Analysis and Numerical Solutions of Fragmentation Equation with Transport

by

Poka David Wetsi

14 December, 2012

Submitted in fulfilment of the academic
requirements for the degree of
Master of Science in Applied Mathematics,
University of KwaZulu-Natal (UKZN),
School of Mathematical Sciences,
Westville Campus,
Durban, 4000.

As the candidate's supervisors we have approved this thesis for submission.

Signature: __________ Name: Prof. J. Banasiak Date: 14 \12\2012

Signature: __________ Name: Dr. S.K. Shindin Date: 14 \12\2012

Preface

The work described in this dissertation was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, from September 2011 to December 2012, under the supervision of Prof. J. Banasiak and Dr S.K. Shindin.

This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

Poka David Wetsi

Declaration 1 - Plagiarism

I, Poka David Wetsi, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other personal data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

4. This thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:

    a. Their words have been re-written but the general information attributed to them has been referenced

    b. Where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.

5. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed ________________________________

Acknowledgment

I owe my deepest gratitude to Prof. Jacek Banasiak and Dr Sergey Shindin for their supervision, encouragement, guidance and support from the start to the end and enabling me to develop an understanding on transport-fragmentation equations. I am sure it would have been an insurmountable task without their help.

I am truly indebted and thankful to the University of KwaZulu Natal for making my study in Durban conducive, with the up-to-date facilities. To my sponsors for giving me the bursary to do the research. Also, thank my colleagues, who contributed in any form to the completion of the dissertation. Not forgetting my parents, brothers and sisters, my wife Matabane and daughter Joalane for their prayers and support.

Above all, I thank God for giving me good health, protection and strength to this stage. Khotso, Pula, Nala!

Abstract

Fragmentation equations occur naturally in many real world problems, see [ZM85, ZM86, HEL91, CEH91, HGEL96, SLLM00, Ban02, BL03, Ban04, BA06] and references therein. Mathematical study of these equations is mostly concentrated on building existence and uniqueness theories and on qualitative analysis of solutions (shattering), some effort has be done in finding solutions analytically. In this project, we deal with numerical analysis of fragmentation equation with transport. First, we provide some existence results in Banach and Hilbert settings, then we turn to numerical analysis. For this approximation and interpolation theory for generalized Laguerre functions is derived. Using these results we formulate Laguerre pseudospectral method and provide its stability and convergence analysis. The project is concluded with several numerical experiments.

Contents

Introduction

The study of fragmentation and coagulation has received significant attention in the past few years, see [ZM85, ZM86, HEL91, CEH91, HGEL96, SLLM00, Ban02, BL03, Ban04, BA06] and references therein. Attempts to describe and model these processes have shown relevance not only in fragmentation and coagulation problems but in other fields. Application are ranging from chemical engineering (polymerization/depolymerization) to marine biology and population dynamics.

One of the earliest work on the fragmentation equations was done by Ziff and McGrady[ZM86], where they showed analytic solutions can be found for certain type of the breakup function (known in the literature as fragmentation kernels). They further provided some numerical simulations. Also, in [ZM85] the same problem was solved using a statistical argument where the breakup was independent of the length of the substance. In this situation the problem reduces to an integro-differential equation that can be solved explicitly. Qualitatively, in the work [BL03, HGEL96, HEL91], it is shown that there exist a solution for the model but under some restrictions on the coefficients in the fragmentation with mass loss, both in the discrete and continuous cases. Huang et al [HEL91, HGEL96], explored the same problem and found general solutions and observed scaling violation for continuous mass loss. In this work the Laplace transforms is used to construct the solutions.

Due to a growing number of application of fragmentation models, we decided to contribute to the numerical side. In the project we develop a Laguerre pseudospectral method for solving the transport-fragmentation equation. One of the advantage of the method is that it allows us to treat unbounded domains directly, another advantage is that the scheme imposes very mild restrictions on the growth rate of the coefficients of the model at infinity. However its stability and convergence analysis requires wellposedness of the transport-fragmentation equation in a Hilbert space. In the project the latter is studied in the same way as in [BA06].

The project is organized as follows: Chapter I contains some classical results form the semigroup theory. Theory presented in Chapter I is applied in Chapter II to the transport-fragmentation equations in L^1 settings. Main results of our research are presented in Chapter III. First, we provide a wellposedness analysis of the transport fragmentation equation in Hilbert settings. Next, we derive several approximation and interpolation estimates for generalized Laguerre functions. Finally, we introduce the Laguerre pseudo-spectral method and analyze its stability and convergence. Chapter III is concluded with several numerical experiments.

Chapter I

Semigroup Theory

It is important to understand a problem thoroughly in general before trying to get a solution either analytically, numerically or using any other method. Sometimes a solution can be obtained but it being significantly of no use to real world application it is needed for. We found it instructive to collect tools that are needed to study the existence of a semigroup solution of transport-fragmentation equations.

1 Abstract Cauchy Problem

The problem of how processes evolve in time is crucial to a number of applications in social, physical and natural sciences. The evolution models are typically described in terms of Partial Differential Equations (PDEs). A PDE is an equation that contains partial derivatives of an unknown function u. The classical examples are:

- The heat equation $u_t = Ku_{xx}$.
- The Laplace equation $u_{xx} + u_{yy} = 0$.

Both equations have a wide range of practical applications. The first one is used to model the heat conduction in bars and solids, evolution of probability distributions in random processes, etc. The second one appears in the study of steady states of various processes.

There is a high need in finding solutions of PDEs. However, there is no unified theory (like in the case of ODEs) which relies on the fundamental existence and uniqueness theorems. In search of the solutions to a given PDE one can deploy a number of well developed methods such as semigroup approach, weak compactness approach, techniques based on the use of the maximum principle and its modifications, etc. In this project we focus on the semigroup approach. In order to apply this technique, a partial differential equation must be reformulated as an abstract Cauchy problem (ACP).

Definition I.1.1. *[EN00] Let X be a Banach space. The initial value problem of the form*

$$u_t = Au, \quad \text{for} \quad t > 0, \quad u(0) = u_0, \tag{I.1.1}$$

is termed an abstract homogeneous Cauchy problem in X associated with $(A, D(A))$ and initial value u_0. Here $u : \mathbb{R}_+ \to X$ and $D(A) \subset X$ is the domain of A.

Definition I.1.2. *[EN00] Let X be a Banach space. The initial value problem of the form*

$$u_t = Au + f(t), \quad \text{for} \quad t > 0, \quad u(0) = u_0, \tag{I.1.2}$$

is termed an abstract inhomogeneous Cauchy problem in X associated with $(A, D(A))$ and initial value u_0.

Note that the nature of the operator A in (I.1.1) and (I.1.2) is not explicitly specified. The operator can be linear, nonlinear, differential, integral or something else. For instance, A could be $\dfrac{\partial^2}{\partial x^2}(\cdot)$, $\int_x^\infty (\cdot)dx$, $\sin(\cdot)$, e.t.c. In the sequel we always assume that operator A is linear.

The crucial question arises — does equation (I.1.1) have a solution? At least formally (I.1.1) looks exactly like a system of linear ODEs whose solution is known to be $e^{At}u_0$. It is natural to search for the solution of abstract problem (I.1.1) in the form $u(t) = e^{At}u_0$ as well. Of course the exponent e^{At} must be given an appropriate interpretation. For this we need the notion of semigroups of linear operators.

2 Semigroups of Linear Operators

2.1 Semigroups

Definition I.2.1. *A semigroup $(S, *)$ is a non-empty set S together with an associative binary operation $*$; for $x, y, z \in S$,*

$$x * (y * z) = (x * y) * z.$$

There are different types of semigroups. For instance, S is a finite semigroup if it has a finite number of elements. Also, S is a commutative semigroup if the operation $*$ is commutative. In the sequel we make use of semigroups of linear operators.

Definition I.2.2. *[BA06] A family of $(S(t))_{t\geq 0}$ of bounded linear operator on X is called a strongly continuous semigroup or C_0-semigroup if*

(i) $S(0) = I$;

(ii) *for every $s, t \geq 0$, the composition satisfies $S(t)S(s) = S(t+s)$; (the semigroup property)*

(iii) $\lim_{t\to 0^+} S(t)x = x, \quad \forall x \in X.$

$S(t)$ is a semigroup of type r if in addition to (i)—(iii),

$$\|S(t)\| \leq e^{tr}, \quad t \geq 0.$$

We should note that if $r = 0$, then the semigroup is called contractive. In this case $\|S(t)\| \leq 1$, for $t \geq 0$ and,

$$\|S(t)u - S(t)v\| \leq \|u - v\|, \quad \text{for} \quad u, v \in X \quad t \geq 0.$$

2.2 Generators of Semigroups

Definition I.2.3. *The operator*

$$Au = \lim_{t\to 0^+} \frac{S(t)u - u}{t} \tag{I.2.1}$$

is called the infinitesimal generator of a semigroup $(S(t))_{t\geq 0}$. *Its domain* $D(A)$ *is the set of all* $u \in X$ *for which the limit of* (I.2.1) *exists.*

It is clear that every C_0-semigroup has a generator. The converse is not true, not every linear operator A generates a semigroup. To be a generator, operator A must satisfy certain conditions:

Theorem I.2.1. *[Bre12, BA06] Let* $(S(t))_{t\geq 0}$ *be a semigroup, and* A *be its generator. Then*

(i) the domain of A *is dense in* X*;*

(ii) the operator A *is closed (or closable).*

Proof. See [Bre12, BA06]. □

A connection between $(S(t))_{t\geq 0}$ and its generator A is stated in the theorem below:

Theorem I.2.2. *[Bre12] Let* $(S(t))_{t\geq 0}$ *be a* C_0*-semigroup and* A *be its generator. Assume that* $u_0 \in D(A)$*, then*

(i) $\forall t \geq 0$*, one has* $S(t)u_0 \in D(A)$ *and* $AS(t)u_0 = S(t)Au_0$*;*

(ii) map $t \to u(t) = S(t)u_0$ *is continuous, differentiable and provides a solution to the Cauchy problem* (I.1.1).

Proof. For proof see [Bre12]. □

2.3 Resolvents and Hille-Yosida Theorem

A very important link between a generator A and a semigroup $(S(t))_{t\geq 0}$ is given by the resolvent of operator A. The resolvent is an analytic function of λ, except when λ lies in the spectrum of the operator. Before we formally define the resolvent we need the following terminology:

Definition I.2.4. *Let* A *be an operator in* X*.*

(i) $\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is bijective}\}$ *is the resolvent set and its complement* $\sigma(A) = \mathbb{C} \setminus \rho(A)$ *is called the spectrum of* A*;*

(ii) $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ *is called the spectral radius of the operator;*

(iii) $s(A) = \sup\{\Re\lambda : \lambda \in \sigma(A)\}$ *is called the spectral bound of the operator.*

For an unbounded operator A the role of the spectral radius is played by the spectral bound.

Definition I.2.5. *[EN00] Let* A *be a linear operator on a Banach space. For* $\lambda \in \rho(A)$ *the resolvent operator* $R_\lambda : X \to X$ *is defined by*

$$R(\lambda, A)u = (\lambda I - A)^{-1}u.$$

We note that an operator $(A, D(A))$ is resolvent positive if there exist r such that $(r, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > r$.

2.3.1 Hille-Yosida Theorem

In the development of semigroup theory and its applications to physical problems there are important results that have remained pillars of semigroup theory since mid 1900's till present. One of the results comes from the works of Hille and Yosida.

Theorem I.2.3. *[Paz83] A linear (unbounded) operator* A *is the infinitesimal generator of a* C_0*-semigroup of contractions* $(S(t))_{t\geq 0}$*, if and only if*

(i) A *is closed and* $\overline{D(A)} = X$*;*

(ii) the resolvent set $\rho(A)$ *of* A *contains* $\mathbb{R}_+$ *and for every* $\lambda > 0$*,* $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$.

Proof. For proof see [Paz83]. □

2.4 Dissipative Operators and Lumer-Phillips Theorem

Sometimes Hille-Yosida theorem is not easy to apply because one needs an explicit estimate on the resolvent of A. The situation is simplified if it is known that A is dissipative.

Definition I.2.6. *[BA06, EN00] A linear operator $(A, D(A))$ on a Banach space X is called dissipative if*

$$\|(\lambda - A)x\| \geq \lambda\|x\|,$$

for all $\lambda > 0$ and $x \in D(A)$.

2.4.1 Properties of Dissipative operators

Let $(A, D(A))$ be dissipative, then (see [EN00])

(i) $\lambda - A$ is injective for all $\lambda > 0$ and

$$\|(\lambda - A)^{-1}z\| \leq \frac{1}{\lambda}\|z\|,$$

for all z in range $rg(\lambda - A) = (\lambda - A)D(A)$.

(ii) $\lambda - A$ is surjective for some $\lambda > 0$ if and only if it is surjective for each $\lambda > 0$. In that case, one has $(0, \infty) \subset \rho(A)$.

(iii) A is closed if and only if the range $rg(\lambda - A)$ is closed for some (hence for all) $\lambda > 0$.

(iv) If $rg(\lambda - A) \subseteq \overline{D(A)}$ and if A is densely defined, then A is closable. Its closure $\overline{A}$ is again dissipative and satisfies $rg(\lambda - \overline{A}) = \overline{rg(\lambda - A)}$ for all $\lambda > 0$.

Theorem I.2.4. *[Paz83] Let A be a linear operator with dense domain $D(A)$ in X.*

(i) If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $rg(\lambda_0 I - A)$ of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

(ii) If A is the infinitesimal generator of a C_0-semigroup of contractions on X then $rg(\lambda - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x), \Re\langle Ax, x^*\rangle \leq 0,$ where $F(x) = \{x^* : x^* \in X^*$ and $\langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}$.

Proof. For proof see [Paz83] □

2.5 Abstract Cauchy Problem(ACP) and Semigroups

The semigroup approach has received significant attention in the literature related to the PDEs, see [BL03] and references therein. The theory was successfully applied to the wide range of problems in physics, chemistry, biology, etc. The popularity of the method is explained by the fact that many interesting practical models can be rewritten in the form of an ACP. In the sequel we mention classical results concerning solvability of the ACP by means of semigroup.

There are generally two types of solutions associated with an ACP, namely classical and mild. The classical solutions are obtained when initial data is in the domain of the generator.

Definition I.2.7. *A function $u : \mathbb{R}_+ \to X$ is a classical solution of* (I.1.1), (I.1.2) *on $[0, \infty)$ if u is continuous $(0, \infty)$, continuously differentiable on $(0, \infty)$, $u(t) \in D(A)$ for $t \geq 0$ and* (I.1.1), (I.1.2) *are satisfied.*

Theorem I.2.5. *[EN00] Let $A : D(A) \subset X \to X$ generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X, then for all $u_0 \in D(A)$,* (I.1.1) *has a unique classical solution given by $u(t) = S(t)u_0$.*

Definition I.2.8. *[Paz83] Let A be an infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ in X. Let $u_0 \in X$ and $f \in L^1((0, \infty), X)$. The function $u \in C([0, \infty), X)$ given by*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\mathrm{d}s$$

is called a mild solution of the ACP (I.1.2) *on $[0, \infty)$.*

If we assume that $u_0 \in D(A)$ then the mild solution becomes classical. Converse is not true, not all mild solutions are classical solutions. In the case when $f = 0$ the mild solution of (I.1.1) is given by $S(t)u_0$, for $u_0 \in X$.

Theorem I.2.6. *[Paz83] Let A be the infinitesimal generator of C_0-semigroup $S(t)$. Let $f \in L^1((0, \infty) : X)$ be continuous on $(0, \infty)$ and let*

$$v(t) = \int_0^t S(t-s)f(s)\mathrm{d}s, \quad t \geq 0.$$

Then (I.1.2) *has a classical solution* u *on* $(0, \infty)$ *for every* $u_0 \in D(A)$ *if one of the following conditions is satisfied:*

(i) $v(t)$ *is continuously differentiable on* $(0, \infty)$*;*

(ii) $v(t) \in D(A)$ *for* $t \geq 0$ *and* $Av(t)$ *is continuous on* $(0, \infty)$.

If (I.1.2) *has a classical solution* u *on* $(0, \infty)$ *for some* $u_0 \in D(A)$ *then* v *satisfies both (i) and (ii).*

3 Positive and Sub-stochastic Semigroups

In many evolution problems (including the one we are dealing with), the positivity of solutions is of utmost importance. For example, the unknown function may define a number of particles, species in a population, etc. This immediately dictates that the semigroup that solves the problem should be positive.

3.1 Banach Lattices

Definition I.3.1. *A vector space* V *equipped with a partial order "*$\leq$*" is called a vector lattice if for each pair* $a, b \in V$,

(i) there is a smallest element c *for which* $a \leq c$ *and* $b \leq c$*;*

(ii) there is a largest element d *for which* $d \leq a$ *and* $d \leq b$*;*

(iii) if $a \leq b$ *then* $a + c \leq b + c$ *for* $c \in V$*;*

(iv) if $a \leq b$ *and* $k \in \mathbb{R}_+$ *then* $ka \leq kb$.

Note that a vector lattice is a Riez space. A non-empty subset K of a vector space V is said to be a cone if it satisfies the following properties [Tou00]:

(i) $K + K \subseteq K$;

(ii) $\alpha K \subseteq K$ for all $\alpha \geq 0$;

(iii) $K \cap (-K) = 0$.

Definition I.3.2. *If E is a vector lattice, then we denote by $E_+ = \{x \in E : 0 \leq x\}$ a positive cone of E.*

Definition I.3.3. *[ABB90] A norm $\|\cdot\|$ on a vector lattice is said to be a lattice norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.*

Definition I.3.4. *[BA06] A Banach lattice is a real Banach space[1] X endowed with an ordering "$\leq$" such that $(X, \leq)$ is a vector lattice and the norm of X is a lattice norm.*

Definition I.3.5. *[BA06] A Banach lattice X is a KB-space (Kantorovic-Banach space) if every increasing, norm bounded sequence of elements of X_+ converges in norm in X.*

3.2 Positive Semigroups

Definition I.3.6. *Let X be a Banach lattice. We say that the semigroup $(S(t))_{t\geq 0}$ on X is positive if for any $x \in X_+$ and $t \geq 0$,*

$$S(t)x \geq 0.$$

The following theorem provides a characterization of positive semigroups;

Theorem I.3.1. *[EN00] A strongly continuous semigroup $(S(t))_{t\geq 0}$ on a Banach lattice X is positive if and only if the resolvent $R(\lambda, A)$ of its generator A is positive for all sufficiently large λ.*

3.3 Stochastic and Sub-stochastic Semigroups

A semigroup describing an evolution that allows the amount of the described quantity to decrease is called strictly sub-stochastic. Formally these semigroups are defined in the following way:

Definition I.3.7. *[BA06] Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. $(S(t))_{t\geq 0}$ is sub-stochastic if for any $t \geq 0$ and $x \geq 0$, $S(t)x > 0$ and $\|S(t)x\| \leq \|x\|$.*

[1] That is a complete normed vector space.

