AN APPLICATION OF MODERN ANALYTICAL SOLUTION TECHNIQUES TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Many physics and engineering problems are modeled by differential equations. In many instances these equations are nonlinear and exact solutions are difficult to obtain. Numerical schemes are often used to find approximate solutions. However, numerical solutions do not describe the qualitative behaviour of mechanical systems and are insufficient in determining the general properties of certain systems of equations. The need for analytical methods is self-evident and major developments were seen in the 1990's. With the aid of faster processing equipment today, we are able to compute analytical solutions to highly nonlinear equations that are more accurate than numerical solutions.

In this study we discuss solutions to nonlinear partial differential equations with focus on non-perturbation analytical methods. The non-perturbation methods of choice are the homotopy analysis method (HAM) developed by Shijun Liao and the variational iteration method (VIM) developed by Ji-Huan He. The aim is to compare

the solutions obtained by these modern day analytical methods against each other focusing on accuracy, convergence and computational efficiency.

The methods were applied to three test problems, namely, the heat equation, Burgers equation and the Bratu equation. The solutions were compared against both the exact results as well as solutions generated using the finite difference method, in some cases. The results obtained show that the HAM successfully produces solutions which are accurate, faster converging and requires less computational resources than the VIM. However, the VIM still provides accurate solutions that are also in good agreement with the closed form solutions of the test problems. The FDM also produced good results which were used as a further comparison to the analytical solutions. The findings of this study is in agreement with those published in the literature.

Declaration

The dissertation is my original work except where due reference and credit is given. The study has not been submitted for any qualification of this, or any other university or institution.

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Dedication

To my loving wife, Ashlene and amazing son, Arshan J.

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Chapter 1

Introduction

Many physics and engineering problems are modeled by partial differential equations. In many instances these equations are nonlinear and exact solutions are difficult to obtain. Numerical methods were developed over a period of time in order to find approximate solutions to these nonlinear equations. However, numerical solutions are insufficient to determine general properties of certain systems of equations and thus analytical and semi-analytical methods have been developed. These methods have transformed numerical analysis and we are now able to provide both qualitative and quantitative analysis to complex mathematical problems.

In this study we discuss solutions to nonlinear partial differential equations with focus on non-perturbation analytical methods. The aim is to compare these modern day analytical methods against each other focusing on accuracy, convergence and computational efficiency.

1.1 Early methods of solution

We begin the study by reviewing some popular approaches to solving ordinary and partial differential equations. The earliest attempts, dating back to the 1930's, began with purely numerical schemes based on discretization of the independent and dependent variables in the original equation. The three most popular numerical discretization schemes, the finite difference, finite element and finite volume method, are discussed briefly below.

1.1.1 The method of finite differences

The method of finite differences (FDM) is one of the oldest numerical schemes that has been used to solve a variety of differential equations. The method is based on discretizing derivatives using finite difference approximations. The method dates back to the early 1930's, Thomee [99], where it was used to determine solutions to Dirichlet problems and the biharmonic equation.

The scheme is based on discretization of the independent variable(s), such as the space x (and/or the time t), using a step-width h (and/or k). The problem is then solved using a set of grid points derived using the step width's above.

The original equation is then replaced by finite difference approximations which are derived from Taylor's theorem, LeVeque [70]. Depending on the order of the equation, the Taylor approximation of the same order is obtained and this creates an iterative scheme which generates a sequence of solutions at each point on the grid.

There are three main versions of the FDM, namely;

1. The explicit FDM,
2. The implicit FDM and
3. The Crank-Nicholson scheme.

The main difference between the methods is the manner in which the Taylor series is expanded with respect to the independent variable. The explicit method uses a forward time difference approximation while the implicit method uses a backward time difference and the Crank-Nicholson scheme uses a central time difference approximation. The delicate balance between the different methods is a trade off between the speed of convergence of the solution against the ease of implementation. The fastest converging and hardest to implement scheme is the Crank-Nicholson scheme while the explicit method is slower converging but is fairly easier to implement.

The main disadvantage of the FDM method arises from the errors at each step due to discretization of the independent variables. This has been shown to cause divergence in complex problems. However, the FDM has been shown to be relatively easy to implement and produces accurate results if the step width's are chosen appropriately. The convergence of the explicit FDM is discussed further in Chapter 2.

The FDM is a widely used method and provides a starting point for the numerical solution of many problems in applied mathematics and physics. Some popular problems which have been solved using the FDM include; the Helmholtz equation, Wong and Li [108], the heat equation, Recktenwald [95] and the Schrödinger equation, Kurtinaitis and Ivanauskas [68]. The ease of implementation and the history behind the method is the reason it has been chosen as a test method in this dissertation. It also provides direction and validation of solutions for problems where analytical solutions to do not exist.

1.1.2 The finite element method

The finite element method (FEM) was formulated by Courant [37] in 1943. The method did not receive notable attention until the 1950's when it appeared in engineering literature and thereafter found its way back into mathematics in the 1960's with important advances in results obtained by Zlámal [111].

The main difference between the FDM and the FEM is in the discretization of the domain. The FDM uses a square network of lines (grid) upon which the differential equation is discretized upon, while the FEM uses different geometric shapes, termed finite elements. Therefore, this method has the ability to solve problems with complex geometries and boundary conditions.

The two common formulations of the FEM are the Galerkin formulation and the Ritz formulation. The Galerkin formulation approximates variables using continuous piecewise functions inside the element. The approximations generates a residual when applied to the original equation. In order to reduce the residual to zero, which provides the true solution, the weighted residual is set to zero and solved for the approximate solution. In the Ritz formulation, the original problem is converted into integral form by applying calculus of variation. Thereafter, the approximate solution is determined by substitution into the integral equation and then extremized using partial derivatives.

The Galerkin method has been used to solve popular problems such as the Maxwell equations, Cohen et al. [34], Burgers equation, Dogan [40] and the two-dimensional Helmholtz equation, Thompson and Pinsky [100]. The Ritz formulation has been used to discretize problems involving fracture and delamination in solids, Chowdhury and Narasimhan [33] as well as the approximation of the Navier-Stokes equation, Boncut [26].

As with most discretization schemes, there are drawbacks when using the FEM. The method suffers from low accuracy in problems with complex geometries. The choice of the ideal domain for a given problem is not always apparent which may cause divergence in the solutions. The method is also harder to implement as compared to the FDM, which uses simple difference equations.

1.1.3 The finite volume method

The finite volume method (FVM) is a discretization scheme, similar in principle to the FEM and FDM, and has been used to solve various conservation laws in fluid mechanics. The method dates back to the 1980's where it was used to solve the two-dimensional Euler equation, Jameson et al. [60, 61].

The idea behind the method is to discretize the domain into grid cells of adjacent control volumes. Using conservation laws, the partial differential equation can be converted to an integral equation. The integrals are then evaluated in each cell and the approximate cell average, integral divided by the volume of the cell, is determined, LeVeque [71]. These averages are then interpolated which results in an equation which provides an approximate solution to the problem.

The FVM can be used to solve problems with complex geometries and has been used to solve popular equations such as; the Euler equation, Uygun and

Kirkköprü [101], and convection-diffusion problems, Shukla et al [97]. There are challenges in higher dimensions due to the method requiring three levels of approximation, interpolation, integration and differentiation. Due to the complexity of the algorithm, the FVM will not be used to generate any numeric solutions to the problems in this study.

There are several other improved discretization schemes in the literature which have been developed over the years in order to enhance solutions to problems. However, the above three methods laid the foundation and changed numerical analysis as we know it. Nonetheless, numerical solutions do not tell us much about the qualitative behaviour of systems and the need to obtain analytical solutions remains. The first attempts at analytical solutions were to apply perturbation techniques to obtain approximate analytical solutions and is discussed below.

1.1.4 Perturbation methods

A traditional approach used by mathematicians to solve nonlinear equations is the application of perturbation techniques to obtain approximate analytical solutions. These include methods such as the δ-expansion method, Bender et al. [24], Jones [63], Lyapunov's artificial small parameter method, Lyapunov [78], and the method of multiple scales, Nayfeh [89], to name a few. These methods rely heavily on the availability of a perturbation parameter, ϵ, which forms part of the equation and/or the boundary conditions, Liao [73]. The requirement is that the solution at $\epsilon = 0$ must be known and the corresponding approximate solutions would then be generated as the parameter is expanded. Liao [73], showed that the approximate solutions are dependent on the chosen parameter ϵ and not the independent variable, thus placing additional restrictions on the problem.

The solutions generated by the above methods proved to be useful in describing both quantitative and qualitative properties of the problem, which is an advantage compared to numerical solutions. However, there were several draw backs for complex equations due to either the non-existence of small or large perturbation parameters or such parameters cause a divergence of solutions as the quantities increase or decrease. In problems where these quantities do not exist, the parameter has to be artificially introduced which may lead to incorrect results, Holmes [53]. Perturbation techniques are therefore found to be mainly useful for weakly nonlinear problems.

To overcome some of the restrictions of the perturbation parameters in perturbation techniques, some non-perturbation techniques were later developed.

1.2 Non-perturbation methods

These include methods such as the Adomian decomposition method (ADM), Adomian [12], the differential transform method (DTM), Zhou [110], the variational iteration method (VIM), He [51], the homotopy perturbation method (HPM), He [49], and the homotopy analysis method (HAM), Liao [73]. These methods remove the requirement for the presence of small parameters in the equation to be solved and are discussed in detail below.

1.2.1 The Adomian decomposition method

In the 1980's, George Adomian introduced a powerful method for solving nonlinear equations, now commonly known as the Adomian decomposition method (ADM), Adomian [10, 11, 12]. In recent times, the ADM has proved to be more efficient than the Taylor series method and Picard's method, Wazwaz [103], and has been used to generate analytical solutions to a wide class of linear and nonlinear differential equations. The method does not require linearization or discretization and produces solutions which are closed form.

The idea is to separate the equation into its linear and nonlinear components. The highest order derivative of the linear part is inverted into the corresponding integral and applied to the equation resulting in the approximate solution. The constant of integration is determined by the initial or boundary condition depending on whether the problem is an initial or boundary value problem.

The Adomian decompostion method assumes that the unknown function can be expressed as an infinite series and the nonlinear operator can be decomposed into a special series of polynomials referred to as Adomian polynomials. These polynomials can be constructed using recurrence relations for all classes of non-linearity as shown by Adomian [10, 11]. The solutions given by ADM have been shown to converge rapidly, Cherruault et al. [32], and are valid for strongly nonlinear partial differential equations.

The ADM has been used successfully to solve problems such as the Falkner-Skan equation, Alizadeh et al. [16], the Klein-Gordon equation, Basak et al. [21], the KdV equation, Wazwaz [107], the Riccati equation, Gbadamosi et al. [44] and nonlinear equations in non-Newtonian flows, Siddiqui et al. [96].

The advantage of the Adomian decomposition method as shown by Wazwaz [103], is its simplicity and ease of implementation as well as the high convergence rate as compared to methods based on the Taylor series expansion. However, there are certain limitations when compared to modern methods such as the variational iteration method and the homotopy analysis method, Wazwaz [104]. The main difficulty arises with the computation of the Adomian polynomials which are based on power series expansions and may have small convergence regions. These can be tedious to compute. Further modifications have been made to the ADM by Abassy [2], Wazwaz and El-Sayed [105] and Dehghan et al. [38] to name a few.

1.2.2 The differential transform method

The differential transform method (DTM) was first introduced by Zhou [110] in 1986. It is an iterative technique initially designed to solve linear and nonlinear problems in electric circuit analysis. In 1999, Chen and Ho [30], developed a two-dimensional DTM which can be used for solving differential and integral equations. This method generates an analytical solution based on Taylor series expansions.

The DTM is based primarily on the Taylor series method. However, at higher orders, the DTM differs from the Taylor series method in the way the coefficients are computed. The Taylor series method requires computing coefficients using the initial data and the differential equation which requires more computational work while the DTM iteratively obtains the Taylor series equations.

The principle behind the method is to apply a differential transform to the original equation. Thereafter, the equation is simplified by applying certain theorems of the differential transform theory, Kangalgil and Ayaz [64]. Finally, an inverse differential transform is applied to the simplified equation resulting in an iteration formula for the problem. Ayaz [19], showed that the DTM is better equipped to solve highly nonlinear problems than the Taylor series method. The DTM does not require linearization or discretization and, like the ADM, produces closed form solutions, Chen and Ho [30].

The DTM has been used to solve various problems in applied mathematics and physics such as systems of differential equations, Ayaz [18], Kanth and Aruna [66], the Schrödinger equation, Kanth and Aruna [65], the KdV and mKdV equations, Kangalgil and Ayaz [64] and the Emden-type equations, Mukherjee et al. [88].

The drawbacks of the DTM are the small convergence regions of the truncated series solutions and does not exhibit periodic behaviour. Several improvements have been made over the years by Odibat and Momani [92], who generalised the method in order to improve convergence using the Caputo fractional derivative. Momani and Ertürk [85], applied Laplace transforms and Padé approximations to the DTM in order to study the periodic behaviour of the solutions and improve the accuracy of the DTM solution in a larger region. The modifications made above have provided more accurate series solutions as compared to the original ADM and other methods, Odibat and Momani [92].

1.2.3 The variational iteration method

In 1999, Ji-Huan He [51, 52], proposed the variational iteration method (VIM). This method is a modification of the general Lagrange multiplier method and provides analytical solutions to linear, nonlinear, initial and boundary value problems.

The principle behind the method, He [52], is to apply a correction functional to the problem which is constructed using a Lagrange multiplier, λ. The initial approximation is determined by the initial and/or boundary conditions. The optimal Lagrange multiplier for the problem is determined by applying the stationary condition to the correction functional and λ is chosen to produce a solution that is superior to the initial approximation. The solution procedure is iterative and is improved at each iteration using the previous solution. This generates an infinite series solution which generally converges to the exact solution to the problem.

Several problems in fluid mechanics have been solved using the VIM such as, the Euler-Bernoulli beam, Liu and Gurram [76], the evolution equations, Mohyud-Din [83], the gas dynamic equation, Mayinfar et al. [82], the KdV equation, Mohyud-Din and Noor [84], the Sawada-Kotera equations, Jafari [57] and the Sturm-Liouville equations, Altintan [17]. The method has also been used as a test method and the solutions have been compared to other methods such as the Adomian decomposition method and the homotopy analysis method, Wazwaz [104].

The advantages of the variational iteration method, just like the ADM and the DTM are that the problem can be solved without any discretization or transformation and is free from round-off errors. Another important advantage of the VIM is that the method provides successive approximate solutions iteratively as compared to the ADM and the DTM, which generates components of the approximate solution and require summation to provide the series approximate solution, Wazwaz [104].

The VIM also requires calculation of the Lagrange multiplier and evaluation of the correction functional while the ADM requires evaluation of the Adomian polynomials, which has been shown to be a tedious task for certain problems. Thus, the VIM solution is straightforward while the ADM requires subsequent steps. The ease of computing the correction functional in the VIM as compared to applying the differential transform theorems in the DTM shows that the VIM is a simpler and more efficient method.

The disadvantage of the VIM is the limited convergence region of the truncated series solution, Abassy et al. [4]. This issue has also been observed in the ADM and DTM methods as stated previously. Abassy et al. [5], also showed for severely nonlinear problems, that VIM may produce unnecessary terms or unneeded computations which may cause a divergence of the solution and increases computation time. These limitations have been addressed by some author's and modified variational iteration methods have been developed over time, such as using the Padé technique and Laplace transforms to eliminate unnecessary computations, Abassy et al. [1, 3], Noor and Mohyud-Din [91]. These modifications have made the VIM one of the most useful methods in order to obtain exact solutions to a variety of problems. In this dissertation, the standard VIM has been chosen to solve the test problems and will be compared to other methods.

1.2.4 The homotopy perturbation method

The homotopy perturbation method (HPM) was developed by Ji-Huan He. The method was initially proposed in 1999, He [50] and revised in 2003, He [49]. The method is derived from Liu's artificial parameter method, Liu [77] and Liao's homotopy analysis method, Liao [75] and generates analytical solutions for linear and nonlinear differential equations.

The principle behind the method is to construct a homotopy of the original equation, Liao [73], using an embedding parameter, p. The general linear operator is then split into a linear and a nonlinear component. As p changes from zero to one, the approximate solution approaches the exact solution in a process referred to as deformation in topology. The embedding parameter can be considered as a small parameter and by the artificial parameter method, the approximate solution can be expressed as a series solution of the power of p. This series is then substituted into the homotopy equation and solved recursively to obtain the exact solution.

The HPM has been used to solve various problems in fluid flow theory such as the Blasius equation, He [48], nonlinear free vibration of systems, Danaee Barforoushi et al. [20], the Helmholtz equation [25], the Brinkman momentum equation, Ezzati and Mousavi [42], as well as a test method for solving nonlinear partial differential equations, He [47].

