LIE GROUP ANALYSIS OF EXOTIC OPTIONS

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As the candidate's supervisor, I have approved this dissertation for submission.

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Abstract

Exotic options are derivatives which have features that makes them more complex than vanilla traded products. Thus, finding their fair value is not always an easy task. We look at a particular example of the exotic options - the power option - whose payoffs are nonlinear functions of the underlying asset price. Previous analyses of the power option have only obtained solutions using probability methods for the case of the constant stock volatility and interest rate. Using Lie symmetry analysis we obtain an optimal system of the Lie point symmetries of the power option PDE and demonstrate an algorithmic method for finding solutions to the equation. In addition, we find a new analytical solution to the asymmetric type of the power option.

We also focus on the more practical and realistic case of time dependent parameters: volatility and interest rate. Utilizing Lie symmetries, we are able to provide a new exact solution for the terminal pay off case.

We also consider the power parameter of the option as a time dependent factor. A new solution is once again obtained for this scenario.

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- The University of KwaZulu-Natal for financial support, through a doctoral scholarship.

Dedication

I dedicate this work to Him that I call:

Kabiyesi eledunmare, arugbo ojo, oba tose baba fun awon alaini baba, oba tinje emi ni maseberu, oba tinmu ileri se, adagba maparo oye, eleti gbaroye, okan soso ajanaku, alewi lese, alese lewi, alawo tele orun, eru jeje leti okun pupa, oyinkin yinki oba mimo, aterere kari aye, oba lana, loni titi ayeraye, oba tin tele bi eniteni, oba tin gba alai lara, oba toso ti enikan oleso, oba ibere ati opin.

THE MIGHTY GOD.

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Details of contribution to publications that form part and/or include research presented in this thesis.

1. **Publication 1**:

   Okelola M. O., Govinder K. S. and O'Hara J. G., Algorithmic solution of the Power option by the Lie group approach, *Preprint*

2. **Publication 2**

   Okelola M. O., Govinder K. S. and O'Hara J. G., Solving a PDE associated with the pricing of power options with time dependent parameters, *Preprint*

3. **Publication 3**

   Okelola M. O., Govinder K. S. and O'Hara J. G., Application of the Lie group analysis to time dependent PDEs in Financial Mathematics, *Preprint*

Contents

Chapter 1

Introduction

One of the challenges that scientists deal with in their research is how to translate a physical or economic phenomenon into a set of equations which describes it. It is usually impossible to describe a phenomenon totally, so one usually strives for a set of equations which describes the physical system adequately. In general, once the set of equations is built, data generated by the equations is compared to real data collected from the system. If the two sets of data 'agree' (or are close), then it can safely be concluded that the set of equations will lead to a good description of the real-world system. This description of real life situations by a set of equations is called modeling and the set of equations used in this description – for our purposes – are called differential equations (DEs).

The first step in building a model is to clearly state the assumptions on which the model will be based. These assumptions should describe the relationships among the quantities to be studied. Then a complete description of the parameters and variables to be used in the model follows. Using the assumptions – from the first step – to derive mathematical equations relating the parameters and variables is the next step. The process concludes with the solving of the equations derived.

The study of DEs began very soon after the invention of the differential and integral calculus, to which it formed a natural sequel. Newton in 1676 solved a differential equation (DE) by the use of an infinite series, but the results were not published until 1693 (the same year in which a DE occurred for the first time in the work of Leibniz) [31].

1.1 Historical background of the emergence of Lie groups

Sophus Lie is credited with developing this general technique to the solution of DEs. In the late nineteenth century, Lie made the discovery that all these special methods of solving DEs were in fact special cases of a general integration procedure based on the invariance of the DE under a continuous group of symmetries (Here, a symmetry refers to a group of transformations that transforms the set of all solutions of the differential equation to itself.). During the plenary meeting of the Royal Saxon Society of the Sciences, Friedrich Engel [23] ranked him as one of the foremost mathematicians of all time, possessing the true measures of a great mathematician - capacity for discovery and original mathematical thinking. Twenty-five years later, Eduard

Study [61] - one of Lie's former colleagues - described him as an autodidactic[1], but possessing the attributes of one of the most brilliant mathematicians who ever lived. His works have indeed outlived him. Their importance and relevance are still of note today.

Ironically, Lie algebras were an area Sophus Lie had little interest in initially pursuing. He instead hoped to develop the equivalent of the Galois theory to DEs. To this end, together with Friedrich Engel, he completed the third and final volume of the massive treatise *Theorie der Transformationsgruppen* [36]. By 1884, he had obtained all of his principal results [28].

The applications of Lie groups have now had a profound effect on all areas of mathematics and mathematically-based sciences [43, 56]. As for his original idea of developing the equivalent theory of Galois theory to DEs, one researcher notes that 'the remarkable range of applications of Lie groups to DEs in geometry, in analysis, in physics, and in the engineering over the past 40 years has resurrected Lie's original vision into one of the most active and rewarding fields of contemporary research' [44].

In this work, we introduce the Lie group theoretic approach to the solution of DEs, particularly in the field of Financial Mathematics. We do this on the basis of the successes it has achieved, particularly in areas where other techniques have failed.

[1] a self-taught person

1.2 Financial Mathematics

Financial Mathematics is a field of Applied Mathematics concerned with financial markets. It is closely related to the discipline of financial economics (which is concerned with the underlying theories in finance). Concisely, what financial mathematicians do is to derive and extend the mathematical or numerical models suggested by financial economists. Thus, for example, while a financial economist might study the structural reasons why a company may have a certain share price, a financial mathematician may take the share price as a given, and attempt to use DEs to obtain the fair value of derivatives of the stock.

A society improves its welfare through investing their savings. The financial market provides the bridge between saving and investment. Thus, while savers can earn high returns from their saving, borrowers on the other hand can execute their investment plans to earn future profits. There are many kinds of financial markets. We have the stock markets, bond markets, currency markets, foreign exchange markets, commodity markets (oil, wheat, gold), futures and options markets. In the futures or options market in particular, more complex contracts than simple buy/sell trades have been introduced. These are called financial derivatives, and will be the subject of our work.

1.2.1 History of the Lie group technique in Financial Mathematics

The Lie group method has been successfully applied in the field of Financial Mathematics. Importantly, it provides a systematic way of obtaining solutions to DEs in the field as opposed to the probabilistic or *ansatz* routes previously traversed. Gazizov and Ibragimov [24] arguably pioneered the employment of the Lie group technique in the resolution of problems in Financial Mathematics. Using Lie groups, they made an analysis of the Black-Scholes model and showed that the popular solution to the PDE is actually an invariant solution. Subsequently, this approach has yielded significant results [12–14, 62]

1.3 Lie group analysis

Definition 1.3.1. A group is a set G, together with a group operation, such that for any two elements $g, h \in G$, the product $g \cdot h$ is again an element of G. This group action is also required to satisfy the following properties:

Associativity: If g, h and k are elements of G, then

$$g \cdot (h \cdot k) = (g \cdot h) \cdot k. \tag{1.1}$$

Identity element: For every element $g \in G$, there is an element $e \in G$ such that

$$e \cdot g = g = g \cdot e. \tag{1.2}$$

Inverse property: For each $g \in G$, there is an inverse, denoted g^{-1}, such that

$$g \cdot g^{-1} = e = g^{-1} \cdot g, \tag{1.3}$$

where e is the identity element [43]. □

Definition 1.3.2. A r-parameter Lie group is a group G which is also a finite r-dimensional smooth manifold, and in which the group operations of multiplication

$$m : G \times G \to G, \qquad m(g,h) = g \cdot h, \qquad \forall g, h \in G \tag{1.4}$$

and inversion

$$i : G \to G, \qquad i(g) = g^{-1}, \qquad \forall g \in G \tag{1.5}$$

are smooth maps between manifolds [43]. □

Given a local group of transformations, we can define the symmetry groups of arbitrary subsets of a given manifold, in a general framework.

Definition 1.3.3. Let G be a local group of transformations acting on a manifold M. A subset $\mathfrak{S} \subset M$ is called G-invariant and G is called a symmetry group of $\mathfrak{S}$, if whenever $x \in \mathfrak{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathfrak{S}$ [43]. □

Definition 1.3.4. Let $\mathfrak{S}$ be a system of DEs

$$\Gamma_v(t, y, \partial y, \ldots, \partial^n y) = 0, \qquad v = 1, \cdots, l \tag{1.6}$$

where $t = t^1, \ldots, t^p$ are the independent variables and $y = y^1, \ldots, y^q$ are the dependent variables. A symmetry group, G, of the system of DEs is a local group of transformations acting on an open subset M of the space of T and Y for the system, with the property that whenever $y = f(t)$ is a solution of $\mathfrak{S}$, and whenever $g \cdot f$ is defined for $g \in G$, then $y = g \cdot f(t)$ is also a solution of the system [43]. $\square$

Theorem 1.3.1. *Let M be an open subset of $T \times Y$ and suppose $\Gamma(t, y, \partial y, \ldots, \partial^n y) = 0$ is an n-th order system of DEs defined over M, with a corresponding subvariety*[2] *$\mathfrak{S}_\Gamma \subset M^{(n)}$. Suppose G is a local group of transformations acting on M, whose prolongation leaves $\mathfrak{S}_\Gamma$ invariant. This means that whenever $(t, y, \partial y, \ldots, \partial^n y) \in \mathfrak{S}_\Gamma$, we have*

$$pr^n g \cdot (t, y, \partial y, \ldots, \partial^n y) \in \mathfrak{S}_\Gamma \tag{1.7}$$

for all $g \in G$ such that this is defined. Then G is a symmetry group of the system of equations in the sense of definition 1.3.4 [43]. $\square$

An invariant solution of a differential equation (DE) is a solution of the DE which is also an invariant curve (surface) of a group admitted by the DE. To make this statement clearer, consider an nth order system of DEs:

$$\Gamma_v(t, y, \partial y, \ldots, \partial^n y) = 0, \qquad v = 1, \cdots, l \tag{1.8}$$

where t and y are as defined earlier. The functions Γ_v are assumed to depend smoothly on their arguments for t and y in some open set M of the total space $T \times Y = \mathbb{R}^p \times \mathbb{R}^q$ of independent and dependent variables. Now, let H be a local group of transformations acting on $M \subset T \times Y$. H is said to be a symmetry group of Γ if each element $h \in H$ transforms solutions of Γ to other solutions of Γ. A real

[2]A subset of an algebraic variety which is itself a variety, where an algebraic variety is defined as a generalization to n dimensions of algebraic curves

valued function $\eta(t, y)$ is called an invariant of H if it is unchanged by the group action

$$\eta\left(h \cdot (t, y)\right) = \eta(t, y), \qquad \forall \;\; t, y \in M \tag{1.9}$$

and for all $h \in H$ such that $h \cdot (t, y)$ is defined.

The new system of DEs (Γ/H) for the H-invariant solutions to Γ will involve just the new variables u, w formed from the invariants of H. On 'reduction' (reduction as used here means that a point symmetry of a DE leads, in the case of a PDE, to finding special solutions called invariant or similarity solutions of the DE) of Γ via H, the system of DEs

$$\left(\Gamma/H\right)_v (u, w, \partial w, \ldots, \partial^n w) = 0, \qquad v = 1, \cdots, l \tag{1.10}$$

which has r $(r < p + q)$ fewer variables, constitutes the new system Γ/H. Now, every solution $w = f(u)$, to equation (1.10), gives rise to an H-invariant solution $y = g(t)$ to Γ. This is determined implicitly from the definition of the new variables u and w. For more rigorous explanations on group invariant solutions by symmetry analysis, see [43, 46].

Let us consider an invertible one-parameter group of transformations

$$\tilde{t} = \theta(t; \varepsilon), \tag{1.11}$$

defined for each vector $t = (t_1, \ldots, t_n)$ in domain $D \subset \mathbb{R}^n$. The infinitesimal form of $\tilde{t}$ can be approximately estimated, via a Taylor series expansion, as

$$\tilde{t} \approx t + \varepsilon \tau(t; \varepsilon). \tag{1.12}$$

The expansion is performed with respect to the parameter ε in the neighborhood $\varepsilon = 0$, and

$$\tau(t) = \frac{\partial}{\partial \varepsilon}\theta(t;\varepsilon)\bigg|_{\varepsilon=0}.$$

Consider a DE

$$F(t, y, \partial y, \ldots, \partial^n y) = 0, \tag{1.13}$$

with independent variables $t = (t^1, \ldots, t^n)$ and a dependent variable $y(t)$. A differential operator

$$G = \sum_i \xi_i(t, y)\frac{\partial}{\partial t_i} + \eta(t, y)\frac{\partial}{\partial y}, \tag{1.14}$$

is said to be a symmetry of equation (1.13) if

$$G^{[n]}F|_{F=0} = 0, \tag{1.15}$$

i.e. the action of the n^{th} extension of G on F is zero when the original equation is satisfied. The above definition for Lie symmetries is according to the theorem called *'The infinitesimal criterion of invariance under a one-parameter Lie group of point transformations'* [8, Theorem 1.2.6, page 17].

Several software packages have been developed to help find the symmetries of DEs [20, 29]. Usually these packages have to be supplemented by further analysis.

Once G is known, we can obtain the global form of the transformation by solving the system

$$\tau(t) = \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\varepsilon},$$

subject to the initial conditions,

$$t = \tilde{t}, \tag{1.16}$$

when $\varepsilon = 0$.

1.4 Tool chest for application of Lie symmetries in Differential Equations

Definition 1.4.1. The Jacobi–Lie bracket or simply Lie bracket, of two infinitesimal generators X and Y is an operator $[,]$ such that [54]

$$[X, Y](f) = X(Y(f)) - Y(X(f)). \tag{1.17}$$

$\square$

Definition 1.4.2. A Lie algebra is a vector space $\mathfrak{g}$ together with the bilinear operator (called the Lie bracket)

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \tag{1.18}$$

and satisfying the following axioms:

(1) Bilinearity

$$\begin{aligned} [cX + c'X', Y] &= c[X, Y] + c'[X', Y], \\ [X, cY + c'Y'] &= c[X, Y] + c'[X, Y'], \end{aligned} \tag{1.19}$$

for constants $c, c' \in \mathbb{R}$.

(2) Skew-Symmetry

$$[X, Y] = -[Y, X]. \tag{1.20}$$

(3) Jacobi identity

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0, \tag{1.21}$$

for all $X, X', Y, Y', Z \in \mathfrak{g}$ [43]. $\square$

It is often more convenient to view all the Lie bracket operations of the point symmetries of a DE in one table called a commutator table. This allows one to easily identify the class of Lie algebra.

Definition 1.4.3. A Lie algebra, $\mathfrak{g}$, is Abelian, if for all vectors X and Y in $\mathfrak{g}$, the following property holds:

$$[X,Y]=0, \tag{1.22}$$

where the relation $[,]$ is the Lie bracket [43].

Definition 1.4.4. A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace which is closed under the Lie bracket, so $[X,Y] \in \mathfrak{h}$ whenever $X, Y \in \mathfrak{h}$ [43]. $\square$

Theorem 1.4.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $H \subset G$ is a Lie subgroup, its Lie algebra is a subalgebra of $\mathfrak{g}$. Conversely, if $\mathfrak{h}$ is any s-dimensional subalgebra of $\mathfrak{g}$, there is a unique connected s-parameter Lie subgroup H of G with Lie algebra $\mathfrak{h}$ [43].* $\square$

1.4.1 Optimal system of Symmetries

One of the difficulties faced in obtaining group invariant solutions of DEs is that the reduction process may include any linear combination of the admitted symmetries.

One is then faced with the dilemma of choosing the best (or optimal) selection. This optimal system provides a non-unique list that represents all possible combinations of the subalgebras (and their corresponding symmetries). An optimal system of s-parameter group-invariant solutions to a system of DEs is a collection of solutions $u = f(x)$ with the following properties:

(1) Each solution in the list is invariant under some s-parameter symmetry group of the system of DEs.

(2) If $u = \tilde{f}(x)$ is any other solution, which is invariant under an s-parameter symmetry group, then there is a further symmetry $\mathfrak{g}$ of the system which maps $\tilde{f}$ to a solution $f = g \cdot \tilde{f}$ on the list.

Theorem 1.4.2. *Let G be the full symmetry group of a system of PDEs Δ. Let $\{H_\alpha\}$ be an optimal system of s-parameter subgroups of G. Then the collection of all $\{H_\alpha\}$-invariant solutions, for $\{H_\alpha\}$ in the optimal system, forms an optimal system of s-parameter group invariant solutions to Δ [43].* $\square$

The Baker–Campbell–Hausdorff formula

The Baker–Campbell–Hausdorff formula is named for Henry Frederick Baker, John Edward Campbell, and Felix Hausdorff. It was first noted in print by Campbell [15]; elaborated by Henri Poincaré [50] and Baker [2]. It was then systematized geometrically, and linked to the Jacobi identity by Hausdorff [27].

Definition 1.4.5. If X and Y are elements of a Lie algebra $\mathfrak{g}$, defined over any field of characteristic zero, then

$$\log\left(\exp(X), \exp(Y)\right), \tag{1.23}$$

can be written as an infinite sum of elements of $\mathfrak{g}$ [27]. $\square$

A standard lemma called the Hadamard lemma [37] can be utilized in producing this infinite expression of the Baker–Campbell–Hausdorff formula.

Lemma 1.4.1. *Let L_n be the space of all complex $n \times n$ matrices, and let $Adj(X)$ be the linear operator defined by $Adj(X)Y = [X, Y]$ for some fixed $X \in L_n$. We can utilize the adjoint action*

$$\exp(Adj(X))Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots, \tag{1.24}$$

in the construction of the conjugacy classes. $\square$

The expression $[X, Y]$ in Lemma (1.4.1) is the Lie bracket of X and Y, as defined in equation (1.17).

Goursat-Twist method

To obtain such an optimal system of subalgebras we follow the approach of Patera and Winternitz [48], called the Goursat-Twist method . This method is utilizable as long as the group classification of the Lie algebra is known. It involves decomposing the given Lie algebra into indecomposable subalgebras, for which each of the optimal set of subalgebras is known.

Although the Lie symmetries of a PDE leave that equation invariant by construction, we are required to ensure that these symmetries also leave the initial conditions invariant [6]. This is also applicable to the initial itself (or domain of the initial-value problem). We operate with an arbitrary linear combination of the one-dimensional Lie point symmetries on each initial condition and determine the relationship (if any) of those arbitrary parameters that ensure the initial condition is left unchanged.

1.5 History on the emergence of Exotic options

Options are derivatives that give the specifics of a contract between two parties for a future transaction on an asset at a reference price [47]. Vanilla options are options with standard features like a fixed strike price, expiration date and a single underlying asset. It is effective at the current date and when exercised, its payoff equals the difference between the value of the underlying asset and the strike price. In 1973, Black and Scholes aptly articulated the Brownian motion idea of Bachelier [1] by formulating what is now known as the Black-Scholes theory [4]. The motivation behind the Black-Scholes model was the creation of a derivative that could perfectly hedge the option of buying and selling the underlying asset whilst reducing risk to the barest minimum. The features of the Black-Scholes theory are that the payoffs are a linear function of the underlying asset price. Power options were introduced to extend the flexibility of the vanilla option and also take advantage of asset prices that traded within a small range. Cox *et al* [17] complemented their work by presenting a simple discrete time model which examines generalized options and presents the Black-Scholes model as a special limiting case. Options that have their terminal asset price raised to a certain integer power were then considered by Ong [42] and Wilmott [64]. Subsequently, Tompkins considered and produced the complete pricing solutions to options where the asset price is raised to a certain integer power [63]. These options are called power options and are in a generalized class of options called exotic options. Exotic options are derivatives which have features that makes them more complex than commonly traded products. They are usually only over-the-counter (OTC) or embedded in structured notes [11]. Power options have a payoff structure depending on the price, raised to a certain power of the underlying asset at the expiration time. Thus the payoff is a nonlinear function of the underlying asset price.