For stochastic semigroups in Definition I.3.7 we have $\|S(t)x\| = \|x\|$ for $x \in X_+$.

4 Perturbations of Positive Semigroups

Not all problems in science can be defined by one linear operator, they can contain two or more operators. Often it is not an easy task to verify whether a group of operators involved generate a strongly continuous semigroup. Therefore, it is preferable to start building from the simpler case to more complex. Methods available to perform the latter are perturbation and approximation. In this work we shall use the perturbation method. Rephrasing the latter problem: Let $(A, D(A))$ be a generator of a semigroup in a Banach space X and $(B, D(B))$ be another operator on X. Under what conditions does $A + B$ generate a semigroup?

The question is easy to answer if B is bounded. In this case we have the following results:

Theorem I.4.1. *[EN00] Let A be a generator of a positive strongly continuous semigroup $(S(t))_{t\geq 0}$ and let $B \in \mathcal{L}(X)$ be a positive operator on the Banach lattice X. Then, the following holds:*

(i) $A + B$ generates a positive semigroup $(T(t))_{t\geq 0}$ satisfying $0 \leq (S(t))_{t\geq 0} \leq (T(t))_{t\geq 0}$ for all $t \geq 0$;

(ii) $s(A) \leq s(A + B)$ and $R(\lambda, A) \leq R(\lambda, A + B)$ for all $\lambda > s(A + B)$.

The situation is more complicated when B is not bounded. It is important to emphasize that addition of two unbounded operators is a very delicate operation as properties of $A+B$ might be very different from that of A and B. We recall that $D(A+B) = D(A) \cap D(B)$. To circumvent the problem of having $D(A + B) = \emptyset$, we assume that $D(B) \supset D(A)$. In this case we have the following theorem:

Theorem I.4.2 (Kato-Voigt [BA06])**.** *Let X be a KB-space. Let us assume that we have two operators $(A, D(A))$ and $(B, D(B))$ satisfying:*

(i) A generates a positive semigroup of contractions $(S_A(t))_{t\geq 0}$;

(ii) $r(BR(\lambda, A)) < 1$ *for some* $0 < \lambda (= s(A))$*;*

(iii) $Bx > 0$ *for* $x \in D(A)_+$*;*

(iv) $\langle x^*, (A+B)x \rangle \leq 0$ *for any* $x \in D(A)_+$*, where* $\langle x^*, x \rangle = \|x\|, \quad x^* > 0.$

Then there is an extension $(K, D(K))$ *of* $(A+B, D(A))$ *generating a* C_0*-semigroup of contractions, say,* $(G_K(t))_{t \geq 0} \geq 0$*. The generator* K *satisfies,*

$$R(\lambda, K)x = \lim_{n \to \infty} R(\lambda, A) \sum_{k=0}^{n} (BR(\lambda, A))^k x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x, \quad \text{for} \quad \lambda > 0.$$

Proof. See [BA06]. □

Generalised Kato-Voigt Theorem I.4.2 in L_1 setting reads as follows (see [BA06]):

Theorem I.4.3. *Let* $X = L_1$ *and suppose that operators* A *and* B *satisfy:*

(i) $(A, D(A))$ *generates a sub-stochastic semigroup* $(S_A(t))_{t \geq 0}$*;*

(ii) $D(B) \supset D(A)$ *and* $Bu \geq 0$ *for* $u \in D(B)_+$*;*

(iii) for all $u \in D(A)_+$*,*

$$\int_{\omega} (Au + Bu) \mathrm{d}\mu \leq 0,$$

then there exist a smallest sub-stochastic semigroup $(G_k(t))_{t \geq 0}$ *generated by an extension* K *of* $A+B$*.*

In Chapter II we apply the semigroup theory (and in particular Theorem I.4.3) to study wellposedness of transport-fragmentation equations in $L^1(xdx, \mathbb{R}_+)$ settings.

Chapter II

Theoretical Analysis of the Model

1 The Transport-Fragmentation Equation

Fragmentation is the breaking down of a substance into small parts due to external or internal forces. The process can be modelled using discrete and/or continuous equations. This is due to the fact that at times fragmentation can be both discrete and continuous depending on a particular scenario. For instance, a solid can fragment in a continuous time but at one state a burst occurs — this can be classified as a discrete fragmentation. There are different ways to model fragmentation processes, what approach to use depends on the nature of the process. In this study we concentrate on the continuous models. They are derived as a balance equations where we have a gain and loss terms. The gain term captures the mass of small particles obtained from bigger particles breaking, while the loss term describes the decrease of mass of bigger particles breaking into smaller particles.

In this work we consider fragmentation equation with transport:

$$\partial_t u(x,t) = \pm\partial_x \left[r(x)u(x,t)\right] - \mu(x)u(x,t) - a(x)u(x,t) + \int_x^\infty a(y)b(x|y)u(y,t)\mathrm{d}y. \quad \text{(II.1.1)}$$

In (II.1.1) function u is the particle mass distribution; r is the transport coefficient, it describes the migration of particles; a is the rate of fragmentation; μ is the mortality rate; b is the fragmentation kernel, it describes the distribution of particle of mass x spawned by the breaking up of a particle of size y. Generally, the fragmentation process is mass conservative but the transport part is not, hence the overall model has a mass leakage or increase.

In the sequel we adopt assumptions on the coefficients from [BA06]. We start with coefficient a that defines the rate at which the particles or molecules break up into smaller pieces due to internal or external forces. We assume that a is essentially bounded on a compact subset of $(0, \infty)$, i.e.

$$0 \leq a \in L_{1,loc}([0, \infty)). \tag{II.1.2}$$

Mortality denoted by μ describes the annihilation of particles during fragmentation. For instance, in the evolution of phytoplankton, after breaking up some fragments may die. We assume that μ is locally integrable in a bounded interval of $(0, \infty)$, i.e.

$$0 \leq \mu \in L_{1,loc}([0, \infty)). \tag{II.1.3}$$

The transport term r describes the situation in which the size of a particle decreases or increases. If we have $+r$ in (II.1.1) then we deal with decrease (decay) at the rate $\frac{dx}{dt} = -r(x)$. Alternatively, if we have $-r$ in (II.1.1) then we deal with growth at the rate $\frac{dx}{dt} = r(x)$. In both cases absolute continuity[1] and strict positivity of r on the interval $(0, \infty)$ are required,

$$r(x) > 0 \quad \text{on} \quad (0, \infty) \quad \text{and} \quad r \in AC((0, \infty)). \tag{II.1.4}$$

The fragmentation kernel (in the sense of calculus[2]) $b(x|y)$ describes the breaking mechanism of particle of size y into particle of size x. We assume that it is measurable, non-negative in both variables and

$$\begin{aligned} b(x|y) &= 0 \quad \text{for } x > y, \\ b(x|y) &\geq 0 \quad \text{for all time} > 0. \end{aligned} \tag{II.1.5}$$

From the practical point of view function b gives the number of expected particles after breaking of a particle of size y, i.e.

$$\int_0^y b(x|y) \mathrm{d}x.$$

[1] A function f is absolutely continuous if and only if for every $\epsilon > 0$ there is a $\delta > 0$, so that for arbitrary disjoint intervals $I_k = [a_k, b_k], k = 1, \ldots, n$, $\sum_{k=1}^n (b_k - a_k) < \delta$ implies $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon$.

[2] I.e. $b(x|y)$ is a function of two variables that defines an integral transform.

If we assume that the fragmentation process is conservative then the total mass of daughter particles must be equal to the mass of the parent particle. This yields:

$$\int_0^y xb(x|y)\mathrm{d}x = y. \tag{II.1.6}$$

The unknown function u is assumed to be positive for all values of $t \geq 0$. The total mass of the ensemble at time t is given by

$$\int_0^\infty xu(x,t)\mathrm{d}x.$$

2 The Transport-Fragmentation Equation in Abstract Settings

We are going to use the semigroup technique to do the formal analysis of (II.1.1). In the sequel we consider the decay case only. First, we rewrite (II.1.1) as an abstract Cauchy problem (ACP). For this we define the following operators:

$$\begin{aligned} T[u](x) &= \partial_x \left[r(x)u(x,t)\right] - pu(x,t), \quad p = (\mu(x) + a(x)), \\ F[u](x) &= \int_x^\infty a(y)b(x|y)u(y,t)\mathrm{d}y, \end{aligned}$$

then (II.1.1) is equivalent to

$$u_t = T[u] + F[u], \quad u(0) = u_0, \quad t \geq 0. \tag{II.2.1}$$

To apply abstract methods discussed in Chapter I we have to choose an appropriate Banach space for the unknown u. In the study of fragmentation processes two spaces $L_1(\mathbb{R}_+, x\mathrm{d}x)$ and $L_1(\mathbb{R}_+, \mathrm{d}x)$ are commonly used, due to the meaning given by their norms. For $u \geq 0$, $L_1(\mathbb{R}_+, x\mathrm{d}x)$ norm

$$\|u\| = \int_0^\infty u(x)x\mathrm{d}x,$$

gives the total mass of the system, while $L_1(\mathbb{R}_+, \mathrm{d}x)$ norm

$$\|u\| = \int_0^\infty u(x)\mathrm{d}x,$$

represents the total number of particles in the system. In the sequel we use $X = L_1(\mathbb{R}_+, x\mathrm{d}x)$ since the total mass of the system is usually bounded while the number of particles might rapidly grow towards infinity due the fragmentation.

Next, we define domains of our operators T and F. The domains should be chosen so that $D(T+F) \neq \emptyset$. The domain of the operators are constructed as follows:

$$\begin{aligned} D(T) &= \{u \in X | (ru)_x, pu \in X\}, \\ D(F) &= \{u \in X | Fu \in X\}. \end{aligned}$$

Now we are at the point where we can start formal analysis of the problem at hand. We proceed as in [BA06]. First, we alter the problem by assuming that $F[u] = 0$ in (II.2.1) and establish existence of a semigroup generated by T. Second, we consider the whole problem by making $F[u] \neq 0$ and apply the perturbation technique to obtain the semigroup for the sum of the operators T and F. Finally, we make some comments on the uniqueness of the semigroup solutions.

2.1 The Transport Semigroup

We start by assuming that $F[u] = 0$, then equation (II.2.1) reduces to

$$u_t = T[u], \quad u(0) = u_0, \quad t \geq 0. \tag{II.2.2}$$

Our aim here is to show that T generates a strongly continuous semigroup. The approach is standard. First, we demonstrate that for appropriately chosen domain $D(T)$ the resolvent $R(\lambda, T)$ satisfy $\|R(\lambda, T)\| \leq \dfrac{1}{\lambda}$ for all $\lambda > 0$. Second, we show that the operator $(T, D(T))$ is closed and densely defined. Then, the generation result follows easily from the Hille-Yosida theorem.

Let $u_0 \in X$, the resolvent equation reads

$$\lambda u - (ru)_x + pu = u_0, \quad \lambda > 0. \tag{II.2.3}$$

The formal solution is given by

$$u(x) = R_\lambda[u_0](x) := \frac{e^{\lambda R(x) + Q(x)}}{r(x)} \int_x^\infty e^{-\lambda R(y) - Q(y)} u_0(y) dy, \tag{II.2.4}$$

where

$$R(x) = \int_{x_0}^{x} \frac{1}{r(s)} ds \quad \text{and} \quad Q(x) = \int_{x_0}^{x} \frac{p(s)}{r(s)} ds, \tag{II.2.5}$$

and x_0 is a fixed positive number. Assumptions (II.1.2)-(II.1.4) imply that $\frac{1}{r}, \frac{p}{r} \in L_{\mathit{1,loc}}((0,\infty))$, therefore the integrals $R(x)$ and $Q(x)$ are well defined for all $x > 0$.

(i) $R(x)$ and $Q(x)$ are continuous and bounded on compact subintervals of $(0,\infty)$;

(ii) $R(x)$ is strictly increasing and $Q(x)$ is non-decreasing on $(0,\infty)$;

(iii) $e^{sR(x)+Q(x)}$ is positive and absolutely continuous on compact subintervals of $[0,\infty)$ for any fixed real s.

Using this facts we can prove the following

Lemma II.2.1. *[BA06] For $\lambda > 0$ operator R_λ maps X into $D(T)$, i.e. $R_\lambda X \subset D(T)$. Moreover, R_λ is bounded and $\|R_\lambda\| \le \frac{1}{\lambda}$.*

Proof. Let $u \in X$ and $v = R_\lambda u$. We show that $\|v\| < \infty, \|pv\| < \infty$ and $\|(rv)_x\| < \infty$. First,

$$\begin{aligned}
\|v\| &= \int_0^\infty x \frac{e^{\lambda R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-\lambda R(y)-Q(y)} |u(y)| dy dx \\
&= \int_0^\infty y|u(y)| \frac{e^{-\lambda R(y)-Q(y)}}{y} \int_0^y x \frac{e^{\lambda R(x)+Q(x)}}{r(x)} dx dy \\
&\le \|u\| \sup_{x>0} \left[\frac{e^{-\lambda R(x)-Q(x)}}{x} \int_0^x y \frac{e^{\lambda R(y)+Q(y)}}{r(y)} dy \right] := \|u\| \sup_{x>0} A_\lambda(x).
\end{aligned}$$

Since

$$\begin{aligned}
A_\lambda(x) &= \frac{e^{-\lambda R(x)-Q(x)}}{x} \int_0^x y \frac{e^{\lambda R(y)+Q(y)}}{r(y)} dy \le \frac{1}{\lambda} e^{-\lambda R(x)} \int_0^x e^{\lambda R(y)} \frac{\lambda}{r(y)} dy \\
&\le \frac{1}{\lambda} e^{-\lambda R(x)} \left(e^{\lambda R(x)} - \lim_{x \to 0} e^{\lambda R(x)} \right) \le \frac{1}{\lambda},
\end{aligned}$$

it follows that $\|v\| \le \frac{1}{\lambda} \|u\|$.

Second,

$$\begin{aligned}
\|pv\| &= \int_0^\infty p(x)x\frac{e^{\lambda R(x)+Q(x)}}{r(x)}\int_x^\infty e^{-\lambda R(y)-Q(y)}u(y)dydx \\
&= \int_0^\infty yu(y)\frac{e^{-\lambda R(y)-Q(y)}}{y}\int_0^y x\frac{p(x)e^{\lambda R(x)+Q(x)}}{r(x)}dxdy \\
&\leq \|u\|\sup_{x>0}\left[\frac{e^{-\lambda R(x)-Q(x)}}{x}\int_0^x y\frac{p(y)e^{\lambda R(y)+Q(y)}}{r(y)}dy\right] := \|u\|\sup_{x>0}B_\lambda(x).
\end{aligned}$$

In the same way as before we obtain

$$B_\lambda(x) = \frac{e^{-\lambda R(x)-Q(x)}}{x}\int_0^x y\frac{p(y)e^{\lambda R(y)+Q(y)}}{r(x)}dx \leq e^{-Q(x)}\left(e^{Q(x)} - \lim_{x\to 0}e^{Q(x)}\right) \leq 1,$$

and $\|pv\| \leq \|u\|$.

Finally, we note that $(rv)_x = (\lambda + p)v - u$, therefore

$$\|(rv)_x\| = \|(\lambda + p)v - u\| \leq 2\|u\| + \|u\| = 3\|u\|,$$

and it follows that $R_\lambda X \subset D(T)$. $\square$

The operator $R_\lambda : X \to D(T)$ is into but it does not have to be onto. To illustrate the statement consider the homogeneous resolvent equation

$$\lambda u - (ru)_x + pu = 0, \quad \lambda > 0.$$

Its general solution is given by $u = Kv_\lambda$, where K is a constant and

$$v_\lambda(x) = \frac{e^{\lambda R(x)+Q(x)}}{r(x)}.$$

It might occur that for some coefficients r and p the eigenfunction v_λ belongs to $D(T)$. Since $v_\lambda \notin R_\lambda X$ it is clear that in such a situation $\lambda I - T$ is not invertible in $D(T)$ and T cannot be a generator of a semigroup.

The situation is demonstrated by the following example, see [BA06]. Let $r(x) = x^b$ with $b > 2$ and $p(x)$ be bounded and integrable. Since $p(x) \leq M$ for some $M > 0$ we have

$$\|v_\lambda\| \leq \int_0^\infty x^{1-b}\exp\left(\frac{-(\lambda+M)}{(b-1)x^{b-1}}\right)dx = \int_0^\infty t^{\frac{1}{1-b}}\exp\left(\frac{-(\lambda+M)t}{(b-1)}\right)\frac{dt}{b-1} < \infty.$$

In the same way one can show that $pv_\lambda \in X$ and hence $v_\lambda \in D(T)$. We see that the operator $(T, D(T))$ does not generate a semigroup.

Lemma II.2.1 implies that $(\lambda I - T)R_\lambda = I$, i.e. R_λ is the right inverse of $(\lambda I - T)$.

Lemma II.2.2. *Operator R_λ is the left inverse of $\lambda I - T$ in $D(T)$ provided that*

$$\lim_{x\to\infty} \frac{u(x)}{v_\lambda(x)} = 0 \quad \textit{for all} \quad u \in D(T). \tag{II.2.6}$$

Condition (II.2.6) is satisfied automatically if $v_\lambda \notin D(T)$.

Proof. Consider $R(\lambda, T)(\lambda I - T)u$, where $u \in D(T)$. We have

$$\begin{aligned}
R_\lambda(\lambda I - T)u &= \frac{1}{r(x)} e^{\lambda R(x)+Q(x)} \int_x^\infty e^{-\lambda R(y)-Q(y)}(\lambda u(y) - [r(y)u(y)]_x + p(y)u(y))\mathrm{d}y \\
&= \frac{1}{r(x)} e^{\lambda R(x)+Q(x)} \int_x^\infty -\frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-\lambda R(y)-Q(y)} r(y)u(y)\right)\mathrm{d}y \\
&= \frac{1}{r(x)} e^{\lambda R(x)+Q(x)} \left[r(x)u(x)e^{-\lambda R(x)-Q(x)} - \lim_{x\to\infty} \frac{u(x)}{v_\lambda(x)}\right] \\
&= u(x) - v_\lambda(x) \lim_{x\to\infty} \frac{u(x)}{v_\lambda(x)}.
\end{aligned}$$

This proofs the first claim of the Lemma.