The advantage of the HPM is that analytical solutions can be obtained relatively easily for highly nonlinear problems. However, Liao showed that the homotopy perturbation method is in fact a special case of the homotopy analysis method, Liao [74]. The main drawback of the method is in relation to the choice of the initial guess and the auxiliary parameter which may cause the solution to diverge if chosen incorrectly. Liao concluded that the homotopy analysis method is a more powerful method than the homotopy perturbation method, a subject that will be discussed later in the chapter.

1.2.5 The homotopy analysis method

In 1992, Shi-Jun Liao proposed the homotopy analysis method (HAM) as part of his PhD thesis. The method aimed to remove the shortfalls seen with other perturbation techniques and, as shown in Liao's book [73], addresses the following points:

1. The method needs to be valid for strongly nonlinear problems with or without small/large parameters,

2. There has to be a convenient way to adjust the convergence region and rate of approximation series,

3. The ability to use different base functions to approximate a nonlinear problem.

The method, which addressed the above points, is thus a powerful analytical method for nonlinear partial differential equations with strong non-linearity.

The basic principle behind the HAM is to replace the nonlinear equation by a system of ordinary differential equations which can be solved iteratively. The first step is to split the general operator into its linear ($\mathcal{L}$) and nonlinear ($\mathcal{N}$) components. Thereafter, using the concept of a homotopy from topology, a zero-order deformation equation is formed using an embedding parameter, p, an auxiliary parameter, $\hbar$, and an auxiliary function, H, as shown below

$$(1-q)\mathcal{L}\left[\,\phi(x,t;q)-u_0(x,t)\right]=q\hbar\,H(x,t)\,\mathcal{N}[\,\phi(x,t;q)], \tag{1.1}$$

where $u_0(x,t)$ is an initial guess of the solution $u(x,t)$ and $\phi(x,t;q)$ is an unknown function.

The auxiliary parameter and the auxiliary function were introduced by Liao in order to provide a convenient way to adjust or control the convergence region of the solution. As p increases from zero to one, the initial guess approaches the exact solution. The linear operator, auxiliary parameter and initial guess are chosen such that the solution converges at $p = 1$. The solutions generated by HAM are expressed by a set of base functions which can be solved using computer programming software such as Maple and Mathematica.

The HAM relies on certain assumptions such as the following:

1. For $p \in [0, 1]$, there exists a solution of the zero-order deformation equation.

2. The higher order deformation equations all have solutions.

3. All Taylor series expansions in p, converge at $p = 1$.

There are numerous engineering and physics problems that have been solved using the HAM. These include the KdV equations, Jafari and Firoozjaee [58], the Davey-Stewartson equation, Jafari and Alipour [59] and the Drinfield-Sokolov equations, Afrouzi et al. [13]. The HAM has also been used to find solutions of general nonlinear integro-differential equations, Hanan [45].

Liao [73], showed that the HAM is in fact a generalized method and is related to perturbation methods such as the δ expansion method and Lyapunov's artificial small parameter method. He also showed that the ADM is a special case of the HAM. In Liao [74], a comparison was made between HAM and HPM and the results showed that the HPM is also a special case of HAM and that for certain values of $\hbar$, the VIM and the HAM are equivalent as shown in this study. Thus, the homotopy analysis method has been referred to as a unification of non-perturbation methods.

The obvious advantage of the HAM is that like other non-perturbation methods, there is no need for small parameters, discretization or linearization. The main advantage of the HAM over other non-perturbation method is mainly due to the introduction of an auxiliary parameter and auxiliary function. The auxiliary parameter generates the so-called $\hbar$-curves which provide an easy way to control and adjust the region of convergence based on the value of $\hbar$. Further details on $\hbar$-curves will be discussed in Chapter 2 and Chapter 3. The fact that the HAM provides solutions based on a set of base functions also allows the freedom to express solutions using different base functions. Choosing solutions in this way helps improve efficiency and speed convergence when solving problems.

As with any technique, there are limitations to the HAM. There are no concrete methods to determine the initial approximation, the auxiliary parameter and the auxiliary function. Liao [73], suggested some general rules in order to ensure these parameters are determined appropriately. These rules include:

1. The rule of solution expression,
2. The rule of coefficient ergodicity,

3. The rule of solution existence.

The rule of solution expression determines the initial approximation, auxiliary linear operator and the auxiliary function. The rule of coefficient ergodicity and the rule of solution existence assists in determining whether the higher order deformation equations are closed and have solutions. These rules are discussed further in Chapter 2.

1.2.6 Further advances on the homotopy analysis method

The main limitation of the HAM is that the initial approximation, auxiliary linear operator and auxiliary function have to be chosen appropriately in order to obtain convergence of the solution, using the suggested rules above. Incorrectly chosen parameters may result in difficultly solving the higher order deformation equations used to obtain the solution to the problem. In terms of convergence, the plot of $\hbar$-curves as suggested by Liao [73], aids in finding the optimal convergence parameter but in most cases these values are generated by trial and error and can be time consuming to obtain.

In recent times there have been enhancements and improvements made to the HAM. In 2007, Yabushita et al. [109], introduced a modified optimisation method which uses the square residual to determine two optimal convergence control parameters. The use of the square residual in determining the optimal value of the convergence control parameters were also applied by Akyildiz and Vajravelu [15] and Marinca and Herisanu [80, 81]. The results showed that the rate of convergence to the series solution was faster using the optimal method, so-called optimal homotopy asymptotic method. Liao [72], showed that although the optimal asymptotic method does provide improved convergence, it is time consuming to calculate square residuals at higher orders and fails for highly complicated problems. Liao [72], proposed a modification to the HAM which contains up to three convergence control parameters. The method was named the optimal homotopy analysis method (OHAM). The method uses an average residual error to determine the optimal convergence control parameters and has been found to be efficient, easier to apply than previous optimal methods and accelerates the convergence of the series solution to the problem.

The OHAM addressed the convergence of the solution by determining the optimal value of the convergence parameter(s). However, there was a need to improve convergence based on the initial guess. Motsa et al. [86], provided an innovative way to improve the HAM algorithm using the Chebyshev pseudospectral collocation method, Bazan [23].

The method, which is a powerful semi-analytical method, was called the spectral-homotopy analysis method (SHAM). The main advantage of the SHAM over the HAM is that there is no need to conform to the rules of solution expression and coefficient ergodicity. In addition, any form of an initial approximation can be used regardless of the impact it would have on the higher deformation equations. The method also allows for a wider range of linear operators due to the higher order equations being discretized and integrated using the Chebyshev pseudospectral method. The SHAM has been shown to converge faster than the HAM algorithm and has been used to solve the MHD Jeffery-Hamel problem and the Darcy-Brinkman-Forchheimer equation, Motsa [86, 87]. The OHAM and SHAM have proved itself as efficient methods however they have moved away from being a fully analytical method like the HAM.

For purposes of this study, the standard HAM will be used to solve the test problems and compared to the VIM. The idea is to illustrate the effectiveness of purely analytical methods with focus on convergence, accuracy and computational efficiency.

1.3 The test equations

1.3.1 The heat and the Burgers equation

The heat equation is a well known parabolic partial differential equation first described by Joseph Fourier in 1807. The equation describes isotropic diffusion and has been extensively used to verify and compare different numerical techniques over time. The general solution was found by Fourier using a method now commonly referred to as Fourier analysis, Fourier [43]. The heat equation has subsequently been solved analytically using separation of variables as shown in this study and other numerical schemes such as finite differences, Recktenwald [95]. The heat equation has been chosen in this study for two reasons; firstly to verify the results of the analytical and numerical methods chosen for analysis and secondly, it forms the basis for the derivation of Burgers equation.

The second test equation of interest in this study is the Burgers equation. Johannes Martinus Burgers [29], a Dutch physicist derived the equation in 1939 by simplifying the Navier-Stokes equations to exclude the pressure term and external force. He also investigated the equation in one spatial dimension in the form,

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $u = u(x, t)$ is the temperature and $c \in \mathbb{R}$ is the viscosity.

Burgers equation is described as a nonlinear quasi-parabolic partial differential equation. This equation embodies all the main mathematical features of the Navier-Stokes equations in one-dimension, since it possesses both the advection, uu_x, and the diffusion, u_{xx}, terms from the Navier-Stokes equations. Despite its fundamental non-linearity, closed form analytical solutions have been obtained for Burgers equation for a variety of initial and boundary conditions, Cole [36] and

Hopf [54].

Literature suggests that the Burgers equation was initially derived by Bateman [22] in 1915 and later by Burgers. However, the study by Burgers on this equation deserves the name attributed to him. The equation has been extensively used to test numerical algorithms and to explore the phenomena of one-dimensional turbulence, Burgers [29]. The essence of turbulence is embodied in the quadratic, nonlinear convection terms of the general three-dimensional Navier-Stokes equations and is a fully three-dimensional phenomenon and as such can be understood completely only with a three-dimensional view. However, Burgers equation, although one-dimensional, possesses a fundamental quadratic nonlinearity and is viewed as an appropriate starting model for studying turbulence. The models can be used to study other physical processes such as shock waves, traffic flow, acoustic transmission, supersonic flow around airfoils and turbulent flow in a channel.

In 1950, Cole [36] and Hopf [54], independently showed that Burgers equation can be transformed into the linear heat equation. Thus, since the solution to the heat equation was well known, a solution to Burgers equation could be obtained. They also proved that the solution to Burgers equation does not exhibit chaotic behaviour. Therefore, the significance of the Burgers equation has been more geared towards numerical analysis in recent times.

Many analytical methods have been applied to the Burgers equation such as the Adomian decomposition method, Mamaloukas and Spartalis [79], the variational iteration method, Abdou [7], the homotopy perturbation method, Desai and Pradhan [39] and the differential transform method, Abazari [6]. Burgers equation is an important test equation to develop and compare the accuracy and convergence of analytical and numerical methods using different initial and boundary conditions.

1.3.2 The Bratu equations

The third test equation in this study is the Bratu equation. The Bratu problems are nonlinear differential equations of the form:

$$\frac{d^2u}{dx^2} + \gamma\, e^u = 0, \qquad 0 < x < 1,$$

where γ is a constant.

The equation arises from the simplification of the solid fuel ignition model and describes the thermal reaction process in a combustible, non-deformable material of constant density during the ignition period, Jacobsen and Schmitt [56], Cohen and Toledo Benavides [35].

The Bratu equation was first solved in 1914, Bratu [28]. The equation has been known to model various phenomena such as radiative heat transfer and the expansion of the universe, Boyd [27]. The other importance of the Bratu equation is that it is often used as a benchmark equation to test the accuracy and robustness of analytical and numerical methods.

The solutions to the Bratu problem depend on a pre-determined constant, $\gamma_c \approx 3.51$. The problem has two solutions when $\gamma < \gamma_c$, one solution for $\gamma = \gamma_c$ and no solutions for $\gamma > \gamma_c$. The Bratu equations have been solved using the Laplace Adomian decomposition method, Syam and Hamdan [98], Khuri [67], and the differential transform method, Hassan and Ertürk [46].

The main focus in the literature has been placed on the initial and boundary value problems of the Bratu-type. The Bratu-type equations are special cases of the Bratu equation with specific choices of γ. The two particular choices for this study are $\gamma = -\pi^2$ and $\gamma = -2$ which provide difficult nonlinear problems which test even the most robust techniques. For this choice of γ, the Bratu problem has been solved analytically using methods such as the Haar Wavelet method, Venkatesh et al. [102] and the Adomian decomposition method, Wazwaz [106]. The Bratu problem has a unique solution for the chosen boundary and initial conditions which provides a good benchmark for comparison to the analytical and numerical methods used in this study.

1.4 Objectives of this dissertation

This study aims to compare the analytical solutions generated by Liao's homotopy analysis method [73] and He's variational iteration method [51]. The focus will be placed on the accuracy of the method as compared to the exact solution as well as the computation time and the rate of convergence. For problems where exact solutions are not specified the method of finite differences will be used to verify the results obtained from each method. It is understood that the method of finite differences is a fully numerical technique but the idea behind using the method is to provide direction of the solution as well as to prove that the HAM and the VIM are more efficient methods.

The test differential equations used in this study to compare the performance of these methods are:

1. The heat equation,
2. the Burgers equation and
3. the Bratu equation.

1.5 Dissertation outline

The dissertation is organised as follows;

In Chapter 2, we will present the theory of the methods and the derivation of the algorithms. We are particularly interested in the following methods; (1) The homotopy analysis method (HAM), (2) the variational iteration method (VIM) and (3) the method of finite differences (FDM). A brief solution of the heat equation will be presented and the Burgers equation will be derived based on the relationship between the Burgers equation and the heat equation. The Bratu problem will be presented together with the boundary and initial conditions.

In Chapter 3, we will use the HAM, VIM and FDM to solve the test problems. The first iteration of the algorithm is performed by hand and thereafter solved using Maple and Matlab software.

In Chapter 4, we present the results of our numerical simulations in tabulated and graphical form. The results will be discussed in detail noting key aspects such as speed, accuracy and convergence of each method.

In Chapter 5, we present our conclusions and recommendations.

Chapter 2

Theory and Formulation

In this chapter we present the underlying theory behind the selected analytical schemes, the homotopy analysis method and the variational method as well as the finite difference method. This includes a description of the scheme as well as the important convergence theorems. The heat equation is presented together with an analytical solution followed by a brief derivation of the Burgers equation along with the initial conditions. The Bratu equations are also presented with their analytical solutions. The aim of the chapter is to explain how each method works and to provide a brief physical background of the test problems.

2.1 The homotopy analysis method (HAM)

The homotopy analysis method is discussed in detail below. Consider the nonlinear partial differential equation

$$\mathcal{N}[\,u(x,t)] = 0, \tag{2.1}$$

where $\mathcal{N}$ is a nonlinear operator, x and t denote the independent variables and u is an unknown function.

A zeroth-order deformation equation, Liao [73], is constructed from the nonlinear equation (2.1) as follows

$$(1-p)\mathcal{L}\,[\,\phi(x,t;p) - u_0(x,t)] = p\hbar\, H(x,t)\,\mathcal{N}[\,\phi(x,t;p)], \tag{2.2}$$

where $\mathcal{L}$ is an auxiliary linear operator, $H(x,t)$ denotes a non-zero auxiliary function, $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is an auxiliary parameter.

The embedding parameter p has the following impact on equation (2.2). When $p = 0$, we have

$$\mathcal{L}\,[\,\phi(x,t;0) - u_0(x,t)] = 0, \tag{2.3}$$

which simplifies to

$$\phi(x,t;0) = u_0(x,t), \tag{2.4}$$

which is the initial condition. Similarly when $p = 1$ in equation (2.2)

$$\hbar\, H(x,t)\,\mathcal{N}[\,\phi(x,t;1)] = 0. \tag{2.5}$$

Since $H(x,t) \neq 0, \hbar \neq 0$ and using equation (2.1) we have

$$\phi(x,t;1) = u(x,t), \tag{2.6}$$

which is the exact solution to the original problem.

Thus, it is clear that as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$. The parameter p is key in determining a convergent solution and is the basis used to derive the higher order deformation equations which are discussed below.

The Taylor expansion of $\phi(x,t;p)$ with respect to p is

$$\phi(x,t;p) = \phi(x,t;0) + \sum_{m=1}^{\infty} u_m(x,t)\, p^m, \tag{2.7}$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;p)}{\partial p^m}\bigg|_{p=0} \tag{2.8}$$

Using equation (2.4), equation (2.7) reduces to

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\, p^m. \tag{2.9}$$

The convergence of equation (2.9) depends upon the auxiliary parameter $\hbar$. If $\hbar$ is chosen such that a convergent solution exists at $p = 1$, equation (2.9) becomes

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\, p^m, \tag{2.10}$$

which is the solution of the original nonlinear equation, Liao [73].

Now suppose

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\},$$

is a vector of unknown functions. Differentiating the zeroth-order deformation equation (2.2) m-times with respect to p, setting $p = 0$ and then dividing the resulting equation by $m!$, we get the m^{th} order deformation equation

$$\mathcal{L}\,[\,u_m(x,t) - \chi_m\, u_{m-1}(x,t)] = \hbar\, \mathcal{R}_m[\,u_{m-1}(x,t)], \tag{2.11}$$

where

$$\mathcal{R}_m[\,u_{m-1}(x,t)] = \frac{1}{(m-1)!}\, \frac{\partial^{m-1} \mathcal{N}[\,\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0} \tag{2.12}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{2.13}$$

Now suppose the linear operator, $\mathcal{L}$, is invertible, then the resulting equation is

$$u_m(x,t) = \chi_m\, u_{m-1}(x,t) + \hbar\, \mathcal{L}^{-1}\, \mathcal{R}_m[\,u_{m-1}(x,t)]. \tag{2.14}$$

The initial approximation is derived using the boundary and/or initial conditions specified in the problem and thereafter the linear equation (2.14) will be solved to generate $u_m(x,t)$ for $m \geq 1$. This will provide an analytical solution for each iteration of m and thus an analytical solution to the problem.

