UNIVERSITY OF KWAZULU-NATAL

SOME MODELS OF RELATIVISTIC RADIATING STARS

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As the candidate's supervisors, we have approved this dissertation for submission.

Signed:	Prof. K S Govinder	DATE:
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Abstract

In this dissertation we study radiating stars in strong gravitational fields. We generate new classes of exact solutions to the Einstein field equations and the boundary condition applicable to radiating relativistic stars. The model of a radiating star in general relativity, matching to the Vaidya exterior spacetime, is reviewed. The boundary condition is converted to a Riccati equation and we consider both cases involving geodesic and non-geodesic particle trajectories. We present the metrics found previously. We first solve the boundary condition for the geodesic case and find the gravitational potentials which are expanding and shearing. This is a new result. Secondly the boundary condition is analysed for the non-geodesic case and we seek new gravitational potentials which are accelerating, expanding and shearing. We are able to identify only geodesic solutions for this second case; this appears to be a new class of models. The solutions found are presented in terms of elementary functions which are helpful in studying the physical properties. The new solutions found cannot be categorised in existing classes of known solutions; they are examples of a new generic class different from previous studies. The matter variables of the model are generated.

Declaration

I declare that the contents of this dissertation are my original work except where due reference has been made. It has not been submitted before for any degree to any other institution.

Matsimele Ngwalodi Mahlatji

February 28, 2013

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Contents

Chapter 1

Introduction

The theory of general relativity is a geometrical theory of gravitation where gravity is related to the four-dimensional spacetime curvature as developed by Albert Einstein. This means gravity is generated when the geometry curves. It is an extension of special relativity where gravitational and inertial forces are the same. It is widely accepted as the best description of gravity in modern physics. The central feature of the theory are the Einstein field equations. This is a nonlinear system of partial differential equations. Applications to the solutions of the field equations are in cosmology and relativistic astrophysics.

Here we are concerned with radiating relativistic objects in backgrounds with strong gravitational interactions. In the process of gravitational collapse in an astrophysical environment, there is a strong probability that a star emits radiation and particles. The interior spacetime and exterior spacetime described by the Vaidya solution of the collapsing star should match, taking into consideration the heat flow in the interior of the star (Thirukanesh and Maharaj 2010). Exact models for relativistic radiating stars are vital in terms of analysing the cosmic censorship hypothesis and gravitational collapse (Goswami and Joshi 2004a, 2004b). The formulation of the junction conditions for shear-free collapse was performed by Santos (1985) where the interior metric is matched with the exterior Vaidya metric at the boundary of the star. This made it possible to generate exact models. The physical application of this concept assists in the investigation of surface luminosity, dynamical stability, relaxation effects, particle production at the stellar surface, and temperature profiles for radiating stars in general relativity (Thirukanesh and Maharaj 2010). A radiating model of an initial interior static configuration leading to slow gravitational collapse was proposed by De Oliveira *et al* (1985). In a recent treatment by Herrera *et al* (2004), a relativistic radiating model with vanishing Weyl tensor, in a first order approximation, without exactly solving the junction condition was proposed. Then the relevant junction condition was solved exactly by Maharaj and Govender (2005) and Herrera *et al* (2006). They generated classes of solutions in terms of elementary functions which contain the Friedman dust solution as a special case. Several other classes of solutions by transforming the junction condition to the form of an Abel equation of the first kind, were later investigated by Misthry *et al* (2008). These exact models have been proved to be important in analysing relativistic behaviour of a collapsing object in the stellar consideration. Treatments involving shear involve the works of Rajah and Maharaj (2008), Naidu *et al* (2006) and Thirukkanesh *et al* (2012).

In this thesis we generate new classes of exact solutions to the Einstein field equations and the boundary condition applicable to radiating relativistic stars by considering a non-separable form of the the gravitational potential for both the geodesic and non-geodesic cases. The model of a radiating star in general relativity, matching to the Vaidya exterior spacetime, is reviewed. The boundary condition is converted to a Riccati equation by including both the radial and temporal pressures, and we consider both cases involving geodesic and non-geodesic particle trajectories. We first solve the boundary condition for the geodesic case and find the gravitational potentials which are expanding and shearing. Secondly the boundary condition is analysed for the non-geodesic case and we seek new gravitational potentials which are accelerating expanding and shearing. This search leads to geodesic trajectories; however the class of metrics found appear to be new. The solutions found are presented in terms of elementary functions which are helpful in studying the physical properties.

This thesis is organised as follows:

- Chapter 1: Introduction.

- Chapter 2: We present the Einstein field equations and line element for the geodesic case. The boundary condition is found by matching to the Vaidya line element. Earlier models are reviewed. This is achieved by choosing a non-separable form of the gravitational potential and generating a Riccati equation. In addition a new class of solutions is found. We regain a number of results found previously by Thirukkanesh and Maharaj (2010).

- Chapter 3: We obtain the Einstein field equations and line element for accelerating fluids. The boundary condition is derived by matching the interior to the Vaidya metric. Solutions found earlier are reviewed. This is achieved by choosing a non-separable form of the gravitational potential and analysing a Riccati equation. A new class of exact models is generated. We regain a number of results found previously by Thirukkanesh *et al* (2012).

- Chapter 4: The results obtained in this thesis are summarised in the conclusion.

Chapter 2

Geodesic Fluids

2.1 Introduction

A geodesic is defined mathematically as a curve that locally minimizes the distance between two points in a space, including a curved manifold. This concept can be used to study the behaviour of a radiating star where the interior consists of expanding, shearing fluid particles which are moving in geodesic motion. We demonstrate that it is possible to find a new class of exact solutions to the boundary condition. Earlier treatments with geodesic motion include the works of Kolassis *et al* (1988), Grammenos and Kolassis (1992), Tomimura and Nunes (1993), and Zhe *et al* (2008). Recent results in this direction were found by Herrera *et al* (2002), Govender *et al* (1998), Naidu *et al* (2006), Rajah and Maharaj (2008), and Thirukkanesh and Maharaj (2009, 2010). In section 2.2 we outline the model of a radiating star for the geodesic metric using the

Einstein field equations and the Vaidya metric. The boundary conditions are presented in section 2.3. Known solutions are reviewed in section 2.4; the particular metrics are derived. A new solution is given in section 2.5; we derive the metric and present forms for the matter variables.

2.2 Field equations

We consider the case where the fluid trajectories in the interior spacetime of a spherically symmetric collapsing star with nonzero shear are geodesic. The spacetime is described by the following line metric

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

where the quantities B and Y are functions of the temporal and radial coordinates t and r respectively. They represent the gravitational potentials and need to be explicitly determined by solving the Einstein field equations.

The comoving vector $\mathbf{u}$ is represented by $u^a = \delta^a_0$ which is the fluid four-velocity.

The kinematical quantities for the line metric (2.1) become

$$\dot{u}^a = 0, \tag{2.2a}$$

$$\Theta = \frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}, \tag{2.2b}$$

$$\sigma = \frac{1}{3}\left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B}\right), \tag{2.2c}$$

where $\dot{u}^a$ is the four-acceleration vector, Θ is the expansion scalar and σ is the magnitude of the shear scalar. Dots on the potentials B and Y represent differentiation with respect to time t. Note that the acceleration is zero because particle trajectories in the star are moving with constant velocity. The energy momentum tensor for the interior matter distribution is represented by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{2.3}$$

where ρ is the energy density, p is the isotropic pressure, π_{ab} is the stress tensor and q_a is the heat flux vector of the fluid. The mass, luminosity and stability of relativistic spheres are affected by the anisotropy which forms a critical part in gravitational collapse. For the physical relevance of anisotropy in astrophysics see the treatments of Chaisi and Maharaj (2005, 2006), Dev and Gleiser (2002, 2003) and Mak and Harko (2002, 2003). The stress tensor is represented by

$$\pi_{ab} = (p_r - p_t)\left(n_a n_b - \frac{1}{3}h_{ab}\right), \tag{2.4}$$

where p_r is the radial pressure, p_t is the tangential pressure and $\mathbf{n}$ is a unit vector defined by $n^a = \delta^a_1$. The relationship

$$p = \frac{1}{3}(p_r + 2p_t) \tag{2.5}$$

gives the isotropic pressure in terms of the radial and tangential pressures.

The Einstein field equations for the line element (2.1) and the matter distribution equation (2.3) are given by

$$\rho = 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'}{B}\frac{Y'}{Y}\right), \tag{2.6a}$$

$$p_r = -2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - \frac{1}{Y^2} + \frac{1}{B^2}\frac{Y'^2}{Y^2}, \tag{2.6b}$$

$$p_t = -\left(\frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y''}{Y} - \frac{B'}{B}\frac{Y'}{Y}\right), \tag{2.6c}$$

$$q = -\frac{2}{B^2}\left(\frac{\dot{B}}{B}\frac{Y'}{Y} - \frac{\dot{Y}'}{Y}\right), \tag{2.6d}$$

where $q^a = (0, q, 0, 0)$ represents the heat flux which is radially orientated and the prime indicates differentiation with respect to r. The geodesic anisotropic matter distribution in a spherically symmetric gravitational field is represented by the system of equations (2.6) which governs the most general model with geodesic gravitational collapse. We make the observation from the system of equations (2.6) that if the gravitational potentials B and Y are specified, then the expressions for the matter variables ρ, p_r, p_t and q follow by substitution.