To prove the second one we assume that $\lim_{x\to\infty} \dfrac{u(x)}{v_\lambda(x)} > 0$ for some non-negative $u \in D(T)$. Since both u and v_λ are absolutely continuous in compact subintervals of $(0, \infty)$ it follows that the limit is a finite non-negative number or infinity. In both cases there exists $x_0 > 0$ such that $\dfrac{u(x)}{v_\lambda(x)} \geq C_\lambda$ for all $x > x_0$ and for some $C_\lambda > 0$.

If $v_\lambda \notin D(T)$ then either $v_\lambda \notin X$ or $pv_\lambda \notin X$. In the first case we have

$$\int_{x_0}^\infty u(x)xdx = \int_{x_0}^\infty v_\lambda(x)\frac{u(x)}{v_\lambda(x)}xdx \geq C_\lambda \int_{x_0}^\infty v_\lambda(x)xdx,$$

and since $\int_0^{x_0} v_\lambda(x)xdx < \infty$ it follows that

$$\|v_\lambda\| \leq \int_0^{x_0} v_\lambda(x)xdx + \frac{1}{C_\lambda}\int_{x_0}^\infty u(x)xdx < \infty.$$

In the second case

$$\int_{x_0}^\infty p(x)u(x)xdx = \int_{x_0}^\infty p(x)v_\lambda(x)\frac{u(x)}{v_\lambda(x)}xdx \geq C_\lambda \int_{x_0}^\infty p(x)v_\lambda(x)xdx,$$

and since $\int_0^{x_0} p(x)v_\lambda(x)xdx < \infty$ it follows that

$$\|pv_\lambda\| \leq \int_0^{x_0} p(x)v_\lambda(x)xdx + \frac{1}{C_\lambda}\int_{x_0}^\infty p(x)u(x)xdx < \infty.$$

The proof is complete. $\square$

In view of Lemmas II.2.1-II.2.2 we redefine $D(T)$ as follows

$$D(T) = \begin{cases} \{u \in X | (ru)_x, pu \in X\}, & v_\lambda \notin X \text{ or } pv_\lambda \notin X; \\ \left\{u \in X | (ru)_x, pu \in X, \lim_{x\to\infty} \dfrac{u(x)}{v_\lambda(x)} = 0\right\}, & v_\lambda, pv_\lambda \in X. \end{cases} \qquad \text{(II.2.7)}$$

Then it follows at once that $R_\lambda = R(\lambda, T)$, i.e. it is the resolvent of $(T, D(T))$ and $\|R(\lambda, T)\| \leq \dfrac{1}{\lambda}$ for all $\lambda > 0$. It remains to show that the operator $(T, D(T))$ is closed and densely defined in X.

The fact that $D(T)$ is dense in X follows easily from the following result:

Corollary II.2.3. *[Rob87] $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ provided that $1 \leq p < \infty$.*

To check that $(T, D(T))$ is closed we take $u_n \in D(T)$ such that $u_n \to u$ and $Tu_n \to f \in X$ and show that $u \in D(T)$ and $Tu = f$. Let $\lambda > 0$ be fixed, since $u_n \in D(T)$ we have $u_n = R(\lambda, T)v_n$ for some $v_n \in X$. Since $Tu_n = \lambda u_n - v_n$ we obtain $v_n \to v = \lambda u - f \in X$. Operator $R(\lambda, T)$ is continuous, hence $u_n \to u = R(\lambda, T)v$ and it follows that $u \in D(T)$. Finally, $Tu = TR(\lambda, T)v = \lambda u - v = f$.

Now we are ready to state the main result of this section:

Theorem II.2.4. *The operator $(T, D(T))$ is the generator of a strongly continuous, positive semigroup of contractions $(S_T(t))_{t\geq 0}$ on X.*

Proof. The result follows from our analysis, the positivity of $R(\lambda, T)$ and the Hille-Yosida theorem.

□

2.2 The Transport-Fragmentation Semigroup

We have established existence of a strongly continuous semigroup $(S_T(t))_{t\geq 0}$ for the reduced problem. Now we consider the full problem

$$u_t = T[u] + F[u], \quad u(0) = f, \quad t > 0. \qquad \text{(II.2.8)}$$

We shall apply the generalised Kato-Voigt Theorem I.4.3 in L_1 setting. First, we note that the operator $(T, D(T))$ is a generator of a sub-stochastic semigroup $(S_T(t))_{t\geq 0}$. Second, (II.1.6) and

(II.2.7) together imply that $D(F) \supset D(T)$. Third, it is clear that $F[g] \geq 0$ for any $g \in D(F)_+$. We have to look at the last condition of Kato-Voigt theorem. Let $u \in D(T)_+$, $u = (\lambda I - T)^{-1}v$, for $v \in X$.

$$\int_0^\infty (T[u] + F[u])x\mathrm{d}x = \underbrace{\int_0^\infty x(r(x)u(x))_x \mathrm{d}x}_{A_{T_0}} - \underbrace{\int_0^\infty xp(x)u(x)\mathrm{d}x}_{A_{T_1}} + \underbrace{\int_0^\infty xF[u]\mathrm{d}x}_{A_F} \quad \text{(II.2.9)}$$

Considering each independently we obtain,

$$\begin{aligned}
A_{T_0} &= \int_0^\infty x\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\lambda R(x)+Q(x)}\int_x^\infty e^{-\lambda R(y)-Q(y)}v(y)\mathrm{d}y\right]\mathrm{d}x \\
&= \int_0^\infty x\frac{\lambda + p(x)}{r(x)}\left[e^{\lambda R(x)+Q(x)}\int_x^\infty e^{-\lambda R(y)-Q(y)}v(y)\mathrm{d}y\right]\mathrm{d}x - \int_0^\infty yv(y)\mathrm{d}y \\
&= \int_0^\infty e^{-\lambda R(y)-Q(y)}v(y)\int_0^y x\frac{\lambda + p(x)}{r(x)}e^{\lambda R(x)+Q(x)}\mathrm{d}x\mathrm{d}y - \int_0^\infty yv(y)\mathrm{d}y \\
&= \int_0^\infty e^{-\lambda R(y)-Q(y)}v(y)\int_0^y x\frac{\mathrm{d}}{\mathrm{d}x}e^{\lambda R(x)+Q(x)}\mathrm{d}x\mathrm{d}y - \int_0^\infty yv(y)\mathrm{d}y \\
&= \int_0^\infty yv(y)\mathrm{d}y - \int_0^\infty v(y)e^{-\lambda R(y)-Q(y)}\left(\int_0^y e^{\lambda R(y)+Q(x)}\mathrm{d}x\right)\mathrm{d}y - \int_0^\infty yv(y)\mathrm{d}y \\
&= -\int_0^\infty r(x)u(x)\mathrm{d}x.
\end{aligned}$$

and

$$\begin{aligned}
A_F &= \int_0^\infty x\left[\int_x^\infty a(y)b(x|y)u(y)\mathrm{d}y\right]\mathrm{d}x = \int_0^\infty a(y)u(y)\int_0^y xb(x|y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty ya(y)u(y)\mathrm{d}y.
\end{aligned}$$

Finally, combining them together we have,

$$\begin{aligned}
\int_0^\infty (Tu + Fu)x\mathrm{d}x &= -\int_0^\infty r(x)u(x)\mathrm{d}x - \int_0^\infty xp(x)u(x)\mathrm{d}x + \int_0^\infty xa(x)u(x)\mathrm{d}x \\
&= -\int_0^\infty r(x)u(x)\mathrm{d}x - \int_0^\infty x\mu(x)u(x)\mathrm{d}x \leq 0.
\end{aligned}$$

It follows that all conditions of the generalised Kato-voigt theorem are satisfied. Hence, we claim that there is a smallest semigroup $(S_K(t))_{t\geq 0}$ generated by the extension K of $T + F$.

To complete the well-posedness analysis, we stress some point on the extension's uniqueness. The method we deployed, yields the smallest semigroup solution. This solution is a particular realization of the problem. There is a possibility that there exist other solutions to (II.1.1).

Indeed, extension $K = K_{min}$ obtained in Section II.2 correspond to the minimal operator $(T + F, D(T))$. There may exist other extensions that correspond to different realization of the sum $T + F$. For instance, it may happen that $K_{min} \subsetneq K_{max}$, where K_{max} is an extension of $(T + F, D(T + F))$, with $D(T + F) = \{u \in X : (T + F)[u] \in X\}$. In this case the solution of (II.1.1) may not be unique. Examples of such behaviour are well-known in the literature, see [ZM85, Ban02] for particular examples.

Chapter III

Numerical Analysis of the Model

In this chapter we concentrate on the quantitative analysis of the fragmentation equation with transport. We apply a pseudo-spectral method to solve equation (II.1.1) . For our method to work (II.1.1) must be well-posed in a Hilbert space. We choose a suitable working space in which we do error and stability analysis and construct a numerical scheme that solves the problem. We begin with the wellposedness analysis in a Hilbert spaces.

1 Wellposedness in $X = L^2_\alpha(\mathbb{R}_+)$

To establish existence of semigroup solutions in Hilbert settings, we follow the technique of Chapter II with some modifications. We begin with the transport semigroup.

1.1 The Transport Semigroup

In what follows we study wellposedness of our problem in the weighted Hilbert space $X = L^2_\alpha(\mathbb{R}_+) = \{f| \quad \|x^{\alpha/2} f\|_{L^2} < \infty\}$. Let κ be a non-negative parameter, and

$$T_\kappa[u] = (ru)_x - (\mu + a - \kappa^2 a^2)u, \quad D(T_\kappa) = D(T_0) = \{u, T_0 u, au \in X\}, \quad \kappa > 0. \qquad \text{(III.1.1)}$$

Assumptions of Section II.2.1 in Chapter II do not necessarily imply that T_κ generates a C_0-semigroup in $X = L^2_\alpha(\mathbb{R}_+)$. For this to be true the coefficients of the model must satisfy some extra conditions.

Lemma III.1.1. *Assume that for some $\kappa > 0$*

$$\omega_\kappa(x) = \frac{\alpha r}{x} + 2(\mu + a - \kappa^2 a^2) - r_x \geq 0, \qquad \textit{a.e. in } \mathbb{R}_+, \tag{III.1.2}$$

and there exist a measurable set $\Omega \subset \mathbb{R}_+$, such that

$$\max\left\{ \|a|_\Omega\|_\infty, \|\tfrac{a}{\sqrt{r}}|_{\mathbb{R}_+/\Omega}\|_2, \|\tfrac{a^2}{\sqrt{r}}|_{\mathbb{R}_+/\Omega}\|_2, \right\} \leq \Lambda. \tag{III.1.3}$$

Then $(T_\kappa, D(T_0))$ generates a C_0-semigroup of contraction.

Proof. Condition (III.1.2) implies that T_κ is monotone, that is $\langle T_k u, u\rangle \leq 0$, for all $u \in D(T_0)$. For "good" functions the statement can be verified without difficulties using integration by parts and for any other $u \in D(T_\kappa)$ the inequality follows from the closedness of T_κ.
Consider the resolvent equation,

$$(\lambda I - T_\kappa)u = v, \qquad v \in X. \tag{III.1.4}$$

Its formal solution is given by:

$$u(x) = \mathbf{R}_\lambda[v] = \frac{1}{r(x)} e^{R(x)} \int_x^\infty e^{-R(t)} v(t)dt, \quad R(x) = \int_1^x \frac{\lambda + \mu(t) + a(t) - \kappa^2 a^2(t)}{r(t)} dt.$$

Using weighted version of Hardy-type inequality [KP03] we conclude that $\mathbf{R}_\lambda[v]$ is bounded from X to $D(T_\kappa)$ if and only if

$$A = \sup_{x>0} \left(\int_0^x t^\alpha \frac{e^{2R(t)}}{r^2(t)} dt \right)^{1/2} \left(\int_x^\infty \frac{e^{-2R(t)}}{t^\alpha} dt \right)^{1/2} \leq \infty, \tag{III.1.5a}$$

$$B_1 = \sup_{x>0} \left(\int_0^x t^\alpha \frac{a^2(t) e^{2R(t)}}{r^2(t)} dt \right)^{1/2} \left(\int_x^\infty \frac{e^{-2R(t)}}{t^\alpha} dt \right)^{1/2} \leq \infty, \tag{III.1.5b}$$

$$B_2 = \sup_{x>0} \left(\int_0^x t^\alpha \frac{a^4(t) e^{2R(t)}}{r^2(t)} dt \right)^{1/2} \left(\int_x^\infty \frac{e^{-2R(t)}}{t^\alpha} dt \right)^{1/2} \leq \infty. \tag{III.1.5c}$$

We show that (III.1.2) implies (III.1.5a).

From (III.1.2) it follows that,

$$0 \leq 2\lambda x^\alpha \frac{e^{2R(x)}}{r(x)} \leq \frac{d}{dx}\left[x^\alpha \frac{e^{2R(x)}}{r(x)} \right],$$

therefore,

$$\int_0^x t^\alpha \frac{e^{2R(t)}}{r(t)} dt \le \frac{1}{2\lambda}\left[x^\alpha \frac{e^{2R(x)}}{r(x)}\right]. \tag{III.1.6}$$

In a similar way,

$$\frac{e^{-2R(x)}}{x^\alpha} \le -\frac{1}{2\lambda}\frac{d}{dx}\left[r(x)\frac{e^{-2R(x)}}{x^\alpha}\right],$$

and

$$\int_x^\infty \frac{e^{-2R(t)}}{t^\alpha} dt \le \frac{1}{2\lambda}\left[r(x)\frac{e^{-2R(x)}}{x^\alpha}\right]. \tag{III.1.7}$$

Combining (III.1.6) and (III.1.7) together we obtain

$$A \le \frac{1}{2\lambda}, \qquad \text{for } \lambda > 0.$$

In the same way one can show that (III.1.2), (III.1.3) imply (III.1.5b) and (III.1.5c), and

$$\max\{B_1, B_2\} \le \frac{\Lambda^{1/2}}{\sqrt{2\lambda}}. \tag{III.1.8}$$

From the above calculations it follows that for any $v \in X$ and $\lambda > 0$ there exist $u = \mathbf{R}_\lambda[v] \in D(T_\kappa)$, such that $(\lambda I - T_\kappa)u = v$, i.e. the range of $(\lambda I - T_\kappa)$ is X. Since T_κ is maximal monotone Lumer-Phillips theorem implies that $(T_\kappa, D(T_\kappa))$ generates a C_0-semigroup of contraction in X.

□

1.2 The Transport-Fragmentation Semigroup

Now we consider the complete model

$$u_t = (ru)_x - (\mu + a)u + \int_x^\infty a(y)u(y)b(x|y)dy := (T_0 + F)[u]. \tag{III.1.9}$$

It seems to be difficult to establish $L^2_\alpha(\mathbb{R}_+)$ theory for general fragmentation kernels. In the sequel we assume that the kernels are separable, that is:

$$b(x|y) = b_1(x)b_2(y), \quad \text{where} \quad b_2(y) = \frac{y}{\int_0^y x b_1(x)dx}. \tag{III.1.10}$$

Lemma III.1.2. *Assume that for some $\kappa > 0$ conditions of Lemma III.1.1 are satisfied. If, in addition,*

$$C = \sup_{x>0}\left(\int_0^x t^\alpha b_1^2(t)dt\right)^{1/2}\left(\int_x^\infty \frac{b_2^2(y)dy}{y^\alpha}\right)^{1/2} \le \infty, \tag{III.1.11}$$

then $(T_0 + F, D(T_0))$ generates a C_0-semigroup $(G(t))_{t\ge0}$ in X. Moreover, $\|G(t)\|_X \le e^{\lambda_0 t}$ with $\lambda_0 = \max\{0, (\kappa^2+1)8\Lambda C^2, \frac{C^2}{\kappa^2}\}$.

Proof. From Lemma III.1.1, we have $\|\mathbf{R}(\lambda, T_\kappa)\| \leq \frac{1}{\lambda}$. Moreover, using estimate (III.1.8) combined with the best Hardy constant[1] we obtain the following upper bound:

$$\max\{\|a\mathbf{R}(\lambda, T_\kappa)\|, \|a^2\mathbf{R}(\lambda, T_\kappa)\|\} \leq \frac{\sqrt{2\Lambda}}{\sqrt{\lambda}}. \tag{III.1.12}$$

Assumption (III.1.11) implies that $\|F[u]\|_X \leq 2C\|au\|$ for all $u \in D(T_\kappa)$, therefore

$$\|F\mathbf{R}(\lambda, T_\kappa)u\| \leq 2C\|a\mathbf{R}(\lambda, T_\kappa)u\| \leq \frac{2C\sqrt{2\Lambda}}{\sqrt{\lambda}}\|u\|, \tag{III.1.13}$$

that is $\|F\mathbf{R}(\lambda, T_0)\| < 1$, provided that $\lambda > 8\Lambda C^2$. Using estimates (III.1.12) and (III.1.13) it is not hard show that the resolvent equation is solvable. Indeed,

$$(\lambda I - T_0 - F)u = (\lambda I - T_\kappa + \kappa^2 a^2 - F)u = (I + \kappa^2 a^2 \mathbf{R}(\lambda, T_\kappa) - F\mathbf{R}(\lambda, T_\kappa))(\lambda I - T_\kappa)u = v, \quad v \in X.$$

Since for all $\lambda > (\kappa^2 + 1)8\Lambda C^2$ operator $(I + \kappa^2 a^2 \mathbf{R}(\lambda, T_\kappa) - F\mathbf{R}(\lambda, T_\kappa))$ is bounded with a bounded inverse, it follows that $\lambda \in \rho(T_0 + F)$ for all $\lambda > \lambda_0$.

To estimate $\|\mathbf{R}(\lambda, T_0 + F)\|$, we take the scalar product of the resolvent equation with u. This gives

$$\langle(\lambda I - T_0 - F)u, u\rangle = \lambda\|u\|^2 - \langle T_0 u, u\rangle - \langle Fu, u\rangle = \langle v, u\rangle,$$

and

$$\begin{aligned}
\lambda\|u\|^2 - \langle T_0 u, u\rangle &\leq \|u\|\|v\| + \|Fu\|\|u\| \\
&\leq \|u\|\|v\| + 2C\|au\|\|u\| \\
&\leq \|u\|\|v\| + \kappa^2\|au\|^2 + \frac{C^2}{\kappa^2}\|u\|^2.
\end{aligned}$$

The last inequality can be rewritten as

$$\lambda\|u\|^2 - \langle T_\kappa u, u\rangle \leq \|u\|\|v\| + \frac{C^2}{\kappa^2}\|u\|^2.$$

Since T_κ is monotone, it follows that $(\lambda - \frac{C^2}{\kappa^2})\|u\| \leq \|v\|$ or $\|\mathbf{R}(\lambda, T_0 + F)\| \leq \dfrac{1}{\lambda - \frac{C^2}{\kappa^2}} \leq \dfrac{1}{\lambda - \lambda_0}$. By Hille-Yosida theorem we conclude that $(T_0 + F, D(T_0))$ generates a C_0-semigroup $(G(t))_{t \geq 0}$ in X and $\|G(t)\| \leq e^{\lambda_0 t}$, for $t > 0$. □

[1] The best constant C satisfy $A \leq C \leq K(p,q)A$, where A is given by (III.1.5a), $K(p,q) = p^{1/q}(p\prime)^{1/p\prime}$, and $\frac{1}{p} + \frac{1}{q} = 1$ (see [KP03]).