The major use of power options is that they allow the hedging of nonlinear risks that are otherwise difficult to manage using already existing products. An example of this is the hedging of the risks of implied volatility of European options. These options have always been in existence, though under several guises. A financial services company, HSBC Trinkaus & Burkhadt, have offered such warrants since 1995 in the form of squared power options [30]. The DAX share price index and German government bond tranche are other examples of the vehicles on which power options have been offered. They have also been embedded in note structures and derivatives due to their ability to yield curved payoffs. Reed [52, 53], Das [18], Hart and Ross [25], and Shimko [55] give comprehensive analyses and highlight these applications of the power options in greater detail. Power options are also implicit in the structure of constant proportional portfolio insurance (CPPI) products [10].

The nonlinear payoff of the power option is achieved by an appropriate choice of the parameter β (the power factor), giving it the flexibility of increasing or decreasing the leverage in comparison to the corresponding European vanilla option. A reference case to the sensitivity of its payoff is the case of Belgium selling an uncapped power option to an investment bank, Merrill. The bank recorded huge losses from a transaction that was intended to be a profitable one [19]. Although the main purpose of power options is to exaggerate leverage, it can also be used to diminish it. In the latter case, giving a lower price than the corresponding vanilla option. Details of the types of power options can be found in [26, 65].

A need for the availability of more flexibility for the power option prompted Esser [21, 22] to embark on producing solutions to options whose asset price is now raised to any real valued number.

1.5.1 Derivation of the power option PDE

The underlying asset price is characterized by geometric Brownian motion [1] and is given as

$$\frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}W, \tag{1.25}$$

where S is the asset price at time t, W is the Weiner process, μ represents the instantaneous expected return of the stock and σ is its volatility. The geometric Brownian motion equation is used mostly due to its pliability in options pricing and non-allowance of negative pricing.

If we let χ denote the level of marginal utility in the economy at time t, it is safe to assume that the dynamics of χ are determined by

$$\frac{\mathrm{d}\chi}{\chi} = \Delta(\chi, S)\mathrm{d}t + \nabla(\chi, S)\mathrm{d}W, \tag{1.26}$$

where Δ and ∇ are general functions. The pricing kernel of the power derivative can be subsequently determined as [5]

$$\frac{\mathrm{d}\chi}{\chi} = -r\mathrm{d}t - \frac{\mu - r}{\sigma}\mathrm{d}W, \tag{1.27}$$

where the functions Δ and ∇ are now determined as $-r$ and $(r-\mu)/\sigma$ respectively. The stochastic process for S^β (where β is the power factor) is given by

$$\begin{aligned} \mathrm{d}S^\beta &= \beta S^{\beta-1}(\mu S\mathrm{d}t + \sigma S\mathrm{d}W) + \frac{1}{2}\beta(\beta-1)S^{\beta-2}\sigma^2 S^2\mathrm{d}t \\ &= \left(\beta\mu + \frac{1}{2}\beta[\beta-1]\right) S^\beta\mathrm{d}t + \beta\sigma S^\beta\mathrm{d}W. \end{aligned} \tag{1.28}$$

Using Itô's lemma (and on substitution of equation (1.28)), the price of the power derivative is given by

$$\mathrm{d}V = \left(\frac{1}{2}\beta^2\sigma^2 S^{\beta 2}\frac{\partial^2 V}{\partial S^{\beta 2}} + \left[\beta\mu + \frac{1}{2}\beta\{\beta-1\}\sigma^2\right] S^\beta\frac{\partial V}{\partial S^\beta} + \frac{\partial V}{\partial t}\right)\mathrm{d}t + \beta\sigma S^\beta\frac{\partial V}{\partial S^\beta}\mathrm{d}W. \tag{1.29}$$

To calculate the drift of $V\chi$ from equations (1.27) and (1.29) (which is identically equal to zero, since it is a martingale), we again use Itô's lemma to obtain the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\beta^2\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \left[\beta r + \frac{1}{2}\beta\sigma^2(\beta-1)\right] S\frac{\partial V}{\partial S} - rV = 0, \tag{1.30}$$

known as the power option PDE, with a payoff $V(S,T) = \max\left\{\phi(S^\beta - K, 0)\right\}$ at initial $t = T$ for $\phi = 1$ or -1, where K is the strike price (The function $\phi = 1$ refers to the call option while $\phi = -1$ is the put option.). The model parameters denoting the value of the power option and the interest rate are given as $V(S,t)$ and r respectively. Notice that if $\beta = 1$, we revert to the vanilla case.

Observe that the payoff, in the case of the call option, can be interpreted as two conditions:

i) When $S^\beta \leq K$, this implies that $V(S,T) = 0$, and

ii) $S^\beta > K$ implies that $V(S,T) = S^\beta - K$.

1.6 Outline

The subsequent three chapters in this thesis comprise of the three distinct publications that emanated from this work. Thus, the historical background of exotic options and a brief description of the Lie group method will be mentioned in each. These will in effect also bear semblance to some sections in this introductory chapter.

We consider the constant case of the power option PDE in chapter 2. Here, without making any *ansatz* or using probability methods, we show an algorithmic method to obtaining a solution to the power option problem when all the model parameters are constants. We interprete the payoff into two separate conditions and obtain solutions for each.

In chapter 3, our work focuses on the more complex but realistic case of the power option PDE with time dependent parameters. In this chapter, we use the Lie group method to obtain a solution to the terminal condition, when the stock volatility and interest rate are constantly changing with time.

The publication represented by chapter 4 extended the time dependent case in chapter 3 to the scenario of when the power parameter, β (alongside the stock volatility and interest rate), is now changing with time too. We show how the symmetry analysis of the PDE enables us to obtain a solution.

We make concluding remarks in chapter 5 and highlight the significance of the results obtained.

Overall, we aim to make available the means and motivation to apply the Lie group method to PDEs (including time dependent ones) modeling financial derivatives traded in the options market. We further illustrate how these solutions can be made to satisfy the initial conditions of the problem. We hope to leave the reader with a sense that the solution of PDEs in Financial Mathematics cannot only be solved via the already known routes but through this method too – especially for problems for which other methods have been found wanting.

Chapter 2

An algorithmic solution to the constant case of the power option

2.1 Introduction

Black and Scholes [4] pioneered and concentrated on options whose payoffs were linear functions of the underlying asset price. Cox *et al* [17] complemented their work by presenting a simple discrete time model which examines generalized options and presents the Black-Scholes model as a special limiting case. Options that have their terminal asset price raised to a certain integer power were then considered by Ong [42] and Wilmott [64]. Tompkins [63] rounded up these results by presenting the generalized closed form solutions for these 'power' options.

The major use of power options is that they allow the hedging of nonlinear risks that are otherwise difficult to manage using already existing products. An example of this is the hedging of the risks of implied volatility of European options. These options have always been in existence, though under several guises. A financial services company - HSBC Trinkaus & Burkhadt - have offered such warrants since 1995 in the form of squared power options [30]. The DAX share price index and German government bond tranche are other examples of the vehicles on which power options have been offered. They have also been embedded in note structures and derivatives due to their ability to yield curved payoffs. Reed [52,53], Das [18], Hart and Ross [25], and Shimko [55] give comprehensive analyses and highlight these applications of the power options in greater detail. Power options are also explicit in the structure of CPPI (constant proportional portfolio insurance) products, see [10].

The nonlinear payoff of the power option is achieved by an appropriate choice of the parameter β (the power factor), giving it the flexibility of increasing or decreasing the leverage in comparison to the corresponding European vanilla option. A reference case to the sensitivity of its payoff is the case of Belgium selling an uncapped power option to an investment bank - Merrill. The bank recorded huge losses from a transaction that was supposed to bring in commensurate profits [19].

The power option is described by the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\beta^2\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left[\beta r + \frac{1}{2}\beta\sigma^2(\beta - 1)\right] S\frac{\partial V}{\partial S} - rV = 0. \tag{2.1}$$

The model parameters denoting the value of the power option, underlying asset price, interest rate and the volatility are given as $V(S,t)$, S, r and σ respectively. The option has a payoff of

$$V(S,T) = \max\left\{\phi(S^\beta - K, 0)\right\}, \tag{2.2}$$

with initial condition $t = T$ for $\phi = 1$ or -1, where K is the strike price. Notice that if $\beta = 1$, we revert to the vanilla case.

Observe that the payoff, in the case of the call option ($\phi = 1$), can be interpreted into two conditions:

i) When $S^\beta \leq K$, this implies that $V(S,T) = 0$ and

ii) $S^\beta > K$ implies that $V(S,T) = S^\beta - K$.

When $\phi = -1$, the option will be a put.

Except for Esser [21], research on the power option has only addressed the power factor as an integer. Esser extended this idea by considering the case of the power factor as a real number. Her work utilized a probabilistic approach without reference to equation (2.1). Here we attempt to solve equation (2.1) via an algorithmic approach called the Lie group theory.

This method, the Lie group theory, was developed by Sophus Lie. He made the profound and far reaching discovery that all the special methods of solving differential equations (DEs) were in fact special cases of a general integration procedure based on the invariance of the DE under a continuous group of symmetries [28]. This symmetry group is a local group of transformations of a system of DEs

$$\delta_v(x, u^{(1)}, \ldots, u^{(n)}) = 0, \qquad v = 1, \cdots, l \tag{2.3}$$

in p independent variables $x = (x_1, \cdots, x_P)$, acting on the space of the dependent and independent variables, and having the property to transform each solution of the transformed equations to a solution of the system as well (i.e. a one-to-one correspondence between them). Morgan [38] gives a theorem on this correspondence.

The group-invariant solutions are the fixed points of this action. Thus, the method for construction of group-invariant solutions is based on the knowledge of the invariants of the symmetry group of the considered system of differential equations [16]. Papers by Bluman & Cole [7], Olver & Rosenau [45] and Olver [43] and the book by Ibragimov [32] are relevant texts to shed more light on these form of solutions. Essentially, the solutions we obtain in this chapter are group-invariant solutions.

Using group analysis, we will attempt to obtain an optimal system of Lie point symmetries of the power option PDE. This system will enable us transform the PDE to ordinary differential equations (ODEs), which we then solve to obtain a new pricing for the value of the power option.

The employment of Lie group in the resolution of problems in Financial Mathematics have proved successful in recent years [12–14]. It's in the light of these successes that we shall employ this method to the resolution of the problem at hand.

The underlying asset price is characterized by the geometric Brownian motion [1] and it is given as

$$\frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}W, \tag{2.4}$$

where S is the asset price at time t, W is the Weiner process, μ represents the instantaneous expected return of the stock and σ is its volatility. The Brownian motion equation is used mostly, due to its pliability in options pricing, and non-allowance of negative pricing.

Now, letting χ denote the level of marginal utility in the economy at time t. Due to the reasons earlier given, it is safe to assume that the dynamics of χ are determined by

$$\frac{\mathrm{d}\chi}{\chi} = \Delta(\chi, S)\mathrm{d}t + \nabla(\chi, S)\mathrm{d}W, \tag{2.5}$$

where Δ and ∇ are general functions. The pricing kernel can be subsequently be determined as [5]

$$\frac{\mathrm{d}\chi}{\chi} = -r\mathrm{d}t - \frac{\mu - r}{\sigma}\mathrm{d}W. \tag{2.6}$$

The Stochastic process for S^β (where β is the power factor) is given by

$$\begin{aligned}
\mathrm{d}S^\beta &= \beta S^{\beta-1}(\mu S\mathrm{d}t + \sigma S\mathrm{d}W) + \frac{1}{2}\beta(\beta - 1)S^{\beta-2}\sigma^2 S^2\mathrm{d}t \\
&= \left(\beta\mu + \frac{1}{2}\beta[\beta - 1]\right) S^\beta\mathrm{d}t + \beta\sigma S^\beta\mathrm{d}W.
\end{aligned} \tag{2.7}$$

Using Itô's lemma (and on substitution of equation 2.7), the price of the power derivative is given by

$$\mathrm{d}V = \left(\frac{1}{2}\beta^2\sigma^2 S^{\beta 2}\frac{\partial^2 V}{\partial S^{\beta 2}} + \left[\beta\mu + \frac{1}{2}\beta\{\beta - 1\}\sigma^2\right] S^\beta\frac{\partial V}{\partial S^\beta} + \frac{\partial V}{\partial t}\right)\mathrm{d}t + \beta\sigma S^\beta\frac{\partial V}{\partial S^\beta}\mathrm{d}W. \tag{2.8}$$

To calculate the drift of $V\chi$ from equations (2.6) and (2.8) (which is identically equal to zero, since it's a martingale), we again use the Itô's lemma and this now gives the PDE,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\beta^2\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \left[\beta r + \frac{1}{2}\beta\sigma^2(\beta - 1)\right] S\frac{\partial V}{\partial S} - rV = 0, \tag{2.9}$$

known as the power option PDE, with a payoff $V(S,T) = \max\left\{\phi(S^\beta - K, 0)\right\}$ at initial $t = T$.

The chapter is divided into five sections. Section 2.2 is dedicated to a brief overview of the Lie groups method of solving differential equations. We do the Lie group analysis of the problem in section 2.3. In section 2.4, we apply the initial condition to the symmetries obtained, enabling us to get invariant solutions to the PDE. In the last section, we illustrate and discuss the results obtained, before stating the relevance and importance of our results, compared to previously obtained ones.

2.2 The Lie group technique

The strength of this approach lies mainly in the ability of the technique to solve differential equations (DEs) by using their symmetries. By a symmetry, we mean the generator of a transformation which leaves the form of the DE invariant. The main application of Lie point symmetries is for obtaining solutions of a system of DEs which are invariant under a continuous symmetry group. These solutions are found by solving a reduced system of DEs involving fewer independent variables than the original DE.

Let us consider an invertible one-parameter group of transformations

$$\tilde{t} = \theta(t; \varepsilon), \tag{2.10}$$

defined for each vector $t = (t_1, \ldots, t_n)$ in domain $D \subset \mathbb{R}^n$. The infinitesimal form of $\tilde{t}$ can be approximately estimated, via a Taylor series expansion, as

$$\tilde{t} \approx t + \varepsilon\tau(t; \varepsilon). \tag{2.11}$$

The expansion is performed with respect to the parameter ε in the neighborhood $\varepsilon = 0$, and

$$\tau(t) = \frac{\partial}{\partial\varepsilon}\theta(t; \varepsilon)\bigg|_{\varepsilon=0}.$$

Consider a DE

$$F(x, u^{(1)}, u^{(2)}, \ldots, u^{(n)}) = 0, \tag{2.12}$$

with independent variables $x = (x_1, \ldots, x_n)$ and a dependent variable $u(x)$. A differential operator

$$G = \sum_i \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u}, \tag{2.13}$$

is said to be a symmetry of equation (2.12) if

$$G^{[n]}F|_{F=0} = 0, \tag{2.14}$$

i.e. the action of the n^{th} extension of G on F is zero when the original equation is satisfied.

Once G is known, we can obtain the global form of the transformation by solving the system

$$\tau(t) = \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\varepsilon},$$

subject to the initial conditions,

$$t = \tilde{t}, \tag{2.15}$$

when $\varepsilon = 0$.

2.3 Analysis of the problem by the Lie group approach

The power option PDE in equation (2.9) admits the symmetries [20, 29]

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial t}, && (2.16)\\
G_2 &= S\frac{\partial}{\partial S}, && (2.17)\\
G_3 &= V\frac{\partial}{\partial V}, && (2.18)\\
G_4 &= \left(\frac{t}{\beta}-\frac{2rt}{\sigma^2\beta}+\frac{2\log(S)}{\sigma^2\beta^2}\right)V\frac{\partial}{\partial V}+2St\frac{\partial}{\partial S}, && (2.19)\\
G_5 &= \left(4rt+\frac{4r^2t}{\sigma^2}+\sigma^2t+\frac{2\log(S)}{\beta}-\frac{4r\log(S)}{\sigma^2\beta}\right)V\frac{\partial}{\partial V}+4S\log(S)\frac{\partial}{\partial S}\\
&\quad +8t\frac{\partial}{\partial t}, && (2.20)\\
G_6 &= \left(-4t+4rt^2+\frac{4r^2t^2}{\sigma^2}+\sigma^2t^2+\frac{4t\log(S)}{\beta}-\frac{8rt\log(S)}{\sigma^2\beta}+\frac{4\log^2(S)}{\sigma^2\beta^2}\right)V\frac{\partial}{\partial V}\\
&\quad +8St\log(S)\frac{\partial}{\partial S}+8t^2\frac{\partial}{\partial t}, && (2.21)\\
G_7 &= f(t,S)\frac{\partial}{\partial V}, && (2.22)
\end{aligned}
$$

where f is a solution of equation (2.9) and hence G_7 is called the solution symmetry (we exclude G_7 from our subsequent analysis).

	G_1	G_2	G_3	G_4	G_5	G_6
G_1	0	0	0	$AG_3 + 2G_2$	$BG_3 + 8G_1$	$2G_5 - 4G_3$
G_2	0	0	0	$\frac{2}{\sigma^2\beta^2}G_3$	$2AG_3 + 4G_2$	$4G_4$
G_3	0	0	0	0	0	0
G_4	$-AG_3 - 2G_2$	$\frac{-2}{\sigma^2\beta^2}G_3$	0	0	$-4G_4$	0
G_5	$-BG_3 - 8G_1$	$-2AG_3 - 4G_2$	0	$4G_4$	0	$8G_6$
G_6	$-2G_5 + 4G_3$	$-4G_4$	0	0	$-8G_6$	0

Table 2.1: Commutation table for $G_1, \ldots, G_6$ where $A = \left(\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right)$ and $B = \left(4r + \frac{4r^2}{\sigma^2} + \sigma^2\right)$.

Using Table 2.1, we realize that we need to relabel the symmetries as follows:

$$e_1 = -\frac{1}{16}G_6, \tag{2.23}$$

$$e_2 = -\frac{1}{4}G_4, \tag{2.24}$$

$$e_3 = \frac{1}{4}G_3 - \frac{1}{8}G_5, \tag{2.25}$$

$$e_4 = \frac{1}{16}\left(4r + \frac{4r^2}{\sigma^2} + \sigma^2\right)G_3 + \frac{1}{2}G_1, \tag{2.26}$$

$$e_5 = \frac{1}{\sigma^2\beta^2}G_3, \tag{2.27}$$

$$e_6 = \left(\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right)G_3 + 2G_2. \tag{2.28}$$

(Of course, $e_1, \ldots, e_6$ are also symmetries of equation (2.9) since they are linear combinations of $G_1, \ldots, G_6$.) The commutator Table 2.2 for these symmetries shows us that the six-dimensional Lie algebra is decomposed into $A_{3,5}^{\frac{1}{2}} \oplus 3A_1$ [48].

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	0	e_1	$\frac{1}{2}e_3$	0	$-2e_2$
e_2	0	0	$\frac{1}{2}e_2$	$8e_6$	0	0
e_3	$-e_1$	$-\frac{1}{2}e_2$	0	e_4	0	$\frac{1}{2}e_6$
e_4	$-\frac{1}{2}e_3$	$-8e_6$	$-e_4$	0	0	0
e_5	0	0	0	0	0	0
e_6	$2e_2$	0	$-\frac{1}{2}e_6$	0	0	0

Table 2.2: The commutator table of symmetries $e_1, \ldots, e_6$; in which $e_1, \ldots, e_3$ can be represented by $A_{3,5}^{\frac{1}{2}}$ and $e_4, \ldots, e_6$ constitute the Abelian Lie algebra $3A_1$.