The method will be applied to the problems following certain rules suggested by Liao [73], namely;

1. The rule of solution expression,

2. The rule of coefficient ergodicity,

3. The rule of solution existence.

The rule of solution expression is useful for determining the initial approximation, auxiliary linear operator and the auxiliary function. The rule of coefficient ergodicity and the rule of solution existence are used to determine whether the higher order deformation equations are closed and have solutions.

The homotopy analysis method is the fundamental test method in this study and will be applied to all the test equations with emphasis placed on the method providing a convergent analytical solution. The accuracy and computational efficiency will be compared to the other methods in the study.

2.1.1 Convergence of the HAM

There is no concrete proof of convergence in the book by Liao [73]. The auxiliary parameter $\hbar$ does control the rate of convergence but the question still remains with the optimal choice of $\hbar$. Literature states that the optimal value of $\hbar$ can be determined using so-called $\hbar$-curves. The $\hbar$-curves will be plotted against the error term to determine which value of $\hbar$ produces the most accurate and fastest converging solution. The underlying convergence theorems are stated below.

Theorem 1: If the series solution (2.10) is convergent, then it converges to an exact solution of the nonlinear problem (2.1).

The proof of theorem 1 can be found in Liao, [73]. Odibat [93] and Abdulaziz et al. [9], both presented sufficient conditions for convergence which also placed additional focus on the region of the $\hbar$-curves that would provide a convergent solution. Their theorem, with slight modification on notation, is stated below.

Theorem 2: Suppose that $A \subset R$ is a Banach space denoted with a suitable norm $\|.\|$ over which the sequence $u_k(x,t)$ of (2.10) is defined for a prescribed value of $\hbar$. Assume also that the initial approximation $u_0(x,t)$ remains inside the ball of the solution $u(x,t)$. Taking $r \in R$ to be a constant, the following statements hold true:

(i) if $\|u_{k+1}(x,t)\| \leq r\|u_k(x,t)\|$ for all k, given some $0 < r < 1$, then the series solution defined in (2.10) converges absolutely at $q = 1$ to $u(x,t)$ over the domain of definition of t,

(ii) if $\|u_{k+1}(x,t)\| \geq r\|u_k(x,t)\|$ for all k, given some $r > 1$, then the series solution defined in (2.10) diverges absolutely at $q = 1$ over the domain of definition of t.

The proof of theorem 2 can be found in Odibat [93] and provides a sufficient condition for convergence at each k. Let r_k be defined as

$$r_k = \frac{\|u_{k+1}(x,t)\|}{\|u_k(x,t)\|}, \tag{2.15}$$

then by ensuring that $r_k < 1$ at each step k of the algorithm will result in a convergent series solution.

In order for the condition $r_k < 1$ to be satisfied, there may be restrictions on the value of $\hbar$. This test will be useful in determining the region of $\hbar$ which provides a convergent solution.

Now to determine the $\hbar$ value that converges the fastest, an estimate of the error is required.

Theorem 3: Suppose the series solution (2.10) is convergent for a prescribed value of $\hbar$. If the truncated series solution $\sum_{m=0}^{M} u_m(x,t)$ is used as an approximation to the solution $u(t)$ of (2.1), then an upper bound for the error, $E_M(x,t)$, is

$$E_M(x,t) \leq \frac{r^{M+1}}{1-r}\|u_0(x,t)\|. \tag{2.16}$$

The convergence of the test problems are studied further in Chapter 3.

2.2 The variational iteration method

The basic idea of the variational iteration method (VIM) is discussed below. Consider the following partial differential equation

$$\mathcal{L}[\,u(x,t)] + \mathcal{N}[\,u(x,t)] = g(x,t), \tag{2.17}$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator, x and t denote the independent variables and g is an unknown function.

The VIM requires that a correction functional, He [51], be applied to equation (2.17) as follows

$$u_m(x,t) = u_{m-1}(x,t) + \int_0^t \lambda\Big(\mathcal{L}[\,u_{m-1}(x,\tau)] + \mathcal{N}[\,\widetilde{u}_{m-1}(x,\tau)] - g(x,\tau)\Big)d\tau, \tag{2.18}$$

where λ is the general Lagrangrian multiplier, u_{m-1} the $(m-1)^{th}$ approximation of u and $\widetilde{u}_{m-1}$ are the restricted variations such that $\delta\widetilde{u}_{m-1} = 0$.

The method is based on obtaining an initial approximation using the initial and/or boundary conditions. The stationary condition is then applied to the correction functional and the Lagrange multiplier λ is chosen such that the solution is superior to the initial approximation. This creates an infinite series solution to the problem which converges to the exact solution.

It is important to note that the solution at each step in the VIM procedure is an analytical solution to the problem for each value of m as opposed to HAM which creates a solution at each m and then requires a summation to obtain the analytical solution to the problem. The final solution is

$$u(x,t) = \lim_{m\to\infty} u_{m-1}(x,t). \tag{2.19}$$

The variational iteration method will be tested on all the equations in this study as a competitor to the homotopy analysis method. Emphasis will be placed on convergence of the method to an analytical solution as well as the accuracy and time taken to obtain the solution.

2.2.1 Convergence of the VIM

A convergence theorem similar to that stated for the homotopy analysis method will be applied for the variational iteration method. Odibat [94], published a paper on the convergence of the VIM and the important theorem is stated below.

Theorem 1: Suppose that $A \subset R$ is a Banach space denoted with a suitable norm $\|.\|$ over which the sequence $u_k(x,t)$ of (2.19) is defined. Assume also that the initial approximation $u_0(x,t)$ remains inside the ball of the solution $u(x,t)$. Taking $r \in R$

be a constant, the following statements hold true:

(i) if $\|u_{k+1}(x,t)\| \leq r\|u_k(x,t)\|$ for all k, given some $0 < r < 1$, then the solution defined in (2.19) converges absolutely $u(x,t)$ over the domain of definition of t,

(ii) if $\|u_{k+1}(x,t)\| \geq r\|u_k(x,t)\|$ for all k, given some $r > 1$, then the solution defined in (2.19) diverges absolutely over the domain of definition of t.

The proof of the theorem 1 above can be found in Odibat [94]. The theorem provides a sufficient condition for convergence at each k. Similarly, let r_k be defined as

$$r_k = \frac{\|u_{k+1}(x,t)\|}{\|u_k(x,t)\|}, \tag{2.20}$$

then by ensuring the $r_k < 1$ at each step of the algorithm will result in a convergent series solution.

2.3 Method of finite differences

The method of finite differences (FDM) is a numerical method based on Taylor's theorem, LeVeque [70], and has been extensively used to solve differential equations dating as far back as the early 1930's. The method is based on discretization of derivatives using finite difference approximations and is discussed below.

Suppose we partition the $[x,t]$ space into $x_0, ..., x_i$ in steps of h and $t_0, ..., t_j$ in steps of k. Taylor's theorem is as follows

$$u(x+h,t) = u(x,t) + \frac{\partial u(x,t)}{\partial x}\frac{h}{1!} + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{h^2}{2!} + ... + \frac{\partial^m u(x,t)}{\partial x^m}\frac{h^m}{m!} + O(h^{m+1}). \quad (2.21)$$

An approximation for the first derivative of u with respect to x can be obtained using the first two terms from the above expansion

$$\frac{\partial u(x,t)}{\partial x} = \frac{u(x+h,t) - u(x,t)}{h} + O(h). \quad (2.22)$$

Similarly an approximation for the first derivative of u with respect to t is

$$\frac{\partial u(x,t)}{\partial t} = \frac{u(x,t+k) - u(x,t)}{k} + O(k). \quad (2.23)$$

The second derivative with respect to x can be approximated as follows

$$\frac{\partial u(x,t)}{\partial x} = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2). \quad (2.24)$$

The idea behind the method is to replace the derivatives of the nonlinear partial differential equation with finite difference approximations and thus, using a basic iteration scheme, approximate solutions can be determined using a suitable initial approximations and reasonable sized step widths, h and k.

The solution generated by the FDM is purely numerical and will be used as a guideline or starting point for problems which have no general analytical solution. In this study, it will be applied to the heat equation and thereafter to the Burgers equation. This will provide a benchmark for the analytical methods, HAM and

VIM and the comparison will centre on accuracy, computational efficiency and convergence of these analytical methods.

2.3.1 Consistency, stability and convergence of the FDM

The study of consistency, stability and convergence has been a broad discussion over the years and is problem specific. For our purposes, the general definitions, found in Chern [31] and LeVeque [70], are given below.

Definition 1 (Consistency): A finite difference method is consistent if its local truncation error, τ, satisfies

$$\|\tau_{h,k}\| \to 0 \quad \text{as} \quad h, k \to 0. \tag{2.25}$$

If a scheme has reasonable discretization such that $\|\tau_{h,k}\| = O(h^p) + O(k^q)$ for some integer $p, q > 0$, then the scheme is most definitely consistent.

Definition 2 (Stability): A finite difference method in the form, $u_i^{j+1} = (A_{h,k}\, u^j)_i$ is stable under the norm $\|.\|$ in a region $(h, k) \in \mathcal{R}$ if

$$\|A^n_{h,k}\, u\| \leq \|u\|, \tag{2.26}$$

for all n with h, k fixed. To show stability in general may require a tedious amount of work especially for nonlinear partial differential equations. Von Neumann analysis, which is a necessary condition for stability, has been applied to many linear partial differential equations and sometimes provides restrictions on the step sizes used in the scheme, Chern [31]. These restrictions provide direction when determining numerical results using Matlab or Maple software.

Definition 3 (Convergence): A finite difference method is convergent if the error, E, satisfies

$$\|E_{h,k}\| \to 0 \quad \text{as} \quad h, k \to 0. \tag{2.27}$$

It has been shown in the literature, Chern [31] and LeVeque [70], that

$$\text{stability} + \text{consistency} \implies \text{convergence.}$$

The issue of consistency, stability and convergence will be discussed in Chapter 3 with application to the heat equation.

2.4 The heat equation in one dimension

Consider an object with temperature $u(x, t)$ at time t. Let $x \in [0, \ell]$ and $t \geq 0$ be the space and time variables. The heat equation is defined as

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\,, \tag{2.28}$$

where $c \in \mathbb{R}$ is the viscosity of the body.

The corresponding initial and boundary conditions are

$$\begin{aligned} u(x,0) &= u_0(x)\,, \\ u(0,t) &= 0\,, \\ u(\ell,t) &= 0\,, \end{aligned} \tag{2.29}$$

where $u_0(x)$ is an arbitrary function of x only.

The general solution to the heat equation can be obtained by Fourier analysis. Since the equation of interest is bounded by a finite domain, a simple method of separation of variables can be used to determine the exact solution. To proceed, we assume the function can be split into the product

$$u(x,t) = X(x)\,T(t)\,. \tag{2.30}$$

Applying the necessary derivatives and substituting into (2.28) gives

$$X(x)\,T'(t) = c^2\,X''(x)\,T(t).$$

Rearranging the above equation yields

$$\frac{1}{c^2}\,\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2\,.$$

Since the left-hand side (LHS) of the equation is a function of t only and right-hand side (RHS) is a function of x only, they must both be equal to a constant $(-\lambda^2)$.

Simplifying the above, we get two linear differential eigenvalue equations

$$\frac{1}{c^2} T'(t) = -\lambda^2 \, T(t)\,, \tag{2.31}$$

$$X''(x) + \lambda^2 \, X(x) = 0\,. \tag{2.32}$$

Integrating and using the method of undetermined coefficients and thereafter applying the necessary boundary conditions, the solution to the heat equation is as follows

$$u(x,t) = \sum_{n=0}^{\infty} B_n \, \sin\left(\frac{n\pi \, x}{\ell}\right) \, \exp\left(\frac{c^2 \, n^2 \, \pi^2 \, t}{\ell}\right)\,, \tag{2.33}$$

and B_k can be obtained from the Fourier series of $u(x,t)$

$$B_k = \frac{1}{\ell} \int_{-\ell}^{\ell} u_0(m) \, \sin\left(\frac{k\pi m}{\ell}\right) \, dm\,. \tag{2.34}$$

The heat equation is linear and will be used as a benchmark equation in this dissertation to verify and compare the solutions obtained by the analytical and numerical methods. The problem selected for comparison is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \tag{2.35}$$

with initial condition,

$$u(x,0) = \sin \pi x, \tag{2.36}$$

and $x \in (0,1)$.

The exact solution to equation (2.35) using the solution (2.33) is

$$u_e(x,t) = \sin(\pi x) \, e^{-\pi^2 t}. \tag{2.37}$$

2.5 The Burgers equation

Johannes Martinus Burgers [29], derived and performed extensive work on this nonlinear equation. The equation is of importance in applied mathematics as it exhibits similar characteristics to the Navier-Stokes equation and possesses both advection and diffusion terms. These terms make computing analytical solutions more difficult and has challenged mathematicians to obtain solutions using a variety of methods. These methods include, to name a few, the Adomian decomposition method, Mamaloukas and Spartalis [79], the homotopy perturbation method, Desai and Pradhan [39], the differential transform method, Abazari and Borhanifar [6], exact-explicit finite difference method, Kutluay et al. [69] and spectral/spline methods, El-Hawary and Abdel-Rahman [41].

The derivation below is based on an inverse Hopf-Cole transformation, Cole [36], Hopf [54], which is applied to the heat equation (2.35). The Hopf-Cole transform is given by

$$v(x,t) = -2c^2 \frac{1}{u} \frac{\partial u(x)}{\partial x} \,, \tag{2.38}$$

Taking the heat equation as defined above

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \,, \tag{2.39}$$

where $c \epsilon \mathbb{R}$ and $c \neq 0$ is the viscosity of the body.

Using the inverse form of equation (2.38), let

$$u(x,t) = \exp\left(-\frac{1}{2\,c^2}\; v(x,t)\right).$$

Now taking the necessary derivatives of $u(x,t)$ gives

$$\frac{\partial v}{\partial t} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 = c^2 \frac{\partial^2 v}{\partial x^2}\,. \tag{2.40}$$

Equation (2.40) is known as the potential form of the Burgers equation.

Now make a second substitution

$$v(x,t) = \int w\, dx\,.$$

Applying the above to equation (2.40) gives

$$\int \frac{\partial w}{\partial t}\, dx + \frac{1}{2}\, w^2 = c^2 \frac{\partial w}{\partial x}\,. \tag{2.41}$$

and taking the derivative with respect to x gives

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = c^2 \frac{\partial^2 w}{\partial x^2}\,, \tag{2.42}$$

which is known as the Burgers equation (BE). The main challenge in obtaining solutions to the BE is due to non-linearity of the advection term ww_x which poses challenges with integration in most analytical/numerical schemes. There are exact solutions to the equation in the literature for a variety of initial and boundary conditions as shown by Cole [36] and Hopf [54]. This study looks at two different initial conditions of the BE, namely;

1. The test initial condition

$$u(x,0) = \frac{\alpha + \beta + (\beta - \alpha)e^{\eta}}{1 + e^{\eta}}\,, \tag{2.43}$$

   where $\eta = \alpha(x/\nu)$ and α, β and ν are arbitrary constants.

   The exact solution to equation (2.42), Abdou and Soliman [8], is

$$u(x,t) = \frac{\alpha + \beta + (\beta - \alpha)e^{\xi}}{1 + e^{\xi}}\,, \tag{2.44}$$

   where $\xi = (\alpha/\nu)(x - \beta t)$.

2. The common initial condition

$$u(x,0) = \sin \pi x. \tag{2.45}$$

The first condition (2.43) has been chosen due to the availability of an exact solution which is easily computable and will provide guidance to the approximate analytical solutions generated by the HAM and the VIM. The second condition is a more popular initial condition which has been referenced numerously in the literature. The exact solution can be obtained by Fourier analysis, Fourier [43] which has shown to be a tedious task. The HAM and the VIM algorithms perfected in this study using initial condition (2.43), will be used to determine the solution for the common initial condition (2.45). The FDM will also be used on condition (2.45) to provide a comparison between the analytical solutions and the numerical solution.

2.6 The Bratu equation

The one-dimensional Bratu equation arises from the simplification of a solid fuel ignition model that describes the thermal reaction process in a non-deformable material of constant density during the ignition period, Jacobsen and Schmitt [56], Cohen and Toledo Benavides [35]. The equation was named after Bratu [28] who solved it in 1914. The significance of the equation in applied mathematics is that it is used as a benchmark in order to compare and test various numerical and analytical methods due to its non-linearity. Mathematicians have used various methods to solve the equation such as the Laplace Adomian decomposition method, Syam and Hamdan [98], Khuri [67], and the differential transform method, Hassan and Ertürk [46].

