Applications of Embedding Theory in Higher Dimensional General Relativity

by

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As the candidate's supervisor I have approved this thesis for submission.

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Abstract

The study of embeddings is applicable and significant to higher dimensional theories of our universe, high-energy physics and classical general relativity. In this thesis we investigate local and global isometric embeddings of four-dimensional spherically symmetric spacetimes into five-dimensional Einstein manifolds. Theorems have been established that guarantee the existence of such embeddings. However, most known explicit results concern embedded spaces with relatively simple Ricci curvature. We consider the four-dimensional gravitational field of a global monopole, a simple non-vacuum space with a more complicated Ricci tensor, which is of theoretical interest in its own right, and occurs as a limit in Einstein-Gauss-Bonnet Kaluza-Klein black holes, and we obtain an exact solution for its embedding into Minkowski space. Our local embedding space can be used to construct global embedding spaces, including a globally flat space and several types of cosmic strings. We present an analysis of the result and comment on its significance in the context of induced matter theory and the Einstein-Gauss-Bonnet gravity scenario where it can be viewed as a local embedding into a Kaluza-Klein black hole. Difficulties in solving the five-dimensional equations for given four-dimensional spaces motivate us to investigate which embedded spaces admit bulks of a specific type. We show that the general Schwarzschild-de Sitter spacetime and the Einstein Universe are the only spherically symmetric spacetimes that can be embedded into an Einstein space with a particular metric form, and we discuss their five-dimensional solutions. Furthermore, we determine that the only spherically symmetric spacetime in retarded time coordinates that can be embedded into a particular Einstein bulk is the general Vaidya-de Sitter solution with constant mass. These analyses help to provide insight to the general embedding problem. We also consider the conformal Killing geometry of a five-dimensional Einstein space that embeds a static spherically symmetric spacetime, and we show how the Killing geometry of the embedded space is inherited by its bulk. The study of embedding properties such as these enables a deeper mathematical understanding of higher dimensional cosmological models and is also of physical interest as conformal symmetries encode conservation laws.

Preface

The work described in this thesis was carried out in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, from January 2009 to April 2012. This thesis was completed under the supervision of Dr. Gareth Amery.

The research contained in this thesis represents original work done by the author and has not been submitted in any form for any degree or diploma to another tertiary institution. This thesis as a whole has not been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

J. Moodley

_____

2012

DECLARATION 1 - PLAGIARISM

I, <u>Jothi Moodley</u>, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

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DECLARATION 2 - PUBLICATIONS

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part of and/or include research presented in this thesis.

Publication 1:
Moodley J. and Amery G., An investigation of embeddings for spherically symmetric spacetimes into Einstein manifolds, *Pramana - Journal of Physics* **77**, 3, 533 (2011).
The candidate performed most of the calculations, which were checked by the supervisor. The analysis and discussion of results were jointly done. The paper was mostly written by the candidate with some advice from the supervisor. This publication consists of research presented in chapter 4, sections 4.2–4.4.

Publication 2:
Amery G., Moodley J. and Londal J. P., Isometric embeddings in cosmology and astrophysics, *Pramana - Journal of Physics* **77**, 3, 415 (2011).
The candidate performed some calculations pertaining to the section on Killing geometry inheritance, and the discussion of those results, as well as the write-up, were jointly done. This work is presented in chapter 5. The example provided in this paper for embedding Ricci flat cosmic string exterior metrics is reproduced in chapter 2, section 2.5.

Publication 3:
Moodley J. and Amery G., Gravitational field of a four-dimensional global monopole embedded in a five-dimensional vacuum, *To be submitted to Pramana – Journal of Physics* (2013).
The candidate performed most of the calculations, which were checked by the supervisor. The analysis and discussion of results were jointly done. The paper is mostly being written by the candidate with input from the supervisor. This research is the focus of chapter 3.

Publication 4:

Amery G. and Moodley J., Exact solutions for isometric embeddings of pseudo-Riemannian manifolds, Proceedings of the International Conference on Gravitation and Cosmology (December 2011, India), *Submitted to J. Phys.: Conf. Ser.* (2012).

This paper proceeds from a poster designed by the candidate with some input from the supervisor, and was presented at the International Conference on Gravitation and Cosmology. The paper was jointly written, and consists of research presented in chapters 3 and 4.