2 The Laguerre-type Spectral Methods

Numerical methods for solving PDE's can be categorized as local and global methods [STW11, HGG07]. Local methods use nearby grid points to approximate a function at a given instant, while in global methods calculations at any given point depend on the information from the whole spatial domain. Typical representatives of local methods are Finite Elements (FEM), Finite Volumes (FVM) and Finite Differences (FDM) methods. The examples of global methods are spectral method and their modifications. The advantage of global methods is their superior accuracy, while local methods enjoy domain flexibility [STW11].

In the past three decades there has been a growing interest in spectral methods that resulted in a well-developed theory [CQHZ06, HGG07, STW11, Tre00]. A spectral method is defined in terms of two sets of functions: the set of basis functions $\{\phi_n\}_{n\geq 0}$ and the set of test or trial functions $\{\psi_n\}_{n\geq 0}$. The exact solution is approximated by a finite linear combination of the basis functions

$$u(x) \approx u_N(x) = \sum_{n=0}^{N} \hat{u}_n \phi_n(x). \tag{III.2.1}$$

The test functions are used to determinethe unknown spectral coefficients $\hat{u}_n$. Three of the earliest types of spectral schemes, Galerkin, Collocation and Tau, differ by the choice of the test functions [CQHZ06]. We emphasize that unlike FEM and FEM-like methods, spectral methods employ bases that are not compactly supported in spatial domains. In most applications the basis and the test functions are analytic or entire. Typical examples are trigonometric functions, orthogonal polynomials and their modifications, see [STW11].

Convergence and stability of a spectral methods are intimately connected with the approximation properties of the basis functions. A choice of the basis depends on many factors, and in particular, on the type of PDE and on the spatial domain. In our project we study Transport-Fragmentation equation (II.1.1) that is posed in an unbounded spatial domain. There are a three general ways to deal with such domains. In the first one, the domain is truncated. The approach works well for linear problems, where it is relatively easy to impose artificial boundary conditions. The second one make use of classical orthogonal polynomials combined with algebraic maps. This approach is quite popular in the numerical literature, see [Boy00, CQHZ06] and references therein. The third approach employs functions that are orthogonal in unbounded intervals.

In the project we follow the third approach. Since our domain is half line, it is natural to use generalized Laguerre functions. They are denoted by

$$\hat{L}_n^\alpha(x) = e^{\frac{-x}{2}} L_n^{(\alpha)}(x), \quad n \geq 0,$$

where $L_n^{(\alpha)}(x)$ are generalized Laguerre polynomials. For $\alpha > -1$ the collection $\{\hat{L}_n^\alpha\}_{n\geq 0}$ forms an orthogonal basis in $L^2_\alpha(\mathbb{R}_+)$ (see [Sze67]) and

$$\int_0^\infty x^\alpha \hat{L}_n^\alpha(x)\hat{L}_m^\alpha(x)dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}. \tag{III.2.2a}$$

The Laguerre functions satisfy many important identities [GR07], we list some of them below:

$$\hat{L}_n^\alpha(x) = \hat{L}_{n-1}^\alpha(x) + \hat{L}_n^{\alpha-1}(x), \quad \alpha \in \mathbb{C}, \tag{III.2.2b}$$

$$\hat{L}_n^\alpha(x) = \frac{1}{x}\left[(n+\alpha)\hat{L}_n^{\alpha-1}(x) - (n+1)\hat{L}_{n+1}^{\alpha-1}(x)\right], \quad \alpha \in \mathbb{C}, \tag{III.2.2c}$$

$$\hat{L}_n^\alpha(x) = \hat{L}_n^{\alpha+1}(x) - \hat{L}_{n-1}^{\alpha+1}(x), \quad \alpha \in \mathbb{C}, \tag{III.2.2d}$$

$$\frac{d}{dx}\hat{L}_n^\alpha(x) = -\frac{1}{2}\hat{L}_n^\alpha(x) - \sum_{s=0}^{n-1}\hat{L}_s^\alpha(x), \quad \alpha \in \mathbb{C}, \tag{III.2.2e}$$

$$\hat{L}_{n+1}^\alpha(x) = \frac{1}{x}\left[(x-n)\hat{L}_n^\alpha(x) + (\alpha+n)\hat{L}_{n-1}^\alpha(x)\right], \quad \alpha \in \mathbb{C}, \tag{III.2.2f}$$

$$\hat{L}_n^\alpha(0) = \binom{n+\alpha}{n}, \quad \text{and} \quad \sum_{m=0}^{n}\hat{L}_m^\alpha(x)\hat{L}_{n-m}^\beta(y) = \hat{L}_n^{\alpha+\beta+1}(x+y), \quad \alpha, \beta \in \mathbb{C}. \tag{III.2.2g}$$

Note that in view of (III.2.2e) constant coefficient linear PDE's can be solved *exactly*. In addition, thanks to the exponential multiplier in $\hat{L}_n^\alpha(x)$, the Laguerre basis allows the treatment of problems with rapidly growing coefficients. The next subsection contains more details on the approximation properties of Laguerre basis.

2.1 The Scale of Bessel Potential Spaces

It is convenient to study Laguerre basis in weighted Bessel potential spaces. They are similar to Sobolev spaces and give an advantage to handle fractional and integer derivatives of functions. We describe the weighted Bessel potential space in terms of fractional integrals, this approach avoids use of space interpolation and allows to obtain sharper error estimates. We begin by defining left and right Bessel fractional integrals of order $\beta > 0$ in $\mathbb{R}_+$. Let $g(x)$ be a measurable function, we define the former by

$$J_+^\beta[g](x) = \int_0^x e^{\frac{t-x}{2}}(x-t)^{\beta-1}g(t)dt, \tag{III.2.3}$$
$$J_-^\beta[g](x) = \int_x^\infty e^{\frac{x-t}{2}}(t-x)^{\beta-1}g(t)dt. \tag{III.2.4}$$

We also define the Riemann-Liouville and the Weyl fractional integrals,

$$I_+^\beta[g](x) = \int_0^x (x-t)^{\beta-1}g(t)dt, \tag{III.2.5}$$
$$I_-^\beta[g](x) = \int_x^\infty (t-x)^{\beta-1}g(t)dt, \tag{III.2.6}$$

then the Bessel fractional integrals can be expressed as follows: $J_+^\beta = e^{\frac{-x}{2}}I_+^\beta e^{\frac{x}{2}}$ and $J_-^\beta = e^{\frac{x}{2}}I_-^\beta e^{\frac{-x}{2}}$.

We describe the weighted Bessel potential spaces by means of operator J_-^β. Let u and v be almost everywhere positive measurable function in $\mathbb{R}_+$ and

$$L_u^p(\mathbb{R}_+) = \{f | \|u^{\frac{1}{p}}f\|_p = \|f\|_{u,p} < \infty\} \quad \text{and} \quad L_v^p(\mathbb{R}_+) = \{f | \|v^{\frac{1}{p}}f\|_p = \|f\|_{v,p} < \infty\},$$

be two corresponding weighted Lebesque spaces with $1 < p < \infty$. We set $u_- = ue^{\frac{pt}{2}}, v_- = ve^{\frac{pt}{2}}, \frac{1}{p} + \frac{1}{p'} = 1$ and

$$A_{p,\beta}(u,v) = \sup_{x>0}\left(I_+^{p(\beta-1)+1}[u_-](x)\right)^{1/p}\left(I_-^1[v_-^{1-p'}](x)\right)^{1/p'}, \tag{III.2.7}$$
$$\tilde{A}_{p,\beta}(u,v) = \sup_{x>0}\left(I_-^{p'(\beta-1)+1}[v_-^{1-p'}](x)\right)^{1/p'}\left(I_+^1[u_-](x)\right)^{1/p}, \tag{III.2.8}$$
$$B_{p,\beta}(u,v) = \sup_{x>0}\left(I_+^\beta[v_-^{1-p'}(I_+^\beta[u_-])^{p'}](x)\right)^{1/p'}\left(I_+^\beta[u_-](x)\right)^{-1/p'}, \tag{III.2.9}$$
$$\tilde{B}_{p,\beta}(u,v) = \sup_{x>0}\left(I_-^\beta[u_-(I_-^\beta[v_-^{1-p'}])^p](x)\right)^{1/p}\left(I_-^\beta[v_-^{1-p'}](x)\right)^{-1/p}. \tag{III.2.10}$$

It is shown in [KP03], that the operator J_-^β, is continuous from $L_v^p(\mathbb{R}_+)$ to $L_u^p(\mathbb{R}_+)$ if and only if $A_{p,\beta}(u,v) < \infty$ and $\tilde{A}_{p,\beta}(u,v) < \infty$, $\beta \geq 1$. Recently in [PS02] for $0 < \beta < 1$, J_-^β, it is proved that J_-^β is continuous provided that $B_{p,\beta}(u,v) < \infty$ or $\tilde{B}_{p,\beta}(u,v) < \infty$, $0 \leq \beta \leq 1$. Hence, when (u,v) are sufficiently regular, the following Hardy-type inequality holds:

$$\|J_-^\beta[f]\|_{u,v} \leq c_{u,v,p,\beta}\|f\|_{v,p}, \quad 1 \leq p \leq \infty, \quad \beta > 0, \tag{III.2.11}$$

where constant $c_{u,v,p,\beta}$ satisifies

$$c_{u,v,p,\beta} \sim \max\{A_{p,\beta}(u,v), \tilde{A}_{p,\beta}(u,v)\}, \quad \beta \geq 1,$$

and

$$c_{u,v,p,\beta} \leq \min\left\{\frac{p}{\Gamma(\beta)}B_{p,\beta}(u,v), \frac{p\prime}{\Gamma(\beta)}\tilde{B}_{p,\beta}(u,v)\right\}, \quad 0 < \beta < 1.$$

In view of (III.2.11) we define the following class $H_v^{p,\beta}(\mathbb{R}_+) = J_-^{\beta}\left(L_v^p(\mathbb{R}_+)\right)$, $1 < p < \infty$, $\beta > 0$. It can be shown [SKM93], that there exists for operator $J_-^{-\beta}$ with the properties:

(i) $J_-^{-\beta}$ maps $H_v^{p,\beta}(\mathbb{R}_+)$ onto $L_v^p(\mathbb{R}_+)$;

(ii) $J_-^{-\beta}[J_-^{\beta}] = I$ in $L_v^p(\mathbb{R}_+)$ and $J_-^{\beta}[J_-^{-\beta}] = I$ in $H_v^{p,\beta}(\mathbb{R}_+)$;

(iii) $J_-^{-\beta}[f] = 0$ if and only if $f = 0$ in $L_v^p(\mathbb{R}_+)$;

(iv) $f \in H_v^{p,\beta}(\mathbb{R}_+)$ if and only if $J_-^{-\beta}[f] \in L_v^p(\mathbb{R}_+)$ and $f \in L_u^p(\mathbb{R}_+)$ where the pair (u,v) satisfies (III.2.7)–(III.2.10).

From (i)—(iv) the class $H_v^{p,\beta}(\mathbb{R}_+)$ equipped with the norm $\|f\|_{v,p,\beta} = \|J_-^{-\beta}[f]\|_{v,p}$ is isometrically isomorphic to $L_v^p(\mathbb{R}_+)$, and hence is a Banach space. It is called the weighted Bessel potential space.

We study approximation properties of Laguerre functions in the Hilbert spaces

$$H_{\alpha,\beta}(\mathbb{R}_+) = H_{x^\alpha}^{2,\beta}(\mathbb{R}_+), \quad \alpha > -1, \qquad \beta > 0,$$

where the scalar product and the induced norm are given by

$$\langle f,g\rangle_{\alpha,\beta} = \int_{\mathbb{R}_+} x^\alpha J_-^{-\beta}[f](x)\overline{J_-^{-\beta}[g](x)}dx, \quad \|f\|_{\alpha,\beta} = \langle f,g\rangle_{\alpha,\beta}^{1/2}.$$

It is not difficult to verify that for $H_{\alpha,\beta}(\mathbb{R}_+)$ inequality (III.2.11) takes the form:

$$\|f\|_\gamma \leq c_{\alpha,\beta,\gamma}\|f\|_{\alpha,\beta}, \tag{III.2.12}$$

where parameters α, β and γ satisfy

$$\alpha > -1, \quad \beta > 0, \quad \gamma > -1 \quad \text{and} \quad \alpha - 2\beta \leq \gamma \leq \alpha.$$

To conclude we mention several useful results related to Bessel fractional integrals and derivatives. For $f \in L^p(\mathbb{R}_+), g \in L^{p'}(\mathbb{R}_+)$, the formula of fractional integration by parts holds:

$$\int_{\mathbb{R}_+} f J^{\beta}_{\pm}[g]dx = \int_{\mathbb{R}_+} g J^{\beta}_{\mp}[f]dx, \qquad \beta > 0. \tag{III.2.13}$$

The formula of fractional differentiation by parts reads:

$$\int_{\mathbb{R}_+} f J^{-\beta}_{\pm}[g]dx = \int_{\mathbb{R}_+} g J^{-\beta}_{\mp}[f]dx, \qquad \beta > 0, \tag{III.2.14}$$

where $f, J^{-\beta}_{\pm} f \in L^p(\mathbb{R}_+)$ and $g, J^{-\beta}_{\pm} g \in L^{p'}(\mathbb{R}_+)$ for some $1 \le p \le \infty$, [SKM93]. For the generalized Laguerre functions the following holds [PBM92, p. 462-463 formulas 2.19.2.2 and 2.19.3.7]:

$$J^{\beta}_{+}[x^{\alpha}\hat{L}^{\alpha}_n(x)] = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\beta+n+1)} x^{\alpha+\beta}\hat{L}^{\alpha+\beta}_n(x), \quad J^{-\beta}_{-}[\hat{L}^{\alpha}_n(x)] = \hat{L}^{\alpha+\beta}_n(x), \quad \beta > 0. \tag{III.2.15}$$

2.2 Laguerre Spectral Approximation in $H_{\alpha,\beta}(\mathbb{R}_+)$

It is natural to approximate elements of a Hilbert space by their truncated Fourier series. Let N be a positive integer, $\mathbb{P}_n$ be a subspace of $L^2_{\alpha}(\mathbb{R}_+)$ spanned by $\{e^{\frac{-x}{2}} x^n | 0 \le n \le N\}$, and $P^{\alpha}_N : L^2_{\alpha}(\mathbb{R}_+) \to \mathbb{P}_n$ be the orthogonal projector onto $\mathbb{P}_n$, i.e. $P^{\alpha}_N f = \sum_{n=0}^{N} \hat{f}_n \hat{L}^{\alpha}_n(x)$, $f \in L^2_{\alpha}(\mathbb{R}_+)$. Then, we have the following estimate:

Lemma III.2.1. *Let $\alpha > -1$ and $\beta \ge 0$, then*

$$\|(I - P^{\alpha}_N) f\|_{\alpha} \le c N^{\frac{-\beta}{2}} \|f\|_{\alpha+\beta,\beta}. \tag{III.2.16}$$

The positive constant c is independent of N and f.

Proof. We have

$$\begin{aligned}
\|(I-P_N^\alpha)f\|_\alpha^2 &= \sum_{N<n}\frac{n!}{\Gamma(\alpha+n+1)}(a_n^\alpha)^2 = \sum_{N<n}\frac{n!}{\Gamma(\alpha+n+1)}\langle \hat{L}_n^\alpha, f\rangle^2 \\
&= \sum_{N<n}\frac{n!}{\Gamma(\alpha+n+1)}\left[\int_{\mathbb{R}_+} J_+^{-\beta}J_+^{\beta}[x^\alpha \hat{L}_n^\alpha]f(x)dx\right]^2 \quad \text{(since } J_+^{-\beta}J_+^{\beta}=I) \\
&= \sum_{N<n}\frac{n!}{\Gamma(\alpha+n+1)}\left[\int_{\mathbb{R}_+} J_+^{\beta}[x^\alpha \hat{L}_n^\alpha]J_-^{-\beta}[f(x)]dx\right]^2 \quad \text{(by } (III.2.14)) \\
&= \sum_{N<n}\frac{\Gamma(\alpha+n+1)n!}{\Gamma^2(\alpha+\beta+n+1)}\left[\int_{\mathbb{R}_+} x^{\alpha+\beta}\hat{L}_n^{\alpha+\beta}J_-^{-\beta}[f(x)]dx\right]^2 \quad \text{(by}(III.2.15)) \\
&\le \frac{\Gamma(\alpha+N+1)}{\Gamma(\alpha+\beta+N+1)}\|f\|_{\alpha+\beta,\beta}^2
\end{aligned}$$

Using Stirling's formula and taking the square root we obtain the result. □

Then Lemma III.2.1 enables us to proof the following results.

Theorem III.2.2. *Let $\alpha > -1$, $\beta > -1$, $\gamma \ge 0$ and $\delta \ge 0$ satisfy*

$$\beta \le \alpha+\gamma, \quad \delta \ge \alpha+2\gamma-\beta, \tag{III.2.17a}$$

then

$$\|(I-P_N^\alpha)f\|_{\beta,\gamma} \le cN^{\gamma+(\alpha-\beta-\delta)/2}\|f\|_{\alpha+\delta,\delta}, \tag{III.2.17b}$$

where positive constant c is independent of N and f.

Proof. By (III.2.15) $J_-^{-\gamma}P_N^\alpha[f] = P_N^{\alpha+\gamma}J_-^{-\gamma}[f]$, therefore

$$\begin{aligned}
\|(I-P_N^\alpha)f\|_{\beta,\gamma} &= \|(I-P_N^{\alpha+\gamma})J_-^{-\gamma}[f]\|_\beta \\
&\le c\|(I-P_N^{\alpha+\gamma})J_-^{-\gamma}[f]\|_{2(\alpha+\gamma)-\beta,\alpha+\gamma-\beta} \quad \text{(by (III.2.12))} \\
&= c\|(I-P_N^{2(\alpha+\gamma)-\beta})J_-^{-(\alpha+2\gamma-\beta)}[f]\|_{2(\alpha+\gamma)-\beta} \\
&\le c_1N^{-\xi/2}\|J_-^{-(\alpha+2\gamma-\beta)}[f]\|_{2(\alpha+\gamma)-\beta+\xi,\xi} \quad \text{(by Lemma III.2.1)} \\
&= c_1N^{-\xi/2}\|f\|_{2(\alpha+\gamma)-\beta+\xi,\xi+\alpha+2\gamma-\beta} \\
&= c_1N^{\gamma+(\alpha-\delta-\beta)/2}\|f\|_{\alpha+\delta,\delta},
\end{aligned}$$

where $\delta = \xi+\alpha+2\gamma-\beta$. The proof is complete. □

Now we estimate the approximation error for $\beta \geq \alpha + \gamma$.