The Bratu equation is defined as

$$\frac{\partial^2 u}{\partial x^2} + \gamma\, e^u = 0, \qquad 0 < x < 1, \tag{2.46}$$

where $u = u(x)$ and γ is a constant.

The corresponding boundary conditions

$$u(0) = u(1) = 0. \tag{2.47}$$

The exact solution, Wazwaz [106], to (2.46) is given by

$$u(x) = 2\,\ln\left[\frac{\cosh(0.5(x-0.5)\,\theta)}{\cosh(0.25\,\theta)}\right], \tag{2.48}$$

where θ satisfies

$$\theta = \sqrt{2\gamma}\,\cosh(0.25\,\theta). \tag{2.49}$$

The above equation has zero, one or two solutions depending whether $\gamma > \gamma_c$, $\gamma = \gamma_c$ or $\gamma < \gamma_c$. The critical value of γ is determined by solving the equation

$$1 = \frac{1}{4}\sqrt{2\gamma_c}\,\sinh(0.25\,\theta_c), \tag{2.50}$$

which has been calculated as

$$\gamma_c = 3.513830719.$$

The above discussion is used as a basis to introduce the two Bratu-type equations chosen for the analysis in this study. In order to obtain the Bratu-type equations from the Bratu equation (2.46), the value of γ has been chosen as, $\gamma = -\pi^2$ and $\gamma = -2$. The reason for the choices of γ are due to the severe non-linearity of these problems as shown in the literature as well as the existence of unique analytical solutions which provides comparison between the methods, Wazwaz [106].

2.6.1 The Bratu equation when $\gamma = -\pi^2$

The boundary value problem when $\gamma = -\pi^2$ together with the boundary conditions are defined as follows:

$$\begin{gathered} \frac{\partial^2 u}{\partial x^2} - \pi^2\, e^u = 0\,, \\ u(0) = u(1) = 0\,. \end{gathered} \tag{2.51}$$

The analytical solution, as found by Wazwaz [106], is:

$$u_e(x) = -\ln\left(1 + \cos\left[\left(\frac{1}{2} + x\right)\pi\right]\right) \tag{2.52}$$

This solution will be used for comparative purposes to the approximate solutions determined by the HAM and the VIM.

2.6.2 The Bratu equation when $\gamma = -2$

The boundary value problem when $\gamma = -2$ together with the boundary conditions are defined as follows:

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2} - 2\,e^u = 0\,, \\ u(0) = u(1) = 0\,.\end{gathered} \tag{2.53}$$

The analytical solution, as found by Wazwaz [106], is:

$$u_e(x) = -2\,\ln(\cos x) \tag{2.54}$$

This solution will be used for comparative purposes to the approximate solutions.

In summary we presented the theory behind the analytical and numerical methods used in this study along with important convergence theorems. The test problems were also discussed with the corresponding initial conditions. In Chapter 3 we provide the modifications on the test equations in order to apply the analytical and numerical methods before implementation in Maple and Matlab. The first iteration of each method is performed by hand in order to demonstrate the schemes and noting the assumptions made and parameters chosen.

Chapter 3

Solution of the test problems

In this chapter we apply the analytical and numerical methods to the test problems. The test problems are presented with the modifications required for implementation as well as their corresponding initial conditions. A few iterations of each method are performed by hand to illustrate the algorithms and address the convergence of the method. The higher order iterations are then obtained using Maple and Matlab software. We begin with the homotopy analysis method (HAM).

3.1 The HAM applied to the test equations

The HAM procedure is applied using the suggested guidelines by Liao [73]. The method requires the following operators to be defined from the test equations:

- The linear operator, $\mathcal{L}$.

- The nonlinear operator, $\mathcal{N}$.

- The right hand side of the m^{th} order deformation equation (2.11), $\mathcal{R}_m$ which includes $\hbar$.

Using the definitions above, the iteration formula (2.14) can be determined for the underlying test equations.

3.1.1 Application to the heat equation

As shown in Chapter 2, the heat test problem is as follows

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \tag{3.1}$$

with initial condition

$$u(x, 0) = \sin(\pi x). \tag{3.2}$$

The linear operator, $\mathcal{L}$, for the above equation (3.1) is chosen as

$$\mathcal{L}[\,u(x, t)] = \frac{\partial u(x, t)}{\partial t}\,, \tag{3.3}$$

with the property

$$\mathcal{L}[\,c_1] = 0, \tag{3.4}$$

where c_1 is a constant. The linear operator is time dependent only and thus allows for a simpler inverse operator for the problem. The inverse linear operator is simply

$$\mathcal{L}^{-1}[\,u(x, t)] = \int_0^t u(x, \tau)\, d\tau\,. \tag{3.5}$$

The nonlinear operator is the total operator of the problem and for this problem it is in fact linear

$$\mathcal{N}[\,u(x,t)] = \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u}{\partial x^2}\,, \tag{3.6}$$

and thus using the HAM methodology

$$\mathcal{R}_m[\,u_{m-1}(x,t)] = \frac{\partial u_{m-1}}{\partial t} - \frac{\partial^2 u_{m-1}}{\partial x^2}\,. \tag{3.7}$$

Finally, the iterative formula for the heat equation is as follows

$$u_m = \chi_m\, u_{m-1} + \hbar \int_0^t \left(\frac{\partial u_{m-1}}{\partial \tau} - \frac{\partial^2 u_{m-1}}{\partial x^2} \right) d\tau, \tag{3.8}$$

Using the initial condition (3.2), the initial guess is chosen as

$$u_0 = \sin(\pi x). \tag{3.9}$$

The first two steps, $m = 1$ and 2, of the HAM will be computed by hand and thereafter in Maple. Applying the initial approximation (3.9) and using the iterative formula (3.8) with $m = 1$ gives

$$u_1 = \chi_1\, u_0 + \hbar \int_0^t \left(\frac{\partial u_0}{\partial \tau} - \frac{\partial^2 u_0}{\partial x^2} \right) d\tau,$$

$$\begin{aligned} u_1 &= \hbar \int_0^t \pi^2 \sin(\pi x) d\tau, \\ u_1 &= \hbar\, \pi^2 \sin(\pi x)\, t, \\ u_1 &= \hbar\, \pi^2\, t\, u_0. \end{aligned} \tag{3.10}$$

Similarly for $m = 2$

$$u_2 = \hbar\,\pi^2 \sin(\pi x)\,t + \hbar\left(\hbar\,\pi^2 \sin(\pi x)\,t + \frac{\hbar\,\pi^4 \sin(\pi x)\,t^2}{2}\right),$$
$$u_2 = \left(1 + \hbar + \frac{\hbar\,\pi^2\,t}{2}\right) u_1. \tag{3.11}$$

The HAM procedure does become complex even for a simple test equation to compute by hand for $m > 2$.

Convergence

The convergence theorems for the HAM have been presented in Chapter 2. Applying the sufficient condition from theorem 2 to equation (3.10) gives:

$$\frac{\|u_1\|}{\|u_0\|} = |\hbar\,\pi^2\,t| \tag{3.12}$$

Similarly for equation (3.11):

$$\frac{\|u_2\|}{\|u_1\|} = \left|1 + \hbar + \frac{\hbar\,\pi^2\,t}{2}\right|. \tag{3.13}$$

The above restrictions will be monitored closely at each iteration and the values of $\hbar$ and t will be chosen such that the norm of the error is less than unity which will ensure convergence. The optimal value of $\hbar$ will be determined using $\hbar$-curves and will be examined further in Chapter 4.

3.1.2 Application to the Burgers equation

The Burgers equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = c^2\frac{\partial^2 u}{\partial x^2} \tag{3.14}$$

where $c \in \mathbb{R}$. For our purposes we take $c = 1$ which simplifies the algebra.

Similar to the heat equation, the linear operator for equation (3.14) has been chosen as

$$\mathcal{L}[\, u(x,t)] = \frac{\partial u(x,t)}{\partial t}\,, \tag{3.15}$$

with the property

$$\mathcal{L}[\, c_1] = 0, \tag{3.16}$$

where c_1 is a constant. The operator is chosen this way in order to satisfy the suggested rules by Liao [73]. This ensures that the operator is easily invertible and results is simpler deformation equations to be solved as m gets larger. The inverse operator is therefore defined as

$$\mathcal{L}^{-1}[\, u(x,t)] = \int_0^t u(x,\tau)\, d\tau\,. \tag{3.17}$$

The nonlinear operator is

$$\mathcal{N}[\, u(x,t)] = \frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u}{\partial x^2}\,, \tag{3.18}$$

and thus $\mathcal{R}_m$ is defined by

$$\mathcal{R}_m[\, u_{m-1}(x,t)] = \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} + \sum_{i=0}^{m-1} u_i(x,t)\frac{\partial u_{m-1-i}(x,t)}{\partial x}. \tag{3.19}$$

The iterative formula for Burgers equation is

$$u_m = \chi_m \, u_{m-1} + \hbar \int_0^t \left(\frac{\partial u_{m-1}}{\partial \tau} - \frac{\partial^2 u_{m-1}}{\partial x^2} + \sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial x} \right) d\tau. \qquad (3.20)$$

The above iteration formula will be computed in Maple. For $m < 5$, solutions can be computed efficiently in Maple. However, as m increases, there is a need for more powerful processing power in order to compute the higher order integrals.

Two separate initial conditions are chosen to illustrate the HAM on Burgers equation.

- The test initial condition

$$u(x, 0) = \frac{\alpha + \beta + (\beta - \alpha)e^{\eta}}{1 + e^{\eta}} \qquad (3.21)$$

  where $\eta = \alpha(x/\nu)$ and α, β and ν are arbitrary constants.

- The common initial condition

$$u(x, 0) = \sin \pi x. \qquad (3.22)$$

To illustrate the method and sufficient condition for convergence, one iteration is performed using initial condition (3.22). Using equation (3.22) and $m = 1$ in equation

(3.20) gives

$$u_1 = \chi_1 \, u_0 + \hbar \int_0^t \left(\frac{\partial u_0}{\partial \tau} - \frac{\partial^2 u_0}{\partial x^2} + u_0 \, \frac{\partial u_0}{\partial x} \right) d\tau,$$

$$u_1 = \hbar \int_0^t \left(\pi^2 \sin(\pi x) + \pi \sin(\pi x) \cos(\pi x)\right) d\tau, \tag{3.23}$$

$$u_1 = \hbar \, \pi \, t(\pi \sin(\pi x) + \sin(\pi x) \cos(\pi x)),$$

$$u_1 = \hbar \, \pi \, t(\pi + \cos(\pi x)) \, u_0.$$

Convergence

Using the same approach as the heat equation, the sufficient condition for convergence for equation (3.49) is

$$\frac{\|u_1\|}{\|u_0\|} = |\hbar \, \pi \, t(\pi + \cos(\pi x))|. \tag{3.24}$$

The value above will be monitored at each step and for each value of $\hbar$ also taking into account the error at each step to determine the optimal value of $\hbar$. The values of $\hbar$ will be plotted on the so-called $\hbar$-curves and discussed further in Chapter 4.

3.1.3 Application to the Bratu equation

The Bratu equations of interest in this study have the form

$$\frac{\partial^2 u}{\partial x^2} + \gamma \, e^u = 0, \qquad 0 < x < 1, \tag{3.25}$$

where $u = u(x)$ and $\gamma = -\pi^2$ or $\gamma = -2$.

The linear operator for the above equation (3.25) is

$$\mathcal{L}[\, u(x,t)] = \frac{\partial^2 u}{\partial x^2}\,, \tag{3.26}$$

with the property

$$\mathcal{L}[\, c_1] = 0, \tag{3.27}$$

where c_1 is a constant. The inverse operator is thus a double integral of the form

$$\mathcal{L}^{-1}[\, u(x)] = \int_0^x \int_0^s u(\phi)\, d\phi\, ds\,. \tag{3.28}$$

The nonlinear operator is

$$\mathcal{N}[\, u(x)] = \frac{\partial^2 u}{\partial x^2} + \gamma\, e^{u}\,, \tag{3.29}$$

and thus

$$\mathcal{R}_m[\, u_{m-1}(x,t)] = \frac{\partial^2 u_{m-1}(x)}{\partial x^2} + \gamma\, e^{u_{m-1}(x)}\,. \tag{3.30}$$

The iterative formula for the Bratu-type equation is

$$u_m = \chi_m\, u_{m-1} + \hbar \int_0^x \int_0^s \left(\frac{\partial^2 u_{m-1}}{\partial \phi^2} + \gamma\, e^{u_{m-1}} \right) d\phi\, ds. \tag{3.31}$$

A problem arises with the double integral due to the exponential function $e^{u(x)}$. Clearly equation (3.31) with any arbitrary initial condition will result in an undetermined integral in the next step. For example, suppose $u_0(x) = 0$ then applying the iterative formula (3.31) yields the following

$$u_1(x) = \frac{\hbar}{2}\,\gamma\, x^2.$$

The next iteration would result in the term containing $e^{u_{m-1}}$ being

$$\int_0^x \int_0^s \left(\frac{\hbar}{2} \gamma \, e^{h \, \gamma \, \phi^2} \right) d\phi \, ds,$$

which has no real solution due to integral $\int e^{\phi^2} d\phi$.

To overcome this limitation an approximation is imposed on $e^{u(x)}$. Using Taylor's theorem, the second order approximation of $e^{u(x)}$ is

$$e^{u(x)} = [1 + u(x) + \frac{1}{2!}(u(x))^2].$$

Substituting the Taylor approximation into equation (3.31) gives a second order Taylor modified iterative formula the for Bratu equation

$$u_m = \chi_m \, u_{m-1} + \hbar \int_0^x \int_0^s \left(\frac{\partial^2 u_{m-1}}{\partial \phi^2} + \gamma \left[1 + u_{m-1} + \frac{1}{2!}(u_{m-1})^2 \right] \right) d\phi \, ds. \quad (3.32)$$

Similarly a fifth order Taylor modified iterative formula for the Bratu equation is

$$\begin{aligned} u_m = \chi_m \, u_{m-1} + \hbar \int_0^x \int_0^s \left(\frac{\partial^2 u_{m-1}}{\partial \phi^2} + \gamma \left[1 + (u_{m-1}) \right.\right. \\ \left.\left. + \frac{1}{2!}(u_{m-1})^2 + \frac{1}{3!}(u_{m-1})^3 + \frac{1}{4!}(u_{m-1})^4 + + \frac{1}{5!}(u_{m-1})^5 \right] \right) d\phi \, ds. \end{aligned} \quad (3.33)$$

This subsequently allows for the evaluation of the double integral. The second and fifth order expansions have been chosen in order to compare the accuracy and impact of the Taylor approximation to the final solution.

There are two separate problems considered in this study with different values of γ, namely;

1. $\gamma = -\pi^2$, with the initial condition

$$u_0(x) = \pi x, \tag{3.34}$$

2. $\gamma = -2$, with the initial condition

$$u_0(x) = 0. \tag{3.35}$$

To illustrate the method by we apply the initial condition (3.35) as the initial approximation to the solution for $\gamma = -2$ and letting $m = 1$ in equation (3.32), the first iteration is

$$u_1 = \chi_1 \, u_0 + \hbar \int_0^x \int_0^s \left(\frac{\partial^2 u_0}{\partial \phi^2} - 2 \left[1 + u_0 + \frac{1}{2!} (u_0)^2 \right] \right) d\phi \, ds, \tag{3.36}$$

$$u_1 = \hbar \int_0^x \int_0^s -2 \, d\phi \, ds.$$

Using basic integration, equation (3.36) reduces to

$$u_1 = -\hbar x^2. \tag{3.37}$$

Convergence

As shown earlier, the exact solutions to the Bratu problems are known and the $\hbar$-curves will be compared to the solution that provides the smallest absolute error when compared to the exact solution. Convergence will be monitored by evaluating the absolute error at each iteration.

Higher order iterations of the HAM solutions are discussed in Chapter 4. We now move our attention to the variational iteration method (VIM).