Publication 5:

Moodley J. and Amery G., An embedding analysis for spherically symmetric spacetimes in retarded time coordinates, *In preparation* (2013).

The candidate computed the results, which were discussed with the supervisor. The paper is mostly being written by the candidate with some input from the supervisor. This work is presented in chapter 4, section 4.6.

Publication 6:

Okelola M., Govinder K. S., Amery G. and Moodley J., *In preparation* (2013).

The focus of this publication is a solution presented in chapter 4, section 4.4.2. The Lie analysis work was carried out by Okelola and Govinder. The physical interpretation of the result was done by the candidate and the supervisor. The paper is being written by all four authors.

Publication 7:

Moodley J. and Amery G., Global embeddings of pseudo-Riemannian spaces, *Submitted to J. Math. Phys.* (2012).

The publication involves a revision of results arising from the candidate's M.Sc. study. Although it is not part of the candidate's Ph.D. study, it is included in this list since it is presented as the appendix of this thesis for the reader's interest. That work as well as its write-up were jointly done.

Signed: _____

Acknowledgements

Contents

Chapter 1

Introduction

Through Einstein's theory of general relativity it has become widely accepted that our universe is well-modelled by a four-dimensional spacetime — comprising of three spatial and one temporal dimensions (Einstein 1920). This idea provides useful descriptions of objects and phenomena that are of astrophysical and cosmological interest, for example, black holes, topological defects and various cosmologies. A major unsolved problem in physics is the unification of the known forces, and over the past century attempts have been made to understand this problem with the use of more than four dimensions. In the 1920's Kaluza (1921) and Klein (1926) tried to unify general relativity and electromagnetism by proposing the existence of a fifth dimension that was compactified (i.e. small and 'curled up'), and therefore unnoticeable. Later, string theory (Green et al. 1987; Becker et al. 2007; Vilenkin and Shellard 1994) emerged as a way to combine gravity and quantum theory, and posits that fundamental particles of matter are characterized by one-dimensional strings with vibrational patterns. It gave rise to five superstring theories that each specify ten dimensions for the universe. The theory of supergravity involves the problem of unifying general relativity and supersymmetry. P-branes, objects having length in p dimensions (Green et al. 1987), were found to be possible solutions in supergravity theory in eleven dimensions. It was then realized that the five superstring theories and supergravity are just different representations of an underlying eleven-dimensional theory called M-theory.

In Horava-Witten (1996) theory, the dimensionality is reduced to five by compactifying six of the eleven dimensions. This has led to a great deal of interest in five-dimensional brane-world models such as those of Arkani-Hamed-Dimopoulos-Dvali (1998; 1999), Randall-Sundrum (1999a; 1999b) and Dvali-Gabadadze-Porrati (2000). A brane-world is a particular three-dimensional spatial or four-dimensional spacetime hypersurface (the brane) embedded into a higher dimensional space, referred to as the bulk, where gravity propagates freely. Randall and Sundrum (1999a,b) considered a four-dimensional Riemannian spacetime embedded in a five-dimensional Anti-de Sitter bulk ($\text{AdS}_{(5)}$) where the cosmological constant is negative. Dvali et al. (2000) presented a model in which four-dimensional Newtonian gravity emerges on a three-dimensional brane embedded in five-dimensional Minkowski space where the extra dimension is infinite. Other higher dimensional theories include induced matter theory (Wesson and Ponce de Leon 1992; Wesson et al. 1996; Overduin and Wesson 1997; Wesson 1999), where matter is described as arising from higher dimensional effects, D-brane

models (Polchinski 1995, 1997; Johnson 2003) of high-energy physics, and Einstein-Gauss-Bonnet (EGB) gravity (Dadhich 2005, 2007), which is a modified view of gravity involving higher order metric derivatives. Thus, the study of higher dimensional theories is a popular and fertile field with the aim of improving our understanding of the universe.