2.4 Application of the initial conditions

Since the power option PDE (2.9) is invariant under equations (2.23)–(2.28), we can use these symmetries to generate group invariant solutions [43]. However, we need to solve the PDE in conjunction with its initial conditions. As a result, we wish to ensure that these symmetries also leave the initial condition invariant. In order to do this, we now operate with an arbitrary linear combination of the six Lie point symmetries

$$\Gamma = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6, \tag{2.29}$$

on the initial condition and determine the constraints (if any) on those arbitrary parameters $\alpha_1, \ldots, \alpha_6$. The expanded form of equation (2.29) is

$$\begin{aligned}
\Gamma \;=\; & \left(-\frac{1}{2}\alpha_1 t^2 - t\alpha_3 t + \frac{1}{2}\alpha_4\right)\partial_t + \left(-\frac{1}{2}\alpha_1 St\log(S) - \frac{1}{2}\alpha_2 St - \frac{1}{2}\alpha_3 S\log(S) + 2\alpha_6 S\right)\partial_S \\
& \left(\alpha_5\frac{1}{\sigma^2\beta^2} + \alpha_1\left[\frac{t}{4} - \frac{rt^2}{4} - \frac{r^2t^2}{4\sigma^2} - \frac{\sigma^2 t^2}{16} - \frac{t\log(S)}{4\beta} + \frac{rt\log(S)}{2\sigma^2\beta} - \frac{\log^2(S)}{4\sigma^2\beta^2}\right]\right. \\
& - \alpha_2\left[\frac{t}{4\beta} - \frac{rt}{2\sigma^2\beta} + \frac{\log(S)}{2\sigma^2\beta^2}\right] + \alpha_4\left[\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16}\right] + \alpha_6\left[\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right] \\
& \left. + \alpha_3\left[\frac{1}{4} - \frac{rt}{2} - \frac{r^2t}{2\sigma^2} - \frac{\sigma^2 t}{8} - \frac{\log(S)}{4\beta} + \frac{r\log(S)}{2\sigma^2\beta}\right]\right)V\partial_V. \qquad (2.30)
\end{aligned}$$

Since the initial condition can be split into two domains, we consider each in turn.

2.4.1 $S^\beta \leq K$:

We will now operate with Γ on both the left and right hand sides of the pay-off $V(S,T) = 0$.

For the left hand side

$$\begin{aligned}
& \Gamma V\Big|_{V=0} \\
= \; & \left(\alpha_5\frac{1}{\sigma^2\beta^2} + \alpha_1\left[\frac{t}{4} - \frac{rt^2}{4} - \frac{r^2t^2}{4\sigma^2} - \frac{\sigma^2 t^2}{16} - \frac{t\log(S)}{4\beta} + \frac{rt\log(S)}{2\sigma^2\beta} - \frac{\log^2(S)}{4\sigma^2\beta^2}\right]\right. \\
& - \alpha_2\left[\frac{t}{4\beta} - \frac{rt}{2\sigma^2\beta} + \frac{\log(S)}{2\sigma^2\beta^2}\right] + \alpha_4\left[\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16}\right] + \alpha_6\left[\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right] \\
& \left. + \alpha_3\left[\frac{1}{4} - \frac{rt}{2} - \frac{r^2t}{2\sigma^2} - \frac{\sigma^2 t}{8} - \frac{\log(S)}{4\beta} + \frac{r\log(S)}{2\sigma^2\beta}\right]\right)V\Bigg|_{V(S,T)=0} \\
\equiv \; & 0,
\end{aligned}$$

which equals the right hand side.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	e_1	e_2	$\epsilon e_1 + e_3$	$\frac{\epsilon^2}{4}e_1 + \frac{\epsilon}{2}e_3 + e_4$	e_5	$-2\epsilon e_2 + e_6$
e_2	e_1	e_2	$\frac{\epsilon}{2}e_2 + e_3$	$e_4 + 8\epsilon e_6$	e_5	e_6
e_3	$\exp(-\epsilon)e_4$	$\exp\left(\frac{-\epsilon}{2}\right)e_2$	e_3	$\exp(\epsilon)e_4$	e_5	$\exp\left(\frac{\epsilon}{2}\right)e_6$
e_4	$\frac{\epsilon^2}{4}e_4 - \frac{\epsilon}{2}e_3 + e_1$	$e_2 - 8\epsilon e_6$	$e_3 - \epsilon e_4$	e_4	e_5	e_6
e_5	e_1	e_2	e_3	e_4	e_5	e_6
e_6	$e_1 + 2\epsilon e_2$	e_2	$e_3 - \frac{\epsilon}{2}e_6$	e_4	e_5	e_6

Table 2.3: Adjoint table for $e_1, \ldots, e_6$

We see that the result from the LHS equals that of the RHS, thereby placing no restrictions on the arbitrary parameters for its solution. This implies that reduction of the power option will be via all the symmetries admitted with no restrictions whatsoever. All group invariant solutions obtained will therefore not violate the pay-off $V(S,T) = 0$.

We will now proceed to obtain the optimal system of symmetries that will enable us find invariant solutions to the power option PDE. This optimal system gives us a list that represents all the relevant possible combinations of the symmetries, that will provide invariant solutions. Solutions from other combinations can be obtained via this optimal system. The Baker–Campbell–Hausdorff formula will be utilized in this process of obtaining the optimal system of symmetries. This necessitates the creation of an adjoint table (Table 2.3) via the Hadamard lemma.

The decomposition of the Lie algebra of the power option PDE into an Abelian identifiable subalgebra, $A_{3,5}^{\frac{1}{2}} \oplus 3A_1$, allows us to use an algorithm called the Goursat-Twist method [48], in the determination of its optimal symmetries. We can decompose $A_{3,5}^{\frac{1}{2}} \oplus 3A_1$ into the subalgebras $A_{3,5}^{\frac{1}{2}}$, A_1, A_1 and A_1. To show that our application of the method is correct, we shall now go ahead to obtain the one-dimensional optimal symmetries for $A_{3,5}^{\frac{1}{2}} \oplus A_1$, before checking these with the set of subalgebras obtained in [48].

Let $\{e_1, e_2, e_3\}$ and $\{e_4\}$ be the generators for the subalgebras $A = A_{3,5}^{\frac{1}{2}}$ and A_1 respectively. Obtaining the optimal set of symmetries for $A_{3,5}^{\frac{1}{2}} \oplus A_1$ proceeds in three steps.

Step 1:

We list all the known subalgebras A_i of A, including those of the factor algebra i.e. $\{\}$ and $\{A_n\}$ (where $i = 1, \ldots, n$). According to [48], these are

$$\begin{array}{lllll} \{\} & \{e_1, e_2, e_3\} & \{e_1, e_3\} & \{e_2, e_3\} & \{e_1\} \\ \{e_2\} & \{e_3\} & \{e_1 + \epsilon e_2\}, & & \end{array}$$

where $\epsilon = \pm 1$.

Step 2:

The optimal list should also include the direct sums of all the subalgebras listed in step 1 to the generator e_4.

$$\begin{array}{lllll} \{e_4\} & \{e_1, e_2, e_3, e_4\} & \{e_1, e_3, e_4\} & \{e_2, e_3, e_4\} & \{e_1, e_4\} \\ \{e_2, e_4\} & \{e_3, e_4\} & \{e_1 + \epsilon e_2, e_4\}. & & \end{array}$$

Step 3:

The next step is to compile twisted subalgebras. These are constructed by adding arbitrary scalar multiples of the generator $\{e_4\}$ to the subalgebras in step 1. We must then confirm that these subalgebras are closed under commutation.

We can classify these subalgebras by dimensions:

a) For the 1-dimensional subalgebras, we have

$$\{e_1 + a_1 e_4\} \qquad \{e_2 + a_2 e_4\} \qquad \{e_3 + a_3 e_4\} \qquad \{e_1 + \epsilon e_2 + a_4 e_4\}.$$

These are trivially taken to be true subalgebras of the optimal system since the Lie bracket operation needed to fulfill the commutation property requires more than one element. We can further simplify two of these subalgebras via the adjoint table in Table 2.3 by using the adjoint representation of e_3. To show how this done, we illustrate the case of $X_1 = \{e_1 + a_1 e_4\}$

$$\begin{aligned} \operatorname{Adj}\left(\exp(\epsilon e_3) X_1\right) &= \exp(-\epsilon) e_1 + a_1 \exp(\epsilon) e_4 \\ &= e_1 + a_1 \exp(2\epsilon) e_4 \\ &= e_1 + \varepsilon e_4, \end{aligned} \tag{2.31}$$

where $\varepsilon = \pm 1$ (This results because $\exp(2\epsilon)$ is positive and the arbitrary parameter a_1 can have any sign.). A similar simplification can be performed for $\{e_2 + a_2 e_4\}$ to become $\{e_2 + \varepsilon e_4\}$. No other subalgebra in this group can be further simplified.

The 1-dimensional subalgebras from the three steps are now

$$\begin{array}{llllll} \{e_1\} & \{e_2\} & \{e_3\} & \{e_4\} & \{e_1 + \varepsilon e_2\} & \{e_1 + \varepsilon e_4\} \\ \{e_2 + \varepsilon e_4\} & \{e_3 + a_3 e_4\} & \{e_1 + \varepsilon e_2 + a_4 e_4\}. \end{array} \tag{2.32}$$

We can combine some of these subalgebras to obtain

$$
\begin{aligned}
&\{e_1\} \qquad \{e_2\} \qquad \{e_4\} \qquad \{e_1 + \varepsilon e_4\} \\
&\{e_2 + \varepsilon e_4\} \qquad \{e_3 + a_3 e_4\} \qquad \{e_1 + \varepsilon e_2 + a_4 e_4\}.
\end{aligned} \tag{2.33}
$$

This is the same set of 1-dimensional subalgebras obtained in [48].

The 2-dimensional and 3-dimensional sets can be obtained in like manner. We repeat the algorithm to obtain the sets for $A_{3,5}^{\frac{1}{2}} \oplus 2A_1$ and then once again for $A_{3,5}^{\frac{1}{2}} \oplus 3A_1$. We eventually have the optimal set of 1-dimensional subalgebras (one parameter symmetries) as

$$
\begin{aligned}
&\{e_1\} \qquad \{e_6\} \qquad \{e_1 + \varepsilon e_6\} \qquad \{e_5 + m_1 e_6\} \\
&\{e_1 + \varepsilon e_5 + m_2 e_6\} \quad \{e_2 + \varepsilon e_5 + m_3 e_6\} \quad \{e_4 + \varepsilon e_5 + m_4 e_6\} \\
&\{e_1 + \varepsilon e_4 + m_5 e_5 + m_6 e_6\} \quad \{e_2 + \varepsilon e_4 + m_7 e_5 + m_8 e_6\} \\
&\{e_3 + \varepsilon e_4 + m_9 e_5 + m_{10} e_6\} \; \{e_1 + \varepsilon e_2 + m_{11} e_4 + m_{12} e_5 + m_{13} e_6\},
\end{aligned}
$$

where m_i, $i = 1, ..., 13$ are arbitrary constants and $\epsilon = \pm 1$.

In the original variables, the 1-dimensional optimal system of symmetries will now be

$$
\begin{aligned}
Y_1 &= \left(\frac{t}{2} - \frac{rt^2}{2} - \frac{r^2t^2}{2\sigma^2} - \frac{\sigma^2 t^2}{8} - \frac{t\log(S)}{2\beta} + \frac{rt\log(S)}{\sigma^2\beta} - \frac{\log^2(S)}{\sigma^2\beta^2}\right) V\frac{\partial}{\partial V} + t^2\frac{\partial}{\partial t} \\
&\quad + St\log(S)\frac{\partial}{\partial S,} \qquad (2.34) \\
Y_2 &= \left(\frac{1}{2\beta} - \frac{r}{\sigma^2\beta}\right) V\frac{\partial}{\partial V} + S\frac{\partial}{\partial S}, \qquad (2.35) \\
Y_3 &= \left(\frac{t}{4} - \frac{rt^2}{4} - \frac{r^2t^2}{4\sigma^2} - \frac{\sigma^2 t^2}{16} - \frac{t\log(S)}{4\beta} + \frac{rt\log(S)}{2\sigma^2\beta} - \frac{\log^2(S)}{4\sigma^2\beta^2}\right. \\
&\quad \left. + a\left\{\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right) V\frac{\partial}{\partial V} + \left(b - \frac{1}{2}t\log(S)\right) S\frac{\partial}{\partial S} - \frac{1}{2}t^2\frac{\partial}{\partial t}, \qquad (2.36) \\
Y_4 &= \left(\frac{1}{\sigma^2\beta^2} + a\left\{\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right) V\frac{\partial}{\partial V} + bS\frac{\partial}{\partial S}, \qquad (2.37) \\
Y_5 &= \left(\frac{t}{4} - \frac{rt^2}{4} - \frac{r^2t^2}{4\sigma^2} - \frac{\sigma^2 t^2}{16} - \frac{t\log(S)}{4\beta} + \frac{rt\log(S)}{2\sigma^2\beta} - \frac{\log^2(S)}{4\sigma^2\beta^2} + \frac{1}{\sigma^2\beta^2}\right. \\
&\quad \left. + a\left\{\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right) V\frac{\partial}{\partial V} + \left(b - \frac{1}{2}t\log(S)\right) S\frac{\partial}{\partial S} - \frac{1}{2}t^2\frac{\partial}{\partial t}, \qquad (2.38)
\end{aligned}
$$

$$
\begin{aligned}
Y_6 &= \left(\frac{-t}{4\beta} + \frac{rt}{2\sigma^2\beta} - \frac{\log(S)}{2\sigma^2\beta^2} + \frac{1}{\sigma^2\beta^2} + a\left\{\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right) V\frac{\partial}{\partial V} \\
&\quad + \left(b - \frac{t}{2}\right) S\frac{\partial}{\partial S}, \qquad (2.39) \\
Y_7 &= \left(\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16} + \frac{1}{\sigma^2\beta^2} + a\left\{\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right) V\frac{\partial}{\partial V} + bS\frac{\partial}{\partial S} + \frac{1}{2}\frac{\partial}{\partial t}, \qquad (2.40) \\
Y_8 &= \left(\frac{t}{4} - \frac{rt^2}{4} - \frac{r^2t^2}{4\sigma^2} - \frac{\sigma^2 t^2}{16} - \frac{t\log(S)}{4\beta} + \frac{rt\log(S)}{2\sigma^2\beta} - \frac{\log^2(S)}{4\sigma^2\beta^2} + \frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16}\right. \\
&\quad \left. + a\left\{\frac{1}{\sigma^2\beta^2} + \frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right) V\frac{\partial}{\partial V} + \left(b - \frac{1}{2}t\log(S)\right) S\frac{\partial}{\partial S} \\
&\quad + \left(\frac{1}{2} - \frac{t^2}{2}\right)\frac{\partial}{\partial t}, \qquad (2.41)
\end{aligned}
$$

$$
\begin{aligned}
Y_9 &= \left(\frac{-t}{4\beta} + \frac{rt}{2\sigma^2\beta} - \frac{\log(S)}{2\sigma^2\beta^2} + \frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16} + a\left\{\frac{1}{\sigma^2\beta^2} + \frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right) V\frac{\partial}{\partial V} \\
&\quad + \left(b - \frac{t}{2}\right) S\frac{\partial}{\partial S} + \frac{1}{2}\frac{\partial}{\partial t}, \qquad (2.42) \\
Y_{10} &= \left(\frac{1}{4} - \frac{rt}{2} + \frac{r^2 t}{2\sigma^2} - \frac{\sigma^2 t}{8} - \frac{\log(S)}{4\beta} + \frac{r\log(S)}{2\sigma^2\beta} + \frac{r}{4} + a\left\{\frac{1}{\sigma^2\beta^2} + \frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right. \\
&\quad \left. + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16}\right) V\frac{\partial}{\partial V} + \left(b - \frac{\log(S)}{2}\right) S\frac{\partial}{\partial S} + \left(\frac{1}{2} - t\right)\frac{\partial}{\partial t}, \qquad (2.43) \\
Y_{11} &= \left(\frac{t}{4} - \frac{rt^2}{4} - \frac{r^2 t^2}{4\sigma^2} - \frac{\sigma^2 t^2}{16} - \frac{t\log(S)}{4\beta} + \frac{rt\log(S)}{2\sigma^2\beta} - \frac{\log^2(S)}{4\sigma^2\beta^2} - \frac{t}{4\beta} + \frac{rt}{2\sigma^2\beta}\right. \\
&\quad \left. - \frac{\log(S)}{2\sigma^2\beta^2} + a\left\{\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16} + \frac{1}{\sigma^2\beta^2} + \frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right\}\right) V\frac{\partial}{\partial V} \\
&\quad + \left(b - \frac{1}{2}t\log(S) - \frac{t}{2}\right) S\frac{\partial}{\partial S} + \left(\frac{c}{2} - \frac{t^2}{2}\right)\frac{\partial}{\partial t}, \qquad (2.44)
\end{aligned}
$$

where a, b and c are arbitrary constants for each symmetry.

Using these optimal symmetries, we reduce the original PDE and list the results of the ODEs in Tables 2.4 and 2.5.

The resulting DE from the last set of subalgebras in Table 4 was not simple to solve, and hence not stated. Also, constraining the constants in the solutions obtained (for the satisfaction of the initial condition) only results in the trivial solution, prompting further analysis.

We note that the symmetries admitted by equation (2.9) generate global transformations that leave the equation invariant. We pursue this line of analysis to investigate the existence of non-trivial solutions to equation (2.9), satisfying the initial conditions. Since the infinitesimal form of the symmetries admitted is already known, we can obtain the global form by solving some simple ODEs.