Theorem III.2.3. *Let $\alpha > -1$, $\beta > -1$, $\gamma \geq 0$ and $\delta \geq 0$ satisfy*

$$\beta \geq \alpha + \gamma, \quad \delta \geq \beta - \alpha, \tag{III.2.18a}$$

then

$$\|(I - P_N^\alpha)f\|_{\beta,\gamma} \leq cN^{(\beta-\alpha-\delta)/2}\|f\|_{\alpha+\delta,\delta}, \tag{III.2.18b}$$

where positive constant c is independent of N and f.

Proof. Using the same approach as in Theorem III.2.1 we obtain

$$\begin{aligned}
\|(I - P_N^\alpha)f\|_{\beta,\gamma} &= \|(I - P_N^{\alpha+\gamma})J_-^{-\gamma}[f]\|_\beta \\
&\leq c\|(I - P_N^{\alpha+\gamma})J_-^{-\gamma}[f]\|_{\beta,\beta-\alpha-\gamma} \quad \text{(by (III.2.12))} \\
&= c\|(I - P_N^\beta)J_-^{-(\beta-\alpha)}[f]\|_\beta \\
&\leq c_1 N^{-\xi/2}\|J_-^{-(\beta-\alpha)}[f]\|_{\beta+\xi,\xi} \quad \text{(by Lemma III.2.1)} \\
&= c_1 N^{-\xi/2}\|f\|_{\beta+\xi,\xi+\beta-\alpha} = c_1 N^{(\beta-\alpha-\delta)/2}\|f\|_{\alpha+\delta,\delta},
\end{aligned}$$

where $\delta = \xi + \beta - \alpha$. $\square$

Theorems III.2.2 and III.2.3 provide us the complete description of Laguerre spectral approximation error in spaces $H_{\alpha,\beta}(\mathbb{R}_+)$.

2.3 Laguerre Interpolation at Gauss and Radau Nodes

The major problem with spectral approximations is that one has to calculate all the integrals $\langle f, \hat{L}_n^\alpha\rangle$ exactly, which is not always possible. The simplest solution is to replace all the integrals with quadrature formulas. For Gaussian quadratures this approach is equivalent to the standard polynomial interpolation (the ancient technique first used by John Wallis in 1655 [GS00]).

Given an ordered set of points $X = \{x_n\}_{n=0}^N, 0 \leq x_0 \leq \cdots \leq x_N \leq \infty$, we define the Lagrange interpolation operator $I_{X,N}^\alpha : L_\alpha^2 \to \mathbb{P}_N$, by the formula

$$e^{-x_n} I_{X,N}^\alpha[e^{x/2}f](x_n) = f(x_n), \qquad 0 \leq n \leq N. \tag{III.2.19}$$

Properties of operator $I^{\alpha}_{X,N}$ are completely determined by the set X, in the sequel we consider Gaussian and Radau abscissas only.

The set of Gaussian abscissas is denoted by $X_{\alpha,G} = \{x_{\alpha,n,N}\}_{n=0}^{N}$, where $x_{\alpha,n,N}$ are zeros of $L^{(\alpha)}_{N+1}(x)$. The Radau abscissas are given by $X_{\alpha,R} = \{0\} \cup \{x_{\alpha+1,n,N-1}\}_{n=0}^{N-1}$. The classical theory of Gaussian quadratures implies that

$$\|I_{X,N}[f]\|^2_{\alpha} = \sum_{n=0}^{N} w_{\alpha,n} e^{x_n} f^2(x_n), \quad f \in \mathbb{P}_N, \tag{III.2.20}$$

where X is either $X_{\alpha,G}$ or $X_{\alpha,R}$ and $w_{\alpha,n}$ are the corresponding quadrature weights. Gauss-Laguerre abscissas satisfy (see [Sze67, p. 129] or [MM08, p. 141] and references therein):

$$\left(\frac{3\pi}{16}\right)^2 \frac{(n+1)^2}{N+1} < x_{\alpha,n,N} < 4\frac{(n+1)^2}{N+1}, \quad 0 \le n \le N, \tag{III.2.21}$$

and

$$c\left(\frac{x_{\alpha,n,N}}{4(N+1) - x_{\alpha,n,N}}\right)^{1/2} \le h_{\alpha,n,N} \le C\left(\frac{x_{\alpha,n,N}}{4(N+1) - x_{\alpha,n,N}}\right)^{1/2}, \quad 0 \le n \le N-1, \tag{III.2.22}$$

where $h_{\alpha,n,N} = x_{\alpha,n+1,N} - x_{\alpha,n,N}$. Associated Christoffel numbers are given explicitly by

$$w_{\alpha,n,N} = \frac{\Gamma(\alpha+n+1)}{n! x_{\alpha,n,N} (L^{(\alpha)\prime}_n(x_{\alpha,n,N}))^2} \quad 0 \le n \le N. \tag{III.2.23}$$

The following estimate can be found in [MM08, p. 144]

$$c x^{\alpha}_{\alpha,n,N} e^{-x_{\alpha,n,N}} h_{\alpha,n,N} \le w_{\alpha,n,N} \le C x^{\alpha}_{\alpha,n,N} e^{-x_{\alpha,n,N}} h_{\alpha,n,N}, \quad 0 \le n \le N. \tag{III.2.24}$$

For Laguerre-Radau quadrature associated Christoffel numbers can be expressed in terms of Gaussian weights as follows:

$$w_{\alpha,0,N} = \frac{\Gamma^2(\alpha+2)\Gamma(N+1)}{(\alpha+2)\Gamma(N+\alpha+2)}, \quad w_{\alpha,n,N} = \frac{w_{\alpha+1,n-1,N-1}}{x_{\alpha,n-1,N-1}}, \quad 1 \le n \le N.$$

Some pointwise estimates and inverse inequalities that are needed to study the interpolation operator $I^{\alpha}_{X,N}$ are discussed in the following subsection.

2.3.1 Pointwise Estimates and Inverse Inequalities

We begin with the pointwise estimates.

Lemma III.2.4. *Let* $f \in H_{\alpha,\beta}(\mathbb{R}_+), \quad \alpha > -1$ *and* $1/2 < \beta \leq 1$*, then for* $x, h > 0$

$$|f(x)| \leq ch^{\beta-1/2}\|\phi\|_{[x,x+h]} + \frac{(1-e^{-h})^{1/2}}{h}\|\hat{f}\|_{[x,x+h]}, \tag{III.2.25}$$

where $\hat{f} \in H_{\alpha,\beta}(\mathbb{R}_+)$ *is defined by means of the identity* $\hat{f} = J_-^{\beta}[|J_-^{-\beta}f|]$ *and positive constant* c *does not depend on* $f, \hat{f}, x$ *and* h.

Proof. For $\tau > 0$ and $\phi = J_-^{-\beta}[f]$ we have

$$\begin{aligned}|f(x)| &\leq \frac{1}{\Gamma(\beta)}\left(\int_x^{x+\tau} + \int_{x+\tau}^{\infty}\right)(t-x)^{\beta-1}e^{(x-t)/2}|\phi(t)|dt \\ &\leq \frac{1}{\Gamma(\beta)}\int_x^{x+\tau}(t-x)^{\beta-1}e^{(x-t)/2}|\phi(t)|dt + e^{(-\tau/2)}\hat{f}(x+\tau).\end{aligned}$$

Integrating both sides with respect to τ from 0 to h and using the Cauchy-Schwarz inequality we get

$$\begin{aligned}h|f(x)| &\leq \int_0^h \frac{1}{\Gamma(\beta)}\int_x^{x+\tau}(t-x)^{\beta-1}e^{(x-t)/2}|\phi(t)|dtd\tau + \int_0^h e^{(-\tau/2)}\hat{f}(x+\tau)d\tau \\ &\leq h^{\beta+1/2}\frac{B^{1/2}(2\beta-1,3)}{\Gamma(\beta)}\|\phi\|_{[x,x+h]} + (1-e^{-h})^{1/2}\|\hat{f}\|_{[x,x+h]}.\end{aligned}$$

The proof is complete. □

For inverse inequalities we have the following result:

Lemma III.2.5. *Let* $\alpha > -1, \beta \geq 0$ *and* $p_N \in \mathbb{P}_N$*, then*

$$\|p_N\|_{\alpha+\beta,\beta} \leq cN^{\beta/2}\|p_N\|_{\alpha}, \tag{III.2.26a}$$

$$\|p_N\|_{\alpha} \leq cN^{|\alpha-\beta|/2}\|p_N\|_{\beta}, \tag{III.2.26b}$$

$$\|p_N\|_{\alpha,\beta} \leq cN^{\beta-\min\{0,\alpha\}}\|p_N\|_{\alpha}. \tag{III.2.26c}$$

In each inequality positive constant c *do not depend on* N *and* p_N.

Proof. (i) Let $p_N = \sum_{n=0}^{N} \frac{n!}{\Gamma(\alpha+n+1)} a_n \phi_n^\alpha(x)$, then using Gautschi inequality[2] and (III.2.15) we obtain

$$\|p_N\|_{\alpha+\beta,\beta}^2 = \sum_{n=0}^{N} \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} \frac{n!}{\Gamma(\alpha+n+1)} a_n^2 \leq cN^\beta \sum_{n=0}^{N} \frac{n!}{\Gamma(\alpha+n+1)} a_n^2 = cN^\beta \|p_N\|_\alpha^2.$$

(ii) By (III.2.15) and generalised Hardy-type inequality,

$$\|p_N\|_\alpha \leq c\|p_N\|_{\alpha,\alpha-\beta} \leq c_2 N^{(\beta-\alpha)/2} \|p_N\|_\beta,$$

if $\alpha \geq \beta$. On the other hand,

$$\|p_N\|_\alpha \leq c\|p_N\|_{2\beta-\alpha,\beta-\alpha} \leq c_2 N^{(\alpha-\beta)/2} \|p_N\|_\beta,$$

if $\alpha \leq \beta$.

(iii) Using (III.2.15) and (III.2.2g)

$$J_-^{-\beta}[p_N] = \sum_{n=0}^{N} \frac{n!}{\Gamma(\alpha+n+1)} \phi_n^\alpha(x) \sum_{m=n}^{N} \frac{m!\Gamma(n+\alpha+1)}{n!\Gamma(m+\alpha+1)} L_{m-n}^{\beta-1}(0) a_m.$$

Then,

$$\|p_N\|_{\alpha,\beta} \leq |D^{-1}UD| \|p_N\|_\alpha,$$

where $|.|$ is the standard euclidean norm, $D = \operatorname{diag}\left\{\frac{1}{\Gamma^{1/2}(\alpha+1)}, \ldots, \frac{(N!)^{1/2}}{\Gamma^{1/2}(\alpha+N+1)}\right\}$, U is the upper triangular Toeplitz matrix of the form $U = \sum_{n=0}^{N} L_n^{\beta-1}(0) I_n$, where I_n are matrices with unit on the n-th upper diagonal. Let $\alpha \geq 0$, then

$$|D^{-1}UD| \leq \sum_{n=0}^{N} L_n^{\beta-1}(0)|D^{-1}I_nD| \leq \sum_{n=0}^{N} L_n^{\beta-1}(0) = L_n^{\beta-1}(0) = \binom{N+\beta}{N} \leq cN^\beta.$$

Similarly, for $-1 < \alpha < 0$, we obtain

$$\begin{aligned} |D^{-1}UD| &\leq \sum_{n=0}^{N} L_n^{\beta-1}(0)|D^{-1}I_nD| \leq \sum_{n=0}^{N} L_n^{\beta-1}(0) \frac{n!\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \\ &\leq c\sum_{n=0}^{N} (n+1)^{\beta-\alpha-1} \leq C_1 N^{\beta-\alpha}. \end{aligned}$$

The proof is complete. □

[2]Which says that $n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq (n+1)^{1-s}$ for all $n \in \mathbb{N}$ and $0 \leq s \leq 1$, see [Gau59].

2.3.2 Interpolation at Gaussian Abscissas

Using results of Section III.2.3.1 we can prove the following stability estimate:

Lemma III.2.6. *Let*

$$\alpha > -1, \quad 1/2 < \gamma \leq 1, \quad \beta \geq 0 \quad \textit{and} \quad \alpha + \gamma \leq \beta \leq \alpha + 2\gamma, \tag{III.2.27a}$$

then

$$\|I^{\alpha}_{X_{G,N}}[f]\|_{\alpha} \leq c\|f\|_{\beta,\gamma}. \tag{III.2.27b}$$

Positive constant c does not depend on N and f.

Proof. Identity (III.2.20) and Lemma III.2.4 yield,

$$\begin{aligned}\|I^{\alpha}_{X_{G,N}}[f]\|^2_{\alpha} \leq 2c_1 \sum_{n=0}^{N} w_{\alpha,n,N} e^{x_{\alpha,n,N}} h^{2\gamma-1}_{\alpha,n,N} \int_{x_{\alpha,n,N}}^{x_{\alpha,n+1,N}} \phi^2(x)dx \\ + 2\sum_{n=0}^{N} w_{\alpha,n,N} e^{x_{\alpha,n,N}} \frac{1-e^{-h_{\alpha,n,N}}}{h^2_{\alpha,n,N}} \int_{x_{\alpha,n,N}}^{x_{\alpha,n+1,N}} \hat{f}^2(x)dx = A + B,\end{aligned}$$

where we set $h_{\alpha,N,N} = h_{\alpha,N-1,N}, h_{\alpha,N+1,N} = x_{\alpha,N,N} + h_{\alpha,N,N}$ and $J^{\gamma}_{-}[\phi] = f$. Using (III.2.24) and taking into account that $\beta \geq 0$ we obtain

$$\begin{aligned} A &\leq 2c_1 \sum_{n=0}^{N} x^{\alpha}_{\alpha,n,N} h^{2\gamma}_{\alpha,n,N} \int_{x_{\alpha,n,N}}^{x_{\alpha,n+1,N}} \phi^2(x)dx \\ &\leq 2c_1 \sum_{n=0}^{N} x^{\alpha-\beta}_{\alpha,n,N} h^{2\gamma}_{\alpha,n,N} \int_{x_{\alpha,n,N}}^{x_{\alpha,n+1,N}} x^{\beta}\phi^2(x)dx \\ &\leq 2c_2 \max_{0\leq n\leq N} x^{\alpha-\beta}_{\alpha,n,N} h^{2\gamma}_{\alpha,n,N} \|f\|^2_{\beta,\gamma}. \end{aligned}$$

The estimates (III.2.21) and (III.2.22) imply,

$$\begin{aligned}\max_{0\leq n\leq N} x^{\alpha-\beta}_{\alpha,n,N} h^{2\gamma}_{\alpha,n,N} &\leq c_2 \max_{0\leq n\leq N-1} \frac{x^{\alpha-\beta+\gamma}_{\alpha,n,N}}{(4(N+1) - x_{\alpha,n,N})^{\gamma}} \\ &\leq c_3 N^{\max\{\alpha-\beta+\gamma, \beta-\alpha-2\gamma\}}.\end{aligned}$$

We follow the same procedure to obtain $B \leq c_4\|\hat{f}\|^2_{\alpha}$, where

$$c_4 \geq 2\left(\frac{16}{\pi}\right)^{-2\alpha} \geq 2 \max_{0\leq n\leq N} \left(\frac{x_{\alpha,n,N}}{x_{\alpha,n+1,N}}\right)^{\alpha} \frac{1-e^{-h_{\alpha,n,N}}}{h_{\alpha,n,N}},$$

when $-1 < \alpha < 0$ and

$$c_4 \geq 2 \geq 2 \max_{0 \leq n \leq N} \frac{1 - e^{-h_{\alpha,n,N}}}{h_{\alpha,n,N}},$$

when $\alpha \geq 0$. By (III.2.12) and definition of $\hat{f}$, $\|\hat{f}\|_\alpha \leq c_5 \|\hat{f}\|_{\beta,\gamma}$. This gives the result. □

Using the stability estimate and approximation properties of Section III.2.2 it is not difficult to obtain the following two theorems:

Theorem III.2.7. *Let $\alpha > -1$, $\beta > -1$, $\gamma \geq 0$, $1/2 < \xi \leq 1$ and $\delta \geq 0$, then*

$$\|(I - I^\alpha_{X_{G,N}})f\|_{\beta,\gamma} \leq c N^{\gamma + (\alpha - \beta - \delta + \xi)/2} \|f\|_{\alpha+\delta,\delta}, \quad \gamma - 1 < \beta \leq \alpha + \gamma, \tag{III.2.28a}$$

$$\|(I - I^\alpha_{X_{G,N}})f\|_{\beta,\gamma} \leq c N^{(\beta - \alpha - \delta + \xi)/2} \|f\|_{\alpha+\delta,\delta}, \quad \beta \geq \alpha + \gamma. \tag{III.2.28b}$$

Positive constant c does not depend on N and f.

Proof. We apply the triangular inequality, Lemma III.2.5 and Lemma III.2.6, to obtain

$$\begin{aligned}
\|(I - I^\alpha_{X_{G,N}})f\|_{\beta,\gamma} &\leq \|(I - P^\alpha_N)f\|_{\beta,\gamma} + \|(P^\alpha_N - I^\alpha_{X_{G,N}})f\|_{\beta,\gamma} \\
&\leq \|(I - P^\alpha_N)f\|_{\beta,\gamma} + cN^{\gamma/2} \|(P^\alpha_N - I^\alpha_{X_{G,N}})f\|_{\beta-\gamma} \quad \text{by (III.2.26a)} \\
&\leq \|(I - P^\alpha_N)f\|_{\beta,\gamma} + cN^{\gamma/2} \|I^\alpha_{X_{G,N}}(I - P^\alpha_N)f\|_{\beta-\gamma} \quad \text{by } [I^\alpha_X P^\alpha_X = P^\alpha_X] \\
&\leq \|(I - P^\alpha_N)f\|_{\beta,\gamma} + c_1 N^{(\gamma + |\alpha - \beta + \gamma|)/2} \|I^\alpha_{X_{G,N}}(I - P^\alpha_N)f\|_\alpha \quad \text{by (III.2.26b)} \\
&\leq \|(I - P^\alpha_N)f\|_{\beta,\gamma} + c_1 N^{(\gamma + |\alpha - \beta + \gamma|)/2} \|(I - P^\alpha_N)f\|_{\alpha+\xi,\xi} \quad \text{by (III.2.27b).}
\end{aligned}$$

The results follows from Theorems III.2.2 and III.2.3. □

Theorem III.2.8. *Let $\alpha > -1$, $\gamma \geq 0$, $-1 \leq \beta \leq \gamma - 1$, $1/2 < \xi \leq 1$ and $\delta \geq 0$, then*

$$\|(I - I^\alpha_{X_{G,N}})f\|_{\beta,\gamma} \leq c N^{\gamma - \min\{0,\beta\} + (|\alpha - \beta| - \delta + \xi)/2} \|f\|_{\alpha+\delta,\delta}. \tag{III.2.29}$$

Positive constant c does not depend on N and f.