The concept of an embedding is a significant aspect in the study of higher dimensions. To proceed with this study, one requires an understanding of how to embed a manifold into another manifold of higher dimension, and so it has led to a renewal of interest in embedding theory. An embedding is a map of a manifold to the higher dimensional space in which it is embedded, with the conditions that it is a homeomorphism onto its image and that there is a one-to-one relationship between the tangent spaces of the two manifolds (Goenner 1980). The lower dimensional space is referred to as the embedded space and the higher dimensional space as the embedding space or the bulk. The embedding can be local, where one maps neighbourhoods in the embedded space, or global, where one maps all of the embedded space. In addition, this mapping may be isometric (distance preserving) and/or analytic.

Since the nineteenth century isometric embeddings into higher dimensions have been explored extensively in geometry (Janet 1926; Cartan 1927; Friedman 1961; Nash 1954, 1956; Clarke 1970; Greene 1970; Greene and Jacobowitz 1971; Gunther 1989; Campbell 1926; Magaard 1963; Anderson and Lidsey 2001; Dahia and Romero 2002a,b; Anderson et al. 2003), with more focus on non-Euclidean spaces in recent times. The main idea in these studies was to determine the minimum number of extra dimensions needed to isometrically embed a pseudo-Riemannian manifold into a higher dimensional manifold. This minimum value is referred to as the codimension. In the case of Euclidean embedding spaces, the codimension is typically large (Stephani et al. 2003). However, when curvature is introduced in the bulk, the codimension is drastically reduced to one. The Campbell-Magaard (1926; 1963) theorem provided the first existence result for a local isometric embedding into a non-Euclidean space, and it has led to several generalizations (Anderson and Lidsey 2001; Dahia and Romero 2002a,b; Anderson et al. 2003). Of particular importance are the theorems given by Dahia and Romero which prove that there exists a local isometric embedding of any analytic pseudo-Riemannian manifold into any Einstein space (2002a), and also into a more general pseudo-Riemannian space (2002b). Furthermore, it has been shown (Katzourakis 2005b; Moodley and Amery 2012) that global embeddings can be constructed from these local ones. This particular construction of a global embedding relies significantly on the relevant local embeddings, and so it motivates one to explicitly determine local embeddings for spaces of interest.

In this thesis we are concerned with codimension-one isometric embeddings between pseudo-Riemannian manifolds. The technique used to determine the embedding space for a given pseudo-Riemannian manifold is to solve a form of the higher dimensional field equations known as the Gauss, Codazzi and propagation equations, such that the higher dimensional metric reduces to the given lower dimensional metric along a hypersurface. This hypersurface is taken to be one in which the coordinate of the extra dimension is a constant; so it is orthogonal to the extra dimension. The extrinsic curvature of a manifold is a second order tensor that measures the curvature of a manifold in relation to the space in which it is embedded. The Gauss and Codazzi

equations are expressed in terms of the extrinsic curvature of the embedded space, and Dahia and Romero (2002a) prove that these equations need only be solved on the hypersurface we embed. The propagation equation specifies the rest of the bulk via the higher dimensional metric.

Despite results for the existence of embeddings, the embedding equations, even for specific cases, are not so easy to solve. Explicit solutions have been found for the embeddings of Einstein spaces into Einstein (Anderson and Lidsey 2001) and Ricci flat (Lidsey et al. 1997) spaces, for embedding n-dimensional plane wave backgrounds and Ricci flat spacetimes into $(n+1)$-dimensional manifolds sourced by massless scalar fields (Anderson and Lidsey 2001), and for embedding n-dimensional Einstein spaces into $(n + 1)$-dimensional manifolds sourced by self-interacting scalar fields (Anderson et al. 2003). It should be stressed that embedding theory is complementary to a brane-world perspective in which some global metric and topology are assumed — often with a (possibly unphysical (Amery et al. 2011)) δ-function energy-momentum tensor. From an embedding perspective, in order to study a connection to four-dimensional gravity, one should embed the perturbed metric.