3.2 The VIM applied to the test equations

3.2.1 Application to the heat equation

A similar approach as used in the HAM algorithm is used to determine the linear and nonlinear operators. The operators are also shown in the literature and are as follows

$$\mathcal{L}[\, u(x,t)] = \frac{\partial u}{\partial t}\,, \tag{3.38}$$

and

$$\mathcal{N}[\, \widetilde{u}(x,t)] = \frac{\partial^2 \widetilde{u}}{\partial x^2}\,, \tag{3.39}$$

Using equation (3.1) above, the correctional functional for the heat equation is

$$u_m(x,t) = u_{m-1}(x,t) + \int_0^t \lambda\left(\frac{\partial u_{m-1}}{\partial \tau} - \frac{\partial^2 \widetilde{u}_{m-1}}{\partial x^2}\right) d\tau, \tag{3.40}$$

To calculate the optimal Lagrange multiplier, λ, a variation is applied to equation (3.40)

$$\delta u_m = \delta u_{m-1} + \int_0^t \lambda\Big((\delta u_{m-1})_\tau - (\delta \widetilde{u}_{m-1})_{xx}\Big) d\tau. \tag{3.41}$$

Since $\delta \widetilde{u}_{m-1} = 0$, equation (3.41) reduces to

$$\delta u_m(x,t) = \delta u_{m-1}(x,t) + \int_0^t \lambda(\delta u_{m-1})_\tau d\tau. \tag{3.42}$$

Integrating by parts reveals the stationary conditions

$$\lambda'(\tau)|_{\tau=t} = 0, \tag{3.43}$$

$$1+\lambda(\tau)|_{\tau=t} = 0. \tag{3.44}$$

Solving the above equations yields $\lambda = -1$ and therefore the correctional functional for the heat equation reduces to

$$u_m = u_{m-1} - \int_0^t \left(\frac{\partial u_{m-1}}{\partial \tau} - \frac{\partial^2 u_{m-1}}{\partial x^2}\right) d\tau. \tag{3.45}$$

Using the initial approximation (3.9) and letting $m = 1$, the first iteration is

$$u_1 = u_0 - \int_0^t \left(\frac{\partial u_0}{\partial \tau} - \frac{\partial^2 u_0}{\partial x^2}\right) d\tau,$$

$$u_1 = \sin(\pi x) - \int_0^t \pi^2 \sin(\pi x) d\tau, \tag{3.46}$$

$$u_1 = \sin(\pi x) - \pi^2 \sin(\pi x)\, t,$$

$$u_1 = (1 - \pi^2\, t)\, u_0.$$

Convergence

The convergence theorems of the VIM has been shown in Chapter 2. Applying the sufficient condition from theorem 1 to equation (3.46) gives

$$\frac{\|u_1\|}{\|u_0\|} = |1 - \pi^2\, t|. \tag{3.47}$$

The ratio above will be monitored closely ensuring it does not exceed one. The remaining iterations for the heat equation will be computed using Maple software.

3.2.2 Application to the Burgers equation

Using equation (3.14) and applying the same methodology above, noting that $\lambda = -1$ for Burgers equation as well, the correctional functional is

$$u_m = u_{m-1} - \int_0^t \left(\frac{\partial u_{m-1}}{\partial \tau} + u_{m-1}\frac{\partial u_{m-1}}{\partial x} - \frac{\partial^2 u_{m-1}}{\partial x^2} \right) d\tau. \qquad (3.48)$$

To illustrate the method, we take the initial condition (3.22) and $m = 1$ in (3.48) which gives

$$u_1 = u_0 - \int_0^t \left(\frac{\partial u_0}{\partial \tau} + u_0\frac{\partial u_0}{\partial x} - \frac{\partial^2 u_0}{\partial x^2} \right) d\tau,$$

$$u_1 = \sin(\pi x) - \int_0^t \left(\pi^2 \sin(\pi x) + \pi \sin(\pi x)\cos(\pi x)\right) d\tau, \qquad (3.49)$$

$$u_1 = \sin(\pi x) - \pi\, t(\pi \sin(\pi x) + \sin(\pi x)\cos(\pi x)),$$

$$u_1 = [1 - \pi^2\, t - \pi\, t\cos(\pi x)]\, u_0.$$

Convergence

Similarly to the convergence theorem above, we monitor the ratio below for each iteration of the VIM

$$\frac{||u_1||}{||u_0||} = |1 - \pi^2\, t - \pi\, t\cos(\pi x)|. \qquad (3.50)$$

The remaining iterations for the BE equation will be computed using Maple.

3.2.3 Application to the Bratu equation

The correctional functional for the Bratu-type equation is

$$u_m(x) = u_{m-1}(x) + \int_0^x \lambda \left[\frac{\partial^2 u_{m-1}}{\partial \phi^2} + \gamma \, e^{\widetilde{u}_{m-1}} \right] d\phi. \tag{3.51}$$

To calculate the optimal Lagrange multiplier, λ, we apply a variation to (3.51)

$$\delta u_m = \delta u_{m-1} + \delta \int_0^x \lambda \left[\frac{\partial^2 u_{m-1}}{\partial \phi^2} + \gamma \, e^{\widetilde{u}_{m-1}} \right] d\phi. \tag{3.52}$$

Since $\delta \widetilde{u}_{m-1} = 0$, all terms involving $\delta \widetilde{u}_{m-1}$ are set to zero and (3.52) becomes

$$\delta u_m = \delta u_{m-1} + \delta \int_0^x \lambda \left[\frac{\partial^2 u_{m-1}}{\partial \phi^2} \right] d\phi. \tag{3.53}$$

Integrating by parts gives

$$\delta u_m = \delta u_{m-1}(1 - \lambda') + \delta u_{m-1} \lambda + \int_0^x \delta u_{m-1} \lambda'' d\phi, \tag{3.54}$$

which results in the following stationary conditions:

$$\begin{aligned} (1 - \lambda')|_{\phi = x} &= 0\,, \\ \lambda|_{\phi = x} &= 0\,, \\ \lambda''|_{\phi = x} &= 0\,. \end{aligned} \tag{3.55}$$

Solving (3.55) yields:

$$\lambda = \phi - x, \tag{3.56}$$

and thus the variational iteration formula for the Bratu equation becomes:

$$u_m = u_{m-1} + \int_0^x (\phi - x)\left[\frac{\partial^2 u_{m-1}}{\partial \phi^2} + \gamma\, e^{u_{m-1}}\right] d\phi. \tag{3.57}$$

The above iteration formula requires evaluation of a single definite integral and as we did in the HAM formulation we need to approximate the exponential function using Taylor's theorem.

In the same manner as before, the second and fifth order Taylor modified iteration formula for the Bratu equation is

$$u_m = u_{m-1} + \int_0^x (\phi - x)\left(\frac{\partial^2 u_{m-1}}{\partial \phi^2} + \gamma\left[1 + u_{m-1} + \frac{1}{2!}(u_{m-1})^2\right]\right) d\phi, \tag{3.58}$$

and

$$\begin{aligned} u_m = u_{m-1} + \int_0^x (\phi - x)\Bigg(\frac{\partial^2 u_{m-1}}{\partial \phi^2} + \gamma\Bigg[1 + u_{m-1} \\ + \frac{1}{2!}(u_{m-1})^2 + \frac{1}{3!}(u_{m-1})^3 + \frac{1}{4!}(u_{m-1})^4 + + \frac{1}{5!}(u_{m-1})^5\Bigg]\Bigg) d\phi. \end{aligned} \tag{3.59}$$

For illustration purposes we apply the initial condition (3.35) as the initial approximation to the solution for $\gamma = -2$ and letting $m = 1$ in equation (3.58), the first iteration is

$$\begin{aligned} u_1 &= -2\int_0^x (\phi - x)\, d\phi, \\ u_1 &= x^2. \end{aligned} \tag{3.60}$$

Convergence

The convergence of the method will be monitored using the absolute error between the VIM solution as compared to the exact solution of the problem at each iteration.

Now that we have applied our analytical schemes, we move onto the numerical scheme used in this study - the method of finite differences (FDM).

3.3 The FDM applied to the heat and Burgers equations

The FDM method will only be applied to the heat and Burgers equation. The application to the heat equation is purely for test purposes in order to verify that all the computer codes are running correctly. This method is primarily used to provide contrast between the analytical methods and to provide direction in determining whether the solutions are converging or diverging.

The application to the heat equation is as follows. Suppose u is represented by:

$$u = u(x_i, t_j) = u_i^j.$$

Applying the approximations of the first and second derivatives to equation (3.1) gives

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} \tag{3.61}$$

which is simplified to

$$u_i^{j+1} = (1 - 2r)u_i^j + r(u_{i-1}^j + u_{i+1}^j) \tag{3.62}$$

where $r = k/h^2$. It has been shown in literature that for a convergent solution, the value of r must be less than or equal to a half.

Similarly, applying the necessary derivatives to Burgers equation (3.14) yields

$$\frac{u_i^{j+1} - u_i^j}{k} = \left(\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} \right) - u_i^j \left(\frac{u_{i+1}^j - u_i^j}{h} \right), \tag{3.63}$$

which simplifies to

$$u_i^{j+1} = \frac{k}{h^2} \left(u_{i+1}^j - 2u_i^j + u_{i-1}^j \right) - \frac{k}{h} u_i^j \left(u_{i+1}^j - u_i^j \right) + u_i^j. \tag{3.64}$$

The iteration schemes above are easily solved using Matlab and discussed further in Chapter 4 with emphasis around convergence, accuracy and speed of each of the methods.

In summary, we have presented algorithms for finding solutions to the test equations in this chapter and illustrated how the analytical methods are used to generate solutions. We also showed the modifications required for the Bratu problem in order for the integration to be applied at higher orders. In Chapter 4 we present the results obtained using these schemes and the computational software, Matlab and Maple.

Chapter 4

Results and Discussion

As shown in Chapter 3, the analytical methods used in this study are fairly complex to apply by hand for more than one or two iterations. It must also be noted that a single iteration is sometimes not sufficient to give an accurate solution to the test problems and further iterations may need to be done. We thus require mathematical software to program the algorithms to obtain higher order results.

The homotopy analysis method (HAM) and the variational iteration method (VIM) were coded in Maple while the method of finite differences (FDM) was coded in Matlab. The Maple software package develops closed form solutions and is the reason it has been chosen to compute the analytical methods. Matlab was used for the FDM method as it is a proven array based software which develop numerical results. The graphs were plotted in Matlab due to convenience. The results were then compared to the exact solution, where possible, and measured in terms of the convergence rate, accuracy and computation speed. The findings are presented below in graphical and tabulated form .

4.1 The solution to the heat equation

The problem to be solved is

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

subject to

$$u(x,0) = \sin(\pi x). \tag{4.2}$$

The results will be compared to the exact solution

$$u_e = \sin(\pi x)\, e^{-\pi^2 t}. \tag{4.3}$$

The initial approximation for all three schemes is chosen as

$$u_0 = \sin(\pi x). \tag{4.4}$$

Since the problem is linear, the computer algorithms proved to be computationally efficient and all three schemes were run until the exact solution was reached at $t =$ 0.1s.

HAM Solution

The optimum value of the auxiliary parameter $\hbar$ for the HAM was determined by trial and error. We began by comparing the absolute error that is, the difference between the approximate solution and the exact solution, using the $\hbar$-values, $\hbar = \{-0.25, -0.50, -0.75, -1.00, -1.25\}$. The number of iterations before the method converged up to order seven, consistent with measurements in the literature, are shown in Table 4.1 below.

Table 4.1: The heat equation: Comparison of the HAM and the exact solutions of different $\hbar$-values at $t = 0.1$s.

x	HAM $\hbar = -0.25$	HAM $\hbar = -0.5$	HAM $\hbar = -0.75$	HAM $\hbar = -1.0$	HAM $\hbar = -1.25$	Exact Solution
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.1151731	0.1151731	0.1151731	0.1151731	0.1151730	0.1151731
0.2	0.2190723	0.2190722	0.2190722	0.2190722	0.2190722	0.2190722
0.3	0.3015272	0.3015270	0.3015270	0.3015270	0.3015270	0.3015270
0.4	0.3544665	0.3544662	0.3544662	0.3544662	0.3544662	0.3544662
0.5	0.3727081	0.3727079	0.3727078	0.3727079	0.3727078	0.3727078
0.6	0.3544665	0.3544662	0.3544662	0.3544662	0.3544662	0.3544662
0.7	0.3015272	0.3015270	0.3015270	0.3015270	0.3015270	0.3015270
0.8	0.2190723	0.2190722	0.2190722	0.2190722	0.2190722	0.2190722
0.9	0.1151731	0.1151731	0.1151731	0.1151731	0.1151731	0.1151731
1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
Time (s)	0.234	0.078	0.047	0.015	0.094	0.000
No. of steps	43	21	12	10	23	1
Conv. test ratio	0.881	0.443	0.259	-0.099	-0.386	-

Table 4.1 shows that the fastest convergence region lies between $\hbar = -0.75$ and $\hbar = -1.00$. The convergence test ratio was used at each step ensuring that it remained below unity for convergence. Now that we have isolated the region of fastest convergence, further values of $\hbar$ were tried and it was found for $\hbar = -0.95$, the solution converged after 8 iterations. The error curves were plotted showing the absolute error after evaluating the solution at every value of $\hbar$ after 7 iterations. This is shown in Figures 4.1 and 4.2 below. Figure 4.1 shows the error curves for $\hbar = \{-0.25, -0.50, -1.00, -1.25\}$ and Figure 4.2 shows the refinement where $\hbar = \{-0.75, -0.85, -0.95, -1.00\}$.

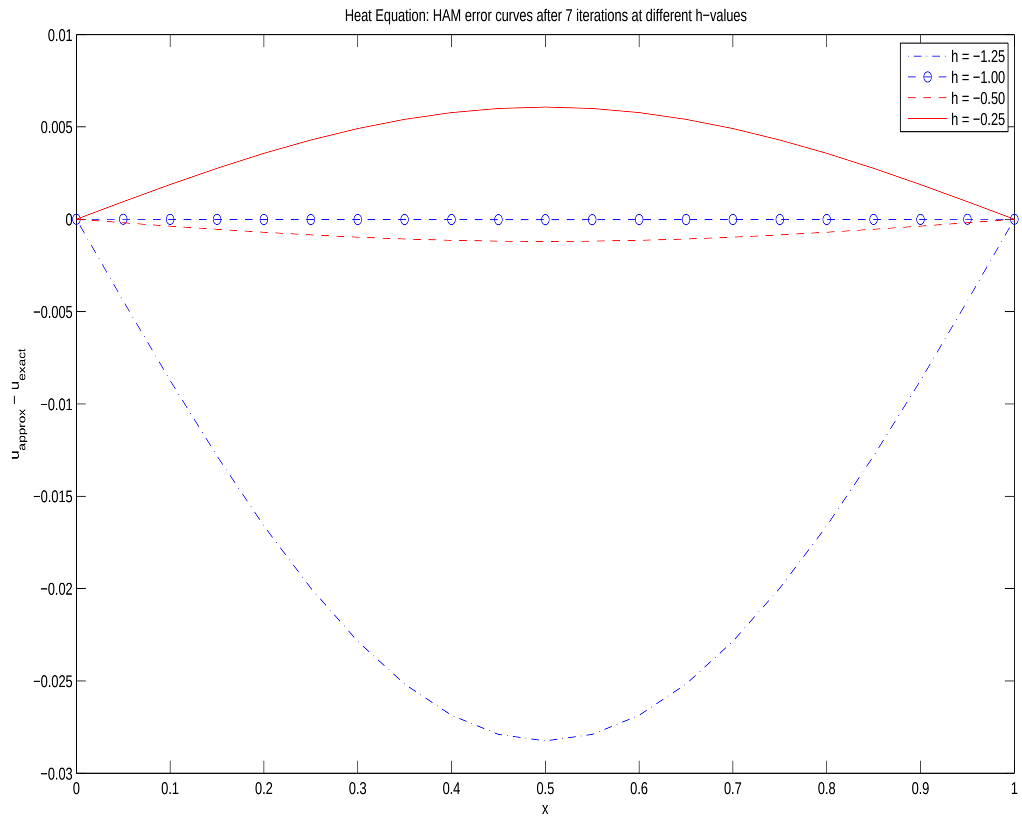

Figure 4.1: The heat equation: HAM error curves at different values of $\hbar$ for $t = 0.1$s.