The interior spacetime has to match across the boundary Σ to the exterior spacetime for a relativistic star. The exterior spacetime is described by

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.7}$$

which represents the familiar Vaidya metric. An observer at infinity measures $m(v)$ as the mass of the radiating star. The line element (2.7) is used to indicate incoherent null radiation which flows in a radial orientation relative to the hypersurface Σ which is the boundary of the star.

2.3 Junction conditions

The junction conditions are generated by the matching of the interior spacetime (2.1) and exterior spacetime (2.7) and the matching of the extrinsic curvature components on the surface Σ. This leads to the following set of junction conditions on the stellar surface Σ:

$$dt = -\left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \tag{2.8a}$$

$$Y(R_{\Sigma,t}) = R_\Sigma(v), \tag{2.8b}$$

$$m(v)_\Sigma = \left[\frac{Y}{2}\left(1 + \dot{Y}^2 - \frac{Y'^2}{B^2}\right)\right]_\Sigma, \tag{2.8c}$$

$$(p_r)_\Sigma = (qB)_\Sigma. \tag{2.8d}$$

Equation (2.8d) describes the nonvanishing of the radial pressure at the boundary Σ. Note that (2.8d) is an extra constraint which has to be satisfied along with the system of equations (2.6). The junction condition (2.8d) was first derived by Santos (1985) in the case of shear-free spacetimes. In order to incorporate spacetimes with nonzero shear the metric (2.1) was extended by Glass (1989). Substituting equations (2.6b) and (2.6d) into (2.8d) we obtain the following equation

$$2Y\ddot{Y} + \dot{Y}^2 - \frac{Y'^2}{B^2} + \frac{2}{B}YY' - 2\frac{\dot{B}}{B^2}YY' + 1 = 0, \tag{2.9}$$

which has to be satisfied on the boundary Σ. The gravitational behaviour of the radiating anisotropic star with nonzero shear and no acceleration is governed by (2.9). This equation is difficult to solve without some simplifying assumption as it is highly nonlinear in nature. The potentials B and Y are two unknown functions in (2.9); to find a solution we have to specify one of the functions, or introduce another condition, so that the resulting equation is integrable.

2.4 Review of known solutions

2.4.1 A Riccati approach

We can write (2.9) in the form of a Riccati equation in the gravitational potential B:

$$\dot{B} = \left[\frac{\ddot{Y}}{Y'} + \frac{\dot{Y}^2}{2YY'} + \frac{1}{2YY'}\right]B^2 + \frac{\dot{Y}'}{Y'}B - \frac{Y'}{2Y}. \tag{2.10}$$

This Riccati equation has to meet the necessary conditions on the stellar boundary Σ and is difficult to solve in general. Nogueira and Chan (2004) obtained numerical solutions in their analysis of a collapsing star. Solutions can be obtained in an ad hoc fashion by making the assumption that the gravitational potential Y is a separable function and specifying the temporal evolution of the model as obtained by Rajah and Maharaj (2008). Thirukkanesh and Maharaj (2010) showed that it is possible to transform the Riccati equation into separable form by a suitable transformation which leads to new solutions. Hence it is possible to discover new exact solutions systematically without assuming separable forms for the function Y. By introducing the transformation

$$B = ZY', \tag{2.11}$$

we can show that equation (2.10) becomes

$$\dot{Z} = \frac{1}{2Y}[FZ^2 - 1], \tag{2.12}$$

where

$$F = 2Y\ddot{Y} + \dot{Y}^2 + 1.$$

Observe that equation (2.12) is integrable provided F is independent of the variable t. In this approach we emphasize that we have not made any hypothesis about the separability of the metric coefficients B and Y or have restricted the t-dependence of the model. This approach leads to new exact solutions.

2.4.2 Solution I

We make the assumption

$$F = 1. \tag{2.13}$$

As a result, we need to solve

$$2Y\ddot{Y} + \dot{Y}^2 = 0.$$

The solution yields

$$Y(r,t) = [R_1(r)t + R_2(r)]^{\frac{2}{3}}, \tag{2.14}$$

where $R_1(r)$ and $R_2(r)$ are arbitrary functions of r. Then substituting (2.14) into (2.12) we obtain

$$\dot{Z} = \frac{[Z^2 - 1]}{2[R_1(r)t + R_2(r)]^{\frac{2}{3}}}. \tag{2.15}$$

Integrating (2.15) we get

$$Z = \frac{1 + f(r)\exp(3(R_1 t + R_2)^{\frac{1}{3}}/R_1)}{1 - f(r)\exp(3(R_1 t + R_2)^{\frac{1}{3}}/R_1)}, \tag{2.16}$$

where *f(r)* is the function of integration. By substituting (2.14) and (2.16) into (2.11) we get

$$B = \frac{2}{3}\left[\frac{1 + f(r)\exp[3(R_1 t + R_2)^{\frac{1}{3}}/R_1]}{1 - f(r)\exp[3(R_1 t + R_2)^{\frac{1}{3}}/R_1]}\right]\frac{[R_1' t + R_2']}{[R_1 t + R_2]^{\frac{1}{3}}} \tag{2.17}$$

which was first found by Thirukkanesh and Maharaj (2010). The line element has the form

$$
\begin{aligned}
ds^2 &= -dt^2 + \frac{4}{9}\left[\frac{1+f(r)\exp[3(R_1 t + R_2)^{1/3}/R_1]}{1-f(r)\exp[3(R_1 t + R_2)]^{1/3}/R_1}\right]^2 \\
&\quad \times \frac{R_1' t + R_2'}{[R_1 t + R_2]^{2/3}} dr^2 \\
&\quad + [R_1(r)t + R_2(r)]^{4/3}(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.18)
\end{aligned}
$$

for the assumption (2.13).

For particular forms of the arbitrary functions we can generate models found previously. When

$$
\begin{aligned}
R_1 &= 0, \\
R_2 &= r^{3/2},
\end{aligned}
$$

then the equivalent of (2.18) is

$$
ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.19)
$$

which is the Minkowski metric. If we let

$$
\begin{aligned}
R_1 &= R^{3/2}, \\
R_2 &= aR^{3/2},
\end{aligned}
$$

then (2.18) becomes

$$\begin{aligned} ds^2 &= -dt^2 + (t+a)^{4/3} \\ &\quad \times \left\{ R'^2 \left[\frac{1 + f(r)\exp[3(t+a)^{1/3}/R]}{1 - f(r)\exp[3(t+a)^{1/3}/R]} \right]^2 dr^2 \right\} \\ &\quad + (t+a)^{4/3} \left\{ R^2(d\theta^2 + \sin^2\theta d\phi^2) \right\}, \end{aligned} \tag{2.20}$$

which was discussed by Rajah and Maharaj (2008). If we set

$$\begin{aligned} a &= 0, \\ R &= r, \end{aligned}$$

then (2.20) becomes

$$\begin{aligned} ds^2 &= -dt^2 + t^{4/3} \\ &\quad \times \left\{ \left[\frac{1 + f(r)\exp[3(t)^{1/3}/r]}{1 - f(r)\exp[3(t)^{1/3}/r]} \right]^2 dr^2 \right\} \\ &\quad + t^{4/3} \left\{ R^2(d\theta^2 + \sin^2\theta d\phi^2) \right\} \end{aligned} \tag{2.21}$$

which was first found by Naidu *et al* (2006). Also if we set

$$\begin{aligned} R_1 &= r^{3/2}, \\ R_2 &= 0, \\ f(r) &= 0, \end{aligned}$$

then we get

$$ds^2 = -dt^2 + t^{4/3}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.22}$$

which corresponds to the Friedmann metric. It is a pleasing feature of the class of spacetimes (2.18) that it contains a number of known solutions.