The increased attention given to higher dimensional models indicates that it has become important to obtain descriptions of objects (for example, stars, black holes and topological defects) that are astrophysically and theoretically interesting in the context of higher dimensions — cf. astrophysically derived constraints on Z_2-symmetric higher dimensional models (Deruelle and Katz 2001; Londal 2005; Amery et al. 2011). In addition to 'stacking' type results (such as the 'black string' (Horowitz and Strominger 1991; Gregory and Laflamme 1993, 1994)) and model dependent numerical analyses (such as that of Wiseman (2002)), some work has been done on embedding, for example, the exterior Schwarzschild black hole (Dahia and Romero 2002a), and other non-topological vacuum spacetimes into Einstein (Anderson and Lidsey 2001) and Ricci flat (Lidsey et al. 1997) spaces, and embedding Friedmann-Lemaître-Robertson-Walker models of our universe into flat space (Ponce de Leon 1988; Wesson 1992, 1994). This is relatively easy as the embedded manifolds have comparatively simple Ricci curvature, which results in commensurately simpler embedding equations. The exterior field to a gauge topological defect (such as a local string) with a δ-function energy-momentum may be simply embedded (Amery et al. 2011; Anderson and Lidsey 2001). Spacetimes having non-trivial Ricci curvature, such as global topological defects, are more difficult to deal with.

The overall objective in this thesis is to apply the Dahia-Romero (2002a) theorem to investigate isometric embeddings of four-dimensional (4D) spherically symmetric (SS) spacetimes into five-dimensional (5D) Einstein spaces, and to study the geometries of the resulting embedding spaces. Spherically symmetric spacetimes provide good descriptions of many structures in our universe, and the embeddings of these astrophysical/cosmological objects into higher dimensions may yield further insight into the properties of such objects. We focus our attention on Einstein spaces because of their role in high-energy physics (Randall and Sundrum 1999a,b; Howe et al. 1998; Lavrinenko et al. 1998), and since it is a reasonable place to start.

First we choose to embed the gravitational field of a four-dimensional global monopole, motivated (in part) by the fact that it represents one of the simpler spacetimes with non-trivial energy-momentum. It is therefore of mathematical interest as

a case study for embedding more complex spacetimes. Despite controversy about stability (Vilenkin and Shellard 1994; Goldhaber 1989; Rhie and Bennett 1991; Perivolaropoulos 1992) and severe cosmological constraints on their number density (Vilenkin and Shellard 1994), global monopoles, as examples of topological defects, are interesting in their own right since they are common in symmetry breaking theories of high-energy physics and cosmology (Vilenkin and Shellard 1994; Kibble 1976; Vilenkin 1985; Barriola and Vilenkin 1989; Sakellariadou 2007; Bezerra de Mello 2001; Bronnikov et al. 2002). The global monopole metric is also of theoretical interest appearing as the $r \to 0$ limit of a Kaluza-Klein black hole solution in EGB gravity (Maeda and Dadhich 2006). There it regularizes the metric and weakens the singularity sufficiently for mass to vanish at $r = 0$. The natural appearance of this metric in this context may well be at least as significant physically as the original context for which it is named. In the context of EGB gravity, by embedding the global monopole metric, we are embedding the Gauss-Bonnet-weakened singularity of a Kaluza-Klein black hole (Maeda and Dadhich 2006). Once a local embedding is determined, it can be used to build various global embedding spaces (noting that the global monopole metric is only valid for $r > r_c$).

Next we focus on the general embedding problem for spherically symmetric spacetimes. Since the usual method of solving the five-dimensional metric for a chosen four-dimensional space leads to difficulties, we take a slightly different approach in order to gain insight into this problem. We restrict the metric of the Einstein bulk to a particular form, and we investigate the kind of spherically symmetric spacetimes that may embed into it. We also consider spherically symmetric spacetimes in retarded time coordinates, which describe radiating solutions such as the Vaidya-de Sitter model, and examine the embeddings of these spacetimes into Einstein bulks with special metric types.

Another objective in this thesis is to study the conformal and Killing geometries of a five-dimensional Einstein bulk that embeds a static spherically symmetric (SSS) spacetime. Conformal and Killing geometries are important tools in general relativity that can help cast light on the structure of a spacetime such as its symmetries, and provide information on conservation laws. Thus, it is of both mathematical and astrophysical interest to investigate the relationship between the conformal Killing geometries of embedded and embedding spaces.

The scope of this thesis is as follows.