VIM and FDM solution

It was found that the VIM algorithm produced equivalent results as the HAM algorithm when $\hbar = -1$. This is due to the value of the Lagrange multiplier at the stationary point which was found to be $\lambda = -1$. This does not come as a surprise as Liao [73, 74] has shown that various other analytical methods, such as the Adomian decomposition method and the homotopy perturbation method, are also special cases of the HAM depending on the value of $\hbar$. Therefore, a

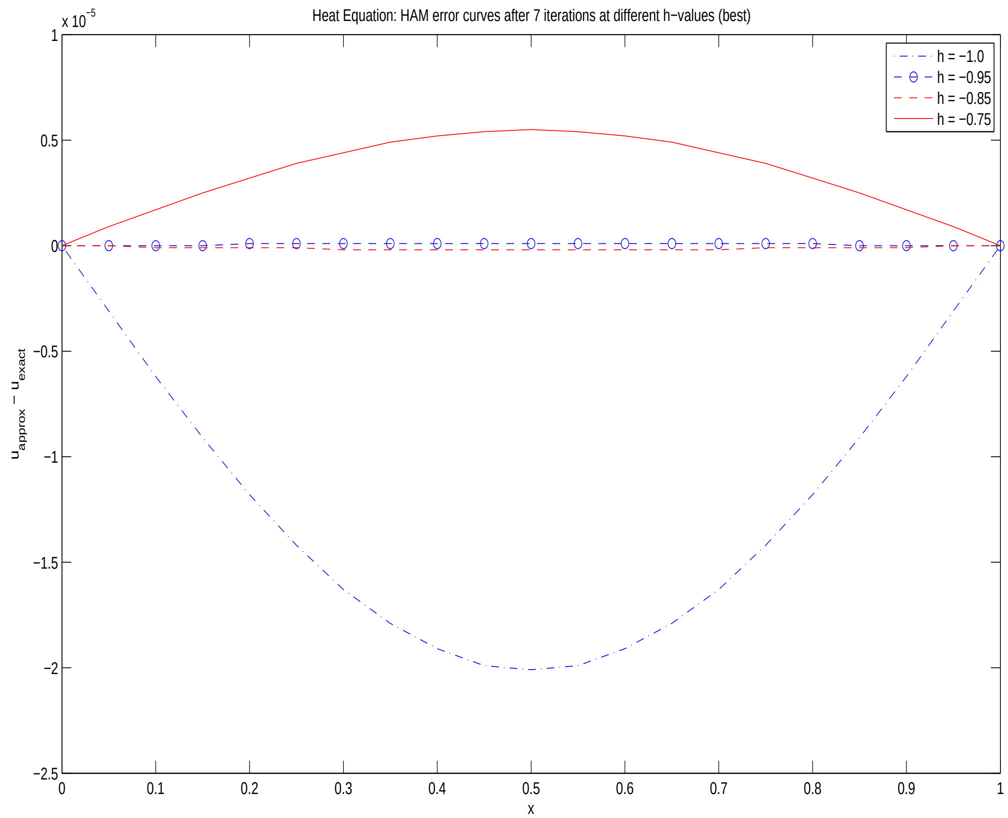

Figure 4.2: The heat equation: HAM error curves at different values of $\hbar$ for $t = 0.1$s (refined).

finding made in this study is that the VIM is also a special case of the HAM for the heat equation. Thus like the HAM, the VIM solution converged after 10 iterations.

The FDM was also run to determine whether an accurate solution, up to order 7, could be generated from the numerical scheme. The scheme converged after 240 steps in t and 20 steps in x ($r = 0.2$). The final results of all the schemes are summarized in Table 4.2 below.

Table 4.2: The heat equation: Comparison of the HAM, VIM and FDM solution for $t = 0.1s$.

x	Exact Solution	HAM $\hbar = -0.95$	VIM $\lambda = -1$	FDM	Absolute Error Exact − FDM
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.1151731	0.1151731	0.1151731	0.1151732	0.0000001
0.2	0.2190722	0.2190722	0.2190722	0.2190724	0.0000002
0.3	0.3015270	0.3015270	0.3015270	0.3015273	0.0000003
0.4	0.3544662	0.3544663	0.3544662	0.3544666	0.0000004
0.5	0.3727079	0.3727079	0.3727079	0.3727083	0.0000004
0.6	0.3544662	0.3544662	0.3544662	0.3544666	0.0000004
0.7	0.3015270	0.3015270	0.3015270	0.3015273	0.0000003
0.8	0.2190722	0.2190722	0.2190722	0.2190724	0.0000002
0.9	0.1151731	0.1151731	0.1151731	0.1151732	0.0000001
1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
Time (s)	0.000	0.015	0.015	0.016	-
No. of steps	1	8	10	240/20	-

The results in Table 4.2 show that all three schemes converged to the exact solution relatively easily. The HAM with $\hbar = -0.95$ converged in the fewest number of iterations, 8. The HAM and VIM are both more accurate than the numerical scheme, FDM. However, the FDM also produced excellent results for the heat equation. In terms of computational efficiency, all three schemes compiled in roughly the same amount of time, 0.015s, which is practically instantly. A plot of the FDM solution alongside the exact solution is shown in Figure 4.3 below.

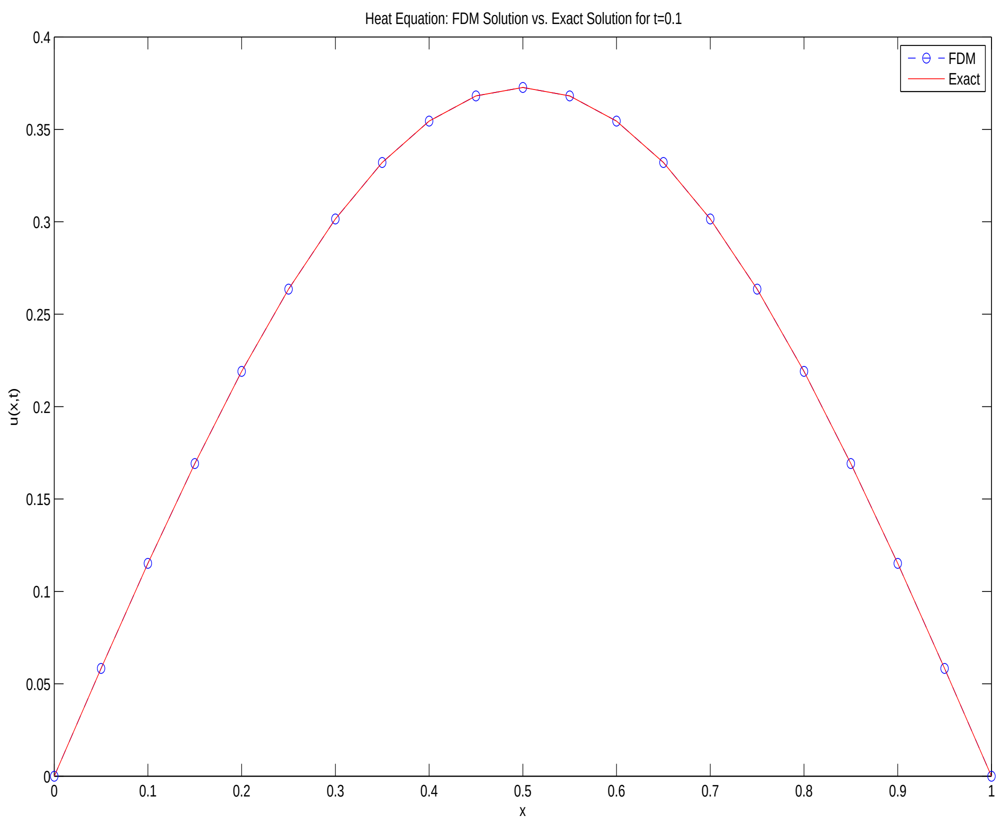

Figure 4.3: The heat equation: FDM solution vs. exact solution for $t = 0.1$s.

We now turn our attention to solving the partial differential equations in this study, beginning with the Burgers equation.

4.2 The solution to the Burgers equation

The Burgers equation was solved using two different initial conditions. This is due to the existence of an exact solution on one condition which will be used to verify the accuracy of the code used to generate the HAM and VIM solutions. The equation to be solved is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = c^2\frac{\partial^2 u}{\partial x^2}\,, \tag{4.5}$$

where $c = 1$ for our purposes.

4.2.1 The Burgers equation with test initial condition

The initial condition used to test the programs is given by

$$u(x,0) = \frac{\alpha + \beta + (\beta - \alpha)e^{\eta}}{1 + e^{\eta}}\,, \tag{4.6}$$

where $\eta = \alpha(x/\nu)$ and α, β and ν are arbitrary constants.

The exact solution to Burgers equation with initial condition (4.6), Abdou and Soliman [8], is

$$u_e(x,t) = \frac{\alpha + \beta + (\beta - \alpha)e^{\xi}}{1 + e^{\xi}}\,, \tag{4.7}$$

where $\xi = (\alpha/\nu)(x - \beta t)$.

The values of the constants were chosen randomly in order to obtain numerical values for comparison and are $\alpha = -2$, $\beta = 1$ and $\nu = 1$.

HAM solution

The optimum value of the auxiliary parameter $\hbar$ was determined by trial and error. We began with $\hbar = -1.0$ and compiled the program. The solution converged to order 6 after 5 iterations with the 5^{th} iteration requiring a large amount of resources. The convergence to order 5 is used in most literature concerning Burgers equation and thus sixth order convergence using the HAM is a very good approximation of the analytical solution. We examined different choices of $\hbar$ in order to determine if the convergence rate could be improved. Table 4.3 shows the solution after 5 iterations with $\hbar$-values $\{-0.25, -0.50, -0.75, -1.00, -1.25\}$.

Table 4.3: Burgers equation: Comparison of HAM solutions at different $\hbar$-values at $t = 0.1$s.

x	HAM $\hbar = -0.25$	HAM $\hbar = -0.5$	HAM $\hbar = -0.75$	HAM $\hbar = -1.0$	HAM $\hbar = -1.25$	Exact Solution
0.00	1.1524701	1.1934168	1.1992077	1.1993360	1.1993896	1.1993360
0.10	0.9523460	0.9937703	0.9998449	1.0000000	1.0000238	1.0000000
0.20	0.7531800	0.7942465	0.8004828	0.8006639	0.8006654	0.8006640
0.30	0.5588791	0.5987815	0.6050449	0.6052492	0.6052357	0.6052494
0.40	0.3729739	0.4109969	0.4171517	0.4173746	0.4173523	0.4173748
0.50	0.1983786	0.2339453	0.2398668	0.2401020	0.2400755	0.2401021
0.60	0.0372436	0.0699416	0.0755252	0.0757656	0.0757382	0.0757657
0.70	-0.1090920	-0.0795051	-0.0743378	-0.0740992	-0.0741255	-0.0740991
0.80	-0.2400574	-0.2136671	-0.2089661	-0.2087355	-0.2087597	-0.2087356
0.90	-0.3557425	-0.3325024	-0.3282909	-0.3280735	-0.3280949	-0.3280735
1.00	-0.4567556	-0.4365182	-0.4327964	-0.4325957	-0.4326142	-0.4325957
Time (s)	15.959	16.006	16.037	16.489	15.990	0.000
No. of steps	5	5	5	5	5	1

The table above shows the fastest convergent solution lies between $\hbar = -0.75$ and $\hbar = -1.25$. Further analysis on different values of $\hbar$ revealed that the optimal value of

$\hbar$ was in fact $\hbar = -1.0$. The error curves below show the absolute error between the approximate solution and the exact solution for $\hbar = \{-0.75, -0.85, -0.95, -1.00\}$.

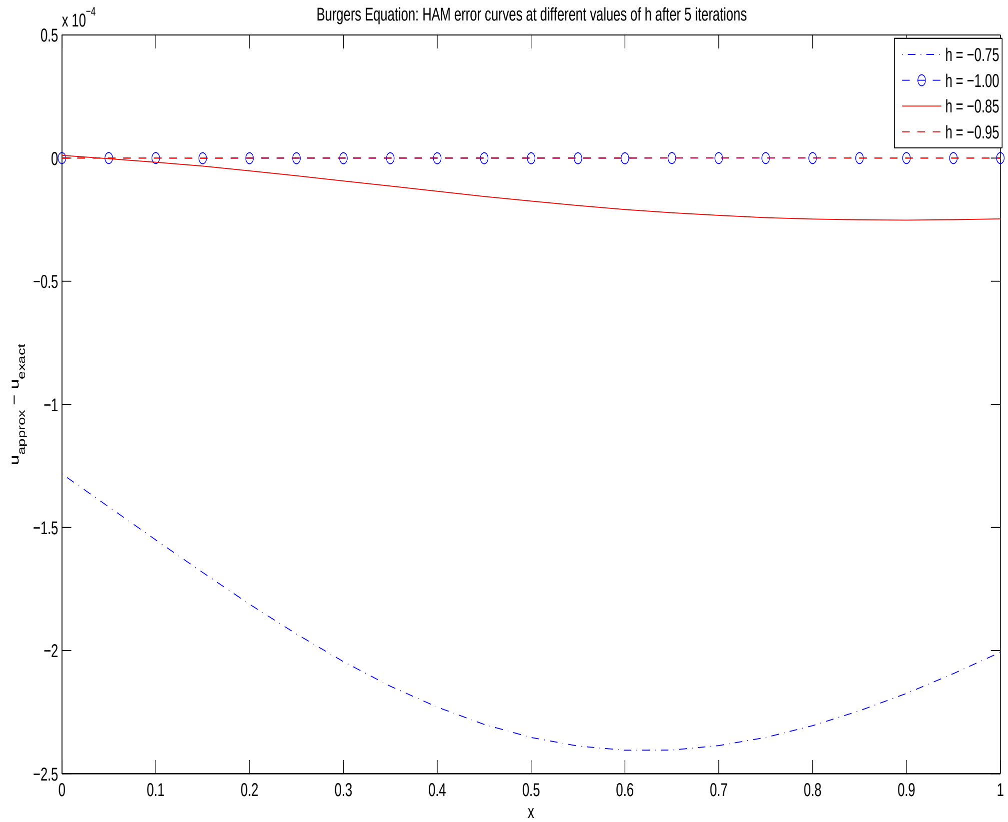

Figure 4.4: Burgers equation: HAM error curves at different values of $\hbar$ for $t = 0.1$s.

VIM Solution

The HAM program converged to the exact solution after 5 iterations using $\hbar = -1.0$. The VIM program with 4 iterations gave an accurate solution up to order 4 but the program required further resources was unable to provide a 5^{th} iteration after compiling for over 60 minutes. The summarized results are shown in Table 4.4 below.

Table 4.4: Burgers equation: Comparison of the HAM and the VIM solution for $t = 0.1$s.

x	Exact Solution	VIM $\lambda = -1.0$	HAM $\hbar = -1.0$	HAM $\hbar = -1.0$	Error VIM - Exact
0.00	1.1993360	1.1993925	1.1993333	1.1993360	0.0000566
0.10	1.0000000	1.0000670	0.9999975	1.0000000	0.0000670
0.20	0.8006640	0.8007232	0.8006621	0.8006639	0.0000592
0.30	0.6052494	0.6052861	0.6052482	0.6052492	0.0000367
0.40	0.4173748	0.4173836	0.4173745	0.4173746	0.0000088
0.50	0.2401021	0.2400870	0.2401025	0.2401020	-0.0000151
0.60	0.0757657	0.0757364	0.0757666	0.0757656	-0.0000293
0.70	-0.0740991	-0.0741322	-0.0740979	-0.0740992	-0.0000331
0.80	-0.2087356	-0.2087648	-0.2087343	-0.2087355	-0.0000293
0.90	-0.3280735	-0.3280953	-0.3280724	-0.3280735	-0.0000217
1.00	-0.4325957	-0.4326094	-0.4325948	-0.4325957	-0.0000136
Time (s)	0.000	25.646	0.827	16.489	-
No. of steps	1	4	4	5	-

The HAM solution after 4 iterations is shown in Table 4.4 above to demonstrate that the HAM is a faster converging and a more accurate method as compared to VIM for the Burgers equation. In terms of computational efficiency, the HAM also takes a shorter time and requires less processing power and is able to provide more iterations than the VIM. However, the VIM solution is still a good approximation to the exact solution as shown in Figure 4.5 below.