2.4.3 Solution II

We now make the assumption

$$F = 1 + R_1^2(r). \tag{2.23}$$

We need to solve

$$2Y\ddot{Y} + \dot{Y}^2 = R_1^2(r).$$

A particular solution is given by

$$Y(r,t) = [R_1(r)t + R_2(r)], \tag{2.24}$$

where $R_1(r)$ and $R_2(r)$ are arbitrary functions of r. This results in the following expression

$$\dot{Z} = \frac{[R_1^2+1]}{2[R_1 t + R_2]}\left[Z^2 - \frac{1}{[R_1^2+1]}\right], \tag{2.25}$$

after substituting (2.24) into (2.12). The solution of (2.25) after integration becomes

$$Z = \frac{1}{\sqrt{R_1^2+1}}\left[\frac{1 + g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}{1 - g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}\right], \tag{2.26}$$

where $g(r)$ is a function of integration. Substituting (2.26) and (2.24) into (2.11) we find

$$B = \frac{1}{\sqrt{R_1^2+1}}\left[\frac{1 + g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}{1 - g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}\right][R_1' t + R_2'], \tag{2.27}$$

as given by Thirukkanesh and Maharaj (2010). The line element has the form

$$\begin{aligned} ds^2 &= -dt^2 + \frac{1}{R_1^2+1}\left[\frac{1+g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}{1-g(r)[R_1 t + R_2]^{\sqrt{R_1^2+1}/R_1}}\right]^2 \\ &\quad \times [R_1' t + R_2']^2 dr^2 + [R_1(r) + R_2(r)]^2 (d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{2.28}$$

for the choice (2.23).

We can regain previously known models from (2.28). When

$$\begin{aligned} R_1 &= R, \\ R_2 &= aR. \end{aligned}$$

then (2.28) becomes

$$\begin{aligned} ds^2 &= -dt^2 + (t+a)^2 \left\{\frac{R'^2}{[R^2+1]}\left[\frac{1+h(r)[t+a]^{\sqrt{R^2+1}/R}}{1-h(r)[t+a]^{\sqrt{R^2+1}/R}}\right]^2 dr^2\right\} \\ &\quad +(t+a)^2\left\{R^2(d\theta^2+\sin^2\theta d\phi^2)\right\} \end{aligned} \tag{2.29}$$

which is contained in the Rajah and Maharaj (2008) class of models. Once again, a class of spacetimes was obtained which contained previously known spacetimes as special cases.

We observe that (2.24) is not the general solution of

$$2Y\ddot{Y} + \dot{Y}^2 = R_1^2(r).$$

If we set

$$\begin{aligned} u &= Y, \\ v &= \dot{Y}, \end{aligned}$$

we can write

$$\frac{dv}{du} = \frac{\ddot{Y}}{\dot{Y}} = \frac{\ddot{Y}}{v}.$$

Therefore our equation reduces to the first order equation

$$2uvv' + v^2 = R_1^2(r).$$

This can be integrated to yield

$$v^2 = R_1^2 + \frac{C_1(r)}{u}, \tag{2.30}$$

were $C_1(r)$ is an arbitrary function of integration. Reverting to the original variables we have

$$\dot{Y}^2 = R_1^2(r) + \frac{C_1(r)}{Y},$$

and so

$$\int \frac{dY}{\pm\sqrt{R_1^2(r) + \frac{C_1(r)}{Y}}} = t + C_2(r). \tag{2.31}$$

From (2.30) it can be seen that

$$C_1(r) = 0$$

yields

$$\dot{Y}^2 = R_1^2(r),$$

and so we have

$$Y(r,t) = R_1 t + R_2(r)$$

as before. In our case (2.31), we have the general solution with three arbitrary functions of r. Unfortunately evaluating the quadrature in (2.31) results in a function of $Y(t,r)$ that cannot be easily inverted to yield $Y(t,r)$ explicitly. Nonetheless, this form of solution can be combined with results obtained earlier in this section to provide new (albeit implicit) solutions.

2.5 New Solution

It is possible to find other new exact solutions to the boundary condition (2.10). To determine this we make the assumption

$$\dot{Y}' = 0. \tag{2.32}$$

Note that this assumption leads to a new class of solutions that do not contain the cases considered in section 2.4 as those solutions do not satisfy (2.32).

For the assumption

$$Y(r,t) = [R_1(r)t + R_2(r)]^{\frac{2}{3}}$$

used in section 2.4.2, we obtain

$$\dot{Y}' = \frac{2}{3}R_1'(R_1 t + R_2)^{-\frac{1}{3}} - \frac{2}{9}R_1(R_1 t + R_1)^{-\frac{4}{3}}(R_1' t + R_2'). \tag{2.33}$$

This can only equal zero when

$$R_1(r) = 0.$$

This forces

$$Y(r,t) = Y(r),$$

i.e., Y cannot depend on t. As a result the solutions we obtain can only reduce to solutions given in section 2.4.2, in the case of static spacetimes.

For assumption

$$Y(r,t) = R_1(r)t + R_2(r) \tag{2.34}$$

considered in section 2.4.3, we obtain

$$\dot{Y}' = R_1'(r).$$

Requiring this to be zero we obtain

$$Y(r,t) = \eta t + R_2(r).$$

Again, only special cases of our solutions can be compared with those in section 2.4.3.

Integrating (2.32) results in

$$Y = a(t) + b(r), \tag{2.35}$$

where a is a function of t and b is a function of r. For this form of Y we indicate that it is possible to find a new exact solution. On substituting (2.32) and (2.35) into (2.10), we get the simpler equation

$$\begin{aligned} \dot{B} &= \left[\frac{\ddot{a}(t)}{b'(r)} + \frac{\dot{a}^2(t)}{2(a(t)+b(r))b'(r)} + \frac{1}{2(a(t)+b(r))b'(r)}\right] B^2 \\ &\quad - \frac{b'(r)}{2(a(t)+b(r))}. \end{aligned} \tag{2.36}$$

This represents the inhomogeneous Riccati equation which has to be integrated.

We introduce the transformation

$$B = wb'(r), \tag{2.37}$$

to obtain

$$\dot{w} = \frac{1}{2(a(t)+b(r))}\left[(2(a(t)+b(r))\ddot{a}(t) + \dot{a}^2(t) + 1)w^2 - 1\right]. \tag{2.38}$$

We introduce the function $f(r)$ expressed as

$$f(r) = 2(a(t)+b(r))\ddot{a}(t) + \dot{a}^2(t) + 1. \tag{2.39}$$

Clearly f can depend only on the radial coordinate r in the above equation to complete the integration. Differentiating (2.39) with respect to r we obtain

$$f'(r) = 2b'(r)\ddot{a}(t).$$

This result implies that

$$f(r) = \alpha b(r) + b_0, \tag{2.40a}$$

$$a(t) = \frac{\alpha}{4}t^2 + a_0 + a_1 t, \tag{2.40b}$$

where α is constant. Then (2.39) yields the condition $\alpha = 0$ for consistency so that we must have

$$a(t) = a_0 + a_1 t. \tag{2.41}$$

Then (2.38) can be expressed as

$$\dot{w} = \frac{1}{2(a_0 + a_1 t + b(r))}[(a_1^2 + 1)w^2 - 1]. \tag{2.42}$$

This is a separable equation in the variables w and t. We integrate w to obtain

$$w = \frac{1 - \exp(tb(r) + 2D + ta_0 + \frac{1}{2}t^2 a_1 + tb(r)a_1^2 + ta_0 a_1^2 + \frac{1}{2}t^2 a_1^3)}{1 + \exp(tb(r) + 2D + ta_0 + \frac{1}{2}t^2 a_1 + tb(r)a_1^2 + ta_0 a_1^2 + \frac{1}{2}t^2 a_1^3)}, \tag{2.43}$$

where D is the constant of integration. Substituting (2.43) into (2.37) results in the following form for the metric function:

$$B = \left(\frac{1 - \exp(tb(r) + 2D + ta_0 + \frac{1}{2}t^2 a_1 + tb(r)a_1^2 + ta_0 a_1^2 + \frac{1}{2}t^2 a_1^3)}{1 + \exp(tb(r) + 2D + ta_0 + \frac{1}{2}t^2 a_1 + tb(r)a_1^2 + ta_0 a_1^2 + \frac{1}{2}t^2 a_1^3)}\right) b'(r). \tag{2.44}$$

The line element has the form

$$\begin{aligned} ds^2 &= -dt^2 \\ &+ \left(\left(\frac{1 - \exp(tb(r) + 2D + ta_0 + \frac{1}{2}t^2 a_1 + tb(r)a_1^2 + ta_0 a_1^2 + \frac{1}{2}t^2 a_1^3)}{1 + \exp(tb(r) + 2D + ta_0 + \frac{1}{2}t^2 a_1 + tb(r)a_1^2 + ta_0 a_1^2 + \frac{1}{2}t^2 a_1^3)}\right) b'(r)\right)^2 dr^2 \\ &+ (a_0 + a_1 t + b(r))^2 (d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{2.45}$$

for the assumption (2.32).

We believe that the metric (2.45) is a new solution to the boundary condition for an expanding, shearing radiating star with particles in geodesic motion. Models found earlier cannot be regained from the metric (2.45). It is remarkable that the solution can be expressed in simple elementary functions and the quantity $b(r)$ is arbitrary.