	Symmetries	Variables	Solution
1	e_1	$m = \frac{\log(S)}{t}$ $V = \exp\left[\frac{\log(S)(\sigma^2-2r)}{2\sigma^2\beta} + \frac{\log^2(S)}{2\sigma^2\beta^2 t} + \frac{(2r+\sigma^2)}{8t\sigma^2} - \frac{\log(t)}{2}\right] U(m)$	$U(m) = \left(\frac{\Phi_2 \log(S)}{t} + \Phi_1\right)$
2	e_6	$m = t$ $V(S,t) = \exp\left[\frac{\log(S)}{\beta}\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)\right] U(m)$	$U(m) = \Phi_1 \exp\left[\frac{t(2r+\sigma^2)^2}{8\sigma^2}\right]$
3	$e_1 + \epsilon e_6$	$m = \frac{S^{\frac{t}{2b}}}{t}$ $V = \exp\left[\frac{t(2r+\sigma^2)^2}{8\sigma^2} + \frac{m(2r-\sigma^2)}{2\sigma^2\beta t} - \frac{12\log(t)\{32+48\log(t)+24m\}}{216\sigma^2\beta^2 t^3}\right.$ $\left. - \frac{4\{32+3m(8+3m)\}}{216\sigma^2\beta^2 t^3} - \frac{\log(t)\left(\sigma^2(4+\beta t-8r)\right)}{2\sigma^2\beta t} + \frac{m(40-3m+9\beta(\sigma^2-2r))}{9\sigma^2\beta^2}\right] U(m)$	$U(m) = \left(\Phi_1 A_i(J) + \Phi_2 B_i(J)\right)$, where $J = \frac{2^{\frac{1}{3}}\left(8-8\beta r+2m(\beta r-2)+\sigma^2\beta(4+\beta-m)\right)}{\sigma^4\beta^4\left[\frac{2\beta r-4-\sigma^2\beta}{\sigma^4\beta^4}\right]^{\frac{2}{3}}}$
4	$e_5 + m_1 e_6$	$m = t$ $V = \log(S)\left\{\frac{1}{b\sigma^2\beta^2} + \frac{a}{b\beta}\left(1 - \frac{2r}{\sigma^2}\right)\right\} U(m)$	$U(m) = \Phi_1 \exp\left[\frac{t(2b\beta r+1+a\beta(\sigma^2-2r))}{2b^2\sigma^2\beta^2}\right.$ $\left.(2a\beta r - 1 + \sigma^2\beta(b-a))\right]$
5	$e_1 + \epsilon e_5 + m_2 e_6$	$m = \frac{S^{\frac{t}{2b}}}{t}$ $V = \exp\left[\frac{2-\beta(2a-b)(2r-\sigma^2)}{\sigma^2\beta^2 t} - \frac{\log(t)}{2} + \frac{m\left\{20b-3(m+6\beta r-3\beta\sigma^2)\right\}}{9\sigma^2\beta^2}\right.$ $\left. + \frac{t(2r+\sigma^2)}{8\sigma^2} - \frac{4b^2}{27\sigma^2\beta^2 t^3} - \frac{\log(S)\left\{4b+9\beta t^2(\sigma^2-2r)+3t\log(S)\right\}}{18\sigma^2\beta^2 t^2}\right] U(m)$	$U(m) = \left(\Phi_1 A_i(N) + \Phi_2 B_i(N)\right)$, where $N = \frac{2^{\frac{1}{3}}\left(2-4a\beta r+2(b-m)(b-\beta r)+\sigma^2\beta(2a+b+\beta-m)\right)}{\sigma^4\beta^4\left[\frac{2\beta r-2b-\sigma^2\beta}{\sigma^4\beta^4}\right]^{\frac{2}{3}}}$
6	$e_2 + \epsilon e_5 + m_3 e_6$	$m = t$ $V = \exp\left[\frac{\log(S)\left(8a\beta r-4-2\beta r t-4a\sigma^2\beta+\sigma^2\beta t+\log(S)\right)}{2\sigma^2\beta^2(t-2b)}\right] U(m)$	$U(m) = \frac{\Phi_1}{\sqrt{t-2b}}\left[\frac{t(2r-\sigma^2)^2}{8\sigma^2} - \frac{\left(\beta(2a-b)(2r-\sigma^2)-2\right)^2}{4\sigma^2\beta^2(2b-t)}\right]$
7	$e_4 + \epsilon e_5 + m_4 e_6$	$m = \log(S) - 2bt$ $V = \exp\left[\frac{\log(S)}{2b}\left(\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16} + \frac{1}{\sigma^2\beta^2} + a\left(\frac{1}{\beta} - \frac{2r}{\sigma^2\beta}\right)\right)\right.$ $\left. + \frac{a(2r-\sigma^2)}{2b\sigma^2\beta} - \frac{(2r+\sigma^2)^2}{32b\sigma^2} - \frac{8\sqrt{2J}}{32b^2\sigma^4\beta^4} - \frac{m\left(16-64b^2+16b\beta(2r-\sigma^2)\right)}{32b\sigma^2\beta^2}\right] U(m)$	$U(m) = \left(\Phi_1 + \Phi_2 \exp\left[\frac{(\log(S)-2bt)\sqrt{Y}}{\sqrt{2}b^2\sigma^4\beta^4}\right]\right)$ where $Y = b^4\sigma^4\beta^4\left[32b^2 - 16 + \beta^2(2r+\sigma^2)^2\right.$ $\left. + (16a\beta - 16b\beta)(2r-\sigma^2)\right]$

Table 2.4: (Φ_1, Φ_2 are constants of integration and a, b, c are arbitrary constants for each symmetry. A_i and B_i are the Airy functions.)

	Symmetries	Variables	Solution
8	$e_1 + \epsilon e_4 + m_5 e_5$ $+ m_6 e_6$	$m = \frac{\log(S)-2bt}{\sqrt{t^2-1}}$ $V = \exp\Big[\frac{t(2r+\sigma^2)}{8\sigma^2} - \frac{\log(1-t^2)}{4}$ $+\frac{\log(S)\left(-4b-\beta(2r-\sigma^2)(t^2-1)+t\log(S)\right)}{2\sigma^2\beta^2(t^2-1)} + \frac{2b^2t}{\sigma^2\beta^2(t^2-1)}$ $-\frac{\tanh^{-1}(t)\left\{2b^2-2(a-2a\beta r+\beta br)+\sigma^2\beta(b-2a)\right\}}{\sigma^2\beta^2}\Big] U(m)$	$U(m) = \left(\Phi_2 D_v\left[\frac{im\sqrt{2}}{\sigma\beta}\right] + \Phi_1 D_w\left[\frac{m\sqrt{2}}{\sigma\beta}\right]\right)$, where $v = \frac{4b^2-\sigma^2\beta^2+2b\beta(\sigma^2-2r)+a(8\beta r-4-4\beta\sigma^2)}{2\sigma^2\beta^2}$ and $w = -v$
9	$e_2 + \epsilon e_4 + m_7 e_5$ $+ m_8 e_6$	$m = \log(S) - 2bt + \frac{t^2}{2}$ $V = \exp\Big[\frac{t}{8\sigma^2}(2r+\sigma^2) - \frac{t^3+3t\log(s)}{3\sigma^2\beta^2} + \frac{2at}{\beta}$ $+\frac{2at}{\sigma^2\beta^2}(1-2\beta r) + \frac{t^2}{4\sigma^2\beta^2}(4b+2\beta r-\sigma^2\beta)\Big] U(m)$	$U(m) = \left(A_i\left[\left(\frac{2}{\sigma^4\beta^4}\right)^{\frac{1}{3}} P\right]\Phi_1 + A_i\left[\left(\frac{2}{\sigma^4\beta^4}\right)^{\frac{1}{3}} P\right]\Phi_2\right)$ $\exp\left[\frac{\left(4b+\beta(\sigma^2-2r)\right)\left(2\log(S)-4bt+t^2\right)}{4\sigma^2\beta^2}\right]$, where $P = 2(b^2-a-bt) + (2\beta r - \sigma^2\beta)(2a-b)$ $+\log(S) + \frac{t^2}{2}$
10	$e_3 + \epsilon e_4 + m_9 e_5$ $+ m_{10} e_6$	$m = \frac{2b-\log S}{\sqrt{1-2t}}$ $V = U(m)\exp\Big[\frac{(2r+\sigma^2)^2}{16\sigma^2}(2t-1) + \frac{2r-\sigma^2}{2\sigma^2\beta}(2b-\log(S))$ $+\frac{4\log(1-2t)}{16\sigma^2\beta^2}\left(4(2a\beta r-a-\beta br)-\sigma^2\beta(4a-2b+\beta)\right)\Big]$	$U(m) = \left(\Phi_1\ H_Q\left[\frac{m}{\sigma\beta}\right] + \Phi_2\ {}_1F_1\left[-Q;\frac{1}{2};\frac{m^2}{\sigma^2\beta^2}\right]\right)$ $\exp\left[-\frac{m^2}{\sigma^2\beta^2}\right]$, where $Q = 4(a-2a\beta r+b\beta r) - \sigma^2\beta(2b+\beta-4a)$
11	$e_1 + \epsilon e_2 + m_{11} e_4$ $+ m_{12} e_5 + m_{13} e_6$	$m = \sqrt{t^2-c}\,(\log(S)+1) - 2b\log(2\sqrt{t^2-c}+2t)$	

Table 2.5: ($H_n[x]$ and ${}_1F_1$ are the Hermite polynomials and Hypergeometric functions respectively)

For symmetry G_1, we see that

$$\left(\frac{\mathrm{d}\bar{t}}{\mathrm{d}\xi_1}\right)_{\xi_1=0} = 1. \tag{2.45}$$

We will then have that

$$\begin{aligned} \bar{t} &= \xi_1 + t, \\ \bar{S} &= S, \\ \bar{V} &= V, \end{aligned} \tag{2.46}$$

is the global transformation that leaves equation (2.1) invariant. Similar to the formalism for G_1, we write out the remaining global transformations in Table 2.6. The combination of these transformations is

$$\bar{t} = \xi_1 + \frac{t\exp(8\xi_5)}{1-8t\xi_6}, \tag{2.47}$$

$$\bar{S} = S\exp\left(2t\xi_4 + \xi_2 + e^{4\xi_5} + e^{8t\xi_6}\right), \tag{2.48}$$

$$\begin{aligned} \bar{V} = {} & V\exp\left(\xi_3 + \left[\frac{t}{\beta} - \frac{2rt}{\sigma^2\beta} + \frac{2\log(S)}{\sigma^2\beta^2}\right]\xi_4 + \right. \\ & \left[4rt + \frac{4r^2t}{\sigma^2} + \sigma^2 t + \frac{2\log(S)}{\beta} - \frac{4r\log(S)}{\sigma^2\beta}\right]\xi_5 \\ & \left.\left[-4t + 4rt^2 + \frac{4r^2t^2}{\sigma^2} + \sigma^2t^2 + \frac{4t\log(S)}{\beta} - \frac{8rt\log(S)}{\sigma^2\beta} + \frac{4\log^2(S)}{\sigma^2\beta^2}\right]\xi_6\right). \end{aligned} \tag{2.49}$$

From equations (2.47)–(2.49), we can now obtain new solutions, V, to equation (2.9) via the solutions presented in Tables 2.4 and 2.5. Due to the presence of the arbitrary parameters $\xi_1, \ldots, \xi_6$ in equations (2.47)–(2.49), we have more freedom than in the original solutions. In all cases, bar one, we still obtain trivial solutions.

Infinitesimal Symmetries	Global transformations
G_2	$\bar{t} = t$ $\bar{S} = S\exp(\xi_2)$ $\bar{V} = V$
G_3	$\bar{t} = t$ $\bar{S} = S$ $\bar{V} = V\exp(\xi_3)$
G_4	$\bar{t} = t$ $\bar{S} = S\exp(2t\xi_4)$ $\bar{V} = V\exp\left(\left[\frac{t}{\beta} - \frac{2rt}{\sigma^2\beta} + \frac{2\log(S)}{\sigma^2\beta^2}\right]\xi_4\right)$
G_5	$\bar{t} = t$ $\bar{S} = S\exp(4\xi_5)$ $\bar{V} = V\exp\left(\left[4rt + \frac{4r^2t}{\sigma^2} + \sigma^2 t + \frac{2\log(S)}{\beta} - \frac{4r\log(S)}{\sigma^2\beta}\right]\xi_5\right)$
G_6	$\bar{t} = \frac{t}{1-8t\xi_6}$ $\bar{S} = S\exp(8t\xi_6)$ $\bar{V} = V\exp\left(\left[-4t + 4rt^2 + \frac{4r^2t^2}{\sigma^2} + \sigma^2t^2 + \frac{4t\log(S)}{\beta} - \frac{8rt\log(S)}{\sigma^2\beta} + \frac{4\log^2(S)}{\sigma^2\beta^2}\right]\xi_6\right)$

Table 2.6: Global transformations of the infinitesimal symmetries $G_2, \ldots, G_6$.

In the case of solution 1 in Table 2.4, we are able to obtain a new solution given by

$$\begin{aligned} V(S,t) &= \sqrt{\frac{t-T}{8(t-T)-1}}\left([\log(S)-\exp(8(t-T))]\,\frac{t-T}{8(t-T)-1}\right) \\ &\quad \exp\left([\log(S)-\exp(8(t-T))]\,\frac{(\sigma^2-2r)}{2\sigma^2\beta}+\left[\frac{9}{2\sigma^2\beta^2}\right][\log(S)-\exp(8(t-T))]^2\right. \\ &\quad +\left[\frac{(2r+\sigma^2)}{8\sigma^2}-4\right]\left[\frac{t-T}{8(t-T)-1}\right]+\left[4r+\frac{4r^2}{\sigma^2}+\sigma^2\right]\left[\frac{t-T}{8(t-T)-1}\right]^2 \\ &\quad \left.+\left[\frac{4}{\beta}-\frac{8r}{\sigma^2\beta}\right]\left[\frac{t-T}{8(t-T)-1}\right][\log(S)-\exp(8(t-T))]\right). \end{aligned} \tag{2.50}$$

2.4.2 $S^\beta > K$:

As earlier noted, this condition implies that $V(S,T) = S^\beta - K$. We again refer to the linear combination of the symmetries $e_1, ..., e_6$ in equation (2.30) and apply it to this pay-off.

For the left hand side

$$\begin{aligned} & \Gamma V\Big|_{V=0} \\ = & \left(\alpha_5\frac{1}{\sigma^2\beta^2}+\alpha_1\left[\frac{t}{4}-\frac{rt^2}{4}-\frac{r^2t^2}{4\sigma^2}-\frac{\sigma^2t^2}{16}-\frac{t\log(S)}{4\beta}+\frac{rt\log(S)}{2\sigma^2\beta}-\frac{\log^2(S)}{4\sigma^2\beta^2}\right]\right. \\ & -\alpha_2\left[\frac{t}{4\beta}-\frac{rt}{2\sigma^2\beta}+\frac{\log(S)}{2\sigma^2\beta^2}\right]+\alpha_4\left[\frac{r}{4}+\frac{r^2}{4\sigma^2}+\frac{\sigma^2}{16}\right]+\alpha_6\left[\frac{1}{\beta}-\frac{2r}{\sigma^2\beta}\right] \\ & \left.+\alpha_3\left[\frac{1}{4}-\frac{rt}{2}-\frac{r^2t}{2\sigma^2}-\frac{\sigma^2t}{8}-\frac{\log(S)}{4\beta}+\frac{r\log(S)}{2\sigma^2\beta}\right]\right)V\Big|_{V=S^\beta-K}. \end{aligned}$$

For the right hand side

$$\begin{aligned} & \Gamma(S^\beta - K) \\ = & \left(\alpha_1\frac{-t\log(S)}{2}-\alpha_2\frac{t}{2}-\alpha_3\frac{\log(S)}{2}+2\alpha_6\right)\left(\beta S^\beta\right). \end{aligned}$$

Equating both sides and for consistency, we have that

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_6 = 0,$$

and

$$\alpha_4 \left[\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16}\right] + \frac{\alpha_5}{\sigma^2\beta^2} = 0.$$

The new symmetry, after the application of initial conditions, will now be

$$G_7 = \sigma^2\beta^2 \left(\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16}\right)\frac{\partial}{\partial t} + \left[\sigma^2\beta^2 \left(\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16}\right)^2 - \frac{1}{\sigma^2\beta^2}\right] V\frac{\partial}{\partial V}. \quad (2.51)$$

Reduction of the power option PDE via this symmetry and subsequent solution of the ODE yields that

$$V(S,t) = S^{\frac{\sigma^2(1-\beta M)-2r}{2\beta\sigma^2}} \left(S^M + \Phi_1\right) \exp\left(Z[\Phi_2 + t]\right), \quad (2.52)$$

where $M = \sqrt{\frac{(2r+\sigma^2)^2 - 8Z\sigma^2}{\beta^2\sigma^4}}$, $Z = \left(\frac{r}{4} + \frac{r^2}{4\sigma^2} + \frac{\sigma^2}{16}\right) - \frac{1}{(\sigma^4\beta^4)\left(\frac{r}{4}+\frac{r^2}{4\sigma^2}+\frac{\sigma^2}{16}\right)}$ and Φ_1, Φ_2 are arbitrary constants of integration.

For the satisfaction of the pay-off $V(S,T) = S^\beta - K$ with initial condition $t = T$, it will require that the arbitrary constants of integration take on the values

$$\Phi_1 = -K, \quad (2.53)$$

and

$$\Phi_2 = -T, \quad (2.54)$$

with constraint

$$\beta = \frac{1}{\sigma}\sqrt{\sigma^2 - 2r}, \quad (2.55)$$

on the power parameter β. No other progress (via global transformations) was possible in this case.

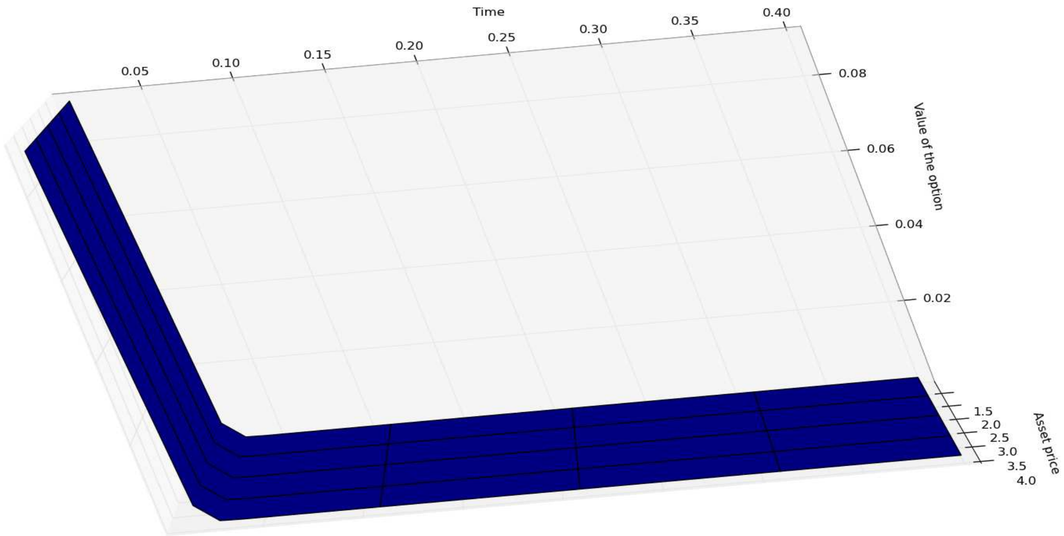

Figure 2.1: 3D plot of the power option pricing against the asset price and time for the terminal condition

2.5 Discussion

We have demonstrated that the power option PDE admits six symmetries (excluding the solution symmetry). By relabeling these symmetries, we showed that they have the Lie algebraic structure of $A_{3,5}^{\frac{1}{2}} \oplus 3A_1$. This is the first time the PDE of this exotic option has been shown to possess such structure. Going further, we used the Goursat-twist method to obtain an optimal system of these symmetries which can be used to reduce the PDE.

For the terminal condition $S^\beta \leq K$, all the symmetries in the optimal system satisfied the initial condition and could be used to obtain group invariant solutions (we succeeded in this, save for one of them). Since these solutions only trivially satisfied the PDE with the initial condition, we went ahead to obtain their global transformations. Via these transformations, we were able to obtain a non-trivial solution. In Figure 2.1, we plotted our solution for a β value of 1.5, where the stock volatility and interest rate are both 5%. The plot shows the value of the option going to zero as we would expect when $S^\beta \leq K$.

We note that the only previous PDE solution available for this terminal condition was [9]

$$V(S,t) = \phi_1 S^{\kappa_{1,2}} e^{-\gamma\kappa_{1,2}t} + \phi_2, \qquad\qquad \phi_1, \phi_2 \in \mathbb{R} \tag{2.56}$$

where

$$\kappa_{1,2} = \tau + \frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta\left(\beta + 4(\tau - 1)\right)}, \tag{2.57}$$

with the constraint

$$\beta = \frac{\tau^2}{2\tau - 1}. \tag{2.58}$$

In fact, this solution was claimed to apply to both initial conditions. However, it is a simple matter to verify that equation (2.56) only applies to the terminal condition when

$$\phi_1 = \phi_2 = 0, \tag{2.59}$$

which forces the solution to be trivial everywhere. This emphasizes the importance of our non-trivial solution as it is the first such solution available. We further note that our solution has no constraints on the parameters.