Proof. Using the triangular inequality, Lemma III.2.5 and Lemma III.2.6, we infer

$$\begin{aligned}
\|(I - I^{\alpha}_{X_{G,N}})f\|_{\beta,\gamma} &\le \|(I - P^{\alpha}_{N})f\|_{\beta,\gamma} + \|(P^{\alpha}_{N} - I^{\alpha}_{X_{G,N}})f\|_{\beta,\gamma} \\
&\le \|(I - P^{\alpha}_{N})f\|_{\beta,\gamma} + cN^{\gamma - \min\{0,\beta\}}\|(P^{\alpha}_{N} - I^{\alpha}_{X_{G,N}})f\|_{\beta} \quad \text{by (III.2.26c)} \\
&\le \|(I - P^{\alpha}_{N})f\|_{\beta,\gamma} + cN^{\gamma - \min\{0,\beta\}}\|I^{\alpha}_{X_{G,N}}(I - P^{\alpha}_{N})f\|_{\beta} \quad \text{by } [P^{\alpha}_{X} I^{\alpha}_{X} = I^{\alpha}_{X}] \\
&\le \|(I - P^{\alpha}_{N})f\|_{\beta,\gamma} + c_1 N^{\gamma - \min\{0,\beta\} + (|\alpha-\beta|)/2}\|I^{\alpha}_{X_{G,N}}(I - P^{\alpha}_{N})f\|_{\alpha} \quad \text{by (III.2.26b)} \\
&\le \|(I - P^{\alpha}_{N})f\|_{\beta,\gamma} + c_1 N^{\gamma - \min\{0,\beta\} + (|\alpha-\beta|)/2}\|(I - P^{\alpha}_{N})f\|_{\alpha+\xi,\xi} \quad \text{by (III.2.27b).}
\end{aligned}$$

Theorem III.2.2 implies (III.2.29). □

2.3.3 Interpolation at Radau Abscissas

The analysis is the same as in the case of Gaussian nodes by this reason we skip proofs.

Lemma III.2.9. *Let* $\alpha > -1$, $1/2 < \beta \le 1$, $\gamma \ge 0$, $\alpha + \beta \le \gamma \le \alpha + 2\beta$ *and* $-1 < \delta < 2\beta - 1$, *then*

$$\|I^{\alpha}_{X_{R,N}}[f]\|_{\alpha} \le c_1\|f\|_{\gamma,\beta} + c_2 N^{-(\alpha+1)/2}\|f\|_{\delta,\beta}. \tag{III.2.30}$$

Positive constant c *does not depend on* N *and* f.

Proof. We know that

$$\|I^{\alpha}_{X_{R,N}}[f]\|^2_{\alpha} = \hat{w}_{\alpha,0,N} f^2(0) + \sum_{n=1}^{N} \hat{w}_{\alpha,0,N} e^{\hat{x}_{\alpha,0,N}} f^2(\hat{x}_{\alpha,0,N}) = A + B.$$

Using Stirling's formula and Gautschi inequality we obtain $A \le \frac{f^2(0)}{N^{1+\alpha}}$. B is estimated in the same way as in Lemma III.2.6 □

Theorem III.2.10. *Let* $\alpha > -1/2$, $\beta > -1$, $\gamma \ge 0$, $0 \le \xi \le 2\min\{1, 1+\alpha\}$ *and* $\delta \ge 0$, *then*

$$\|(I - I^{\alpha}_{X_{R,N}})f\|_{\beta,\gamma} \le cN^{\gamma + (\alpha - \beta - \delta + \xi)/2}\|f\|_{\alpha+\delta,\delta}, \quad \gamma - 1 < \beta \le \alpha + \gamma, \tag{III.2.31a}$$

$$\|(I - I^{\alpha}_{X_{R,N}})f\|_{\beta,\gamma} \le cN^{(\beta - \alpha - \delta + \xi)/2}\|f\|_{\alpha+\delta,\delta}, \quad \beta \ge \alpha + \gamma, \tag{III.2.31b}$$

Positive constant c *does not depend on* N *and* f.

Proof. The same as in Theorem III.2.7 □

Theorem III.2.11. *Let $\alpha > -1/2$, $\beta > -1$, $\gamma \geq 0$, $0 < \xi < 2\min\{1, 1+\alpha\}$ and $\delta \geq 0$, then*

$$\|(I - I^{\alpha}_{X_{R,N}})f\|_{\beta,\gamma} \leq cN^{\gamma - \min\{0,\beta\} + (|\alpha-\beta| - \delta + \xi)/2} \|f\|_{\alpha+\delta,\delta}. \tag{III.2.32}$$

Positive constant c does not depend on N and f.

Proof. The same as in Theorem III.2.8 □

3 Numerical Scheme

3.1 An Alternative Form of (II.1.1)

The transport-fragmentation equation written in the form of (II.1.1) is not suitable for space discretization. For numerical purposes it is necessary to reformulate both the transport and the fragmentation operators. For the transport operator we set

$$T[u] = \frac{1}{2}(ru)_x + \frac{1}{2}r_x u + \frac{1}{2}ru_x - (\mu + a)u. \tag{III.3.1}$$

For the fragmentation part we set

$$\begin{aligned} F[u] &= b_1(x)\int_x^{\infty} b_2(y)a(y)u(y)dy \\ &= -\frac{1}{x}\frac{d}{dx}\int_x^{\infty} yb_1(y)dy \int_y^{\infty} b_2(z)f(z)dz \\ &= -\frac{1}{x}\frac{d}{dx}\int_x^{\infty} b_2(z)a(z)u(z)dz \int_x^{z} yb_1(y)dy \\ &= -\frac{1}{x}\frac{d}{dx}\int_x^{\infty} b_2(z)a(z)u(z)\left[\frac{z}{b_2(z)} - \frac{x}{b_2(x)}\right]dz \\ &= a(x)u(x) + \frac{1}{x}\frac{d}{dx}\left[\frac{x}{b_2(x)}\int_x^{\infty} b_2(z)a(z)u(z)dz\right], \end{aligned}$$

so that

$$F[u] = au + 2v + xv_x, \tag{III.3.2}$$

where

$$x^2 v(x) = \frac{x}{b_2(x)}\int_x^{\infty} b_2(z)a(z)u(z)dz. \tag{III.3.3}$$

Using (III.3.3) and condition (II.1.6) it is not difficult to verify that v satisfies

$$-xv_x + v(xb(x|x) - 2) = au. \tag{III.3.4}$$

The following result provides sufficient conditions that guarantees solvability of (III.3.4).

Lemma III.3.1. *Let the fragmentation kernel satisfy*

$$0 < c \leq xb(x|x) + \frac{\alpha+1}{2} - 2 \leq B < \infty, \tag{III.3.5a}$$

then for any $au \in L^2_\alpha(\mathbb{R}_+)$, there exists a unique solution v of (III.3.4) such that

$$\|v\|_\alpha \leq \frac{1}{c}\|au\|_\alpha, \quad \|xv_x\|_\alpha \leq \frac{2B+2c+\alpha+1}{2c}\|au\|_\alpha. \tag{III.3.5b}$$

Proof. The weak formulation of (III.3.4) reads as follows: find $v \in X_1 := \{f | f, xf_x \in L^2_\alpha(\mathbb{R}_+)\}$ so that

$$a(v,\phi) := \langle -xv_x + v(xb(x|x) - 2), \phi\rangle_\alpha = \langle au, \phi\rangle_\alpha =: f(\phi), \tag{III.3.6}$$

for all $\phi \in X_2 := L^2_\alpha(\mathbb{R}_+)$. Condition (III.3.5a) implies that

$$a(v,v) = \langle (xb(x|x) + \tfrac{\alpha+1}{2} - 2)v, v\rangle_\alpha \geq c\|v\|^2_\alpha,$$

and $a(v,\phi) \leq (\frac{\alpha+3}{2} + B)\|v\|_{X_1}\|\phi\|_{X_2}$, so that by the Lax-Milgram theorem there exists a unique v that solves (III.3.4).

To obtain (III.3.5b) we substitute $\phi = v$ into (III.3.6). This yields the estimate

$$c\|v\|^2_\alpha \leq a(v,v) = \langle au, v\rangle_\alpha \leq \|au\|_\alpha\|v\|_\alpha,$$

which is equivalent to the first inequality in (III.3.5b). Similarly, putting $\phi = xv_x$ into (III.3.6) we infer

$$\begin{aligned}\|xv_x\|^2_\alpha &= \langle (xb(x|x) - 2)v, xv_x\rangle_\alpha - \langle au, xv_x\rangle_\alpha \\ &\leq \frac{2B+\alpha+1}{2}\|v\|_\alpha\|xv_x\|_\alpha + \|au\|_\alpha\|xv_x\|_\alpha.\end{aligned}$$

Combining this with the first estimate of (III.3.5b) yields the result. □

Lemma III.3.1 implies that for separable fragmentation kernels equation (II.1.1) can be rewritten as

$$u_t = \frac{1}{2}(ru)_x + \frac{1}{2}r_x u + \frac{1}{2}ru_x - (\mu + a)u + au + 2v + xv_x, \tag{III.3.7}$$

provided that (III.3.5a) holds. On the other hand from Section III.1 we know that (II.1.1) is wellposed in $L^2_\alpha(\mathbb{R}_+)$ provided that (III.1.11) holds. It is important to emphasize that for the power fragmentation laws (III.1.11) and (III.3.5a) are equivalent.

3.2 The Numerical Scheme

Here we consider slightly more general situation, we assume that equation (III.3.7) is augmented with a source term f.

Let $\alpha > 0$ and conditions (III.1.2), (III.1.3), (III.1.11) and (III.3.5a) be satisfied. To solve the transport-fragmentation equation numerically we apply Laguerre pseudospectral method, i.e. we replace $u(t)$ with $u_N(t) \in \mathbb{P}_N$, and approximate equation (III.3.7) by

$$\begin{aligned}(u_N)_t = {} & \frac{1}{2}I_N^{\alpha-1}[(ru_N)_x] + \frac{1}{2}I_N^{\alpha-1}[r_x u_N] + \frac{1}{2}I_N^{\alpha-1}[r(u_N)_x] - I_N^{\alpha-1}[\mu u_N] \\ & + 2v_N + xP_{N-1}^{\alpha+2}[(v_N)_x] + I_N^{\alpha-1}[f],\end{aligned} \tag{III.3.8a}$$

where $u_N(0) = I_N^{\alpha-1}[u_0]$, $v_N \in \mathbb{P}_N$ satisfies

$$\langle I_N^{\alpha-1}[(xb_1b_2 - 2)v_N], \hat{L}_m^\alpha\rangle_\alpha - \langle (v_N)_x, \hat{L}_m^\alpha\rangle_{\alpha+1} = \langle I_N^{\alpha-1}[au], \hat{L}_m^\alpha\rangle_\alpha, \quad 0 \le m \le N, \tag{III.3.8b}$$

and $I_N^{\alpha-1}$ is the Laguerre interpolation operator with Gaussian abscissas.

In (III.3.8) the transport part is split into four terms, in each term the product of a coefficient and u_N is evaluated at Gaussian abscissas and then interpolated. The derivatives are calculated using formulas (III.2.2e) and (III.2.2g). To evaluate the fragmentation part one has to solve system of linear equations (III.3.8b). Using the same arguments as in Lemma III.3.1 one can show that there exists a unique solution v_N that satisfy

$$\|v_N\|_\alpha \le \frac{1}{c}\|I_N^{\alpha-1}[au]\|_\alpha, \quad \|xP_{N-1}^{\alpha+2}[(v_N)_x]\|_\alpha \le \frac{2B + 2c + \alpha + 1}{2c}\|I_N^{\alpha-1}[au_N]\|_\alpha, \tag{III.3.9}$$

so that the right-hand side of (III.3.8a) makes sense. Discrete equation (III.3.8a) can be viewed as a sort of a collocation scheme, where the numerical solution is required to satisfy (III.3.7) at zeros of $L_{N+1}^{(\alpha-1)}(x)$.

3.2.1 Stability Analysis

First, we show the numerical solution u_N depends continuously on the input data, i.e. semidiscretization (III.3.8) is stable. Since (III.3.8) is linear it is sufficient to show that $\|u_N\|_\alpha$ is uniformly bounded with respect to the discretization parameter N.

Theorem III.3.2. *Let $\alpha > 0$ and conditions (III.1.2), (III.1.3), (III.1.11) and (III.3.5a) be satisfied, then for any fixed $0 < t < \infty$ the following estimate holds:*

$$\|u_N(t)\|_\alpha \leq e^{tM}\|u_N(0)\|_\alpha + \left(\int_0^t e^{2M(t-s)}\|I_N^{\alpha-1}[f](s)\|_\alpha^2 ds\right)^{1/2}, \tag{III.3.10}$$

where M does not depend on N.

Proof. At each time level $0 < t < \infty$ the numerical solution satisfy

$$\langle (u_N)_t, \phi\rangle_\alpha = \langle T_N[u_N], \phi\rangle_\alpha + \langle F_N[u_N], \phi\rangle_\alpha + \langle f_N, \phi\rangle_\alpha \quad \phi \in \mathbb{P}_N, \tag{III.3.11}$$

where

$$\begin{aligned}
T_N[u_N] &= \frac{1}{2}I_N^{\alpha-1}[(ru_N)_x] + \frac{1}{2}I_N^{\alpha-1}[r_x u_N] + \frac{1}{2}I_N^{\alpha-1}[r(u_N)_x] - I_N^{\alpha-1}[(\mu+a)u_N],\\
F_N[u_N] &= I_N^{\alpha-1}[au_N] + 2v_N + xP_{N-1}^{\alpha+2}[(v_N)_x],
\end{aligned}$$

v_N is given by (III.3.8b) and $f_N = I_N^{\alpha-1}[f]$. We estimate each term of (III.3.11) separately by taking its scalar product with u_N.

For the discrete transport operator we have

$$\begin{aligned}
\langle T_N[u_N], u_N\rangle_\alpha = {}& \frac{1}{2}\langle u_N, I_N^{\alpha-1}[(ru_N)_x]\rangle_\alpha + \frac{1}{2}\langle u_N, I_N^{\alpha-1}[r_x u_N]\rangle_\alpha + \frac{1}{2}\langle u_N, I_N^{\alpha-1}[ru_{Nx}]\rangle_\alpha\\
&- \langle u_N, I_N^{\alpha-1}[(\mu+a)u_N]\rangle_\alpha.
\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}
\langle T_N[u_N], u_N\rangle_\alpha = {}& -\frac{1}{2}\lim_{x\to 0} x^\alpha u_N I_N^{\alpha-1}[ru_N] - \frac{1}{2}\langle (u_N)_x, I_N^{\alpha-1}[ru_N]\rangle_\alpha - \frac{\alpha}{2}\langle u_N, I_N^{\alpha-1}[ru_N]\rangle_{\alpha-1}\\
&+ \frac{1}{2}\langle u_N, I_N^{\alpha-1}[r_x u_N]\rangle_\alpha + \frac{1}{2}\langle u_N, I_N^{\alpha-1}[ru_{Nx}]\rangle_\alpha - \langle u_N, I_N^{\alpha-1}[(\mu+a)u_N]\rangle_\alpha\\
= {}& -\frac{\alpha}{2}\langle u_N, I_N^{\alpha-1}[ru_N]\rangle_{\alpha-1} + \frac{1}{2}\langle xu_N, I_N^{\alpha-1}[r_x u_N]\rangle_{\alpha-1} - \langle xu_N, I_N^{\alpha-1}[(\mu+a)u_N]\rangle_{\alpha-1}.
\end{aligned}$$

Since the interpolation $I_N^{\alpha-1}$ is based on the Gaussian nodes each scalar product in the formula above can be replaced with the quadrature formula and we have the following:

$$\begin{aligned}\langle T_N[u_N], u_N\rangle_\alpha &= -\frac{1}{2}\sum_{n=0}^{N-1} w_{\alpha-1,n,N}\big[u_N\big]^2(x_{\alpha-1,n,N})x_{\alpha-1,n,N}\big[2(\mu+a)-r_x+\tfrac{\alpha r}{x}\big](x_{\alpha-1,n,N}) \\ &= -\frac{1}{2}\left\langle u_N, I_N^{\alpha-1}\big[(2(\mu+a)-r_x+\tfrac{\alpha r}{x})u_N\big]\right\rangle_\alpha. \qquad \text{(III.3.12)}\end{aligned}$$

We treat the fragmentation part in the same manner as before:

$$\langle F_N[u_N], u_N\rangle_\alpha = \langle I_N^{\alpha-1}[au_N], u_N\rangle_\alpha + 2\langle v_N, u_N\rangle_\alpha + \langle xP_{N-1}^{\alpha+2}[(v_N)_x], u_N\rangle_\alpha.$$

Using Cauchy-Schwartz and Young inequalities and taking into account (III.3.9) we obtain

$$\begin{aligned}\langle F_N[u_N], u_N\rangle_\alpha &\le \|I_N^{\alpha-1}[au_N]\|_\alpha\|u_N\|_\alpha + 2\|v_N\|_\alpha\|u_N\|_\alpha + \|xP_{N-1}^{\alpha+2}[(v_N)_x]\|_\alpha\|u_N\|_\alpha \\ &\le \frac{2B+3c+\alpha+3}{c}\|I_N^{\alpha-1}[au_N]\|_\alpha\|u_N\|_\alpha \\ &\le \kappa^2\|I_N^{\alpha-1}[au_N]\|_\alpha^2 + \frac{(2B+3c+\alpha+3)^2}{4\kappa^2c^2}\|u_N\|_\alpha^2. \qquad \text{(III.3.13)}\end{aligned}$$

For the source term we have

$$\langle f_N, \phi\rangle_\alpha \le \frac{1}{4}\|u_N\|_\alpha^2 + \|f_N\|_\alpha^2. \qquad \text{(III.3.14)}$$

Combing (III.3.12), (III.3.13) and (III.3.14) we see that

$$\begin{aligned}\frac{d}{dt}\|u_N\|_\alpha^2 &\le -\left\langle u_N, I_N^{\alpha-1}\big[(2(\mu+a)-r_x+\tfrac{\alpha r}{x}-\kappa^2a^2)u_N\big]\right\rangle_\alpha \\ &\quad + \Big(\frac{(2B+3c+\alpha+3)^2}{2\kappa^2c^2}+\frac{1}{4}\Big)\|u_N\|_\alpha^2 + \|f_N\|_2^\alpha \\ &\le \Big(\frac{(2B+3c+\alpha+3)^2}{2\kappa^2c^2}+\frac{1}{4}\Big)\|u_N\|_\alpha^2 + \|f_N\|_2^\alpha.\end{aligned}$$

From Gronwall's inequality it follows that

$$\|u_N(t)\|_\alpha^2 \le e^{2Mt}\|u_N(0)\|_\alpha^2 + \int_0^t e^{2M(t-s)}\|f_N(s)\|_\alpha^2 ds, \quad \text{with} \quad M = \frac{(2B+3c+\alpha+3)^2}{4\kappa^2c^2}+\frac{1}{8}.$$

The proof is complete. □

Theorem III.3.2 says that in finite time intervals $[0,T]$, the numerical solution is bounded by

$$\|u_N(0)\|_\alpha + e^{MT}\|f_N\|_{L^2([0,T],L^2_\alpha(\mathbb{R}_+))} = \|I_N^{\alpha-1}[u_0]\|_\alpha + e^{MT}\|I_N^{\alpha-1}[f]\|_{L^2([0,T],L^2_\alpha(\mathbb{R}_+))}.$$

We note that

$$\|I_N^{\alpha-1}[u_0]\|_\alpha \le \|u_0\|_\alpha + \|(I - I_N^{\alpha-1})[u_0]\|_\alpha,$$
$$\|I_N^{\alpha-1}[f]\|_{L^2([0,T],L^2_\alpha(\mathbb{R}_+))} \le \|f\|_{L^2([0,T],L^2_\alpha(\mathbb{R}_+))} + \|(I - I_N)^{\alpha-1}[f]\|_{L^2([0,T],L^2_\alpha(\mathbb{R}_+))},$$

so that by Theorem III.2.7 both quantities are uniformly bounded with respect to N provided that $u_0 \in H_{\alpha+\beta-1,\beta}(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ and $f \in L^2([0,T], H_{\alpha+\beta-1,\beta}(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+))$, with $\beta > \frac{3}{2}$.