We can now compute the dynamical quantities from the Einstein field equations (2.6). We obtain the expressions

$$\begin{aligned}
\rho &= 2\frac{\mu}{\tau}\frac{a_1}{(a_0+a_1t+b(r))}+\frac{1}{(a_0+a_1t+b(r))^2}+\frac{a_1^2}{(a_0+a_1t+b(r))^2} \\
&\quad -\frac{1}{\kappa}\left(2\frac{b''(r)}{(a_0+a_1t+b(r))}+\frac{b'^2(r)}{(a_0+a_1t+b(r))^2}\right) \\
&\quad -\frac{1}{\kappa}\left(2\frac{\epsilon}{\tau}\frac{b'(r)}{(a_0+a_1t+b(r))}\right), \qquad (2.46a)\\
p_r &= -\frac{(a_1)^2}{(a_0+a_1t+b(r))^2}-\frac{1}{(a_0+a_1t+b(r))^2} \\
&\quad +\frac{1}{\kappa}\frac{b'^2(r)}{(a_0+a_1t+b(r))^2}, \qquad (2.46b)\\
p_t &= -\left(\frac{\Omega}{\tau}+\frac{\mu}{\tau}\frac{a_1}{(a_0+a_1t+b(r))}\right) \\
&\quad +\frac{1}{\kappa}\left(\frac{b''(r)}{(a_0+a_1t+b(r))}-\frac{\epsilon}{\tau}\frac{b'(r)}{(a_0+a_1t+b(r))}\right), \qquad (2.46c)\\
q &= -\frac{2}{\kappa}\left(\frac{\mu}{\tau}\frac{b'(r)}{(a_0+a_1t+b(r))}\right), \qquad (2.46d)
\end{aligned}$$

in terms of the constants a_0, a_1 and the arbitrary function $b(r)$. We have introduced the quantities μ, ϵ, τ and κ for convenience. If we let

$$\Gamma = \exp(tb(r) + 2D + ta_0 + \frac{1}{2}t^2a_1 + tb(r)a_1^2 + ta_0a_1^2 + \frac{1}{2}t^2a_1^3)$$

and

$$\Psi = \exp(2tb(r) + 4D + 2ta_0 + t^2a_1 + 2tb(r)a_1^2 + 2ta_0a_1^2 + t^2a_1^3).$$

These quantities are defined by:

$$\mu = \left(-\frac{\Gamma(1-\Gamma)(b(r)+a_0+ta_1+b(r)a_1^2+a_0a_1^2+ta_1^3)}{(1+\Gamma)^2}\right)b'(r)$$
$$-\left(\frac{\Gamma(b(r)+a_0+ta_1+b(r)a_1^2+a_0a_1^2+ta_1^3)}{1+\Gamma}\right)b'(r), \tag{2.47a}$$

$$\Omega = \left(-\frac{\Gamma(1-\Gamma)(a_1+a_1^3)}{(1+\Gamma)^2}-\frac{\Gamma(a_1+a_1^3)}{1+\Gamma}\right)b'(r)$$
$$+\left(\frac{2\Psi\Gamma(b(r)+a_0+ta_1+b(r)a_1^2+a_0a_1^2+ta_1^3)^2}{(1+\Gamma)^3}\right)b'(r)$$
$$+\left(\frac{2\Psi(b(r)+a_0+ta_1+b(r)a_1^2+a_0a_1^2+ta_1^3)^2}{(1+\Gamma)^2}\right)b'(r)$$
$$-\left(\Gamma(1-\Gamma)(b(r)+a_0+ta_1+b(r)a_1^2+a_0a_1^2+ta_1^3)^2\right)b'(r)$$
$$-\left(\frac{\Gamma(b(r)+a_0+ta_1+b(r)a_1^2+a_0a_1^2+ta_1^3)^2}{1+\Gamma}\right)b'(r) \tag{2.47b}$$

$$\epsilon = \left(\left(-\frac{\Gamma(1-\Gamma)(tb'(r)+ta_1^2b'(r))}{(1+\Gamma)^2}-\frac{\Gamma(tb'(r)+ta_1^2b'(r))}{1+\Gamma}\right)b'(r)\right)b''(r)$$
$$+\left(\frac{1-\Gamma}{1+\Gamma}\right)b''(r), \tag{2.47c}$$

$$\tau = \left(\frac{1-\exp(tb(r)+2D+ta_0+\frac{1}{2}t^2a_1+tb(r)a_1^2+ta_0a_1^2+\frac{1}{2}t^2a_1^3)}{1+\exp(tb(r)+2D+ta_0+\frac{1}{2}t^2a_1+tb(r)a_1^2+ta_0a_1^2+\frac{1}{2}t^2a_1^3)}\right)$$
$$\times b'(r), \tag{2.47d}$$

$$\kappa = \left(\frac{1-\exp(tb(r)+2D+ta_0+\frac{1}{2}t^2a_1+tb(r)a_1^2+ta_0a_1^2+\frac{1}{2}t^2a_1^3)}{1+\exp(tb(r)+2D+ta_0+\frac{1}{2}t^2a_1+tb(r)a_1^2+ta_0a_1^2+\frac{1}{2}t^2a_1^3)}b'(r)\right)^2$$
$$\times(b'(r))^2. \tag{2.47e}$$

A physical analysis of the matter variables in (2.46)-(2.47) will be carried out in future work. We have verified the accuracy of the results in this chapter with the help of Mathematica (Wolfram 2008).

We note that the quantities a_0, a_1 and $b(r)$ are arbitrary quantities in the above solution. Particular choices simplify the expressions found for the gravitational potentials and the matter variables. As an example we make the choice

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ b(r) &= r, \end{aligned}$$

without any loss of generality. Then the line element (2.45) becomes

$$\begin{aligned} ds^2 &= -dt^2 \\ &\quad + \left(\frac{1-\exp(2tr+2D+t^2)}{1+\exp(2tr+2D+t^2)}\right)^2 dr^2 \\ &\quad +(t+r)^2(d\theta^2+\sin^2\theta d\phi^2). \end{aligned} \tag{2.48}$$

The dynamical quantities (2.46) can be written in the form

$$
\begin{aligned}
\rho &= 2\frac{\mu}{\tau}\frac{1}{(t+r)} + \frac{2}{(t+r)^2} - \frac{1}{\kappa}\left(\frac{1}{(t+r)^2}\right) \\
&\quad -\frac{1}{\kappa}\left(2\frac{\epsilon}{\tau}\frac{1}{(t+r)}\right), && \text{(2.49a)} \\
p_r &= -\frac{2}{(t+r)^2} + \frac{1}{\kappa}\frac{1}{(t+r)^2}, && \text{(2.49b)} \\
p_t &= -\left(\frac{\Omega}{\tau} + \frac{\mu}{\tau}\frac{1}{(t+r)}\right) - \frac{1}{\kappa}\left(\frac{\epsilon}{\tau}\frac{1}{(t+r)}\right), && \text{(2.49c)} \\
q &= -\frac{2}{\kappa}\left(\frac{\mu}{\tau}\frac{1}{(t+r)}\right), && \text{(2.49d)}
\end{aligned}
$$

which is much simpler than the general expressions that appeared earlier.

The quantities μ, Ω, τ and κ also take on a simpler form. They are given by

$$\begin{aligned}
\mu &= \left(-\frac{\Gamma(1-\Gamma)(2r+2t)}{(1+\Gamma)^2}\right) \\
&\quad - \left(\frac{\Gamma(2r+2t)}{1+\Gamma}\right), && \text{(2.50a)} \\
\Omega &= \left(-\frac{2\Gamma(1-\Gamma)}{(1+\Gamma)^2} - \frac{2\Gamma}{1+\Gamma}\right) \\
&\quad + \left(\frac{2\Psi\Gamma(2r+2t)}{(1+\Gamma)^3}\right) \\
&\quad + \left(\frac{2\Psi(2r+2t)^2}{(1+\Gamma)^2}\right) \\
&\quad - \left(\Gamma(1-\Gamma)(2r+2t)^2\right) \\
&\quad - \left(\frac{\Gamma(2r+2t)^2}{1+\Gamma}\right) && \text{(2.50b)} \\
\epsilon &= 0, && \text{(2.50c)} \\
\tau &= \left(\frac{1-\exp(2tr+2D+t^2)}{1+\exp(2tr+2D+t^2)}\right), && \text{(2.50d)} \\
\kappa &= \left(\frac{1-\exp(2tr+2D+t^2)}{1+\exp(2tr+2D+t^2)}\right)^2, && \text{(2.50e)}
\end{aligned}$$

where Γ and Ψ become

$$\Gamma = \exp(2tr + 2D + t^2)$$

and

$$\Psi = \exp(6tr + 4D + 2t^2).$$

Chapter 3

Accelerating Fluids

3.1 Introduction

In this chapter we study the behaviour of a relativistic spherically symmetric radiative star with accelerating, expanding and shearing interior matter distribution where there is anisotropic pressure. We integrate the boundary condition to find new solutions. Unlike the previous chapter the boundary condition is more difficult to integrate. Consequently fewer treatments with accelerating particles have been attempted in the past. Nogueira and Chan (2004) modelled shear and bulk viscosity but needed to utilise numerical methods to make progress. Herrera and Santos (2010) and Govender *et al* (2010) found solutions for Euclidean stars where the areal radius and proper radius are equal in the interior of the star. It is possible to write the boundary condition as a linear, Bernoulli, and Riccati equation in standard form as demonstrated by

Thirukkanesh *et al* (2012). In section 3.2 we outline the model of a spherically symmetric collapsing star for the non-geodesic metric using the Einstein field equations and the Vaidya metric. The boundary conditions are presented in section 3.3. Known solutions that have been published are recovered in section 3.4; the relevant particular metrics are derived. A new solution is given in section 3.5; we derive the metric and present forms for the matter variables.