For the condition $S^\beta > K$, our six-parameter symmetry reduces to a one-parameter symmetry. In spite of the reduction in parameters, we were still able to determine an explicit solution. For fixed values of the interest rate and stock volatility, $r = 0.1\%$

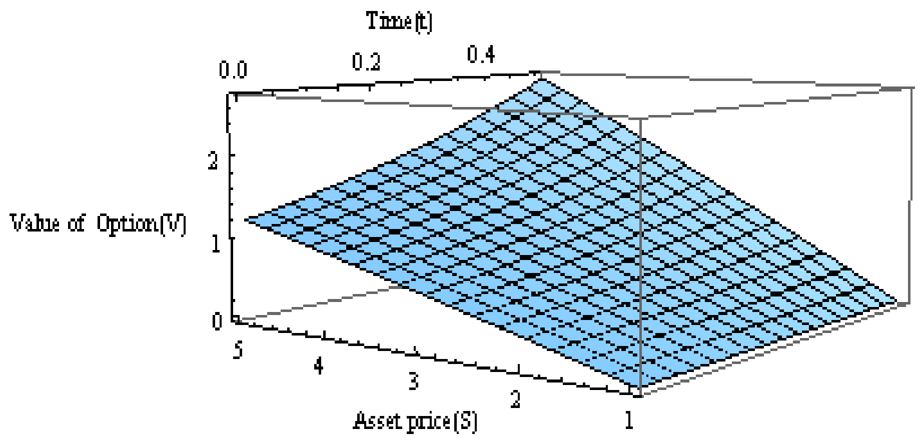

Figure 2.2: 3D plot of the power option pricing against the asset price and time

and $\sigma = 6.3\%$, (β is determined from equation (2.55)), we plotted the value of the option pricing obtained against the asset price (S) and time (t) in Figure 2.2. The plot shows how the value of the power option increases with movement of the underlying asset price in time. However, unlike in the terminal case, we do have the constraint

$$\beta = \frac{1}{\sigma}\sqrt{\sigma^2 - 2r}. \tag{2.60}$$

Fortunately, this is the only constraint. In the case of Bordag *et al* [9], not only do they have

$$\beta = \frac{\tau^2}{2\tau - 1}, \tag{2.61}$$

they also assumed that large traders of the option only use the delta hedging strategy discussed in [3, 57]. We make no such assumption in our work.

We note that previous solutions have only satisfied the $S^\beta > K$ condition whilst only trivially satisfying the terminal case. This is the first time solutions that will non-trivially satisfy both conditions are obtained. It should also be noted that these solutions were obtained via an algebraic approach and not probabilistic, as was previously employed by Esser [22].

As a final remark, we note that a claimed solution to this problem was previously published by Baz & Chacko [5]. They assumed that since the option price (S), interest rate (r) and stock volatility (σ) of the vanilla option PDE was replaced by S^β, $r\beta + \frac{1}{2}(\beta - 1)\sigma^2$ and $\sigma\beta$ respectively in the power option PDE, then the solution will just be a straight forward replacement of these terms in the Black-Scholes solution. This assumption is incorrect because the dynamics of the system changes with these new terms and a simple replacement will not reflect this changes. Their result also did not take the change in payoff into account.

Chapter 3

Solving a PDE associated with the pricing of power options with time dependent parameters

3.1 Introduction

Options are derivatives that give the specifics of a contract between two parties for a future transaction on an asset, at a reference price [47]. In 1973, Black and Scholes aptly articulated the Brownian motion idea of Bachelier [1] by formulating what is now known as the Black-Scholes theory [4]. The motivation behind the Black-Scholes model was the creation of a derivative that could perfectly hedge the option of buying and selling the underlying asset whilst reducing risk to the barest minimum. The major feature of the Black-Scholes theory is that the payoffs are a linear function of the underlying asset price. Complexities in the options market extended beyond this linear functional to a nonlinear dependence of the payoff on the asset price - this was the motivation of the works of Cox and Rubinstein [17] (they studied squared power options). Subsequently, Tompkins considered and produced the complete pricing solutions to options where the asset price is raised to a certain integer power [63]. These options are called power options and are in a generalized class of options called exotic options. Essentially, exotic options are derivatives which have features that makes them more complex than commonly traded products. They are usually only over-the-counter (OTC) or embedded in structured notes [11]. Power options have a payoff structure depending on the price, raised to a certain power of the underlying asset at the expiration time. Thus the payoff is a nonlinear function of the underlying asset price.

A need for the availability of more flexibility for the power option prompted Esser [21, 22] to embark on producing solutions, via the probabilistic route, to options whose asset price is now raised to any real valued number. This case has also been analyzed in [9, 39], showing an algorithmic approach to solving the power option partial differential equation (PDE).

The power option PDE is [63]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\beta^2\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left[\beta r + \frac{1}{2}\beta\sigma^2(\beta - 1)\right] S\frac{\partial V}{\partial S} - rV = 0, \tag{3.1}$$

where β is called the power factor and is responsible for its high risk nature and curved payoffs. The option has a payoff of $V(S,T) = \max\{\phi(S^\beta - K, 0)\}$, where $t = T$ is the initial condition. The model parameters denoting the interest rate and the volatility are given as r and σ respectively, where S is the price of the underlying asset and K is the strike price. The parameter ϕ determines whether the option is a call or a put ($\phi = 1$ refers to the former and $\phi = -1$ to the latter).

A shortcoming of the previous solutions of the power option problem is the non-provision for a scenario whereby σ and r are considered to be time dependent (and hence constantly changing throughout the lifetime of the option). Here, we attempt to solve the power option PDE, with time dependent σ and r, via an algorithmic method called the Lie group method.

Around 1870, Sophus Lie realized that many of the methods for solving DEs could be unified using group theory. This theory, now called the Lie symmetry method, is central to the modern approach for studying nonlinear DEs. It uses the notion of symmetry to generate solutions in a systematic manner. A symmetry of a PDE system is any transformation of its solution manifold into itself, i.e., a symmetry transforms (maps) any solution of a PDE system to another solution of the same system [8].

Gazizov and Ibragimov [24] arguably pioneered the employment of the Lie group technique in the resolution of problems in Financial Mathematics. Using Lie groups, they made an analysis of the Black-Scholes model and showed that the popular solution to the Black-Scholes PDE is actually an invariant solution. More recently, Caister *et al* [12–14] and Taylor & Glasgow [62] also utilized the Lie group method in the resolution of problems in this field.

An invariant solution of a differential equation (DE) is a solution of the DE which is also an invariant curve (surface) of a group admitted by the DE. To make the previous statement clearer, consider an nth order system of DEs,

$$\Gamma_v(t, y, \partial y, \ldots, \partial^n y) = 0, \qquad\qquad v = 1, \cdots, l \tag{3.2}$$

where $t = t^1, \ldots, t^p$ are the independent variables, $y = y^1, \ldots, y^q$ are the dependent variables and ∂^k is the k-th partial derivative of y with respect to t (for $k = 1, \ldots, n$). The functions Γ_v are assumed to depend smoothly on their arguments, t and y, in some open set M of the total space $T \times Y = \mathbb{R}^p \times \mathbb{R}^q$ of independent and dependent variables. Now, let H be a local group of transformations acting on $M \subset T \times Y$. H is said to be a symmetry group of Γ if each element $h \in H$ transforms solutions of Γ to other solutions of Γ. A real valued function $\eta(t, y)$ is called an invariant of H if it is unchanged by the group action

$$\eta\left(h \cdot (t, y)\right) = \eta(t, y), \qquad\qquad \forall t, y \in M \tag{3.3}$$

for all $h \in H$ such that $h \cdot (t, y)$ is defined.

The new system of DEs (Γ/H) for the H-invariant solutions to Γ will involve just the new variables u, w formed from the invariants of H. On reduction (reduction as used here means that a point symmetry of a DE leads, in the case of a PDE, to finding special solutions called invariant or similarity solutions of the DE) of Γ via H, the system of DEs

$$\left(\Gamma/H\right)_v \left(u, w, \partial w, \ldots, \partial^n w\right) = 0, \qquad\qquad v = 1, \cdots, l \tag{3.4}$$

which has r $(r < p + q)$ fewer variables, constitutes the new system Γ/H. Now, every solution $w = f(u)$, to equation (3.4), gives rise to an H-invariant solution

$y = g(t)$ to Γ. This is determined implicitly from the definition of the new variables u and w. For more rigorous explanations on group invariant solutions by symmetry analysis, see [43, 46].

We now derive the PDE that models the power option. Firstly, we note that the power option pays its owner the value of the stock (raised to a pre-specified power) less a strike price, at expiration - only if the value of the underlying is greater than the strike price. This payoff is simply represented as $\max\{S^\beta - K, 0\}$.

The stochastic process for S^β is given by

$$\mathrm{d}S^\beta = \left(\beta\mu + \frac{1}{2}\beta[\beta - 1]\right) S^\beta \mathrm{d}t + \beta\sigma S^\beta \mathrm{d}W, \tag{3.5}$$

and its pricing kernel is characterized by the Brownian motion model as

$$\frac{\mathrm{d}\chi}{\chi} = -r\mathrm{d}t - \frac{\mu - r}{\sigma}\mathrm{d}W. \tag{3.6}$$

W is the Weiner process and μ is the instantaneous expected return of the stock [5]. According to Itô's lemma [34], the price of the power derivative will now be

$$\mathrm{d}V = \left(\frac{1}{2}\beta^2\sigma^2 S^{\beta 2}\frac{\partial^2 V}{\partial S^{\beta 2}} + \left[\beta\mu + \frac{1}{2}\beta\{\beta - 1\}\sigma^2\right] S^\beta \frac{\partial V}{\partial S^\beta} + \frac{\partial V}{\partial t}\right)\mathrm{d}t + \beta\sigma S^\beta \frac{\partial V}{\partial S^\beta}\mathrm{d}W. \tag{3.7}$$

On solving for χ and V in equations (3.6) and (3.7) respectively, their product (since it is a martingale, the drift term must be zero) gives the power option PDE in equation (3.12).

3.2 Brief overview of the Lie group technique

Considering equation (3.2), where

$$(t^*)^i = f^i(t, y; \epsilon) \tag{3.8a}$$

$$(y^*)^\mu = g^\mu(t, y; \epsilon) \tag{3.8b}$$

is a one-parameter Lie group of point transformations, with infinitesimal generator

$$X = \zeta^i(t, y)\frac{\partial}{\partial t^i} + \eta^\mu(t, y)\frac{\partial}{\partial y^\mu}. \tag{3.9}$$

Lie's algorithm for finding the point symmetries of a given PDE system is according to the following theorem [8] called the infinitesimal criterion of invariance under a one-parameter Lie group of point transformations.

Theorem 3.2.1. *Let (3.9) be the infinitesimal generator of the one-parameter Lie group of point transformations (3.8). Then the transformation (3.8) is a point symmetry of the PDE system (3.2) if and only if for each $\alpha = 1, \ldots, l$,*

$$X^{(n)}\Gamma_v(t, y, \partial y, \ldots, \partial^n y) = 0, \tag{3.10}$$

when

$$\Gamma_v(t, y, \partial y, \ldots, \partial^n y) = 0, \qquad v = 1, \cdots, l \tag{3.11}$$

and where $X^{(n)}$ is the n^{th} prolongation of X.

Several software packages have been developed to help find the symmetries of DEs [20, 29]. Usually these packages have to be supplemented by further analysis.

3.3 Analysis of the problem by the Lie group approach

None of the developed software packages - for producing symmetries of a DE - were able to give the one-parameter symmetries of the time dependent case of the power option (as we obtained for the constant case in a previous paper [39]). The best outcome was by engaging Program LIE [29] in interactive mode. From this interaction, we were able to get a form of the general symmetry, but this required further analysis. The general form of the six-parameter symmetry of the time dependent power option PDE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\beta^2\sigma(t)^2S^2\frac{\partial^2 V}{\partial S^2}+\left[\beta r(t)+\frac{1}{2}\beta\sigma(t)^2(\beta-1)\right]S\frac{\partial V}{\partial S}-r(t)V=0, \tag{3.12}$$

is given by

$$\begin{aligned} X_1 &= a(t)\partial_t+\left[\frac{\dot a(t)\log(S)}{2}+\frac{a(t)\log(S)\dot\sigma(t)}{\sigma(t)}+b(t)\right]S\partial_S+\frac{1}{2r(t)\beta-\beta\sigma(t)^2}\left[G(S,t)\right.\\ &\quad \left.+(2r(t)\beta-\beta\sigma(t)^2)F(S,t)\right]\partial_V, \end{aligned} \tag{3.13}$$

where the function $G(S,t)$ is given explicitly as

$$\begin{aligned} G(S,t) &= \left(2Vr(t)\beta-V\beta\sigma(t)^2\right)c(t)-2V\log(S)\dot c(t)+2Vr(t)\log(S)\dot a(t)\\ &\quad+\left\{2Vr(t)\beta(\log(S))^2-2V\beta^2\log S\sigma(t)^2-V\beta(\log(S))^2\sigma(t)^2\right\}\tau\\ &\quad+2V\dot r(t)\log(S)a(t), \end{aligned}$$

and $F(S,t)$ is the solution of the PDE.

The functions $a(t), b(t)$ and $c(t)$ satisfy the system of equations

$$-2\dot{\sigma}(t)^2 a(t) + 2\dot{a}(t)\dot{\sigma}(t)\sigma(t) + \sigma(t)^2\ddot{a}(t) + 2\sigma(t)\ddot{\sigma}(t)a(t) - 4\beta^4\sigma(t)^4\tau = 0, \tag{3.14}$$

$$\begin{aligned}&-8r(t)\beta a(t)\dot{\sigma}(t)^2 + 6r(t)\beta\sigma(t)\dot{a}(t)\dot{\sigma}(t) + 4r(t)\beta\sigma(t)a(t)\ddot{\sigma}(t) - 3\beta\sigma(t)^2\dot{a}(t)\dot{r}(t)\\ &+ 6\beta\sigma(t)a(t)\dot{\sigma}(t)\dot{r}(t) - 2\beta\sigma(t)^2a(t)\ddot{r}(t) - 4\sigma(t)\dot{b}(t)\dot{\sigma}(t) + 2\sigma(t)^2\ddot{b}(t) = 0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}&-8r(t)\beta\sigma(t)a(t)\dot{r}(t) - 4r(t)\beta\sigma(t)^3\dot{a}(t) + 8r(t)\sigma(t)\dot{b}(t) - 4r(t)^2\beta\sigma(t)\dot{a}(t)\\ &+ 8r(t)^2\beta a(t)\dot{\sigma}(t) + 8\beta^3\sigma(t)^5\tau - 4\beta\sigma(t)^3a(t)\dot{r}(t) - \beta\sigma(t)^5\dot{a}(t) + 8\beta\sigma(t)^3\dot{c}(t)\\ &- 2\beta\sigma(t)^4a(t)\dot{\sigma}(t) - 4\sigma(t)^3\dot{b}(t) = 0.\end{aligned} \tag{3.16}$$

The system (3.14)–(3.16) can be written in the simplified form

$$\ddot{a}(t) + 2\frac{\dot{\sigma}(t)}{\sigma(t)}\dot{a}(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\sigma}(t)}{\sigma(t)}\right)a(t) - 4\beta^4\sigma(t)^2\tau = 0, \tag{3.17}$$

$$\begin{aligned}&\ddot{b}(t) - 2\frac{\dot{\sigma}(t)}{\sigma(t)}\dot{b}(t) + \left(\frac{3\beta\dot{\sigma}(t)\dot{r}(t)}{\sigma(t)} - \frac{4r(t)\beta\dot{\sigma}(t)^2}{\sigma(t)^2} + \frac{2r(t)\beta\ddot{\sigma}(t)}{\sigma(t)} - \beta\ddot{r}(t)\right)a(t)\\ &+ \left(\frac{3r(t)\beta\dot{\sigma}(t)}{\sigma(t)} - \frac{3\beta\dot{r}(t)}{2}\right)\dot{a}(t) = 0,\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\dot{c}(t) + \left(\frac{r(t)^2\dot{\sigma}(t)}{\sigma(t)^3} - \frac{r(t)\dot{r}(t)}{\sigma(t)^2} - \frac{\dot{r}(t)}{2} - \frac{\sigma(t)\dot{\sigma}(t)}{4}\right)a(t) + \beta\sigma(t)^2\tau\\ &+ \left(-\frac{r(t)}{2} - \frac{r(t)^2}{2\sigma(t)^2} - \frac{\sigma(t)^2}{8}\right)\dot{a}(t) + \left(\frac{r(t)}{\beta\sigma(t)^2} - \frac{1}{2\beta}\right)\dot{b}(t) = 0.\end{aligned} \tag{3.19}$$

This reconstruction shows us that equation (3.17) is of the form

$$\ddot{a}(t) + \phi(t)\dot{a}(t) + \dot{\phi}(t)a(t) + \varphi(t) = 0. \tag{3.20}$$

We can now solve this to obtain

$$a(t) = \frac{1}{\sigma(t)^2}\left[a_2 + \int \sigma(t)^2\mathrm{dt}\left(a_1 + 2\tau\beta^4\int \sigma(t)^2\mathrm{dt}\right)\right]. \tag{3.21}$$

Having obtained $a(t)$, we now proceed to solve for $b(t)$ in equation (3.18) - which is of the form

$$\ddot{b}(t) + \theta(t)\dot{b}(t) + \alpha(t) = 0. \tag{3.22}$$

Since the function $\alpha(t)$ represented in equation (3.18) is now an explicit function of the model parameters, we can solve for $b(t)$ to yield

$$b(t) = \int \sigma(t)^2 \left(\int -\frac{\alpha(t)}{\sigma(t)^2} \mathrm{dt} + b_1 \right) \mathrm{d}t + b_2, \tag{3.23}$$

where

$$\begin{aligned}
\alpha(t) \;=\; & \frac{1}{\sigma(t)^2}\left(\frac{3\beta\dot{\sigma}(t)\dot{r}(t)}{\sigma(t)} - \frac{4r(t)\beta\dot{\sigma}(t)^2}{\sigma(t)^2} + \frac{2r(t)\beta\ddot{\sigma}(t)}{\sigma(t)} - \beta\ddot{r}(t)\right)\left(a_2 + \int \sigma(t)^2\mathrm{dt}\right. \\
& \left.\left[a_1 + 2\tau\beta^4 \int \sigma(t)^2\mathrm{dt}\right]\right) + \left(\frac{3r(t)\beta\dot{\sigma}(t)}{\sigma(t)} - \frac{3\beta\dot{r}(t)}{2}\right)\left(2\beta^4\tau \int \sigma(t)^2\mathrm{d}t\right. \\
& \left. + \frac{a_1 + 2\beta^4 \int \sigma(t)^2\mathrm{dt}}{\sigma(t)} + 2\frac{\dot{\sigma}(t)}{\sigma(t)^3}\left[a_2 + \left\{a_1 + 2\beta^2\tau \int \sigma(t)^2\mathrm{dt}\right\} \int \sigma(t)^2\mathrm{dt}\right]\right).
\end{aligned}$$

Equation (3.19) is of the simple form

$$\dot{c}(t) + \varrho(t) = 0, \tag{3.24}$$

with solution,

$$c(t) = \int -\varrho(t)\ \mathrm{d}t + c_1, \tag{3.25}$$

where

$$\begin{aligned}
\varrho(t) \;=\; & \frac{1}{\sigma(t)^2}\left(-\frac{r(t)\dot{r}(t)}{\sigma(t)^2} + \frac{r(t)^2\dot{\sigma}(t)}{\sigma(t)^3} - \frac{\dot{r}(t)}{2} - \frac{\sigma(t)\dot{\sigma}(t)}{4}\right)\left[a_2 + \int \sigma(t)^2\mathrm{dt}\right. \\
& \left.\left(a_1 + 2\tau\beta^4 \int \sigma(t)^2\mathrm{dt}\right)\right] + \left(-\frac{r(t)}{2} - \frac{r(t)^2}{2\sigma(t)^2} - \frac{\sigma(t)^2}{8}\right)\left(2\beta^4\tau \int \sigma(t)^2\mathrm{dt}\right. \\
& \left. + \frac{a_1 + 2\beta^4 \int \sigma(t)^2\mathrm{dt}}{\sigma(t)} + 2\frac{\dot{\sigma}(t)}{\sigma(t)^3}\left[a_2 + \left\{a_1 + 2\beta^2\tau \int \sigma(t)^2\mathrm{dt}\right\} \int \sigma(t)^2\mathrm{dt}\right]\right) \\
& + \left(\frac{r(t)}{\beta\sigma(t)^2} - \frac{1}{2\beta}\right)\left(\sigma(t)^2\left[\int -\frac{\alpha(t)}{\sigma(t)^2}\mathrm{dt} + b_1\right]\right) + \beta\sigma(t)^2\tau,
\end{aligned}$$

and $\alpha(t)$ is as previously defined. The parameters a_1, a_2, b_1, b_2 and c_1 are all constants of integration.