3.2.2 Convergence Analysis

From the previous section we know that the sequence $\{u_N\}_{N\ge1}$ is uniformly bounded with respect to N in $L^2_\alpha(\mathbb{R}_+)$ provided that $u_0 \in H_{\alpha+\beta-1,\beta}(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$, with $\beta > \frac{3}{2}$. From this it follows that there exist a weakly convergent subsequence $\{u_{N_k}\}$. Let, $\bar{u}$ be its weak limit, then in a standard way one can show that $\bar{u}$ is a weak solution of (II.1.1) in $L^2_\alpha(\mathbb{R}_+)$. The argument, similar to that employed in [BGS07], then shows that $\bar{u}$, is the strong solution of (II.1.1). This result is not very practical, since it is not clear how to select the weakly convergent subsequence. Moreover, we do not have any control on the rate of convergence.

Better results are obtained if it is known a priori that solutions are regular. Unfortunately, the theory presented in the project does not say what conditions guarantee such a regularity. We do not concentrate on the theoretical study of regularity conditions since it would require us to build entirely new theory. To derive practical error estimate we assume that the exact solution, as well as quantity v, defined by (III.3.3), are sufficiently regular.

Theorem III.3.3. *Let $u_0 \in D(T_0)$, $\delta > 4$, $1/2 < \xi \le 1$ and $\alpha > 0$, then*

$$\|u - u_N\|_{L^\infty([0,T],L^2_\alpha(\mathbb{R}_+))} \le c_1 N^{(1-\delta+\xi)/2} \|u\|_{L^\infty([0,T],H_{\alpha-1+\delta,\delta}(\mathbb{R}_+))} + c_2 N^{(3-\delta+2\xi)/2} \|v\|_{L^2([0,T],H_{\alpha-1+\delta,\delta}(\mathbb{R}_+))}, \quad \text{(III.3.15)}$$

with c_1, c_2 independent on N, provided that the norms in the right-hand side of (III.3.15) are finite.

Proof. First, the numerical solution satisfy

$$(u_N)_t = T_N[u_N] + F_N[u_N] + f_N. \quad \text{(III.3.16)}$$

Let $\bar{u} = I_N^{\alpha-1}[u]$ be the interpolant of the exact solution, then taking into account that

$$I_N^{\alpha-1}[f I_N^{\alpha-1}[g]] = I_N^{\alpha-1}[g I_N^{\alpha-1}[f]] = I_N^{\alpha-1}[fg]$$

for any f and g, we obtain

$$\bar{u}_t = T_N[\bar{u}] + F_N[\bar{u}] + f_N + (I_N^{\alpha-1}F - F_N I_N^{\alpha-1})[u] = T_N[\bar{u}] + F_N[\bar{u}] + f_N + E_N[u]. \quad \text{(III.3.17)}$$

Subtracting (III.3.16) out of (III.3.17) we see that the error $e_N = \bar{u} - u_N$ satisfy

$$(e_N)_t = T_N[e_N] + F_N[e_N] + E_N[u], \quad e_N(0) = 0,$$

so that by Lemma III.3.2 the error is bounded as follows:

$$\|e_N(t)\|_\alpha^2 \leq \int_0^t e^{2M(t-s)} \|E_N[u](s)\|_\alpha^2 ds. \quad \text{(III.3.18)}$$

Second, we estimate $E_N[u]$. For this we rewrite $E_N[u]$ in the form

$$\begin{aligned} E_N[u] &= 2(I_N^{\alpha-1}[v] - v_N) + (I_N^{\alpha-1}[xv_x] - xP_{N-1}^{\alpha+2}[(v_N)_x]) \\ &= 2(I_N^{\alpha-1}[v] - v_N) + xP_{N-1}^{\alpha+2}\big[(I_N^{\alpha-1}[v] - v_N)_x\big] + (I_N^{\alpha-1}[xv_x] - xP_{N-1}^{\alpha+2}[\big(I_N^{\alpha-1}[v]\big)_x]) \\ &=: 2\hat{e} + \tilde{e} + \bar{e}, \end{aligned}$$

where v satisfies (III.3.4) and v_N is the solution of

$$\langle I_N^{\alpha-1}[(xb(x|x) - 2)v_N] - x(v_N)_x, \phi\rangle_\alpha = \langle I_N^{\alpha-1}[au], \phi\rangle_\alpha, \quad \phi \in \mathbb{P}_N.$$

If $\bar{v} = I_N^{\alpha-1}[v]$, then from (III.3.4) we have

$$\langle I_N^{\alpha-1}[(xb(x|x) - 2)\bar{v}] - x\bar{v}_x, \phi\rangle_\alpha = \langle I_N^{\alpha-1}[au], \phi\rangle_\alpha + \left\langle I_N^{\alpha-1}\big[x\big((I - I_N^{\alpha-1})[v]\big)_x\big], \phi\right\rangle_\alpha, \quad \phi \in \mathbb{P}_N.$$

It follows that, $\hat{e}$ satisfies

$$\langle I_N^{\alpha-1}[(xb(x|x) - 2)\hat{e}] - x\hat{e}_x, \phi\rangle_\alpha = \left\langle I_N^{\alpha-1}\big[x\big((I - I_N^{\alpha-1})[v]\big)_x\big], \phi\right\rangle_\alpha, \quad \phi \in \mathbb{P}_N.$$

In the same way as in Lemma III.3.1 we obtain

$$\begin{aligned} \|\hat{e}\|_\alpha &\leq \frac{1}{c} \left\| I_N^{\alpha-1}\big[x\big((I - I_N^{\alpha-1})[v]\big)_x\big] \right\|_\alpha, \\ \|\tilde{e}\|_\alpha &\leq \frac{2B + 2c + \alpha + 1}{2c} \left\| I_N^{\alpha-1}\big[x\big((I - I_N^{\alpha-1})[v]\big)_x\big] \right\|_\alpha. \end{aligned}$$

Now applying the approximation results of Section III.2 we infer

$$
\begin{aligned}
\|\hat{e}\|_\alpha &\le \frac{1}{c}\left\| I_N^{\alpha-1}\big[x\big((I-I_N^{\alpha-1})[v]\big)_x\big]\right\|_\alpha \\
&\le \frac{1}{c}\|(I-I_N^{\alpha-1})[v]\|_{\alpha+2,1} + \frac{1}{c}\left\|(I-I_N^{\alpha-1})\big[x\big((I-I_N^{\alpha-1})[v]\big)_x\big]\right\|_\alpha \\
&\le c_1 N^{(3-\delta+\xi)/2}\|v\|_{\alpha-1+\delta,\delta} \qquad \text{by Theorem III.2.7} \\
&+ c_2 N^{(1-\delta_1+\xi)/2}\|(I-I_N^{\alpha-1})[v]\|_{\alpha+1+\delta_1,\delta_1+1} \qquad \text{by Theorem III.2.7} \\
&\le c_1 N^{(3-\delta+\xi)/2}\|v\|_{\alpha-1+\delta,\delta} \\
&+ c_3 N^{(1-\delta_1+\xi)/2} N^{(2+\delta_1-\delta+\xi)/2}\|v\|_{\alpha-1+\delta,\delta} \qquad \text{by Theorem III.2.7} \\
&\le c_4 N^{(3-\delta+2\xi)/2}\|v\|_{\alpha-1+\delta,\delta},
\end{aligned}
$$

and $\|\tilde{e}\|_\alpha \le c_5 N^{(3-\delta+2\xi)/2}\|v\|_{\alpha-1+\delta,\delta}$, where $1/2 < \xi \le 1$ and all the constants do not depend on N. Similarly,

$$
\begin{aligned}
\|\bar{e}\|_\alpha &\le \|I_N^{\alpha-1}[xv_x] - xP_{N-1}^{\alpha+2}[\big(I_N^{\alpha-1}[v]\big)_x]\|_\alpha \\
&\le \|(I-I_N^{\alpha-1})[xv_x]\|_\alpha + \left\|(I-P_{N-1}^{\alpha+2})\big[\big((I-I_N^{\alpha-1})[v]\big)_x\big]\right\|_{\alpha+2} \\
&\le c_6 N^{(3-\delta+\xi)/2}\|v\|_{\alpha-1+\delta,\delta-1} \qquad \text{by Theorem III.2.7} \\
&+ c_7 N^{-\delta_1/2}\|(I-I_N^{\alpha-1})[v]\|_{\alpha+\delta_1,\delta_1+1} \qquad \text{by Lemma III.2.1} \\
&\le c_6 N^{(3-\delta+\xi)/2}\|v\|_{\alpha-1+\delta,\delta-1} \\
&+ c_8 N^{-\delta_1/2} N^{(1+\delta_1-\delta+\xi)/2}\|v\|_{\alpha-1+\delta,\delta} \qquad \text{by Theorem III.2.7} \\
&\le c_9 N^{(3-\delta+\xi)/2}\|v\|_{\alpha-1+\delta,\delta},
\end{aligned}
$$

where $1/2 < \xi \le 1$ and c_9 does not depend on N. Combining last the three estimates we conclude that

$$
\|E[u](t)\|_\alpha \le cN^{(3-\delta+\xi)/2}\|v(t)\|_{\alpha-1+\delta,\delta}, \tag{III.3.19}
$$

where $v(t)$ is given explicitly by (III.3.3).

Finally, we use (III.3.18) and (III.3.19) to obtain

$$
\begin{aligned}
\|u(t)-u_N(t)\|_\alpha &\le \|u(t)-\bar{u}(t)\|_\alpha + \|e_N(t)\|_\alpha \\
&\le c_1 N^{(1-\delta+\xi)/2}\|u(t)\|_{\alpha-1+\delta,\delta} \qquad \text{by Theorem III.2.7} \\
&+ c_2 N^{(3-\delta+2\xi)/2}\left(\int_0^t e^{2M(t-s)}\|v(s)\|^2_{\alpha-1+\delta,\delta}\right)^{1/2},
\end{aligned}
$$

where $\delta > 4$, $1/2 < \xi \leq 1$ and c_1, c_2 do not depend on N. This completes the proof. $\square$

Proof of Theorem III.3.3 indicates that accuracy of computations depend on the regularity of u and v but not on the individual regularity of the coefficient r, μ, a, $b(x|y)$ or f.

4 Numerical Simulations

In this section we provide some numerical simulations that illustrate theory developed above. In all the computations semidiscrete problem (III.3.8) is integrated using built-in `Matlab` ODE solver `ode23`. The tolerances `RelTol` and `AbsTol` are chosen so that the numerical error is dominated by $\|u - u_N\|_\alpha$.

Example 4.1 In our first example we set

$$r = 1 + x, \quad \mu = 0, \quad a = 1, \quad b(x|y) = \frac{(\gamma+1)(\gamma+2)y(1+x)^\gamma}{(1+y)^{\gamma+1}((\gamma+1)y-1)+1},$$

we have also augmented the right-hand side of (II.1.1) with a source function, so that the exact solution is given by

$$u(x,t) = x^\delta \big((1+x)^{\gamma+1}((\gamma+1)x-1)+1\big)e^{-x-t}.$$

For numerical simulations we took $\delta = 1$ and $\gamma = -1/2$, with this choice of the parameters problem (II.1.1) is wellposed in $L^2_1(\mathbb{R}_+)$.

The results of simulations are shown in Fig. 4.1. The upper left panel of Fig 4.1 illustrates the numerical solution u_N, $N = 64$, in the time interval $[0, 10]$. The pointwise error is shown in the lower left panel. It attains its maximum close to the boundary $x = 0$ and gradually decays as x increases. Observe that the pointwise error is of magnitude 10^{-7} in the whole computational domain.

Parameters of the model are chosen so that both $u(x,t)$ and $v(x,t)$ are regular. By Theorem III.3.3 the error $\|u - u_N\|_{L^\infty([0,T],L^2_\alpha(\mathbb{R}_+))}$ shall decrease faster than algebraically. This is completely confirmed by the work-precision diagram (see the right panel of Fig. 4.1).

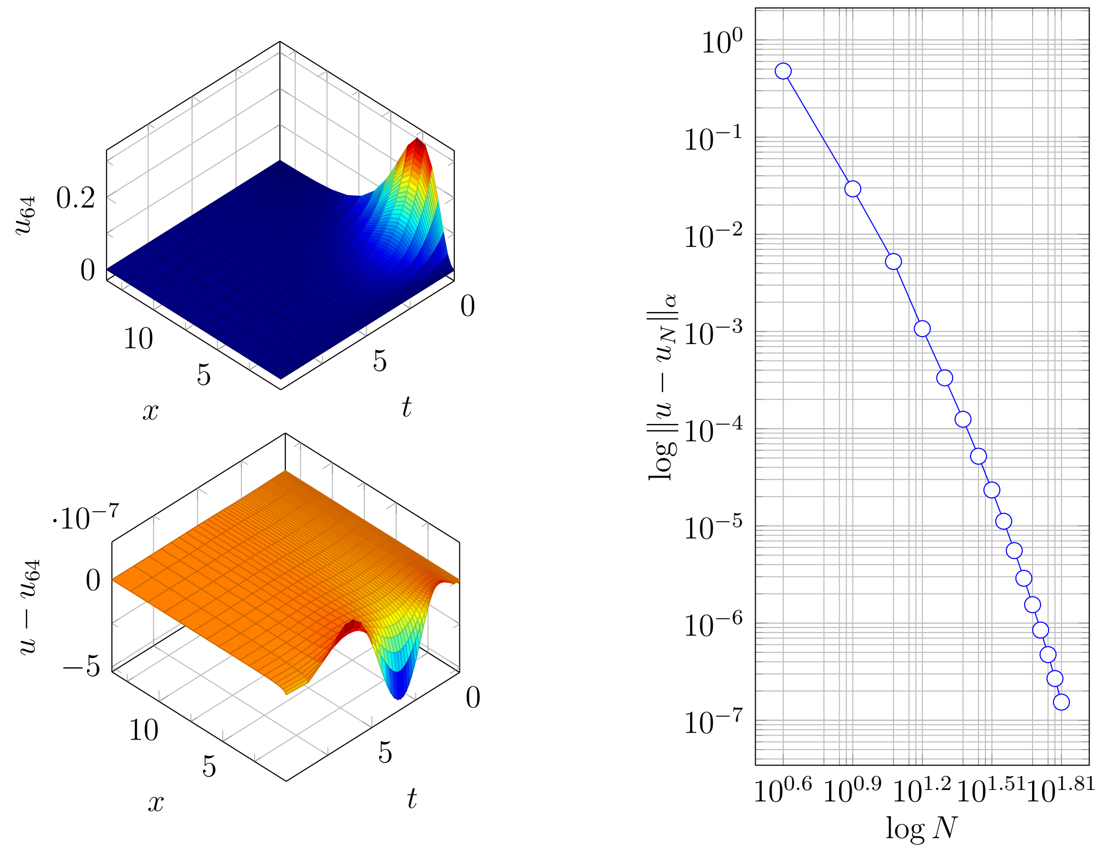

Figure 4.1: The numerical solution of (II.1.1) (upper left panel); the pointwise error (lower left panel); and the work precision-diagram (right panel) with $\alpha = 1$, $\delta = 1$, $\gamma = -1/2$.

Example 4.2 In this example we consider (II.1.1) with an unbounded fragmentation rate. That is, we took the fragmentation rate to be $a(x) = x^\beta$, $\beta = 1/2$, and left all other parameters the same as Example 4.1. With this settings we integrated (II.1.1) in time interval $[0, 10]$. The diagrams shown in Fig. 4.2 has the same meaning as in Example 4.1. Once again the pointwise error is concentrated near the boundary $x = 0$ and the convergence rate is faster than algebraic (see the right panel of Fig. 4.2).