3.2 Field Equations

We now study a spherically symmetric collapsing star which is expanding, accelerating and shearing. The fluid trajectories are no longer geodesic as the particles are accelerating. The interior spacetime is represented by the following general form of the line metric

$$ds^2 = -A^2dt + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}$$

where A, B and Y are functions of both the temporal and radial coordinates t and r respectively. The functions A, B and Y represent the gravitational potentials.

The presence of a four-velocity vector $\mathbf{u}$ enables us to introduce the kinematical quantities

$$\dot{u}^a = u^a_{;b}u^b, \tag{3.2a}$$

$$\Theta = u^a_{;a}, \tag{3.2b}$$

$$\sigma_{ab} = h^c_a h^d_b u_{(c;d)}, \tag{3.2c}$$

where $h_{ab} = g_{ab} + u_a u_b$ $(h_{ab}u^a = 0)$ represents the symmetric projection tensor. The acceleration of the fluid particles relative to the congruences of $\mathbf{u}$ is represented by the acceleration vector $\dot{u}^a(\dot{u}^a u_a = 0)$. The rate of increase of a volume of fluid element is represented by the expansion scalar Θ, and the shear $\sigma_{ab}(\sigma_{ab}u^b = 0 = \sigma^a_a)$ is the tendency of a sphere to distort to an ellipsoid. The comoving fluid four-velocity is $u^a = \frac{1}{A}\delta^a_0$. The acceleration vector $\dot{u}^a$, the expansion scalar Θ and the shear scalar $\text{\ae}$ are given by

$$\dot{u}^a = \left(0, \frac{A'}{AB^2}, 0, 0\right), \tag{3.3a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right), \tag{3.3b}$$

$$\sigma = -\frac{1}{3A}\left(\frac{\dot{B}}{B} - \frac{\dot{Y}}{Y}\right), \tag{3.3c}$$

where the dots represent differentiation with respect to t and primes are differentiation with respect to r on the metric functions A, B and Y. The energy momentum tensor for the interior matter distribution is represented by the following

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{3.4}$$

where æ is the density of the fluid, p is the isotropic pressure, q_a is the heat flux vector and ß$_{ab}$ is the stress tensor. The stress tensor is represented by

$$\pi_{ab} = (p_r - p_t)\left(n_a n_b - \frac{1}{3}h_{ab}\right) \tag{3.5}$$

where p_r and p_t represent the radial pressure and the tangential pressure, and $\mathbf{n}$ is a unit radial vector defined by $n^a = \frac{1}{B}\delta^a_1$. The relationship

$$p = \frac{1}{3}(p_r + 2p_t) \tag{3.6}$$

defines the isotropic pressure in terms of the radial and the tangential pressures.

The Einstein field equations for the line element (3.1) and the matter distribution (3.4) are given by

$$\rho = \frac{2}{A^2}\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{1}{A^2}\frac{\dot{Y}^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'}{B}\frac{Y'}{Y}\right), \tag{3.7a}$$

$$p_r = \frac{1}{A^2}\left(-2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y'^2}{Y^2} + 2\frac{A^2}{A}\frac{Y'}{Y}\right) - \frac{1}{Y^2}, \tag{3.7b}$$

$$p_t = -\frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} - \frac{\dot{A}}{A}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) \tag{3.7c}$$

$$+ \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + \frac{A'}{A}\frac{Y'}{Y} - \frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right), \tag{3.7d}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{Y}'}{Y} + \frac{\dot{B}}{B}\frac{Y'}{Y} + \frac{A'}{A}\frac{\dot{Y}}{Y}\right), \tag{3.7e}$$

where the heat flux represented by $q^a = (0, q, 0, 0)$ comprises only of the nonvanishing radial component. The Einstein field equations govern the general model when characterising matter distributions with anisotropic pressures and heat flux for a spherically symmetric relativistic stellar object. The nonlinear gravitational interactions for a shearing matter distribution which it is expanding and accelerating is described by the above Einstein field equations. The physical relevance of shear in general relativity has been studied by Ivanov (2012), Krasinski (1997) and Stephani *et al* (2003).

The exterior spacetime

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.8}$$

is described by the Vaidya metric for a radiating star where $m(v)$ is the mass of the fluid at infinity as measured by an observer. This represents incoherent null radiation.

3.3 Junction conditions

The flow of null radiation is in the radial direction relative to the hypersurface Σ, which indicates the boundary of the star. The junction conditions on the hypersurface Σ are produced by the matching of the metric potentials and extrinsic curvature components for the interior spacetime (3.1) and the exterior spacetime (3.8). These conditions are represented by the following

$$A(R_\Sigma,t)dt = \left(1-\frac{2m}{R_\Sigma}+\frac{dR_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \tag{3.9a}$$

$$Y(R_\Sigma,t) = R_\Sigma(v), \tag{3.9b}$$

$$m(v)_\Sigma = \left[\frac{Y}{2}\left(1+\frac{\dot{Y}^2}{A^2}-\frac{Y'^2}{B^2}\right)\right]_\Sigma, \tag{3.9c}$$

$$(p_r)_\Sigma = (qB)_\Sigma. \tag{3.9d}$$

Note that the radial pressure is nonzero at the boundary Σ.

Substituting (3.7a) and (3.7d) into (3.9d) we get the boundary condition which has to be satisfied at the stellar surface

$$\begin{aligned} & 2Y\ddot{Y}+\dot{Y}^2-2\left(\frac{\dot{A}}{A}+\frac{A'}{B}\right)Y\dot{Y}+2\frac{A}{B}Y\dot{Y}' \\ & -2\frac{A}{B^2}(A'+B)YY'-\frac{A^2}{B^2}Y'^2+A^2=0. \end{aligned} \tag{3.10}$$

This is the governing equation that describes the gravitational behaviour of the radiating anisotropic star with nonzero shear, acceleration and expansion. Rewriting the above equation by making $\dot{B}$ subject of the formula, we get the following Riccati equation

$$\begin{aligned} & \dot{B}-\left[\frac{\ddot{Y}}{AY'}+\frac{\dot{Y}^2}{2AYY'}-\frac{\dot{A}\dot{Y}}{A^2Y'}-\frac{A}{2YY'}\right]B^2 \\ & -\left[\frac{\dot{Y}'}{Y'}-\frac{A'}{A}\frac{\dot{Y}}{Y'}\right]B+\left[A'+\frac{AY'}{2Y}\right]=0. \end{aligned} \tag{3.11}$$

This can be solved for special cases.

3.4 Review of known solutions

3.4.1 A Riccati approach

In spite of the nonlinearity and complexity of (3.11), we can find particular exact solutions of (3.11) by considering it as a first order differential equation in the variable B and placing restrictions on the bracketed expressions as indicated in the following three cases.

3.4.2 Linear equation

Equation (3.11) becomes a linear equation if we set the following restriction

$$\frac{\ddot{Y}}{AY'} + \frac{\dot{Y}^2}{2AY'} - \frac{\dot{A}\dot{Y}}{A^2Y'} + \frac{A}{2YY'} = 0, \tag{3.12}$$

which can be represented as

$$\dot{A} - \left[\frac{\ddot{Y}}{\dot{Y}} + \frac{\dot{Y}}{2Y}\right] A = \frac{A^3}{2Y\dot{Y}}. \tag{3.13}$$