3.4 Application of the initial conditions

Having obtained the general symmetry in equation (3.13) in its explicit form, we now require the general symmetry to be consistent with the terminal pay-off i.e. $V(S,T) = 0$, when $t = T$. This implies that

$$a(T) = 0 \qquad \text{and} \qquad G(S,T) = 0 \tag{3.26}$$

(which comes from the conditions on time and those on $V(S,T)$) must be satisfied. These conditions introduce constraints to the system. To highlight these constraints, we can separate the function $G(S,T)$ by coefficients of powers of $\log(S)$:

$$(2Vr(T)\beta - V\beta\sigma(T)^2)c(T) = 0, \tag{3.27}$$

$$2Vr(T)\dot{a}(T) - 2V\dot{c} + 2V\dot{r}(T)a(T) - 2V\beta^2\sigma(T)^2\tau = 0, \tag{3.28}$$

$$(2Vr(T)\beta - V\beta\sigma(T)^2)\tau = 0. \tag{3.29}$$

To solve the system of equations (3.27)–(3.29), we begin from equation (3.27) where we will have that $c(T) = 0$. This then implies that $\tau = 0$ (from equation (3.29)) and that

$$r(T)\dot{a}(T) + \dot{r}(T)a(T) = 0, \tag{3.30}$$

from equation (3.28). We solve (3.30) to obtain

$$a(T) = \frac{\chi_2}{r(T)}, \tag{3.31}$$

where χ_2 is an arbitrary constant of integration. Recall though from the dual conditions in equation (3.26) that $a(T) = 0$. This condition forces χ_2 to become zero. The general symmetry now has a leaner appearance, represented by

$$\begin{aligned} X_2 \;=\; & \left[\frac{\dot{a}(t)\log(S)}{2} + \frac{a(t)\log(S)\dot{\sigma}(t)}{\sigma(t)} + b(t)\right] S\partial_S - \frac{1}{2r(t)\beta - \beta\sigma(t)^2}\left[2V\log(S)\dot{c}(t)\right. \\ & - \left(2Vr(t)\beta - V\beta\sigma(t)^2\right)c(t) - 2Vr(t)\log(S)\dot{a}(t) - 2V\dot{r}(t)\log(S)a(t)\left.\right]\partial_V \\ & + a(t)\partial_t. \end{aligned} \quad (3.32)$$

We do not investigate the case of $r(t) = \frac{1}{2}\sigma(t)^2$ in equation (3.27) due to the financial irrelevance of this constraint.

3.5 The characteristic system and solution obtained

The next step will be to obtain the new invariants via the 5-parameter symmetry in equation (3.32). This is not a simple task because it will require the independent variable t to be expressed in a complicated form of the zeroth order differential invariant. This form does not add value to the analysis. We avoid this unnecessary complication by setting $a_1 = a_2 = 0$, eliminating two of the remaining symmetries and leaving us with a 3-parameter symmetry. This has in no way removed the fundamentals of the problem i.e. of σ and r depending on time, whilst also satisfying the initial conditions. Thus, what we will obtain will still be a group invariant solution, though not the most general group invariant solution.

This 3-parameter symmetry has the associated Lagrange system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}S}{Sb(t)} = \frac{2r(t)\beta - \beta\sigma(t)^2\mathrm{d}V}{\left([2r(t)\beta - \beta\sigma(t)^2]c(t) - 2\log(S)\dot{c}(t)\right)V}, \tag{3.33}$$

with new invariants

$$u = t, \tag{3.34}$$

$$Q(u) = \int \frac{-b_1}{S\beta\left(b_1\int\sigma(u)^2\mathrm{d}u + b_2\right)}\left(\int\left[\frac{\sigma(u)^2}{2} - r(u)\right]\mathrm{d}u + \frac{\log(S)}{\beta} + c_1\right)\mathrm{d}S + \log(V), \tag{3.35}$$

and so

$$V = Q(u)\exp\left(\frac{b_1\log(S)\left(2\beta c_1 + 2\beta\int[\sigma(u)^2 - 2r(u)]\,\mathrm{d}u + \log(S)\right)}{2\beta^2\left(b_2 + b_1\int\sigma(u)^2\mathrm{d}u\right)}\right). \tag{3.36}$$

The power option PDE will now be reduced via equation (3.36). This will result in an ODE, whose solution is also admitted by the PDE. The ODE is given by

$$\begin{aligned}
&4r(u)\left(b_2 + b_1\int\sigma(u)^2\mathrm{d}u\right)\left(2b_2 - 2b_1c_1 + 2b_1\int\sigma(u)^2\mathrm{d}u - b_1\int\left[\sigma(u)^2 - 2r(u)\right]\mathrm{d}u\right)Q(u)\\
&- b_1\sigma(u)^2\left(4\left[b_2 - b_2c_1 + b_1c_1^2 - b_1(c_1 - 1)\int\sigma(u)^2\mathrm{d}u\right] - 2\int\left[\sigma(u)^2 - 2r(u)\right]\mathrm{d}u\right.\\
&\left.\left[b_2 - b_1c_1 + b_1\int\sigma(u)^2\mathrm{d}u\right] + b_1\left\{\int\left[\sigma(u)^2 - 2r(u)\right]\mathrm{d}u\right\}^2\right)Q(u)\\
&- 8\left(b_2 + b_1\int\sigma(u)^2\mathrm{d}u\right)^2 Q'(u) = 0.
\end{aligned} \tag{3.37}$$

Equation (3.37) is a first order ODE in Q, which we solve to obtain

$$
\begin{aligned}
Q(u) \;=\; & \Phi \exp\Bigg(\int \Bigg(\Bigg[4r(u)\left\{b_1\int \sigma(u)^2\mathrm{d}u + b_2\right\}\Bigg\{-b_1\int\left[\sigma(u)^2 - 2r(u)\right]\mathrm{d}u \\
& + 2b_1\int\sigma(u)^2\mathrm{d}u - 2b_1c_1 + 2b_2\Bigg\} - b_1\sigma(u)^2\Bigg\{b_1\int\left[\sigma(u)^2 - 2r(u)\right]\mathrm{d}u \\
& + 4\left[(b_1 - b_1c_1)\int\sigma(u)^2\mathrm{d}u + b_1c_1^2 - b_2c_1 + b_2\right] \\
& -2\int\left[\sigma(u)^2 - 2r(u)\right]\mathrm{d}u\left[b_1\int\sigma(u)^2\mathrm{d}u - 2b_1c_1 + b_2\right]\Bigg\}\Bigg]\Bigg/ \\
& \left[8\left\{b_1\int\sigma(u)^2\mathrm{d}u + b_2\right\}^2\right]\Bigg)\,\mathrm{d}u, \qquad (3.38)
\end{aligned}
$$

where Φ is a constant of integration. According to the definition of the invariants in equations (3.34) and (3.35), a solution of the time dependent case of the power option PDE will now be given as

$$
\begin{aligned}
V \;=\; & \Phi \exp\Bigg(\int \Bigg(\Bigg[4r(t)\left\{b_1\int \sigma(t)^2\mathrm{d}t + b_2\right\}\Bigg\{-b_1\int\left[\sigma(t)^2 - 2r(t)\right]\mathrm{d}t \\
& + 2b_1\int\sigma(t)^2\mathrm{d}t - 2b_1c_1 + 2b_2\Bigg\} - b_1\sigma(t)^2\Bigg\{b_1\int\left[\sigma(t)^2 - 2r(t)\right]\mathrm{d}t \\
& + 4\left[(b_1 - b_1c_1)\int\sigma(t)^2\mathrm{d}t + b_1c_1^2 - b_2c_1 + b_2\right] \\
& -2\int\left[\sigma(t)^2 - 2r(t)\right]\mathrm{d}t\left[b_1\int\sigma(t)^2\mathrm{d}t - 2b_1c_1 + b_2\right]\Bigg\}\Bigg]\Bigg/ \\
& \left[8\left\{b_1\int\sigma(t)^2\mathrm{d}t + b_2\right\}^2\right]\Bigg)\,\mathrm{d}t + \\
& + \frac{\log(S)b_1\left(2\beta c_1 + \beta\int\left[\frac{\sigma(t)^2}{2} - r(t)\right]\mathrm{d}t + \log(S)\right)}{2\beta^2\left(b_2 + b_1\int\sigma(t)^2\mathrm{d}t\right)}\Bigg). \qquad (3.39)
\end{aligned}
$$

To interprete the solution in a meaningful way, we need to fix the parameters. We must be careful though because setting $b_1 = 0$ forces the solution to be trivial.

Setting $\Phi = b_1 = 1$ and $b_2 = c_1 = 0$, we obtain

$$\begin{aligned} V &= \exp\Bigg(\int \Bigg(\Bigg[\Bigg\{4r(t)\int \sigma(t)^2\mathrm{d}t\Bigg\}\Bigg\{\int \left[2r(t) - \sigma(t)^2\right]\mathrm{d}t + 2\int \sigma(t)^2\mathrm{d}t\Bigg\} \\ &\quad - \sigma(t)^2\Bigg\{4\Bigg[\int \sigma(t)^2\mathrm{dt}\Bigg] - \int \left[\sigma(t)^2 - 2r(t)\right]\mathrm{d}t\Bigg[2\int \sigma(t)^2\mathrm{dt} - 1\Bigg]\Bigg\}\Bigg] \\ &\quad \Bigg/\Bigg[8\Bigg\{\int \sigma(t)^2\mathrm{dt}\Bigg\}^2\Bigg]\Bigg)\mathrm{d}t + \frac{\log(S)\left(2\beta\int\left[\sigma^2 - 2r(t)\right]\mathrm{d}t + \log(S)\right)}{2\beta^2\left(\int \sigma(t)^2\mathrm{d}t\right)}\Bigg), \end{aligned} \tag{3.40}$$

which is a simpler, but still non-trivial solution.

3.6 Discussion

To illustrate this result, we plot the value of the option against the asset price and time to option expiration in Figure 3.1. This shows the behavior of the option with respect to the value of the asset, in time. In this surface plot, the interest rate (r) and stock volatility (σ) changed dynamically according to the functions $0.0005t+0.1$ and $0.0005t + 0.05$ respectively for a duration of four months and β is taken to be 1.5. Figures 3.2 and 3.3 on the other hand are line plots of the value of the option against time for different values of β. Figure 3.2 is the time dependent case where r and σ are again taken to be $0.0005t + 0.1$ and $0.0005t + 0.05$ respectively while Figure 3.3 is the constant case scenario where r and σ are constants, with values 10% and 5% respectively, throughout the run (see [39] for the constant case scenario solution)

The diagrams show that the price of a power option increases as the parameter β increases. In other words, the power option magnifies the effects of the corresponding

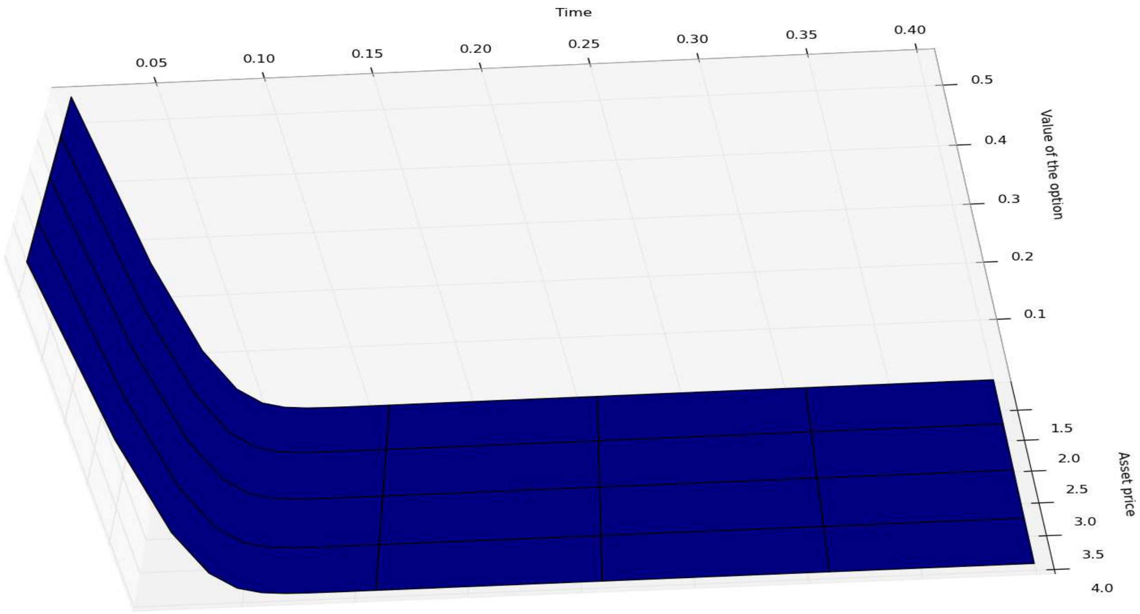

Figure 3.1: Plot of the power option pricing against the underlying asset price and time

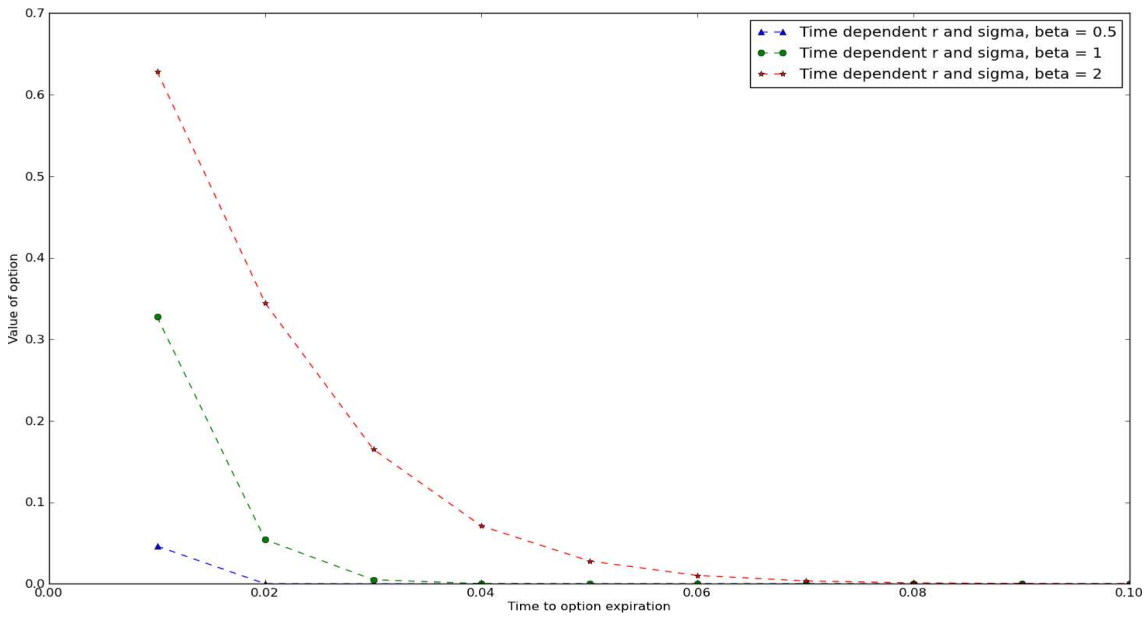

Figure 3.2: Plot of the power option pricing against time

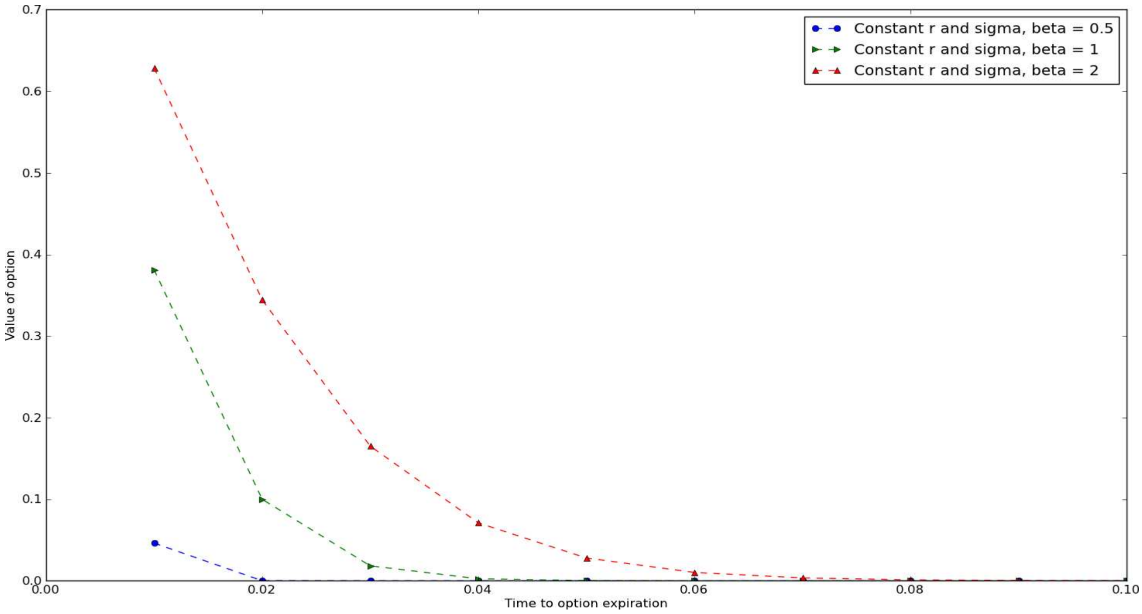

Figure 3.3: Plot of the power option pricing against time

vanilla option. In fact the original use of power options was to exploit this property. However, this led to magnified risk, especially when the structure of this exotic option was not well understood.

Alternately we can understand the power option in another manner. Where an underlying asset traditionally trades within a narrow band of prices, the corresponding option market is lacklustre and it would be impractical to run a derivatives market. The development of power options can create a derivatives market in such a case. Clearly the payoff becomes more attractive and yet the option price can still be reasonable. Of course, there is still the small possibility that the underlying asset price might experience a shock and the risk associated with the option could become intolerable. Although this possibility is slight it is still significant. However, by placing a barrier on the power option we can manage this extreme circumstance [33]. What we are proposing is the creation of a derivatives market on slow moving assets, where the risk can be managed in a reasonable way.