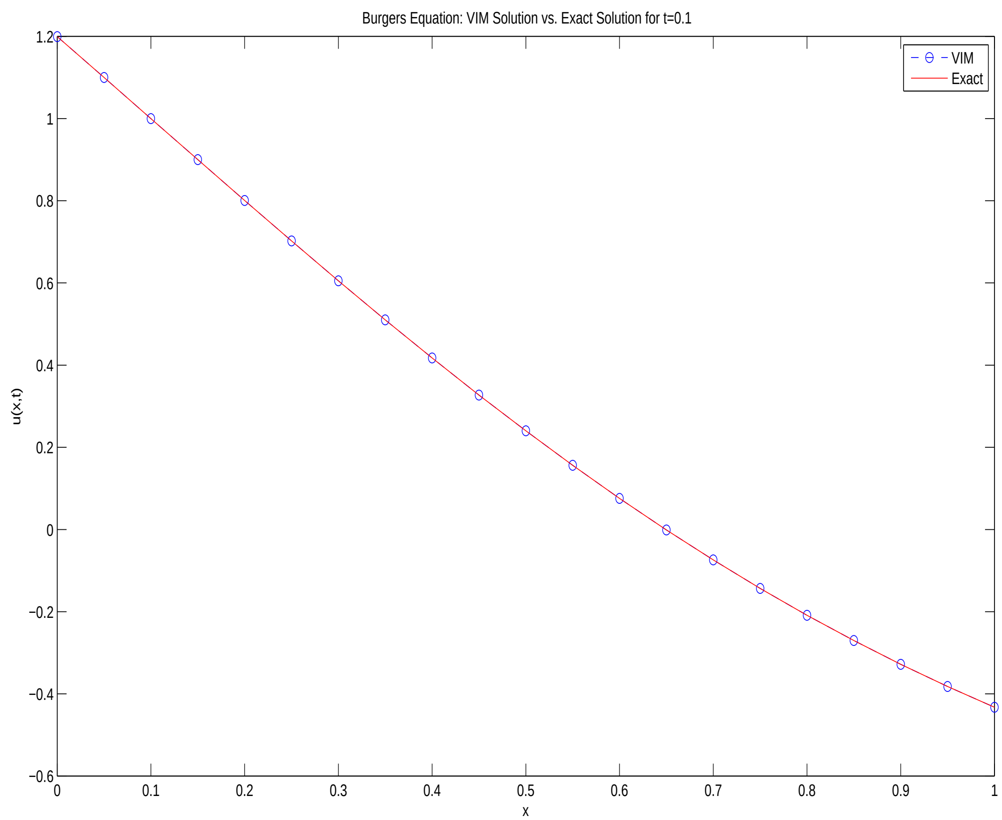

Figure 4.5: Burgers equation: VIM solution vs. exact solution for $t = 0.1$s.

We now turn our attention to the common initial condition.

4.2.2 The Burgers equation with $u(x, 0) = \sin(\pi x)$

We apply a similar approach as before. The HAM and VIM programs will be used to generate an analytical solution which will be then compared against a numerical solution determined by FDM. This will demonstrate the ability of these analytical methods to provide solutions which are accurate and convergent without the need for a complete exact solution to compare to. The complete exact solution for the problem is not computed in this study but can be obtained using Fourier analysis. The numerical values to the exact solution for $t = 0.1$ have been used, Aksan and Ozdes [14], Inc [55], for comparative purposes to the solutions generated by the three schemes.

The HAM solution was generated using different values of $\hbar$. We began with the choice of $\hbar = -1.0$ which gave unfavourable results after 7 iterations. With $\hbar = -0.25$ and $\hbar = -0.5$ better results were obtained and it was evident the optimal $\hbar$ lied in this range. Further analysis resulted in convergence to order 5 after 7 iterations at $\hbar = -0.33$.

Due to limitations on computer processing equipment, the VIM program could not compile for more than 5 iterations and the solution did not converge to the analytical solution. The FDM solution was run for 240 steps in t and 20 steps in x. The resulting solution are shown in Table 4.5 below.

Table 4.5: Burgers equation: Comparison of the HAM, VIM and FDM solution for $t = 0.1$s.

x	Exact Solution	HAM $\hbar = -0.25$	HAM $\hbar = -0.5$	HAM $\hbar = -0.33$	VIM $\lambda = -1$	FDM
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.10	0.10954	0.11283	0.10829	0.10954	-0.27375	0.11080
0.20	0.20979	0.21582	0.20889	0.20945	-0.29944	0.21230
0.30	0.29190	0.29958	0.29078	0.29063	-0.00283	0.29555
0.40	0.34792	0.35578	0.34578	0.34506	0.44059	0.35250
0.50	0.37158	0.37803	0.36996	0.36668	0.73187	0.37668
0.60	0.35922	0.36290	0.35889	0.35226	0.68826	0.36413
0.70	0.31006	0.31091	0.31003	0.30221	0.38466	0.31436
0.80	0.22793	0.22693	0.22743	0.22093	0.07284	0.23111
0.90	0.12069	0.11959	0.12025	0.11658	-0.05106	0.12242
1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Time (s)	0.000	19.891	15.319	17.300	96.495	0.044
No. of steps	1	7	7	7	5	240/20

As shown in the test problem (4.6), the HAM solution is more accurate method and faster converging than the VIM solution. The point is further validated using the different initial condition above. The VIM solution would have most likely converged given further iterations. However, we are interested in the most accurate and fastest converging method. The main disadvantage of the VIM in this problem is the fixed value of $\lambda = -1.0$ which cannot be controlled to improve the convergence rate. We saw a similar issue using the HAM at $\hbar = -1.0$ but the algorithm allowed for modification which improved the convergence to the exact solution.

The FDM solution is in closer relation to the exact solution than the VIM solution and is shown plotted alongside the HAM solution in Figure 4.6 below. The HAM is however, the superior method to the VIM and this is also seen when compared to other methods, Aksan and Soliman [14], which take longer to converge to the exact solution for Burgers equation.

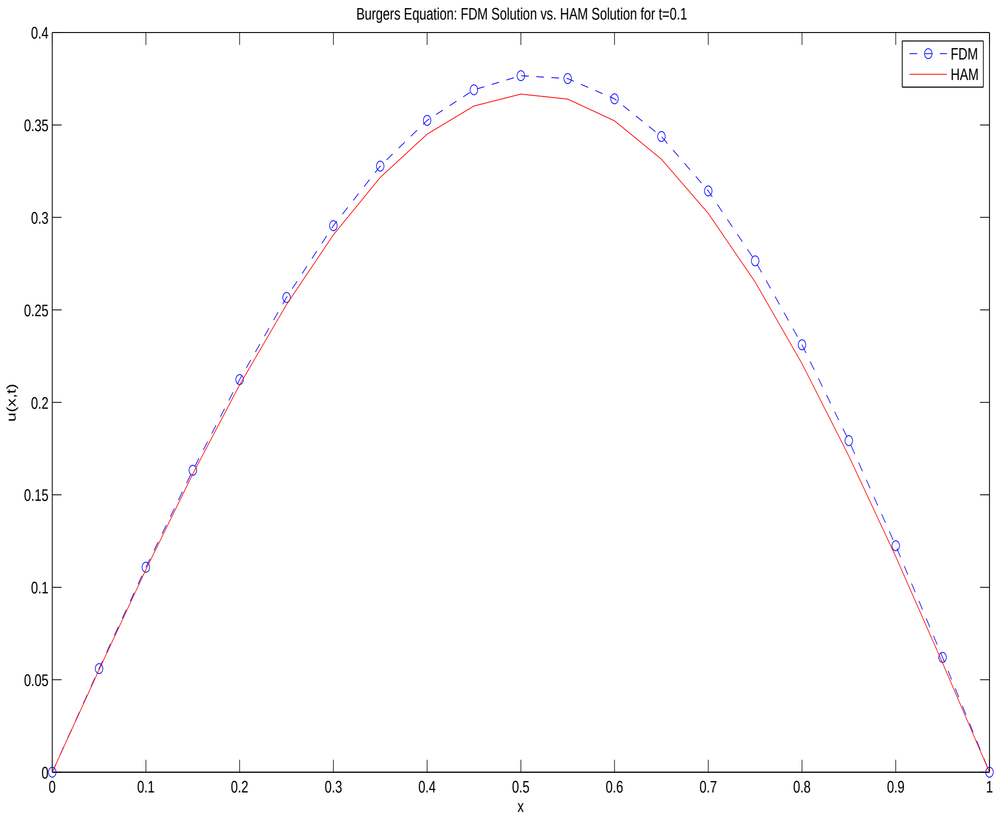

Figure 4.6: Burgers equation: FDM solution vs. HAM solution for $t = 0.1$s.

In order to demonstrate some form of convergence on the VIM algorithm, we looked at the solution for $t = 0.03$s. The VIM and FDM showed good results in relation to the analytical solution found using the HAM. The results are shown in Table 4.6 and graphically in Figure 4.7 below.

Table 4.6: Comparison of the HAM, VIM and FDM solution for $t = 0.03$s.

x	HAM $\hbar = -0.33$	VIM $\lambda = -1.0$	FDM
0.00	0.0000000	0.0000000	0.0000000
0.10	0.2206276	0.2178745	0.2195648
0.20	0.4227149	0.4179439	0.4207804
0.30	0.5885362	0.5825809	0.5859436
0.40	0.7021776	0.6953957	0.6990543
0.50	0.7508569	0.7432256	0.7472499
0.60	0.7266016	0.7183243	0.7226016
0.70	0.6281162	0.6200544	0.6240387
0.80	0.4623707	0.4558789	0.4588610
0.90	0.2451743	0.2415497	0.2430855
1.00	0.0000000	0.0000000	0.0000000
Time (s)	16.786	103.450	0.09
No. of steps	7	5	240/20

We now move onto the Bratu test problems.

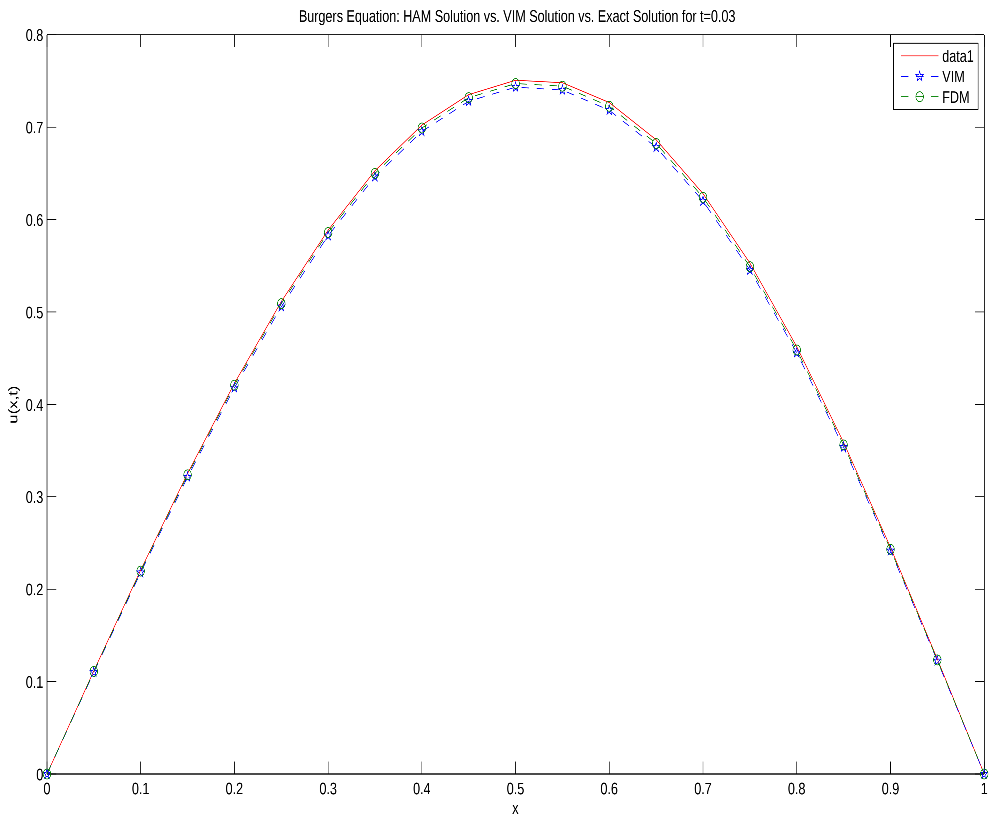

Figure 4.7: Burgers equation: HAM solution vs. VIM solution vs. FDM solution for $t = 0.03$s.

4.3 The solution to the Bratu equations

There are two different problems which will be solved by the HAM and VIM and thereafter compared to the analytical solutions for each method. Since the analytical solutions exist and our study is geared towards analytical methods, the FDM will not be computed for the Bratu equations. The purpose behind using the Bratu problem is to test which method will perform better when measured against the same number of iterations. This test is also limited by the processing limitations of the computer hardware used to generate the analytical solutions.

4.3.1 The solution for $\gamma = -\pi^2$

The problem to be solved is as follows

$$\frac{\partial^2 u}{\partial x^2} - \pi^2 \, e^u = 0 \,, \qquad u(0) = u(1) = 0 \,. \tag{4.8}$$

The solutions generated using the VIM and HAM will be compared to the exact solution, Wazwaz [106],

$$u_e = -\ln\left(1 + \cos\left[\left(\frac{1}{2} + x\right)\pi\right]\right) \tag{4.9}$$

The initial condition is chosen as,

$$u_0(x) = \pi \, x$$

which was obtained from Wazwaz [106] and the region chosen for the analysis was $x \in [-0.4, 0.4]$ due to the fact that the exact solution (4.9) has an infinite value at $x = 0.5$. The HAM and VIM with second and fifth order Taylor approximations were run for 1 iteration. The error curves using the 5^{th} order Taylor approximation on HAM are shown in Figure 4.8 below.

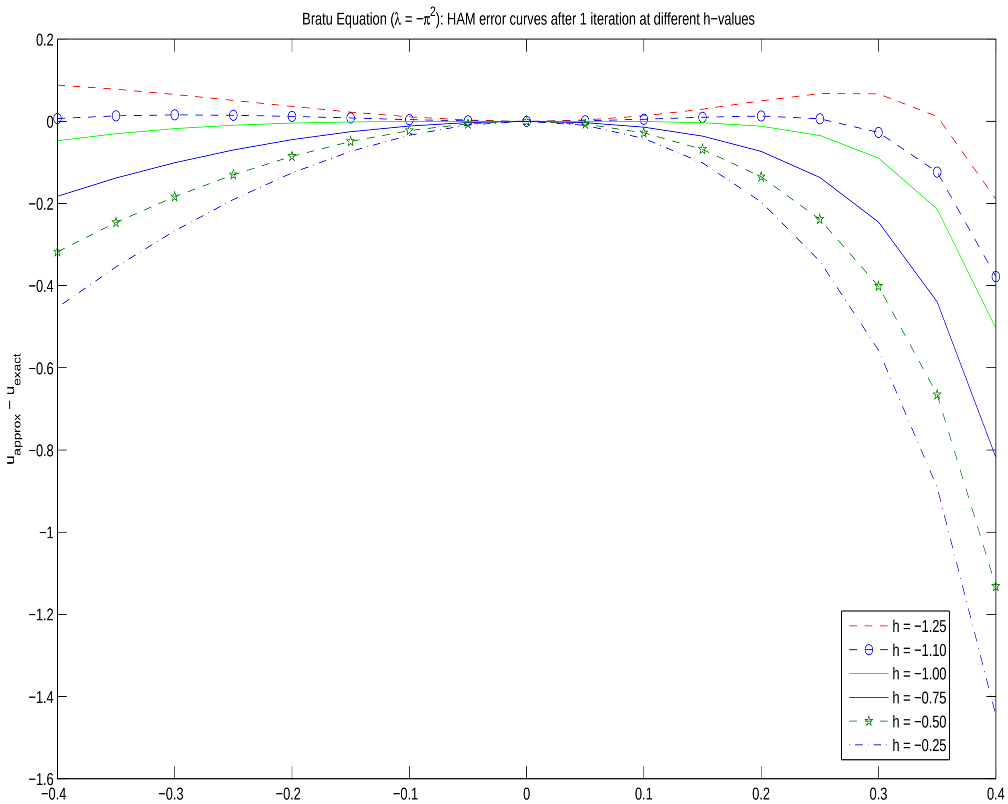

Figure 4.8: Bratu equation ($\lambda = -\pi^2$): HAM error curves at different values of $\hbar$ after 1 iteration.

The best $\hbar$ value was found as $\hbar = -1.10$. The main region where the solution diverged was near the critical value at $x = 0.5$. This is expected as the polynomial form of the solution cannot grow as rapidly as the exact solution which consists of logarithmic and cosine functions. The summarized results are shown in the Table 4.7 below.

Table 4.7: Bratu's equation ($\lambda = -\pi^2$): Comparison of the HAM and the VIM solution.

x	Exact Solution	HAM Taylor 2nd	HAM Taylor 5th	VIM Taylor 2nd	VIM Taylor 5th
-0.4	-0.6683710	-0.6376255	-0.6614144	-0.6938993	-0.7155256
-0.3	-0.5927836	-0.5712503	-0.5771403	-0.6049983	-0.6103528
-0.2	-0.4623401	-0.4495198	-0.4503319	-0.4657742	-0.4665125
-0.1	-0.2692765	-0.2651145	-0.2651411	-0.2695731	-0.2695973
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.3696400	0.3745730	0.3746026	0.3690809	0.3691078
0.2	0.8862108	0.8980690	0.8990691	0.8735463	0.8744554
0.3	1.6555708	1.6206674	1.6286989	1.5590138	1.5663152
0.4	3.0170890	2.6032626	2.6390838	2.4808421	2.5134068
Time (s)	0.000	0.000	0.000	0.000	0.000
No. of steps	0	1	1	1	1

The results can be see graphically in Figure 4.9 below.