This represents a Bernoulli equation with respect to the variable A. Integrating this equation where Y is an arbitrary function we get

$$A^2 = \frac{Y\dot{Y}}{h(r) - Y}, \tag{3.14}$$

where $h(r)$ is a function of integration. Using (3.14), (3.11) becomes

$$\dot{B} - \left[\frac{\dot{Y}'}{Y'} - \frac{A'}{A}\frac{\dot{Y}}{Y'}\right] B + \left[A' + \frac{AY'}{2Y}\right] = 0, \tag{3.15}$$

making it linear in B. Even though A and Y are arbitrary in (3.15), it is possible to obtain B. Hence the solution for the junction condition (3.11) can be represented by

$$
\begin{aligned}
A &= \sqrt{\frac{Y\dot{Y}^2}{h(r)-Y}}, && \text{(3.16a)}\\
B &= Y' \exp\left(-\int \frac{A'\dot{Y}}{AY'}dt\right) \\
&\quad \times \left\{k(r) - \int \left[\left(\frac{A'}{Y'} + \frac{A}{2Y}\right)\exp\left(\int \frac{A'\dot{Y}}{AY'}dt\right)\right] dt\right\}, && \text{(3.16b)}\\
Y &= Y(t,r), && \text{(3.16c)}
\end{aligned}
$$

where $k(r)$ is a function of integration. Equations (3.16) are a new solution to the boundary condition (3.10) as found by Thirukkanesh *et al* (2012). Note that a choice for Y should be made to provide us with a physically reasonable model. The line element is therefore given by

$$
\begin{aligned}
ds^2 &= -\left(\sqrt{\frac{Y\dot{Y}^2}{h(r)-Y}}\right)^2 dt \\
&\quad + \left(Y' \exp\left(-\int \frac{A'\dot{Y}}{AY'}dt\right)\right)^2 \\
&\quad \times \left(\left\{k(r) - \int \left[\left(\frac{A'}{Y'} + \frac{A}{2Y}\right)\exp\left(\int \frac{A'\dot{Y}}{AY'}dt\right)\right] dt\right\}\right)^2 dr^2 \\
&\quad +(Y(t,r))^2(d\theta^2 + \sin^2\theta d\phi^2). && \text{(3.17)}
\end{aligned}
$$

3.4.3 Bernoulli equation

Equation (3.11) becomes a Bernoulli equation if we set

$$A' + \frac{AY'}{2Y} = 0. \tag{3.18}$$

By integrating this equation we get the following result

$$Y = \frac{C_1(t)}{A^2}, \tag{3.19}$$

where $C_1(t)$ is a function of integration. By substituting (3.19) into (3.11) we get the equation

$$\begin{aligned} &\dot{B} - \left[\frac{3}{2}\frac{\dot{C}_1}{C_1} - 4\frac{\dot{A}}{A} + \frac{\dot{A}'}{A'}\right] B \\ &= \left[\frac{7}{2}\frac{\dot{A}\dot{C}_1}{AA'C_1} - 5\frac{\dot{A}^2}{A^2A'} - \frac{\ddot{C}_1^2}{2C_1A'} + \frac{\ddot{A}}{AA'} - \frac{\dot{C}_1^2}{4C_1^2A'} - \frac{A^6}{4C_1^2A'}\right] B^2, \end{aligned} \tag{3.20}$$

which represents a Bernoulli equation in the variable B. Integrating (3.20) we get the solution for B as

$$B = \frac{A'C_1^{\frac{3}{2}}}{A^4[\int I dt + g(r)]}, \tag{3.21}$$

where $g(r)$ is a function of integration. We also define

$$I = -\frac{7}{2}\frac{C_1^{\frac{1}{2}}\dot{A}\dot{C}_1}{A^5} + 5\frac{\dot{A}^2C_1^{\frac{3}{2}}}{A^6} + \frac{\ddot{C}_1C_1^{\frac{1}{2}}}{2A^4} - \frac{\ddot{A}C_1^{\frac{3}{2}}}{A^5} + \frac{\dot{C}_1^2}{4C_1^{\frac{1}{2}}A^4} + \frac{A^2}{4C_1^{\frac{1}{2}}} \tag{3.22}$$

for convenience. Hence, the following functions satisfy the junction condition (3.11):

$$A = A(t,r), \tag{3.23a}$$

$$B = \frac{A'C_1^{\frac{3}{2}}}{A^4[\int I dt + g(r)]}, \tag{3.23b}$$

$$Y = \frac{C_1}{A^2}. \tag{3.23c}$$

Note that $A(t,r)$ is an arbitrary function in this class of solution and once it is specified along with the integration constant C_1, then an explicit form for I can be found as shown by Thirukkanesh *et al* (2012). The line element is therefore given by

$$ds^2 = -(A(t,r))^2 dt + \left(\frac{A'C_1^{\frac{3}{2}}}{A^4[\int I dt + g(r)]}\right)^2 dr^2 + \left(\frac{C_1}{A^2}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{3.24}$$

3.4.4 Inhomogeneous Riccati equation

Equation (3.11) has an inhomogeneous Riccati equation form if we set

$$\frac{\dot{Y}'}{Y'} - \frac{A'}{A}\frac{\dot{Y}}{Y'} = 0. \tag{3.25}$$

Integrating the above equation we get

$$A = \dot{Y}\alpha(t), \tag{3.26}$$

where $\mathit{ff}(t)$ is defined as a function of integration. On substitution, (3.11) becomes

$$\dot{B} = \left[\frac{\dot{Y}(1+\alpha^2)}{2\alpha YY'} - \frac{\dot{\alpha}}{\alpha^2 Y'}\right]B^2 - \left[\dot{Y}'\alpha + \frac{\dot{Y}Y'\alpha}{2Y}\right] \tag{3.27}$$

which is an inhomogeneous Riccati equation which is difficult to solve. By placing particular restrictions on ﬀ and Y, it is possible to integrate the above equation. If we define α to be a real constant and Y to be a separable function, we get the following form

$$Y(t,r) = K(r)C(t), \tag{3.28}$$

where $K(r)$ and $C(t)$ are arbitrary functions of r and t respectively. On substitution (3.27) becomes

$$\dot{B} = \frac{(1+\alpha^2)\dot{C}}{2\alpha K'}\frac{\dot{C}}{C^2}B^2 - \frac{3}{2}\alpha K'\dot{C}. \tag{3.29}$$

To write the Riccati equation (3.29) in standard form, we introduce the following transformation

$$B = wC \tag{3.30}$$

so that it reduces to

$$\left[\frac{2\alpha K'}{(1+\alpha^2)w^2 - 2\alpha K'w - 3\alpha^2 K'^2}\right]\dot{w} = \frac{\dot{C}}{C}. \tag{3.31}$$

Since (3.31) is a separable equation in the variables w and C, it can be integrated if the constant α is defined. Taking $\alpha = -2$, then (3.31) becomes

$$\frac{\dot{w}}{(5w - 6K')(w + 2K')} = -\frac{1}{4K'}\frac{\dot{C}}{C}. \tag{3.32}$$

Integrating the above equation we get

$$w = \frac{2K'[3C^4 + f(r)]}{5C^4 - f(r)}, \tag{3.33}$$

where $f(r)$ is the function of integration. The new solution of the inhomogenous Riccati equation (3.27) is given by

$$A = -2K\dot{C}, \tag{3.34a}$$

$$B = \frac{2K'C[3C^4 + f(r)]}{5C^4 - f(r)}, \tag{3.34b}$$

$$Y = KC, \tag{3.34c}$$

where functions K, C and f are arbitrary. The line element is therefore given by

$$\begin{aligned} ds^2 &= (2K\dot{C})^2 dt \\ &+ \left(\frac{2K'C[3C^4 + f(r)]}{5C^4 - f(r)}\right)^2 dr^2 + (KC)^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{3.35}$$

3.5 New Solution

Equation (3.11) is an inhomogeneous Riccati equation if we set

$$\frac{\dot{Y}'}{Y'} - \frac{A'}{A}\frac{\dot{Y}}{Y'} = 0. \tag{3.36}$$

Integrating the above equation we get

$$A = \dot{Y}\alpha(t), \tag{3.37}$$

where $\alpha(t)$ is a function of integration. On substitution, (3.11) becomes

$$\dot{B} = \left[\frac{\dot{Y}(1+\alpha^2)}{2\alpha Y Y'} - \frac{\dot{\alpha}}{\alpha^2 Y'}\right] B^2 - \left[\dot{Y}'\alpha + \frac{\dot{Y}Y'\alpha}{2Y}\right] \tag{3.38}$$

which is an inhomogeneous Riccati equation which is difficult to solve. However again placing particular restrictions on ff and Y, it is possible to integrate the above equation. We take ff to be a real function and Y to be a sum of two functions of the following form

$$Y(t,r) = a(t) + b(r), \tag{3.39}$$

where $a(t)$ and $b(r)$ are arbitrary functions of k and r respectively. The class of solutions derived here are different from those described in section 3.4 as those models do not satisfy (3.39).