Chapter 4

Application of Lie group analysis to time dependent PDEs in Financial Mathematics

4.1 Introduction

In this work, we utilize the Lie group theoretic approach for the solution of DEs to the field of Financial Mathematics. This we do on the basis of the success it has been able to achieve over the years, particularly in areas where other techniques have failed.

Sophus Lie is credited with developing this general technique to the solution of DEs. During the plenary meeting of the Royal Saxon Society of the Sciences, Friedrich Engel [23] ranked him as one of the foremost mathematicians of all time, possessing the true measures of a great mathematician - capacity for discovery and original mathematical thinking. Twenty-five years later, Eduard Study [61] - one of Lie's former colleagues - described him as an autodidactic, but possessing the attributes of one of the most brilliant mathematicians who ever lived. His works have indeed outlived him. Their importance and relevance are still of note today.

Ironically, Lie algebras were an area Sophus Lie had little interest in initially pursuing. He instead hoped to develop the equivalent of Galois theory for DEs. To this end, together with Friedrich Engel, he completed the third and final volume of the massive treatise *Theorie der Transformationsgruppen* [36]. In the late nineteenth century, Lie made the profound and far reaching discovery that all special methods of solving DEs were in fact special cases of a general integration procedure based on the invariance of the DE under a continuous group of symmetries (Here, a symmetry refers to a group of transformations that transforms the set of all solutions of the differential equation to itself.). By 1884, he had obtained all of his principal results [35].

The applications of Lie groups have had a profound effect on all areas of mathematics and mathematically-based sciences [43,56]. As for his original idea of developing the equivalent theory of Galois theory to DEs, one researcher notes that 'the remarkable range of applications of Lie groups to DEs in geometry, in analysis, in physics, and in the engineering over the past 40 years has resurrected Lie's original vision into one of the most active and rewarding fields of contemporary research' [44].

Of recent, the Lie group method has been applied in the field of Financial Mathematics, yielding solutions but more importantly showing a systematic way of obtaining solutions to DEs in the field instead of the probabilistic or *ansatz* routes previously utilized. Examples of these can be seen in [12–14].

There has been recent interest in analyzing power options. This is one of the exotic options (i.e. derivatives which have more complex features than the commonly known products like the vanilla options). The emergence of these options was due to the increasing need for options which extends beyond the linear payoff offered by the Black-Scholes model [4]. Power options are derivatives whose nonlinear payoffs are characterized by the difference between the asset price (S) raised to a certain power and the strike price (K). The progression of work on this option has been as follows:

1. Obtaining closed form solutions for the power options when the power factor (β) is a certain integer [42, 63, 64].

2. Obtaining solutions to the power option model in a scenario whereby β is any real-valued number [9, 21, 22, 39].

3. We extended the results to a case of the volatility (σ) and the interest rate (r) changing with time for the terminal case [40].

Using the Brownian motion model,

$$\frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}W, \tag{4.1}$$

where W and μ is the Weiner process and expected return of the stock respectively, we can employ stochastic calculus [5] to derive the partial differential equation (PDE) representing the power option to be

$$\frac{\partial V}{\partial t}+\frac{1}{2}\beta^2\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\left[\beta r+\frac{1}{2}\beta\sigma^2(\beta-1)\right]S\frac{\partial V}{\partial S}-rV=0, \tag{4.2}$$

with payoff $V(S,T)=\max\{\phi\left(S^\beta-K\right),0\}$, where $t=T$ is the initial condition ($\phi=1$ or -1 is the call or put option respectively.).

This payoff can be interpreted as two conditions:

i) $S^\beta\leq K$ implies that $V(S,T)=0$, and

ii) $S^\beta>K$ implies that $V(S,T)=S^\beta-K$.

In this work, we attempt to go beyond the previous attempts by looking at a situation in which the power parameter (β) is changing with time. This yields the PDE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\beta(t)^2\sigma(t)^2S^2\frac{\partial^2 V}{\partial S^2}+\left[\beta(t)r(t)+\frac{1}{2}\beta(t)\sigma(t)^2(\beta(t)-1)\right]S\frac{\partial V}{\partial S}-r(t)V=0. \tag{4.3}$$

4.2 The methodology of the Lie group approach

4.2.1 Invariance of differential equations

Given a local group of transformations, we can define the symmetry groups of arbitrary subsets of a given manifold, in a general framework.

Definition 4.2.1. Let G be a local group of transformations acting on a manifold M. A subset $\mathfrak{S}\subset M$ is called G-invariant and G is called a symmetry group of $\mathfrak{S}$, if whenever $x\in\mathfrak{S}$, and $g\in G$ is such that $g\cdot x$ is defined, then $g\cdot x\in\mathfrak{S}$.

With Definition 4.2.1 in mind, a local group of transformations is called a symmetry of the system of DEs if the following definition holds.

Definition 4.2.2. Let $\mathfrak{S}$ be a system of DEs

$$\Gamma_v(x, u^{(n)}) = 0, \qquad v = 1, \ldots, l \tag{4.4}$$

where $x = x^1, \ldots, x^p$ are the independent variables and $u = u^1, \ldots, u^q$ are the dependent variables. A symmetry group, G, of the system of DEs is a local group of transformations acting on an open subset M of the space of X and U for the system, with the property that whenever $u = f(x)$ is a solution of $\mathfrak{S}$, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

The invariance of the subset $\mathfrak{S}$ as defined in Definition 4.2.1 for an n-th system of DEs is explained by the next theorem [43].

Theorem 4.2.1. *Let M be an open subset of $X \times U$ and suppose $\Gamma(x, u^{(n)}) = 0$ is an n-th order system of DEs defined over M, with a corresponding subvariety*[1] *$\mathfrak{S}_\Gamma \subset M^{(n)}$. Suppose G is a local group of transformations acting on M, whose prolongation leaves $\mathfrak{S}_\Gamma$ invariant. This means that whenever $(x, u^{(n)}) \in \mathfrak{S}_\Gamma$, we have*

$$pr^n g \cdot (x, u^{(n)}) \in \mathfrak{S}_\Gamma, \tag{4.5}$$

for all $g \in G$ such that this is defined. Then G is a symmetry group of the system of equations in the sense of the Definition 4.2.2.

[1] A subset of an algebraic variety which is itself a variety - where an algebraic variety is defined as a generalization to n dimensions of algebraic curves

4.2.2 Group invariant solutions of differential equations

An invariant solution of a differential equation (DE) is a solution of the DE which is also an invariant curve (surface) of a group admitted by the DE. To be more precise, consider the nth order system of DEs in equation (4.4). The functions Γ_v are assumed to depend smoothly on their arguments for x and u in some open set M of the total space $X \times U = \mathbb{R}^p \times \mathbb{R}^q$ of independent and dependent variables. Now, let G be a local group of transformations acting on $M \subset T \times Y$. H is said to a symmetry group of Γ if each element $h \in H$ transforms solutions of Γ to other solutions of Γ. A real valued function $\eta(t, y)$ is called an invariant of H if it is unchanged by the group action

$$\eta\left(h \cdot (t, y)\right) = \eta(t, y), \qquad \forall t, y \in M \tag{4.6}$$

and all $h \in H$ such that $h \cdot (t, y)$ is defined.

The new system of DEs (Γ/H) for the H-invariant solutions to Γ will involve just the new variables u, w formed from the invariants of H. On reduction, the original system, Γ, becomes

$$(\Gamma/H)_v\,(u, w^{(n)}) = 0, \qquad v = 1, \ldots, l \tag{4.7}$$

which has r $(r < p+q)$ fewer variables. This constitutes the new system Γ/H. Now, every solution $w = f(u)$, to equation (4.7), gives rise to an H-invariant solution $y = g(t)$ to Γ. This is determined implicitly from the definition of the new variables u and w. See [43, 46] for more literature on obtaining group invariant solutions via symmetry analysis.

4.3 Symmetry analysis of the Power option PDE

The software package, Program LIE [29], gives the general six-parameter symmetry of the PDE in equation (4.3) to be

$$\begin{aligned} G_1 &= \left[\frac{\dot{\beta(t)} f(t)\log(S)}{\beta(t)} + \frac{\dot{f}(t)\log(S)}{2} + \frac{f(t)\log(S)\dot{\sigma}(t)}{\sigma(t)} + g(t)\right] S\partial_S \\ &+ \frac{1}{2r(t)\beta(t) - \beta(t)\sigma(t)^2}\left[M(S,t) + (2r(t)\beta(t) - \beta(t)\sigma(t)^2)F(S,t)\right]\partial_V \\ &+ f(t)\partial_t, \end{aligned} \tag{4.8}$$

where the function $M(S,t)$ is given by

$$\begin{aligned} M(S,t) &= \left(2Vr(t)\beta(t) - V\beta(t)\sigma(t)^2\right)h(t) - 2V\log(S)\dot{h}(t) + 2Vr(t)\log(S)\dot{f}(t) \\ &+ 2V\dot{r}(t)\log(S)f(t) + \left\{2Vr(t)\beta(t)(\log(S))^2 - 2V\beta(t)^2\log(S)\sigma(t)^2 \right. \\ &\left. - V\beta(t)(\log(S))^2\sigma(t)^2\right\}\tau. \end{aligned}$$

The functions $f(t), g(t)$ and $h(t)$ are required to satisfy the system of equations

$$\begin{aligned} &2\beta(t)\sigma(t)^2\dot{\beta}(t)\dot{f}(t) + 2\beta(t)^2\dot{f}(t)\dot{\sigma}(t)\sigma(t) + 2\beta(t)^2\sigma(t)\ddot{\sigma}(t)f(t) + 2\beta(t)\sigma(t)^2 f(t)\ddot{\beta}(t) \\ &- 4\beta(t)^4\sigma(t)^4\tau - 2\sigma(t)^2 f(t)\dot{\beta}(t)^2 + \beta(t)^2\sigma(t)^2\ddot{f}(t) - 2\beta(t)^2\dot{\sigma}(t)^2 f(t) = 0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &-16r(t)\beta(t)^2 f(t)\dot{\sigma}(t)^2 + 12r(t)\beta(t)^2\sigma(t)\dot{f}(t)\dot{\sigma}(t) + 8r(t)\beta(t)^2\sigma(t)f(t)\ddot{\sigma}(t) \\ &+ 6r(t)\beta(t)\sigma(t)^2\dot{f}(t)\dot{\beta}(t) - 4r(t)\beta(t)\sigma(t)f(t)\dot{\sigma}(t)\dot{\beta}(t) + 4r(t)\beta(t)\sigma(t)^2 f(t)\ddot{\beta}(t) \\ &- 6\beta(t)^2\sigma(t)^2\dot{f}(t)\dot{r}(t) + 12\beta(t)^2\sigma(t)f(t)\dot{\sigma}(t)\dot{r}(t) - 4\beta(t)^2\sigma(t)^2 f(t)\ddot{r}(t) \\ &+ 4\beta(t)\sigma(t)^2 f(t)\dot{\beta}(t)\dot{r}(t) - 3\beta(t)\sigma(t)^4\dot{f}(t)\dot{\beta}(t) - 2\beta(t)\sigma(t)^3 f(t)\dot{\beta}(t)\dot{\sigma}(t) \\ &- 8\beta(t)\sigma(t)\dot{g}(t)\dot{\sigma}(t) + 4\beta(t)\sigma(t)^2\ddot{g}(t) - 8\sigma(t)^2\dot{\beta}(t)\dot{g}(t) - 4r(t)\sigma(t)^2 f(t)\dot{\beta}(t)^2 \\ &- 2\beta(t)\sigma(t)^4 f(t)\ddot{\beta}(t) + 2\sigma(t)^4 f(t)\dot{\beta}(t)^2 = 0, \end{aligned} \tag{4.10}$$

$$-8r(t)\beta(t)\sigma(t)f(t)\dot{r}(t) - 4r(t)\beta(t)\sigma(t)^3\dot{f}(t) + 8r(t)\sigma(t)\dot{g}(t) - 4r(t)^2\beta(t)\sigma(t)\dot{f}(t)$$
$$+ 8\beta(t)^3\sigma(t)^5\tau - 4\beta(t)\sigma(t)^3 f(t)\dot{r}(t) - \beta(t)\sigma(t)^5\dot{f}(t) + 8\beta(t)\sigma(t)^3\dot{h}(t)$$
$$- 4\sigma(t)^3\dot{g}(t) + 8r(t)^2\beta(t)f(t)\dot{\sigma}(t) - 2\beta(t)\sigma(t)^4 f(t)\dot{\sigma}(t) = 0. \tag{4.11}$$

The function $F(S,t)$ is the solution of the equation and is henceforth not included in further analysis. We now proceed to obtain an explicit form of the general symmetry since none of the software packages were able to do so. Considering equations (4.9)–(4.11) in turn, we can rewrite equation (4.9) as

$$\ddot{f}(t) + 2\left(\frac{\dot{\sigma}(t)}{\sigma(t)} + \frac{\dot{\beta}(t)}{\beta(t)}\right)\dot{f}(t) + 2\frac{d}{dt}\left(\frac{\dot{\sigma}(t)}{\sigma(t)} + \frac{\dot{\beta}(t)}{\beta(t)}\right)f(t)$$
$$- 4\beta(t)^2\sigma(t)^2\tau = 0. \tag{4.12}$$

This shows that it is an equation of the form

$$\ddot{f}(t) + \phi(t)\dot{f}(t) + \dot{\phi}(t)f(t) + \varphi(t) = 0. \tag{4.13}$$

We solve this to obtain

$$f(t) = \frac{1}{(\sigma(t)\beta(t))^2}\left[f_2 + \int(\sigma(t)\beta(t))^2\mathrm{dt}\left(f_1 + 2\tau\int(\sigma(t)\beta(t))^2\mathrm{dt}\right)\right]. \tag{4.14}$$

Equation (4.10) on another hand can be reconstructed to be

$$\ddot{g}(t) - 2\left(\frac{\dot{\sigma}(t)}{\sigma(t)} + \frac{\dot{\beta}(t)}{\beta(t)}\right)\dot{g}(t) + \left(-\frac{4r(t)\beta(t)\dot{\sigma}(t)^2}{\sigma(t)^2} + \frac{2r(t)\beta(t)\ddot{\sigma}(t)}{\sigma(t)} - \frac{r(t)\dot{\beta}(t)^2}{\beta(t)}\right.$$
$$- r(t)\dot{\sigma}(t)\dot{\beta}(t) + r(t)\beta(t)\ddot{\beta}(t) + \frac{\sigma(t)^2\dot{\beta}(t)^2}{2\beta(t)} + \dot{\beta}(t)\dot{r}(t) - \frac{\sigma(t)\dot{\beta}(t)\dot{\sigma}(t)}{2} - \frac{\sigma(t)^2\ddot{\beta}(t)}{2}$$
$$\left.+ \frac{3\beta(t)\dot{\sigma}(t)\dot{r}(t)}{\sigma(t)} - \beta(t)\ddot{r}(t)\right)f(t) + \left(\frac{3r(t)\dot{\beta}(t)}{2} - \frac{3\beta(t)\dot{r}(t)}{2} + \frac{3r(t)\beta(t)\dot{\sigma}(t)}{\sigma(t)}\right.$$
$$\left.- \frac{3\sigma(t)^2\dot{\beta}(t)}{4}\right)\dot{f}(t) = 0, \tag{4.15}$$

and this is an equation of the form

$$\ddot{g}(t)+\theta(t)\dot{g}(t)+\zeta(t)=0. \tag{4.16}$$

This has a solution given by

$$g(t)=\int(\sigma(t)\beta(t))^2\left(\int\frac{-\zeta(t)}{(\sigma(t)\beta(t))^2}\mathrm{dt}+g_1\right)\mathrm{dt}+g_2, \tag{4.17}$$

where

$$\begin{aligned}\zeta(t)=&\Bigg(-\frac{4r(t)\beta(t)\dot{\sigma}(t)^2}{\sigma(t)^2}+\frac{2r(t)\beta(t)\ddot{\sigma}(t)}{\sigma(t)}-\frac{r(t)\dot{\beta}(t)^2}{\beta(t)}-\frac{r(t)\dot{\sigma}(t)\dot{\beta}(t)}{}+r(t)\beta(t)\ddot{\beta}(t)\\&+\frac{\sigma(t)^2\dot{\beta}(t)^2}{2\beta(t)}+\dot{\beta}(t)\dot{r}(t)-\frac{\sigma(t)\dot{\beta}(t)\dot{\sigma}(t)}{2}-\frac{\sigma(t)^2\ddot{\beta}(t)}{2}+\frac{3\beta(t)\dot{\sigma}(t)\dot{r}(t)}{\sigma(t)}-\beta(t)\ddot{r}(t)\Bigg)\\&\left(\frac{1}{(\sigma(t)\beta(t))^2}\left[a_2+\int(\sigma(t)\beta(t))^2\mathrm{dt}\left(a_1+2\tau\int(\sigma(t)\beta(t))^2\mathrm{dt}\right)\right]\right)+\Bigg(\frac{3r(t)\dot{\beta}(t)}{2}\\&-\frac{3\beta(t)r\dot{(t)}(t)}{2}+\frac{3r(t)\beta(t)\dot{\sigma}(t)}{\sigma(t)}-\frac{3\sigma(t)^2\dot{\beta}(t)}{4}\Bigg)\Bigg(2\tau\int\left(\sigma(t)\beta(t)^2\right)^2\mathrm{d}t\\&-\frac{2\left[a_2+\int\left(\sigma(t)\beta(t)^2\right)^2\mathrm{d}t\left\{a_1+2\tau\int\left(\sigma(t)\beta(t)^2\right)^2\mathrm{d}t\right\}\right]}{\left(\sigma(t)\beta(t)^2\right)^2}\left[\frac{\dot{\beta}(t)}{\beta(t)}+\frac{\dot{\sigma}(t)}{\sigma(t)}\right]\Bigg).\end{aligned}$$

The third equation in the system is given by

$$\begin{aligned}&\dot{h}(t)+\left(-\frac{r(t)\dot{r}(t)}{\sigma(t)^2}+\frac{r(t)^2\dot{\sigma}(t)}{\sigma(t)^3}-\frac{\dot{r}(t)}{2}-\frac{\sigma(t)\dot{\sigma}(t)}{4}\right)f(t)\\&+\left(\frac{r(t)}{\beta(t)\sigma(t)^2}-\frac{1}{2\beta(t)}\right)\dot{g}(t)-\left(\frac{r(t)}{2}+\frac{r(t)^2}{2\sigma(t)^2}+\frac{\sigma(t)^2}{8}\right)\dot{f}(t)\\&+(\beta(t)\sigma(t))^2\tau=0.\end{aligned} \tag{4.18}$$

This is of the simple form

$$\dot{h}(t)+\gamma(t)=0, \tag{4.19}$$

and has straight forward solution

$$h(t) = \int -\gamma(t)\,\mathrm{dt} + h_1, \tag{4.20}$$

where

$$\begin{aligned}
\gamma(t) \;=\; & \frac{1}{(\sigma(t)\beta(t))^2}\left(-\frac{r(t)\dot{r}(t)}{\sigma(t)^2} + \frac{r(t)^2\dot{\sigma}(t)}{\sigma(t)^3} - \frac{\dot{r}(t)}{2} - \frac{\sigma(t)\dot{\sigma}(t)}{4}\right)\left[a_2 + \int(\sigma(t)\beta(t))^2\mathrm{dt}\right.\\
& \left.\left(a_1 + 2\tau\int(\sigma(t)\beta(t))^2\mathrm{dt}\right)\right] - \left(\frac{r(t)}{2} + \frac{r(t)^2}{2\sigma(t)^2} + \frac{\sigma(t)^2}{8}\right)\left(2\tau\int\left(\sigma(t)\beta(t)^2\right)^2\mathrm{d}t\right.\\
& \left. - \frac{2\left[a_2 + \int\left(\sigma(t)\beta(t)^2\right)^2\mathrm{d}t\left\{a_1 + 2\tau\int\left(\sigma(t)\beta(t)^2\right)^2\mathrm{d}t\right\}\right]}{\left(\sigma(t)\beta(t)^2\right)^2}\left[\frac{\dot{\beta}(t)}{\beta(t)} + \frac{\dot{\sigma}(t)}{\sigma(t)}\right]\right)\\
& + \left(\frac{r(t)}{\beta(t)\sigma(t)^2} - \frac{1}{2\beta(t)}\right)\left((\sigma(t)\beta(t))^2\left(\int\frac{-\zeta(t)}{(\sigma(t)\beta(t))^2}\mathrm{dt} + b_1\right)\right)\\
& + (\beta(t)\sigma(t))^2\tau,
\end{aligned} \tag{4.21}$$

where ζ is as defined in equation (4.17).

