta(t)r^2+\gamma(t)]}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \end{aligned}$$

The choice of $B^{-1} = \alpha(t)\sinh x$ yielded the solution

$$\begin{aligned} ds^2 &= -\left[\varrho(t)e^{\sqrt{2}r^2}\operatorname{csch} r^2 + \varsigma(t)e^{-\sqrt{2}r^2}\operatorname{csch} r^2\right]^2 dt^2 \\ &\quad +\alpha(t)^{-2}\operatorname{csch}^2 r^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \end{aligned}$$

which is singular at $r = 0$. For $B^{-1} = \alpha(t)\cosh x$, the resulting metric found was

$$\begin{aligned} ds^2 &= -\left[\phi(t)e^{\sqrt{2}r^2}\operatorname{sech} r^2 + \vartheta(t)e^{-\sqrt{2}r^2}\operatorname{sech} r^2\right]^2 dt^2 \\ &\quad +\alpha(t)^{-2}\operatorname{sech}^2 r^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \end{aligned}$$

which differs from the preceding solution since $\operatorname{sech} r^2$ is regular at the origin. The choice $B^{-1} = \alpha(t)x^2/(\beta(t)x + 1)$ gave the solution

$$\begin{aligned} ds^2 &= -\left[\gamma(t)\left(\frac{r^2}{\beta(t)r^2+1}\right)^{-\frac{1}{2}\left(3+\sqrt{17}\right)} + \varphi(t)\left(\frac{r^2}{\beta(t)r^2+1}\right)^{\frac{1}{2}\left(-3+\sqrt{17}\right)}\right]^2 dt^2 \\ &\quad +\left(\frac{\beta(t)r^2+1}{\alpha(t)r^4}\right)^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \end{aligned}$$

Letting $B^{-1} = \alpha(t)A^{\beta(t)}$ resulted in

$$\begin{aligned} ds^2 &= -\left[\left(\frac{\beta(t)^2 - 2\beta(t) - 1}{\beta(t) - 1}\right)[\zeta(t)r^2 + \tau(t)]\right]^{\frac{2\beta(t)-2}{\beta(t)^2-2\beta(t)-1}} dt^2 \\ &\quad +\alpha(t)^{-2}\left[\left[\frac{\beta(t)^2 - 2\beta(t) - 1}{\beta(t) - 1}\left[\zeta(t)r^2 + \tau(t)\right]\right]^{\frac{\beta(t)-1}{\beta(t)^2-2\beta(t)-1}}\right]^{-2\beta(t)} \\ &\quad \times\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \end{aligned}$$

and $B^{-1} = \alpha(t)(\beta(t)x + A)$ yielded

$$\begin{aligned} ds^2 &= -\left[\eta(t) - \chi(t)\tanh^{-1}\left(\frac{r^2}{\chi(t)}\right)\right]^2 dt^2 \\ &\quad +\alpha(t)^{-2}\left[\beta(t)r^2 + \eta(t) - \chi(t)\tanh^{-1}\left(\frac{r^2}{\chi(t)}\right)\right]^{-2} \\ &\quad \times[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \end{aligned}$$

Finally, a general solution was obtained by integrating the transformed pressure isotropy condition directly. Assuming a functional dependence $A = A(B, t)$, we generated the solution

$$\int\left[\exp\int\left(\frac{\left[B^2\left(\frac{A}{B}\right)_{BB} - 4A/B\right]}{(AB)_B}\right)dB\right]dB = \alpha(t)x + \beta(t).$$

A very brief discussion on the mathematics of generating solutions was provided. It is important to note that our solutions are new and have not been published previously.

- A physical analysis was then performed, in chapter 5, for a special case of the class of solutions with a power law form $B = \alpha r^n$ in the untransformed case as well as for the solution $B^{-1} = \alpha A^\beta$ in the transformed case. The spatial and temporal profiles that were produced for the matter and gravitational variables, indicate that in the untransformed case, the resulting model is consistent with a core-envelope scenario, which is more plausible for a realistic description of a stellar interior. Our results also suggest that closer to the centre of the fluid distribution, the heat flux q dominates the pressure p and energy density ρ by at least two orders of magnitude. Both the models exhibited behaviour for the matter variables which is acceptable in a realistic stellar scenario. We also analysed the energy conditions for the two solutions and, again, found consistent behaviour for the most part, with only the dominant energy condition being violated. Finally, the sound speed was investigated for the untransformed power law model and it was found to be largely reasonable within the interior of the heat conducting fluid distribution.

- In this chapter, the thermal evolution of the fluid was studied in detail. A brief exploration of the Eckart (1940) and Israel and Stewart (Israel 1976, Israel and Stewart 1979) theory was presented as well as a description of the Maxwell-Cattaneo heat transport equation. Expressions were found for the noncausal and causal cases relating to our heat conducting model, and explicit expressions were produced for some solutions in the untransformed scenario. Finally, we generated the temperature profiles for both the noncausal and causal scenarios and remarked on their similarities and differences.

The investigations in this thesis and the results generated form an essential part of a wide array of models that can be used within the framework of general relativity to construct realistic and physically meaningful studies in astrophysics and cosmology. For the purpose of astrophysical modeling, these investigations can be enhanced by including the following features, amongst others, for more accurate descriptions:

- Imposing an equation of state. This would provide a more specific picture of the actual matter content in the interior of the radiating fluid. As a consequence, the extended models that arise can form a strong basis for investigations of compact neutron stars and pulsars, as well as the so-called ultra compact objects like quark stars and quark gluon stars.

- Introducing a finite boundary or surface that localises the fluid distribution and acts as an interface between the interior and exterior spacetimes. This would bring into the model, the relativistic junction conditions that are defined at the stellar surface and which describe the coupling between the evolution of the interior and exterior matter and gravitational fields. The Santos junction condition would have to be solved as a highly nonlinear differential equation at the boundary in order for us to fully examine the physics of the radiating stellar object.

- Investigating the dynamical stability of the dissipating fluid in the context of non-adiabatic gravitational collapse. To achieve this, we would require a detailed analysis of the behaviour of the effective adiabatic index Γ_{eff}, the Weyl tensor C_{abcd} and the rate of collapse $\Theta = u^a{}_{;a}$.

- Including the effects of shearing anisotropic stresses in the radiating fluid. This is a more general and realistic way of depicting stellar fluids in relativistic astrophysics.

These research endeavours will be carried out in future work.

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