By introducing the transformation

$$B = wb'(r), \tag{3.40}$$

and using (3.39), (3.38) becomes

$$\dot{w} = \frac{\alpha\dot{a}}{2(a(t)+b(r))}\left[\left(\frac{(1+\alpha^2)}{\alpha^2} - \frac{2\dot{\alpha}(a(t)+b(r))}{\alpha^3}\right)w^2 - 1\right]. \tag{3.41}$$

We introduce the function $g(r)$ expressed as

$$g(r) = \frac{1+\alpha^2}{\alpha^2} - \frac{2\dot{\alpha}(a(t)+b(r))}{\alpha^3}. \tag{3.42}$$

Note $g(r)$ can be defined only on the coordinate r to complete the integration. Differentiating (3.42) with respect to r we obtain

$$g'(r) = -\frac{2\dot{\alpha}b(r)}{\alpha^3}.$$

This result implies that

$$g(r) = \gamma b(r) + \gamma_0, \tag{3.43a}$$

$$\alpha = \frac{1}{(\gamma_1 + \gamma t)^{\frac{1}{2}}}. \tag{3.43b}$$

For consistency we have α is a constant from (3.42). Then (3.41) can be expressed as

$$\dot{w} = \frac{\alpha \dot{a}(t)}{2(a(t) + b(r))} \left[\frac{1+\alpha^2}{\alpha^2} w^2 - 1 \right]. \tag{3.44}$$

We integrate (3.44) to obtain

$$w(t) = \frac{\alpha \tanh(\frac{1}{4}(-\sqrt{1+\alpha^2} a^2(t) - 2\sqrt{1+\alpha^2} a(t) b(r) - 4\alpha\sqrt{1+\alpha^2} C))}{\sqrt{1+\alpha^2}} \tag{3.45}$$

where C is the constant of integration. Substituting equation (3.45) into (3.40) results in

$$B = \left(\frac{\alpha \tanh(\frac{1}{4}(-\sqrt{1+\alpha^2} a^2(t) - 2\sqrt{1+\alpha^2} a(t) b(r) - 4\alpha\sqrt{1+\alpha^2} C))}{\sqrt{1+\alpha^2}} \right) \times b'(r). \tag{3.46}$$

The line element has the form

$$ds^2 = -(\alpha \dot{a}(t))^2 dt + (\varpi)^2 dr^2 + (a(t) + b(r))^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{3.47}$$

where

$$\varpi = \left(\frac{\alpha \tanh(\frac{1}{4}(-\sqrt{1+\alpha^2} a^2(t) - 2\sqrt{1+\alpha^2} a(t) b(r) - 4\alpha\sqrt{1+\alpha^2} C))}{\sqrt{1+\alpha^2}} \right) b'(r),$$

for the assumption (3.39) and the gravitational potential $A = \dot{Y}\alpha(t)$ from (3.37).

Our objective was to obtain a new solution to the boundary condition for an expanding shearing star with particles in non-geodesic motion. For the metric (3.47) we observe from (3.3a) that the acceleration vanishes; consequently the particles are travelling on geodesic paths. Even though the particle motion is geodesic this model is a new solution to the Einstein field equations. Models obtained earlier cannot be regained from the metric (3.47). We note that the solution has a simple form and the quantities $a(t)$ and $b(r)$ are arbitrary. It is possible that the other choices for the potential $Y(t, r)$ may lead to an accelerating model; this is an objective of future research.

We can now compute the dynamical quantities from the Einstein field equation

(3.7). We obtain the expressions

$$
\begin{aligned}
\rho &= \frac{2}{(\alpha\dot{a}(t))^2}\frac{\zeta}{\varpi}\frac{\dot{a}(t)}{a(t)+b(r)} + \frac{1}{(a(t)+b(r))^2} \\
&\quad + \frac{1}{(\alpha\dot{a}(t))^2}\frac{\dot{a}^2(t)}{(a(t)+b(r))^2} \\
&\quad - \frac{1}{\varphi}\left(2\frac{b''(r)}{a(t)+b(r)} - 2\frac{\nu}{\varpi}\frac{b'(r)}{a(t)+b(r)}\right), \qquad (3.48a)\\
p_r &= \frac{1}{(\alpha\dot{a}(t))^2}\left(-2\frac{\ddot{a}(t)}{a(t)+b(r)} - \frac{\dot{a}^2(t)}{(a(t)+b(r))^2} + 2\frac{\alpha\ddot{a}(t)}{\alpha\dot{a}(t)}\frac{\dot{a}(t)}{a(t)+b(r)}\right) \\
&\quad + \frac{1}{\varphi}\left(\frac{b'^2(r)}{(a(t)+b(r))^2} + 2\frac{(\alpha\dot{a}(t))^2}{\alpha\dot{a}(t)}\frac{b'(r)}{a(t)+b(r)}\right) - \frac{1}{(a(t)+b(r))^2}, \qquad (3.48b)\\
p_t &= -\frac{1}{(\alpha\dot{a}(t))^2}\left(\frac{\vartheta}{\varpi} - \frac{\alpha\ddot{a}(t)}{\alpha\dot{a}(t)}\frac{\zeta}{\varpi} + \frac{\zeta}{\varpi}\frac{\dot{a}(t)}{a(t)+b(r)}\right) \\
&\quad - \frac{1}{(\alpha\dot{a}(t))^2}\left(-\frac{\alpha\ddot{a}(t)}{\alpha\dot{a}(t)}\frac{\dot{a}(t)}{a(t)+b(r)} + \frac{\ddot{a}(t)}{a(t)+b(r)}\right) \\
&\quad - \frac{1}{\varphi}\left(\frac{\nu}{\varpi}\frac{b'(r)}{a(t)+b(r)}\right) \\
&\quad + \frac{1}{\varphi}\left(\frac{b''(r)}{a(t)+b(r)}\right), \qquad (3.48c)\\
q &= -\frac{2}{(\alpha\dot{a}(t))\varpi^2}\left(\frac{\zeta}{\varpi}\frac{b'(r)}{a(t)+b(r)}\right), \qquad (3.48d)
\end{aligned}
$$

in terms of the constants α, C and the arbitrary functions $a(t)$ and $b(r)$. We have introduced the new quantities ζ, ϑ, ν and φ for convenience. These are defined by:

$$
\begin{aligned}
\zeta &= \frac{1}{4\sqrt{1+\alpha^2}} \\
&\times \alpha \operatorname{sech}\left(\frac{1}{4}\left(-\sqrt{1+\alpha^2}a^2(t) - 2\sqrt{1+\alpha^2}a(t)b(r) - 4\alpha\sqrt{1+\alpha^2}C\right)\right)^2 \\
&\times(-2\sqrt{1+\alpha^2}a(t)a'(t) - 2\sqrt{1+\alpha^2}b(r)a'(t))b'(r), \qquad (3.49a)
\end{aligned}
$$

$$
\begin{aligned}
\vartheta &= -\frac{b'(r)}{8\sqrt{1+\alpha^2}} \\
&\times \alpha \operatorname{sech}\left(\frac{1}{4}\left(-\sqrt{1+\alpha^2}a^2(t) - 2\sqrt{1+\alpha^2}a(t)b(r) - 4\alpha\sqrt{1+\alpha^2}C\right)\right)^2 \\
&\times \tanh\left(\frac{1}{4}\left(-\sqrt{1+\alpha^2}a^2(t) - 2\sqrt{1+\alpha^2}a(t)b(r) - 4\alpha\sqrt{1+\alpha^2}C\right)\right) \\
&\times \left(-2\sqrt{1+\alpha^2}a(t)a'(t) - 2\sqrt{1+\alpha^2}b(r)a'(t)\right)^2 + \frac{b'(r)}{4\sqrt{1+\alpha^2}} \\
&\times \alpha \operatorname{sech}\left(\frac{1}{4}\left(-\sqrt{1+\alpha^2}a^2(t) - 2\sqrt{1+\alpha^2}a(t)b(r) - 4\alpha\sqrt{1+\alpha^2}C\right)\right)^2 \\
&\times(-2\sqrt{1+\alpha^2}a'^2(t) - 2\sqrt{1+\alpha^2}a(t)a''(t) - 2\sqrt{1+\alpha^2}b(r)a''(t)), \qquad (3.49b)
\end{aligned}
$$

$$
\begin{aligned}
\nu &= -\frac{1}{2}\alpha a(t) \\
&\times \left(\operatorname{sech}\left(\frac{1}{4}\left(-\sqrt{1+\alpha^2}a^2(t) - 2\sqrt{1+\alpha^2}a(t)b(r) - 4\alpha\sqrt{1+\alpha^2}C\right)\right)^2 b'(r)\right) b'(r) \\
&+ \left(\frac{\alpha \tanh(\frac{1}{4}\left(-\sqrt{1+\alpha^2}a^2(t) - 2\sqrt{1+\alpha^2}a(t)b(r) - 4\alpha\sqrt{1+\alpha^2}C)\right)}{\sqrt{1+\alpha^2}}\right) \\
&\times b''(r), \qquad (3.49c)
\end{aligned}
$$

$$
\begin{aligned}
\varphi &= \left(\frac{\alpha \tanh(\frac{1}{4}\left(-\sqrt{1+\alpha^2}a^2(t) - 2\sqrt{1+\alpha^2}a(t)b(r) - 4\alpha\sqrt{1+\alpha^2}C)\right)}{\sqrt{1+\alpha^2}}\right)^2 \\
&\times b'^2(r). \qquad (3.49d)
\end{aligned}
$$

A physical analysis of the variables in (3.48)-(3.49) will be pursued in the future.