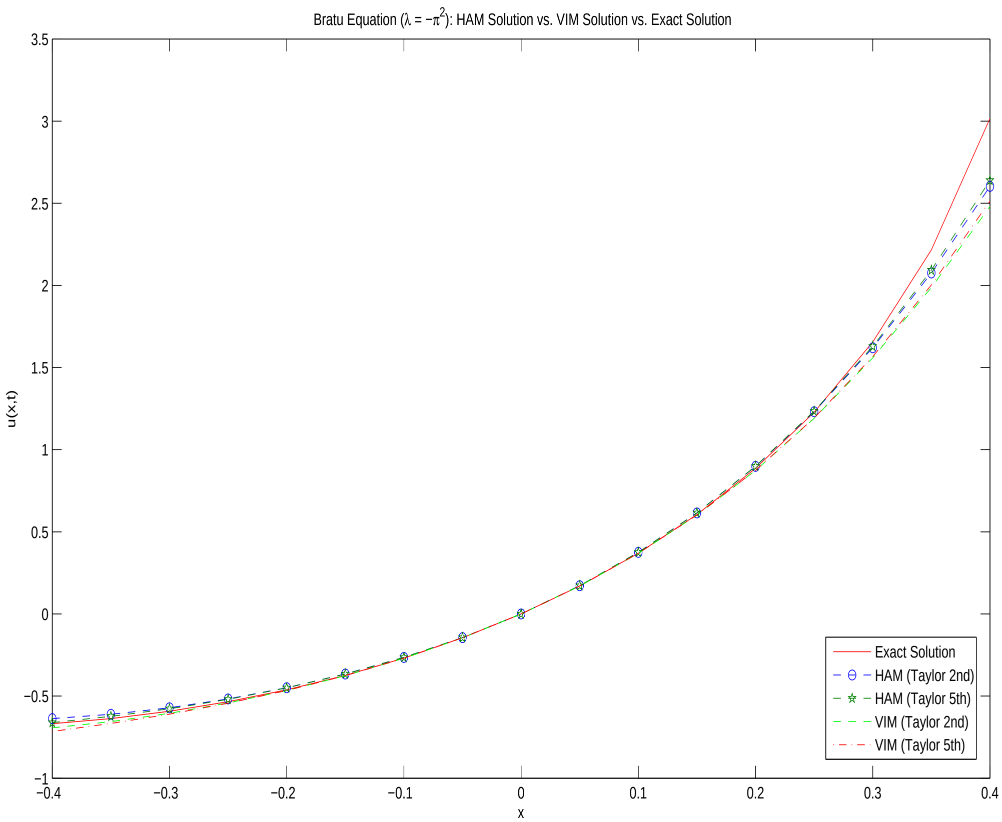

Figure 4.9: Bratu equation ($\lambda = -\pi^2$): HAM solution vs. VIM solution vs. exact solution.

The results after 1 iteration is in good agreement with the exact solution. The absolute error is is shown in the Table 4.8 below.

Table 4.8: Bratu's equation ($\lambda = -\pi^2$), Comparison of the HAM and the VIM absolute error.

x	HAM Taylor 2nd	HAM Taylor 5th	VIM Taylor 2nd	VIM Taylor 5th
-0.4	0.0307455	0.0069566	-0.0255283	-0.0471545
-0.3	0.0215333	0.0156433	-0.0122147	-0.0175692
-0.2	0.0128203	0.0120083	-0.0034341	-0.0041723
-0.1	0.0041620	0.0041354	-0.0002966	-0.0003208
0.0	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.0049330	0.0049626	-0.0005592	-0.0005323
0.2	0.0118582	0.0128583	-0.0126646	-0.0117554
0.3	-0.0349034	-0.0268719	-0.0965570	-0.0892557
0.4	-0.4138265	-0.3780053	-0.5362470	-0.5036822

The absolute error shows that the HAM solution is a better approximation for the Bratu equation for $\gamma = -\pi^2$ after 1 iteration than the VIM solution. Further iterations would result in the analytical solution being obtained as shown by, Wazwaz [106], Jin [62] and Noor and Mohyud-Din [90] but require further computational resources.

We discuss the solution for $\gamma = -2$ next.

4.3.2 The solution for $\gamma = -2$

The problem to be solved is as follows

$$\frac{\partial^2 u}{\partial x^2} - 2\,e^u = 0\,,$$
$$u(0) = u(1) = 0\,. \tag{4.10}$$

The solutions generated using VIM and HAM will be compared to the exact solution, Wazwaz [106], is

$$u_e = -2\;\ln(\cos x) \tag{4.11}$$

The initial condition was chosen as

$$u_0(x) = 0$$

and the region chosen for the analysis was $x \in [-0.5, 0.5]$. Two iterations were performed for each method and the optimal auxiliary parameter, $\hbar$, was determined by trial an error for the HAM. The error curves are shown in Figure 4.10 below.

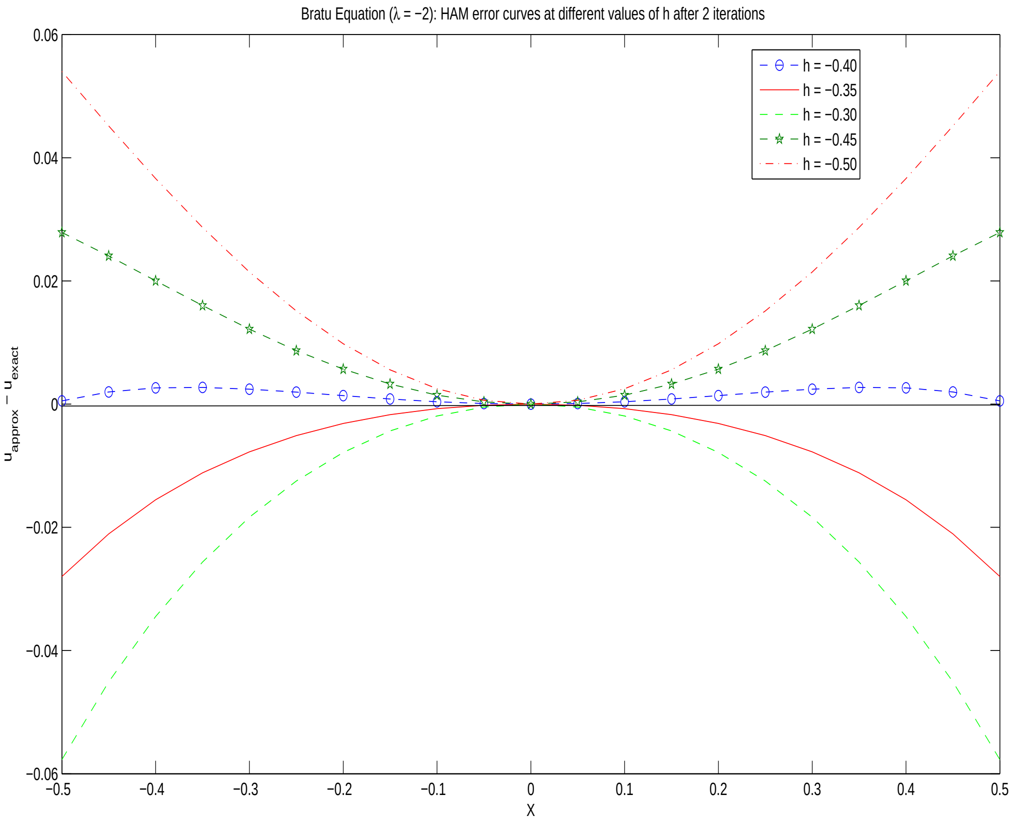

Figure 4.10: Bratu equation ($\lambda = -2$): HAM error curves at different values of $\hbar$ after 2 iterations.

The summarized results are shown in Table 4.9 below.

Table 4.9: Bratu's equation ($\lambda = -2$): Comparison of the HAM and the VIM solution.

x	Exact Solution	HAM Taylor 2nd	HAM Taylor 5th	VIM Taylor 2nd	VIM Taylor 5th
-0.50	0.2611685	0.2617000	0.2617006	0.2609375	0.2609617
-0.40	0.1644580	0.1670914	0.1670915	0.1644032	0.1644072
-0.30	0.0913833	0.0938176	0.0938176	0.0913743	0.0913747
-0.20	0.0402695	0.0416428	0.0416428	0.0402688	0.0402688
-0.10	0.0100167	0.0104027	0.0104027	0.0100167	0.0100167
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.10	0.0100167	0.0104027	0.0104027	0.0100167	0.0100167
0.20	0.0402695	0.0416428	0.0416428	0.0402688	0.0402688
0.30	0.0913833	0.0938176	0.0938176	0.0913743	0.0913747
0.35	0.1250859	0.1278041	0.1278041	0.1250623	0.1250637
0.40	0.1644580	0.1670914	0.1670915	0.1644032	0.1644072
0.45	0.2097277	0.2117112	0.2117115	0.2096112	0.2096215
0.50	0.2611685	0.2617000	0.2617006	0.2609375	0.2609617
Time (s)	0.000	0.015	0.015	0.016	0.016
No. of steps	1	2	2	2	2

The results in Table 4.9 show minor differences between Taylor second and fifth order approximations. A graphical comparison of the HAM and VIM using fifth order Taylor approximation alongside the exact solution is show in Figure 4.11 below. Figure 4.11 shows that after 2 iterations that convergent solutions are obtained for

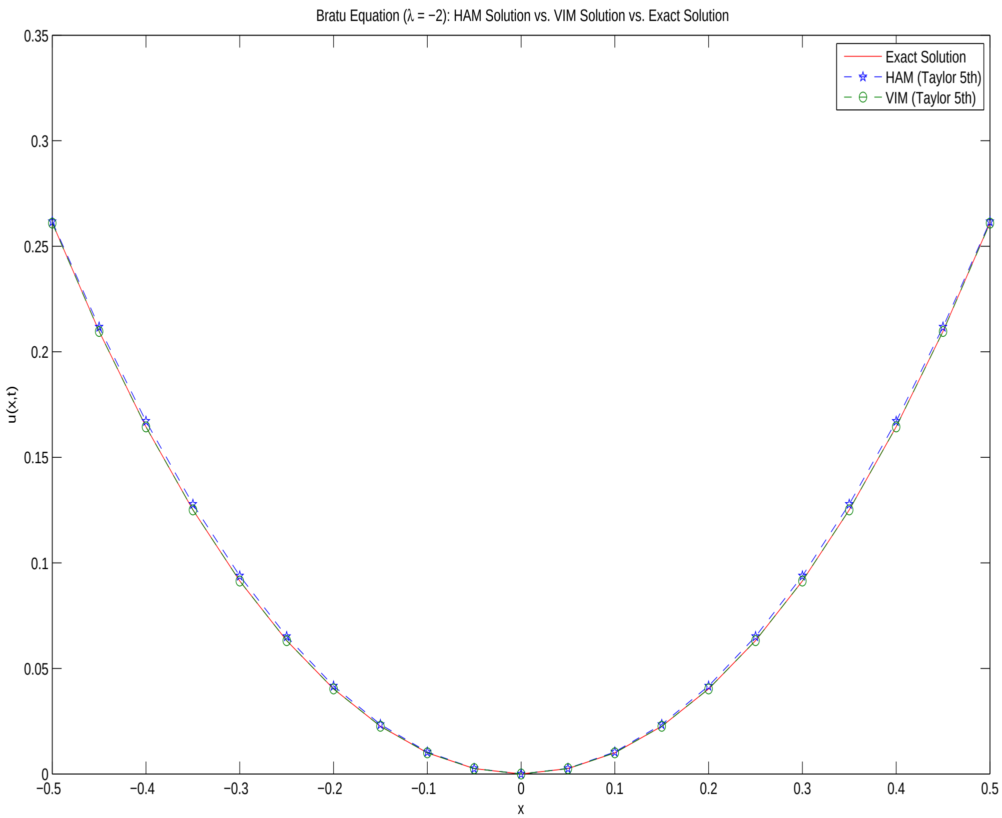

Figure 4.11: Bratu equation ($\lambda = -2$): HAM solution vs. VIM solution vs. exact solution.

both the HAM and VIM. Similarly as shown for $\gamma = -\pi^2$, the exact solutions can be determined after further iterations and can be found in Wazwaz [106], Jin [62] and Noor and Mohyud-Din [91].

Chapter 5

Conclusion

The purpose of this study was to apply analytical techniques to solve linear and nonlinear partial differential equations. We compared two particular techniques, the homotopy analysis method(HAM) and the variational iteration method(VIM) on three test problems. The comparison was focussed on accuracy, speed of convergence and computational efficiency. The test problems were identified as they displayed severe non-linearity which may pose problems for even the most robust techniques. A summary of the findings for each problem is given below.

In Chapter 4.1, we solved the one dimensional heat equation. This was an introductory problem which provided the basic conceptual understanding needed for Burgers equation. The problem is linear in nature and afforded us the opportunity to test each method to ensure that the codes were operating efficiently.

We also introduced the method of finite differences (FDM) to provide contrast between the analytical schemes and to show that even 'old' fashioned methods can provide accurate solutions. The HAM solution with optimum auxiliary parameter, $\hbar = -0.95$, converged after 8 iterations of the algorithm at $t = 0.1$s in Maple. The VIM performed marginally slower converging to the exact solution after 10 iterations. The FDM also converged to the exact solution after a tedious 240 steps in t and 20 steps in h. The HAM definitely stood out as the superior method for the heat equation as it provided an accurate solution compared to the exact solution and converged the fastest. From a computational efficiency side, all three schemes were able to compile almost instantly with out any additional resources being required.

In Chapter 4.2, we solved the Burgers equation subject to two different boundary conditions obtained from the literature. In Chapter 4.2.1, the first boundary condition was chosen as it contained an easily computable exact solution which provided additional guidance to the convergence of the analytical solutions. The HAM solution converged to order 6 after 5 iterations of the algorithm at $t = 0.1$s, with the optimal $\hbar = -1.00$. The VIM algorithm provided fourth order convergence after 4 iterations but was not able to compile further due to limited computational resources. The HAM solution after 4 iterations was also compared to the VIM solution and was found to be more accurate and did not require a large amount of computer processing power to obtain the solution.

In Chapter 4.2.2, the more common initial condition was used. We used the HAM algorithm to obtain the analytical solution consistent with the literature. The FDM method was also applied to determine a numerical approximation. Convergence of the HAM was achieved after 7 iterations at $t = 0.1$s, with $\hbar = -0.33$. We then compiled the VIM to compare if any changes were found in convergence rate and we found that the algorithm was unable to converge after 5 iterations and became tedious to compute with successive runs taking up to 10 minutes. To provide some form of convergence for the VIM scheme we looked at the solution at $t = 0.03$s and compared that to the FDM and the exact solution generated by HAM. It was clearly evident that for Burgers equation the HAM algorithm was more accurate, converged faster and required less processing power to provide analytical solutions.

In Chapter 4.3, we studied the solutions of the Bratu equations for $\gamma = -\pi^2$ and $\gamma = -2$. The analytical solutions were known in the literature, thus we decided to test the HAM and VIM against each other at a common point rather than seeking the complete closed forml solution. The purpose of this decision was based on which method is likely to perform faster in solving problems without known solutions. There was also an underlying issue in computation of the Bratu equations due to integration of the exponential function which required immense amounts of processing power which was not available.

In Chapter 4.3.1, the solution when $\gamma = -\pi^2$ was examined after just one iteration. In order to assist integration, second and fifth order Taylor approximations were applied to the exponentials in each algorithm. The HAM solution using the fifth order Taylor approximation provided up to second order accuracy compared to the VIM which just managed first order. In terms of computational efficiency both methods compiled quickly for one iteration. It must be noted though that the HAM and VIM did require more processing power for further iterations due to the evaluation of double (HAM) and single (VIM) integrals in the higher orders.

In Chapter 4.3.2, the solution when $\gamma = -2$ was examined after two iterations. A similar trend was seen with the HAM providing up to third order accuracy while the VIM lagged behind with second order accuracy. It was clear at this point the HAM algorithm was more accurate and faster converging for the Bratu equations. However, it did require additional resources to provide higher order iterations.

In summary, the purpose of the study was to find and compare analytical solutions generated by the HAM and VIM. The HAM proved to be the more robust method and this is attributed to the freedom to control the auxiliary parameter, $\hbar$. The HAM and the VIM have shown to be robust analytical methods and can be applied to severely nonlinear systems with ease. The FDM also reminded us that discretization methods still can be used to solve nonlinear problems but do not converge as rapidly as the HAM.

However, purely analytical methods do have their shortfalls as discussed earlier in Chapter 2. In this study even further shortfalls were seen mainly due to the large requirements of computer hardware to perform higher order iterations. These shortfalls have been mitigated by methods like SHAM which use numerical schemes to solve the higher order iterations.

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