4.4 Application of the initial conditions

The payoff $V(S,T) = 0$ is actually two conditions. The first condition is that $t = T$ and the second condition is that $V = 0$ when $t = T$. Thus, the general symmetry in equation (4.8) is required to be consistent with the dual conditions $V(S,T) = 0$ and $t = T$. For this to be the case, we will have that

$$f(T) = 0 \qquad \text{and} \qquad M(S,T) = 0. \tag{4.22}$$

At the initial, the additional constraints from the separation of $M(S,T)$ by coefficients of powers of $\log(S)$ are

$$(2Vr(T)\beta(T) - V\beta(T)\sigma(T)^2)h(T) = 0, \tag{4.23}$$

$$(2Vr(T)\beta(T) - V\beta(T)\sigma(T)^2)\tau = 0, \tag{4.24}$$

$$2Vr(T)\dot{f}(T) - 2V\dot{h}(T) + 2V\dot{r}(T)f(T) - 2V\beta(T)^2\sigma(T)^2\tau = 0. \tag{4.25}$$

To solve the system of equations (4.23)–(4.24), we shall proceed from equation (4.23), where we consider the case of $h(T) = 0$. This will imply that τ will also be equal to zero in equation (4.24). Equation (4.25) is now

$$r(t)\dot{f}(t) + \dot{r}(t)f(t) = 0. \tag{4.26}$$

This has solution

$$f(T) = \frac{\chi_2}{r(t)}. \tag{4.27}$$

Recall though that $f(T) = 0$ from equation (4.22). This requirement now makes the arbitrary constant χ_2 to equal zero. The general symmetry invariant under the initial condition will now be

$$\begin{aligned} G_2 \;=\; & f(t)\partial_t + \left[\frac{\dot{\beta}(t)f(t)\log(S)}{\beta(t)} + \frac{\dot{f}(t)\log(S)}{2} + \frac{f(t)\log(S)\dot{\sigma}(t)}{\sigma(t)} + g(t)\right] S\partial_S + \\ & \frac{1}{2r(t)\beta(t) - \beta(t)\sigma(t)^2}\Big[\left(2r(t)\beta(t) - \beta(t)\sigma(t)^2\right)h(t) - 2\log(S)\dot{h}(t) \\ & + 2r(t)\log(S)\dot{f}(t) + 2\dot{r}(t)\log(S)f(t)\Big] V\partial_V, \end{aligned} \tag{4.28}$$

Another route we could have pursued from equation (4.23) could be that $\sigma(T)^2 = 2r(T)$. We do not pursue this because it causes the general symmetry to diverge.

4.5 The associated Lagrange's system and solution obtained

We continue our analysis with the 5-parameter symmetry in equation (4.28) since this is already a linear combination of all the optimal symmetries that can reduce the PDE and it is in itself a symmetry. The Lagrange system will now be given as

$$\frac{\mathrm{d}t}{f(t)} = \frac{\mathrm{d}S}{\left[\frac{\dot{\beta}(t)f(t)\log(S)}{\beta(t)} + \frac{\dot{f}(t)\log(S)}{2} + \frac{f(t)\log(S)\dot{\sigma}(t)}{\sigma(t)} + g(t)\right] S} =$$
$$\frac{\left(2r(t)\beta(t) - \beta(t)\sigma(t)^2\right)\mathrm{d}V}{\left[\left(2r(t)\beta(t) - \beta(t)\sigma(t)^2\right)h(t) - 2\log(S)\dot{h}(t) + 2r(t)\log(S)\dot{f}(t) + 2\dot{r}(t)\log(S)f(t)\right] V}. \tag{4.29}$$

This system though is not easily resolved so we eliminate two of the symmetries by setting $f_1 = f_2 = 0$, thus leaving us with a non-trivial 3-parameter symmetry. This new 3-parameter symmetry has the associated Lagrange system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}S}{Sg(t)} = \frac{\mathrm{d}V}{\left[h(t) - \frac{2\log(S)\dot{h}(t)}{2r(t)\beta(t)-\beta(t)\sigma(t)^2}\right] V} \tag{4.30}$$

with new invariants

$$u = t, \tag{4.31}$$
$$Z(u) = \log(V) - \frac{2h_1 + g_1\int \beta(u)\{\sigma(u)^2 - 2r(u)\}\mathrm{d}u + 2g_1\log(S)}{S\left(2g_2 + 2g_1\int \sigma(u)^2\beta(u)^2\mathrm{d}u\right)}. \tag{4.32}$$

We can write

$$V = Z(u)\exp\left(\frac{\log(S)\left(2h_1 + \log(S) + g_1\left[\int \beta(u)\{\sigma(u)^2 - 2r(u)\}\mathrm{d}u\right]\right)}{2\left(g_2 + g_1\int \sigma(u)^2\beta(u)^2\mathrm{d}u\right)}\right). \tag{4.33}$$

We reduce the PDE in equation (4.3) via (4.31)–(4.32). This yields the ODE

$$\begin{aligned}
&\left(g_1 \int \beta(u)^2\sigma(u)^2\mathrm{d}u + g_2\right)\left(2h_1 + g_1\left[\log(S) + \int \beta(u)\{\sigma(u)^2 - 2r(u)\}\mathrm{d}u\right]\right)\\
&(-2\log(S)\left[r(u)Z(u) - Z'(u)\right]) + 2g_1\beta(u)^2\sigma(u)^2 Z(u)\,(g_2 - 2h_1\log(S)\\
&- g_1\log(S)\left[\int \beta(u)\{\sigma(u)^2 - 2r(u)\}\mathrm{d}u + \log(S)\right] + g_1\int \beta(u)^2\sigma(u)^2\mathrm{d}u\Big)\\
&+ \beta(u)\left(2r(u) - \sigma(u)^2\right)Z(u)\left(2h_1 + g_1\int \beta(u)\{\sigma(u)^2 - 2r(u)\}\mathrm{d}u\right)\\
&\left(g_2 + g_1\int \beta(u)^2\sigma(u)^2\mathrm{d}u\right) = 0, \qquad (4.34)
\end{aligned}$$

which has solution

$$\begin{aligned}
Z(u) \;=\; & \Phi\exp\Bigg[\int\Bigg\{2g_1\beta(u)^2\sigma(u)^2\left(g_1\log(S)\left[\log(S) + \int \beta(u)\{\sigma(u)^2 - 2r(u)\}\mathrm{d}u\right] - g_2\right.\\
& \left. + 2h_1\log(S) - g_1\int \beta(u)^2\sigma(u)^2\mathrm{d}u\right) + 2r(u)\log(S)\left(g_2 + g_1\int \beta(u)^2\sigma(u)^2\mathrm{d}u\right)\\
& \left(2h_1 + g_1\left[\log(S) + \int \beta(u)\{\sigma(u)^2 - 2r(u)\}\mathrm{d}u\right]\right) - \beta(u)\left(2r(u) - \sigma(u)^2\right)\\
& \left(g_2 + g_1\int \beta(u)^2\sigma(u)^2\mathrm{d}u\right)\left(g_1\int \beta(u)\{\sigma(u)^2 - 2r(u)\}\mathrm{d}u + 2h_1\right)\Bigg\}\Bigg/\\
& \Bigg\{2\log(S)\left(g_1\int \beta(u)^2\sigma(u)^2\mathrm{d}u + g_2\right)\left(\left[\int \beta(u)(\sigma(u)^2 - 2r(u))\mathrm{d}u + \log(S)\right]\right.\\
& \left.+ 2h_1\right)\Bigg\}\,\mathrm{d}u\Bigg], \qquad (4.35)
\end{aligned}$$

where Φ is a constant of integration.

The general solution which is invariant under the initial condition is now obtained, via the transformation of the new invariants in equations (4.31)–(4.32), to be

$$\begin{aligned}
V &= \Phi \exp\Bigg[\int \Bigg\{ 2g_1\beta(t)^2\sigma(t)^2 \left(g_1 \log(S) \left[\log(S) + \int \beta(t)\{\sigma(t)^2 - 2r(t)\}\mathrm{d}t\right] - g_2 \right. \\
&\quad \left. + 2h_1 \log(S) - g_1 \int \beta(t)^2\sigma(t)^2\mathrm{d}t \right) + 2r(t)\log(S) \left(g_2 + g_1 \int \beta(t)^2\sigma(t)^2\mathrm{d}t \right) \\
&\quad \left(2h_1 + g_1 \left[\log(S) + \int \beta(t)\{\sigma(t)^2 - 2r(t)\}\mathrm{d}t\right] \right) - \beta(t)\left(2r(t) - \sigma(t)^2\right) \\
&\quad \left(g_2 + g_1 \int \beta(t)^2\sigma(t)^2\mathrm{d}t \right) \left(g_1 \int \beta(t)\{\sigma(t)^2 - 2r(t)\}\mathrm{d}t + 2h_1 \right) \Bigg\} \Big/ \\
&\quad \Bigg\{ 2\log(S) \left(g_1 \int \beta(t)^2\sigma(t)^2\mathrm{d}t + g_2 \right) \left(\left[\int \beta(t)(\sigma(t)^2 - 2r(t))\mathrm{d}t + \log(S) \right] + 2h_1 \right) \Bigg\} \, dt \\
&\quad + \frac{\log(S)\left(2h_1 + \log(S) + g_1\left[\int \beta(t)\{\sigma(t)^2 - 2r(t)\}\mathrm{d}t\right]\right)}{2\left(g_2 + g_1 \int \sigma(t)^2\beta(t)^2\mathrm{d}t\right)} \Bigg]. \qquad (4.36)
\end{aligned}$$

On the other hand, the application of the initial condition $t = T$ when the payoff is given by $V(S,T) = S^\beta - K$ yielded no symmetry so we were unable to pursue this case any further.

4.6 Discussion

In this work, via the Lie group technique, we have been able to obtain an exact solution to the power option PDE. This work is an extension of previous work [40] where we considered the power option PDE with time dependent interest rate and stock volatility. In this work, we have also considered the power factor (β) as a time dependent parameter. This inclusion made the problem more challenging due to the increase in complexities of the determining equations to be solved. Reconstructing these equations though provided a path to their solution.

Several attempts have been made on Lie group analysis of time dependent financial derivatives which have the terminal condition imposed on them [58, 59]. These attempts have had difficulties going past the stage of solving the determining equations that the software packages produce (despite simplifying assumptions in some cases). In this work though, we have been able to reconstruct and solve these determining equations, enabling us to obtain the explicit nature of the six-parameter symmetry of the time dependent power option PDE. On obtaining this symmetry, we now imposed the terminal condition on it so as to obtain a symmetry invariant under the initial condition. We proceeded to construct the Lagrange system of this symmetry, which we utilized in the reduction of the PDE. The reduced PDE gave rise to an ODE which we solved before proceeding to transform the solution via the transformations earlier defined.

Indeed the advent of the Lie group technique in Financial Mathematics is further enhanced by this work, by showing not only an alternative method for problem solving but an algorithmic one which can also be applied to realistic time dependent financial derivatives.

Chapter 5

Conclusion

5.1 Summary

In this work, we have successfully employed the tool of Lie symmetry methods in the understanding of models in Financial Mathematics. We focused mainly on power options, which is a type of the exotic options.

The first chapter highlighted the use of DEs as a tool for modeling real life situations. We then went on to discuss the field of Financial Mathematics and the history of the problem solving approach (the Lie group technique) which we employed. A detailed explanation of the technique then followed before the definitions and theorems relevant to the field of Financial Mathematics were presented. We rounded up the chapter by discussing the major highlight of this work - exotic options. We discussed its emergence and the metamorphosis of an example of it (the power option) to date.

In chapter two, we dealt with the case of the power option PDE having constant model parameters. We were able to show a systematic way, through the Lie group approach, to obtaining its solutions. On splitting the payoff into two conditions, we obtained the following results which also satisfied the initial conditions:

1. For the case of $S^\beta \leq K$, we obtained a number of trivial solutions which are presented in Tables 2.4 and 2.5. Extending these solutions further to the notion of global transformations, we were able to obtain a non-trivial solution. This is given by

$$\begin{aligned} V &= \sqrt{\frac{t-T}{8(t-T)-1}}\left([\log(S)-\exp(8(t-T))]\frac{t-T}{8(t-T)-1}\right) \\ &\exp\left([\log(S)-\exp(8(t-T))]\frac{(\sigma^2-2r)}{2\sigma^2\beta}+\left[\frac{9}{2\sigma^2\beta^2}\right][\log(S)-\exp(8(t-T))]^2\right. \\ &+\left[\frac{(2r+\sigma^2)}{8\sigma^2}-4\right]\left[\frac{t-T}{8(t-T)-1}\right]+\left[4r+\frac{4r^2}{\sigma^2}+\sigma^2\right]\left[\frac{t-T}{8(t-T)-1}\right]^2 \\ &\left.+\left[\frac{4}{\beta}-\frac{8r}{\sigma^2\beta}\right]\left[\frac{t-T}{8(t-T)-1}\right][\log(S)-\exp(8(t-T))]\right), \end{aligned} \tag{5.1}$$

   and is the first such solution obtained.

2. For the case of $S^\beta > K$, we obtained a new analytic solution for the asymmetric power option. This is given by

$$V(S,t) = S^{\frac{\sigma^2(1-\beta M)-2r}{2\beta\sigma^2}}\left(S^M+\Phi_1\right)\exp\left(Z[\Phi_2+t]\right), \tag{5.2}$$

   where $M = \sqrt{\frac{(2r+\sigma^2)^2-8Z\sigma^2}{\beta^2\sigma^4}}$, $Z = \left(\frac{r}{4}+\frac{r^2}{4\sigma^2}+\frac{\sigma^2}{16}\right) - \frac{1}{(\sigma^4\beta^4)\left(\frac{r}{4}+\frac{r^2}{4\sigma^2}+\frac{\sigma^2}{16}\right)}$ and Φ_1, Φ_2 are arbitrary constants of integration. The arbitrary constants now take on values

$$\Phi_1 = -K \qquad \text{and} \qquad \Phi_2 = -T, \tag{5.3}$$

with a constraint

$$\beta = \sqrt{1 - \frac{2r}{\sigma^2}}, \tag{5.4}$$

on the parameter β. While we do not have a constraint on the parameters, this is again, the first time, that such a solution has been determined.

Chapter three dealt with the power option PDE having time dependent volatility and interest rate. We were able to obtain the new analytic solution

$$\begin{aligned} V \;\; = \;\; & \Phi \exp\Bigg(\int \Bigg(\Bigg[4r(t)\left\{b_1 \int \sigma(t)^2 \mathrm{d}t + b_2\right\}\left\{-b_1 \int \left[\sigma(t)^2 - 2r(t)\right]\mathrm{d}t\right. \\ & \left. + 2b_1 \int \sigma(t)^2 \mathrm{d}t - 2b_1 c_1 + 2b_2\right\} - b_1 \sigma(t)^2 \left\{b_1 \int \left[\sigma(t)^2 - 2r(t)\right]\mathrm{d}t\right. \\ & + 4\left[(b_1 - b_1 c_1)\int \sigma(t)^2 \mathrm{d}t + b_1 c_1^2 - b_2 c_1 + b_2\right] \\ & \left. -2\int \left[\sigma(t)^2 - 2r(t)\right]\mathrm{d}t\left[b_1 \int \sigma(t)^2 \mathrm{d}t - 2b_1 c_1 + b_2\right]\right\}\Bigg]\Bigg/ \\ & \left[8\left\{b_1 \int \sigma(t)^2 \mathrm{d}t + b_2\right\}^2\right]\Bigg)\mathrm{d}t + \\ & + \frac{\log(S) b_1 \left(2\beta c_1 + \beta \int \left[\frac{\sigma(t)^2}{2} - r(t)\right]\mathrm{d}t + \log(S)\right)}{2\beta^2 \left(b_2 + b_1 \int \sigma(t)^2 \mathrm{d}t\right)}\Bigg), \end{aligned} \tag{5.5}$$

satisfying the initial condition to the terminal payoff case.

In chapter four, we extended the results in the previous chapter to a scenario where the power factor (β) is now a time dependent parameter, alongside the stock volatility and the interest rate. We sought and obtained a new solution for this case which satisfied the initial condition for the terminal payoff case. This solution is given by

$$\begin{aligned}V \;=\; & \Phi\exp\Bigg[\int\Bigg\{2g_1\beta(t)^2\sigma(t)^2\Bigg(g_1\log(S)\left[\log(S)+\int\beta(t)\{\sigma(t)^2-2r(t)\}\mathrm{d}t\right]-g_2\\ & +2h_1\log(S)-g_1\int\beta(t)^2\sigma(t)^2\mathrm{d}t\Bigg)+2r(t)\log(S)\left(g_2+g_1\int\beta(t)^2\sigma(t)^2\mathrm{d}t\right)\\ & \left(2h_1+g_1\left[\log(S)+\int\beta(t)\{\sigma(t)^2-2r(t)\}\mathrm{d}t\right]\right)-\beta(t)\left(2r(t)-\sigma(t)^2\right)\\ & \left(g_2+g_1\int\beta(t)^2\sigma(t)^2\mathrm{d}t\right)\left(g_1\int\beta(t)\{\sigma(t)^2-2r(t)\}\mathrm{d}t+2h_1\right)\Bigg\}\Bigg/\\ & \left\{2\log(S)\left(g_1\int\beta(t)^2\sigma(t)^2\mathrm{d}t+g_2\right)\left(\left[\int\beta(t)(\sigma(t)^2-2r(t))\mathrm{d}t+\log(S)\right]+2h_1\right)\right\}\mathrm{d}t\\ & +\frac{\log(S)\left(2h_1+\log(S)+g_1\left[\int\beta(t)\{\sigma(t)^2-2r(t)\}\mathrm{d}t\right]\right)}{2\left(g_2+g_1\int\sigma(t)^2\beta(t)^2\mathrm{d}t\right)}\Bigg].\end{aligned} \tag{5.6}$$

5.2 Significance of the results

The new results obtained in this work are of particular significance in the following ways:

- These solutions to the power option have provided a reasonable theoretical framework for further practical implementation of the power option model.
- Obtaining solutions to the realistic scenarios of time dependence parameters will go a long way in aiding the creation and implementation of realistic financial models in Financial Mathematics.
- We have shown a systematic and step by step approach to solving exotic options DEs in Financial Mathematics; as opposed to the probabilistic or *ansatz* route.

- Obtaining new solutions to the time dependent exotic option PDE via this approach has opened new avenues to solving time dependent DEs in Financial Mathematics.

- The possible application of the power parameter as just a real number or a time dependent value, now offers risk hedging traders more options.

- In some cases that the various software packages for obtaining symmetries have failed in solving the determining equations that follow. The reconstruction of these equations done in this work gives more insight into solving them and it is hoped that the developers of these packages can integrate these technique into their packages.

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