We have checked the accuracy of the results in this chapter with the help of Mathematica (Wolfram 2008).

We observe that the quantities α, C, $a(t)$ and $b(r)$ are arbitrary quantities in the solution given above. Particular choices simplify the form of the solution. We illustrate this by making the choice

$$\alpha = 1, \tag{3.50a}$$

$$C = 1, \tag{3.50b}$$

$$a(t) = t, \tag{3.50c}$$

$$b(r) = r, \tag{3.50d}$$

without losing generality. Then the matter variables (3.49) can be written in the simpler form

$$
\begin{aligned}
\rho &= 2\frac{\zeta}{\varpi}\frac{1}{t+r} + \frac{2}{(t+r)^2} \\
&\quad + \frac{1}{\varphi}\left(2\frac{\nu}{\varpi}\frac{1}{t+r}\right), &\text{(3.51a)} \\
p_r &= -\frac{2}{(t+r)^2} \\
&\quad + \frac{1}{\varphi}\left(\frac{1}{(t+r)^2} + 2\frac{1}{t+r}\right), &\text{(3.51b)} \\
p_t &= -\left(\frac{\vartheta}{\varpi} + \frac{\zeta}{\varpi}\frac{1}{t+r}\right) \\
&\quad - \frac{1}{\varphi}\left(\frac{\nu}{\varpi}\frac{1}{t+r}\right), &\text{(3.51c)} \\
q &= -\frac{2}{\varpi^2}\left(\frac{\zeta}{\varpi}\frac{1}{t+r}\right), &\text{(3.51d)}
\end{aligned}
$$

which has a simpler form than the expressions given earlier.

The quantities ζ, ϑ, φ and ν also take on a simple form. They are given by

$$
\begin{aligned}
\zeta &= \frac{1}{4\sqrt{2}} \\
&\quad \times \operatorname{sech}\left(\frac{1}{4}\left(-\sqrt{2}t^2 - 2\sqrt{2}tr - 4\sqrt{2}\right)\right)^2 \\
&\quad \times(-2\sqrt{2}t - 2\sqrt{2}r, \qquad \text{(3.52a)}
\end{aligned}
$$

$$
\begin{aligned}
\vartheta &= -\frac{1}{8\sqrt{2}} \\
&\quad \times \operatorname{sech}\left(\frac{1}{4}\left(-\sqrt{2}t^2 - 2\sqrt{2}tr - 4\sqrt{2}\right)\right)^2 \\
&\quad \times \tanh\left(\frac{1}{4}\left(-\sqrt{2}t^2 - 2\sqrt{2}tr - 4\sqrt{2}\right)\right) \\
&\quad \times \left(-2\sqrt{2}t - 2\sqrt{2}r\right)^2 + \frac{1}{4\sqrt{2}} \\
&\quad \times \operatorname{sech}\left(\frac{1}{4}\left(-\sqrt{2}t^2 - 2\sqrt{2}tr - 4\sqrt{2}\right)\right)^2, \qquad \text{(3.52b)}
\end{aligned}
$$

$$
\begin{aligned}
\nu &= -\frac{t}{2} \\
&\quad \times \left(\operatorname{sech}\left(\frac{1}{4}\left(-\sqrt{2}t^2 - 2\sqrt{2}tr - 4\sqrt{2}\right)\right)^2\right), \qquad \text{(3.52c)}
\end{aligned}
$$

$$
\varphi = \left(\frac{\tanh(\frac{1}{4}\left(-\sqrt{2}t^2 - 2\sqrt{2}tr - 4\sqrt{2}\right))}{\sqrt{2}}\right)^2. \qquad \text{(3.52d)}
$$

Chapter 4

Conclusion

In this thesis we have studied radiating relativistic stars which are expanding, accelerating and shearing. We found that the boundary condition, matching the interior spacetime to the exterior spacetime, can be written as a Riccati equation in general. This is the fundamental equation that governs the radiating star in spherical symmetry. The goal of this thesis was to generate a new solution for the Riccati equation by assuming that the gravitational potential possesses a non-separable form. We first considered the geodesic case with particles travelling with constant velocity. Exact solutions were reviewed for various forms of the gravitational potential and we regained the models of Thirukkanesh and Maharaj (2010). A new solution was generated which results in a new class of Einstein field equations and line element.

We then used the same analogy for the non-geodesic case with accelerating particles. We solved the Riccati equation by using a particular form of the gravitational potential. Exact solutions were reviewed by regaining linear, Bernoulli and inhomogeneous Riccati equations. We regained the known models of Thirukkanesh and Maharaj (2012). A new (non-shearing) solution was found which results in a new class of Einstein field equations and line metric.

We now present an overview of the main results which are achieved during the course of this thesis:

In chapter 2 we considered the case of geodesic motion for the fluid particles. We derived the new line element

$$\begin{aligned} ds^2 &= -dt^2 \\ &\quad + \left(\left(\frac{1-\exp(tb(r)+2D+ta_0+\frac{t^2a_1}{2}+tb(r)a_1^2+ta_0a_1^2+\frac{1}{2}t^2a_1^3)}{1+\exp(tb(r)+2D+ta_0+\frac{t^2a_1}{2}+tb(r)a_1^2+ta_0a_1^2+\frac{1}{2}t^2a_1^3)} \right) b'(r) \right)^2 dr^2 \\ &\quad +(a_0+a_1t+b(r))^2(d\theta^2+\sin^2\theta d\phi^2), \end{aligned}$$

with the assumption $Y(t,r)=a(t)+b(r)$. If we let

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \end{aligned}$$

then the above metric takes the simple form

$$
\begin{aligned}
ds^2 &= -dt^2 \\
&\quad + \left(\left(\frac{1-\exp(tb(r)+2D+\frac{t^2}{2}+tb(r)+\frac{1}{2}t^2)}{1+\exp(tb(r)+2D+\frac{t^2}{2}+tb(r)+\frac{1}{2}t^2)} \right) b'(r) \right)^2 dr^2 \\
&\quad + (t+b(r))^2(d\theta^2+\sin^2\theta d\phi^2).
\end{aligned}
$$

This is a new solution which has not been published elsewhere and it is not contained in the papers of Naidu *et al* (2006), Rajah and Maharaj (2008) and Thirukkanesh and Maharaj (2009, 2010). The simple form of the solution allows us to present expressions for the matter variables energy density, radial and tangential pressures, and heat flux.

In Chapter 3 we considered the case of non-geodesic motion for the fluid particles. We derived the new line element

$$ds^2 = -(\alpha\dot{a}(t))^2 dt + (\varpi)^2 dr^2 + (a(t)+b(r))^2(d\theta^2+\sin^2\theta d\phi^2),$$

where

$$\varpi = \left(\frac{\alpha\tanh(\frac{1}{4}(-\sqrt{1+\alpha^2}a^2(t) - 2\sqrt{1+\alpha^2}a(t)b(r) - 4\alpha\sqrt{1+\alpha^2}C))}{\sqrt{1+\alpha^2}} \right) b'(r),$$

with the assumption $Y(t,r) = a(t) + b(r)$. If we let

$$\begin{aligned} a(t) &= t, \\ b(r) &= r, \\ C &= 1, \\ \alpha &= 1, \end{aligned}$$

then the above metric takes the form

$$ds^2 = -dt + (\varpi)^2 dr^2 + (t+r)^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where

$$\varpi = \left(\frac{\tanh(\frac{1}{4}(-\sqrt{2}t^2 - 2\sqrt{2}tr - 4\sqrt{2}))}{\sqrt{2}} \right).$$

This is a new solution, it has not been published elsewhere, and it is not contained in the papers of Herrera and Santos (2010), Govender *et al* (2010) and Thirukkanesh *et al* (2012). We observe that we were seeking a non-geodesic solution but our solution turns out to be geodesic. However, we note that this still constitutes a new solution. This solution allows us to find simple expressions for the matter variable: energy density, radial and tangential pressures, and heat flux.

In summary, we have generated a new solution of a radiating star when the interior expanding, shearing fluid particles are moving in geodesic motion when the condition of separability in the metric function has been relaxed. Similarly a second new solution has been found, when considering the non-geodesic case, where the gravitational potential has a non-separable form. Therefore two new classes of solutions, satisfying their respective boundary conditions, have been found.

The new exact solutions for geodesic fluids make it possible to investigate the physical features of the model such as luminosity, rate of collapse, particle production, neutrino flux, and temperature profiles as demonstrated by Thirukkanesh and Maharaj (2010). To determine particular explicit forms for the causal temperature, we need to solve the Maxwell-Catteneo heat transport equation

$$\tau h_a^b \dot{q}_a + q_a = -\kappa(h_a^b \nabla_b T + T \dot{u}_a).$$

This is a causal transport equation and generalises the Fourier law as indicated in Naidu *et al* (2006) and Maharaj (2008). This will be pursued in the future.

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