Example 4.3 In this case we took

$$r = 1 + x^\xi, \quad \mu = 0, \quad a = x^\beta, \quad b(x|y) = (\gamma + 2)x^\gamma y^{-\gamma-1},$$

and had chosen the source function to make the exact solution be,

$$u(x,t) = x^\delta e^{-x-t}.$$

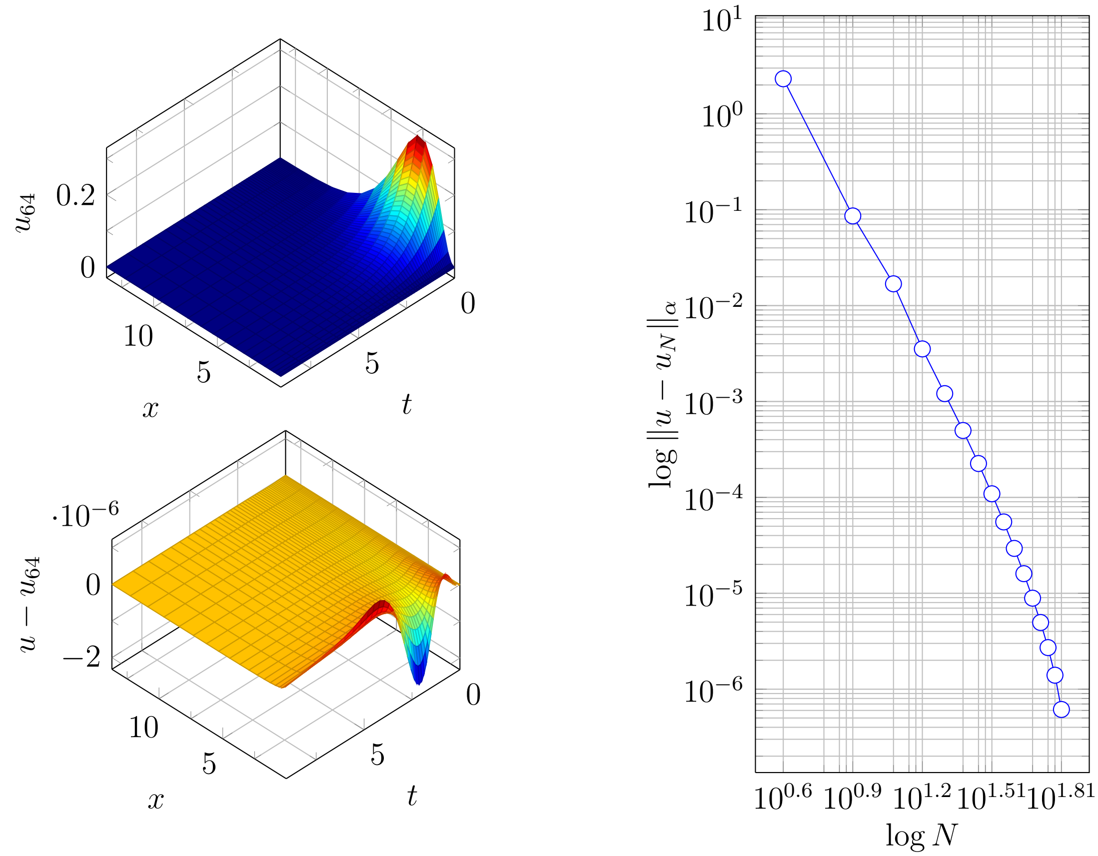

Figure 4.2: The numerical solution of (II.1.1) (upper left panel); the pointwise error (lower left panel); and the work precision-diagram (right panel) with $\alpha = 1$, $\beta = 1/2$, $\delta = 1$, $\gamma = -1/2$.

The results of numerical simulations of (II.1.1) with $\alpha = 2$, $\beta = -1/2$, $\xi = 1/2$, $\delta = 1$, $\gamma = -1/2$ in time interval $[0, 10]$ are shown in Fig. 4.3. We observe that the errors are significantly larger as compared with two previous examples. This is due to the singularity of $v(x,t)$ at $x = 0$. By the same reason the work-precision diagram shows a decline in the convergence rate. The convergence now is only algebraic. To reduce the errors one can move to the higher moment spaces (of course one has to verify that all the conditions that guarantee wellposedness of (II.1.1) are still satisfied).

Example 4.4 In our last example we consider pure fragmentation equation with

$$r = 0, \quad \mu = 0, \quad a = x^\beta, \quad b(x|y) = (\gamma + 2)x^\gamma y^{-\gamma-1},$$

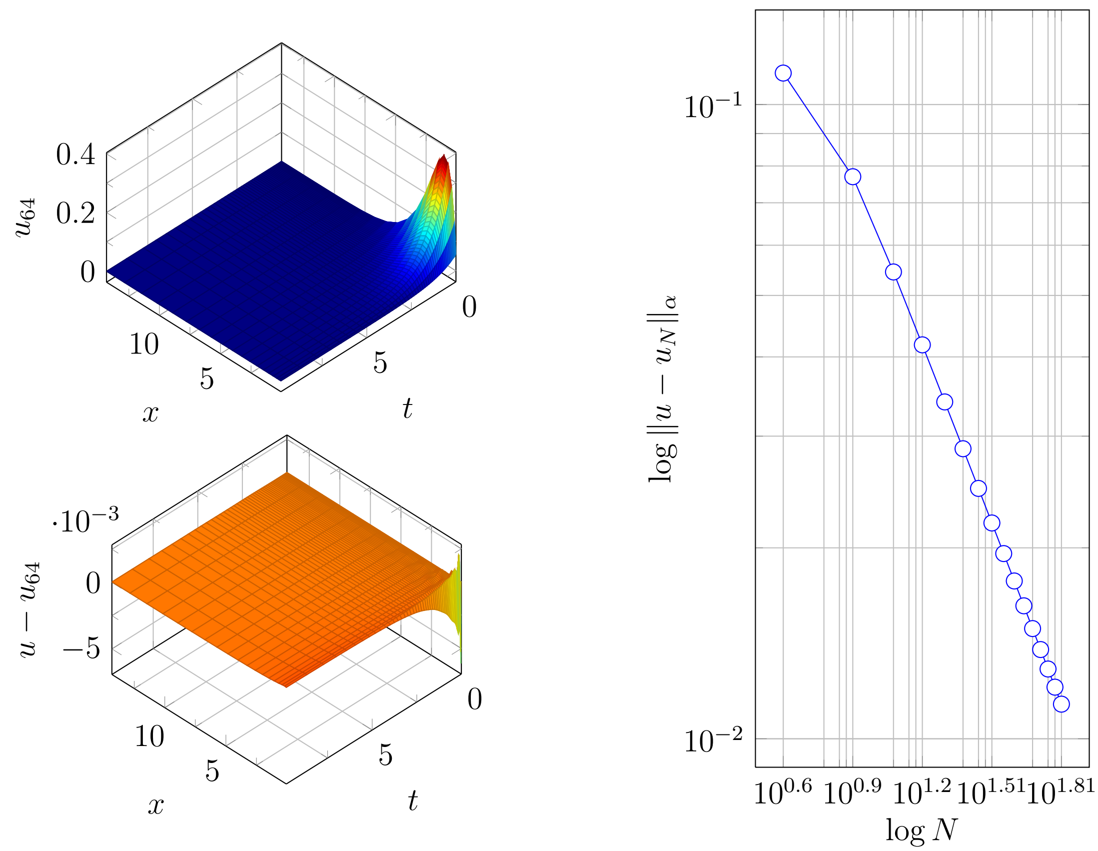

Figure 4.3: The numerical solution of (II.1.1) (upper left panel); the pointwise error (lower left panel); and the work precision-diagram (right panel) with $\alpha = 2$, $\beta = -1/2$, $\xi = 1/2$, $\delta = 1$, $\gamma = -1/2$.

equipped with the following initial condition

$$u_0 = \frac{1}{(1+x)^3}.$$

It is known (see [ZM85, Ban02]) that for $\beta = 1$, $\gamma = 0$, the solution of (II.1.1) is not unique. In particular we have the following two formal solutions

$$u^1(x,t) = e^{-xt}\left(\frac{1}{(1+x)^3} + \int_x^\infty \frac{2t + t^2(y-x)}{(1+y)^3} dy\right),$$
$$u^2(x,t) = \frac{e^t}{(1+x)^3},$$

where the first one is mass conserving and the second one is not. Theoretical results of Chapter III do not guarantee wellposedness of (II.1.1) in this particular case. However, the results of calculations presented in Fig. 4.4 indicate that the numerical solution u_N converges to the physically correct mass conservative solution u^1, though the convergence is quite poor.

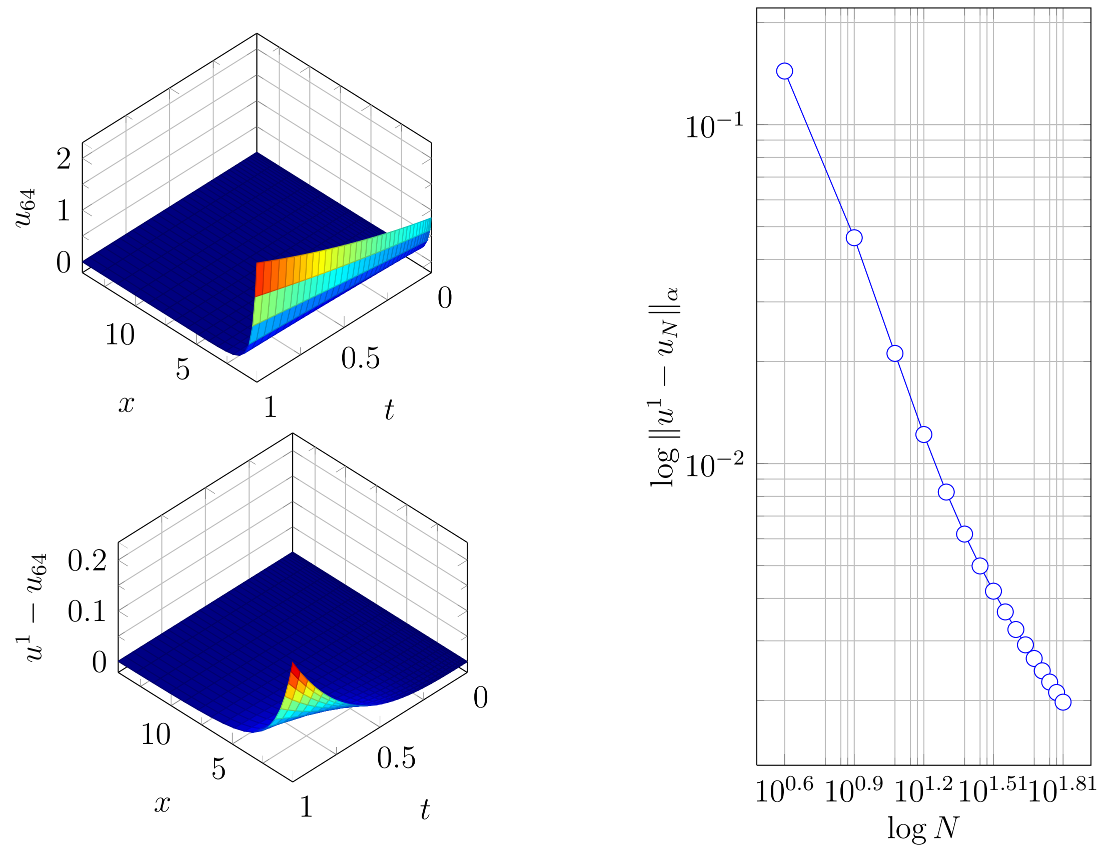

Figure 4.4: The numerical solution of (II.1.1) (upper left panel); the pointwise error (lower left panel); and the work precision-diagram (right panel) with $\alpha = 2$, $\beta = 1$, $\gamma = 0$.

To conclude this section we observe that in all computational experiments presented above the numerical scheme behaves well, provided that both u and v are smooth. The convergence rate may deteriorate when u or v develops singularities. To handle such problems the type of singularities and their locations must be known a priori and numerical scheme (III.3.8) must be reformulated accordingly.

Conclusion

Due to increasing practical applications of the fragmentation equations, they have been an object of extensive study [ZM85, ZM86, HEL91, CEH91, HGEL96, SLLM00, Ban02, BL03, Ban04, BA06]. The model deals with objects whose breaking up or disperse is random like in polymer degradation and population dynamics processes. The model has some application in diffusion processes. In the project we deal with the numerical analysis of the transport-fragmentation equation in the decay case.

In Chapter I we collected some classical results on the semigroup theory which are used later on. In Chapter II we considered transport-fragmentation equations in L^1 settings. Following [BA06] we studied existence of solutions in the decay case. Uniqueness of solutions was not investigated, however, it was noticed in several papers (see [ZM85, Ban02, BA06] and references therein) that pure fragmentation equations might have multiple solutions emanating from the same initial data.

The main results of our research are presented in Chapter III, where we study Laguerre pseudo-spectral semidiscretization of the transport-fragmentation equations with separable fragmentation kernels. Because of the nature of our numerical method we had to provide a wellposedness analysis in Hilbert settings. Existence of $L^2_\alpha(\mathbb{R}_+)$ semigroup solutions was carried out in a similar manner as in Chapter II. However, the approximation technique employed in Chapter III does not preserve the positive cone of $L^2_\alpha(\mathbb{R}_+)$. By this reason positivity does not play much role in our analysis.

Stability and convergence analysis of our numerical scheme required some approximation and interpolation estimates for generalized Laguerre functions. We found that it is natural to derive such estimates in weighted Bessel potential spaces $H_{\alpha,\beta}(\mathbb{R}_+)$, where parameter α controls behaviour of functions at zero and infinity and parameter β controls regularity. The approximation and interpolation theory developed in Section III.2 is quite general and can be used in other applications.

Numerical computations were carried using the Laguerre pseudo-spectral method. The model was reformulated in a way that the stability estimate can be derived, stability of the scheme was proven in Theorem III.3.2. Error estimates were obtained in Theorem III.3.3. From Theorem III.3.3 it follows that the convergence rate depends on the regularity of u and v. The convergence rate is faster than algebraic if $u \in L^\infty([0,T], H_{\alpha-1+\delta,\delta}(\mathbb{R}_+))$, $v \in L^2([0,T], H_{\alpha-1+\delta,\delta}(\mathbb{R}_+))$, for all $\delta > 0$, and algebraic if the above inclusions hold for $4 < \delta \leq D$, with some $D > 4$. The major problem is that we do not know what conditions guarantee such a regularity. Chapter III is concluded with several numerical experiments, which illustrate our theoretical analysis

As our future research, we plan to study models with more general different fragmentation kernels. Another important perspective is to consider the transport-fragmentation-coagulation equations, as in many applications these processes occur together.

Bibliography

[ABB90] C.D. Aliprantis, D.J. Brown, and O. Burkinshaw, *Existence and optimality of competitive equilibria*, Springer-Verlag, 1990.

[BA06] J. Banasiak and L. Arlotti, *Pertubations of positive semigroups with applications*, Springer, 2006.

[Ban02] J. Banasiak, *On a non-uniqueness in fragmentation models*, Math. Meth. Appl. Sci. **25** (2002), 541–556.

[Ban04] ______, *Conservative and shattering solutions for some classes of fragmentation models*, Math. Meth. Appl. Sci. **14** (2004), 483–501.

[BGS07] S. Benzoni-Gavage and D. Serre, *Multidimentional hyperbolic partial differential equations. first-order systems and applications*, Clarendon Press, Oxford, 2007.

[BL03] J. Banasiak and W. Lamb, *On application of substochastic semigroup theory to fragmentation models with mass loss*, Mathematical Analysis and Applications (2003), 1.

[Boy00] J.P. Boyd, *Chebyshev and fourier spectral methods*, Dover Publications, Inc., 2000.

[Bre12] A. Bressman, *Functional analysis and linear partial differential equations*, Lecture notes, February 2012.

[CEH91] M. Cai, B.F. Edwards, and H. Han, *Exact and asymptotic scaling solutions for fragmentation with mass loss*, Journal Phys. Rev. **A43** (1991), 656–662.

[CQHZ06] C. Canuto, A. Quarteroni, M.Y. Hussani, and T.A. Zang, *Spectral methods: Fundamental in single domains*, Springer-verlag. Berlin, Heidelberg, 2006.

[EN00] K.J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer, 2000.

[Gau59] W. Gautschi, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J.Math. and Phys. **38** (1959), 77–81.

[GR07] I.S. Gradshteyn and I.M. Ryzhik, *Chebyshev and fourier spectral methods*, Academic Press, 2007.

[GS00] M. Gasca and T. Sauer, *On the history of multivariate polynomial interpolation*, Journal of Computational and Applied Mathematics **II** (2000), 23–35.

[HEL91] J. Huang, B.F. Edwards, and A.D. Levine, *General solutions and scaling violation for fragmentation with mass loss*, Journal Phys. **A24** (1991), 3967–3977.

[HGEL96] J. Huang, X. Guo, B.F. Edwards, and A.D. Levine, *Cut-off model and exact general solutions for fragmentation with mass loss*, Journal Phys. **A29** (1996), 7377–7388.

[HGG07] J. Hesthaven, S. Gottlieb, and D. Gottlieb, *Spectral method for time-dependent problems*, Cambridge University Press, 2007.

[KP03] A. Kufner and L. Persson, *Weighted inequality of hardy type*, World Scientific Publishing Co. Pte. Ltd, 2003.

[MM08] G. Mastroianni and G.V. Milovanović, *Interpolation processes. basic theory and applications*, Springer-Verlag, 2008.

[Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.

[PBM92] A.P. Prudnikov, U.A. Brychkov, and O.I. Marichev, *Integrals and series: Special functions*, Gordon and Breach Science Publishers, Switzerland, 1992.

[PS02] D.V. Prokhorov and V.D. Stepanov, *On riemann-liouville operators*, Dokl. Akad. Nauk **382** (2002), 452–455.

[Rob87] J.C. Robinson, *An atlas of functions*, Hemisphere Publishing Cooperation, 1987.

[SKM93] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach Science Publishers, Switzerland, 1993.

[SLLM00] A.L. Smith, W. Lamb, M. Langer, and A.C. McBride, *Discrete fragmentation with mass loss*, J. Evol. Equ. **A24** (2000), 3967–3977.

[SO01] J. Spanier and K.B. Oldham, *Infinite-dimensional dynamical systems: An introduction to dissipative parabolic pdes and the theory of global attractions*, Cambridge University Press, 2001.

[STW11] J. Shen, T. Tang, and L. Wang, *Spectral method: Algorithms, analysis and applications*, Springer-verlag. Berlin, Heidelberg, 2011.

[Sze67] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc., New-York, 1967.

[Tou00] R. Tourky, *Cones and duality*, American Mathematical Society, 2000.

[Tre00] L.N. Trefethen, *Spectral methods in matlab*, SIAM, 2000.

[ZM85] R. Ziff and E.D. McGrady, *The kinetics of cluster fragmentation and depolymerisation*, J.Phys.A **18** (1985), 3027–3037.

[ZM86] ______, *Kinetics of polymer degradation*, Macromolecules **19** (1986), 2513–2519.