- Chapter 2: In this chapter we present the framework for isometric embeddings of pseudo-Riemannian manifolds. We begin by reviewing some basic mathematical tools pertinent to this study. In particular we provide definitions of an isometry, the pseudo-Riemannian line element and the curvature tensors derived from it, and highlight the Einstein field equations of general relativity. Extrinsic curvature is an important aspect of an embedding, and so we explain this concept along with intrinsic curvature. Local and global isometric embeddings are formally defined. We introduce the Gauss, Codazzi and Ricci equations for embeddings with arbitrary codimension, and also explain the meaning of a rigid embedding. We provide a simple background of embedding theorems and particular results, before focussing our attention on the Dahia-Romero $(2002a)$ theorem for Einstein

embedding spaces. We review the methodology explained in their theorem as we shall implement it in this thesis to treat our cases of interest. We conclude the chapter with some examples of known embeddings: the embedding of Einstein spaces into Einstein spaces, and the embedding of Ricci flat cosmic strings into Ricci flat bulks.

- Chapter 3: In this chapter we first discuss the global monopole exterior spacetime and some of its properties, and then proceed to consider the embedding of this spacetime. We obtain a Riemann flat solution that embeds the global monopole metric. We further find that the global monopole metric is the typical static spherically symmetric spacetime for embedding into a bulk of a certain form. We analyze the properties of the resulting bulk metric to gain a better understanding of the solution, and verify that the embedding space is related by a coordinate transformation to 5D Minkowski space. From a 5D perspective this embedding implies stability for the global monopole. We briefly discuss various global embeddings, including cosmic string solutions. We also discuss the relevance of the solution in the context of induced matter theory and the EGB gravity scenario where it can be interpreted as a local embedding into a Kaluza-Klein black hole. The results of this chapter are original work and are being prepared for publication (Moodley and Amery 2013*b*).

- Chapter 4: In this chapter we aim to investigate what spherically symmetric spacetimes may embed into particular 5D Einstein spaces. We treat two forms for the 5D metric. In the first form, the unknown functions depend on the extra dimension only, so that the components of this metric are separable with respect to the extra dimension. This form guarantees that the embedding is 'energetically rigid' and that the Killing geometry of the embedded space is inherited by the higher dimensional space (Londal 2005; Amery et al. 2011). The embedding analysis for this metric form leads to two kinds of four-dimensional solutions: the general Schwarzschild-de Sitter spacetime and the Einstein Universe. We present the properties of these spacetimes and discuss solutions for their embedding spaces. The second form that we choose for the bulk metric involves an unknown function that depends on the extra dimension as well as the radial coordinate. The analysis for this metric indicates that there can be no radial dependence, and so it reduces to a special case of the first bulk form. Next we discuss 4D spherically symmetric spacetimes in retarded time coordinates and investigate 5D Einstein bulk metrics that may admit embedded spaces of this form. For our choices of the bulk metric, we obtain the Vaidya-de Sitter model with constant mass as the only four-dimensional solution. Part of the original research presented in this chapter has been published (Moodley and Amery 2011), and other parts are being prepared for publication (Amery and Moodley 2012; Moodley and Amery 2013*a*; Okelola et al. 2013).

- Chapter 5: Here we study the conformal geometry when embedding 4D static spherically symmetric spacetimes into a 5D Einstein bulk with a general form. We review the properties of conformal Killing vectors and note some results for decomposable spaces. Then we outline the method we use and the equations

that must be solved to determine the conformal Killing vectors of the specified 5D bulk. We show that the Killing geometry of a static spherically symmetric spacetime is inherited by its Einstein embedding space. Other possible conformal and Killing vectors of the Einstein bulk are also discussed. It is proved that there are no hypersurface-like Killing vectors other than those inheriting the embedded Killing geometry. In our partial analysis of the general conformal geometry, we find that for a special case of the bulk, there does exist a conformal Killing vector in the direction of the extra coordinate. The results of this chapter are original work and have been published in Amery et al. (2011).

- Chapter 6: We conclude with a summary of the results produced in this study, and also comment on open problems and future work.

- Appendix A: This appendix is complementary to chapter 2 and is provided for the reader's interest and to contextualize the local results reported in this thesis. Here we present previous work carried out by the author on the construction of a global isometric embedding from given local isometric embeddings into an Einstein bulk (Moodley 2008; Moodley and Amery 2012).

We consistently adopt the following notational conventions: Roman lower case indices label the coordinates $(0, \ldots, n-1)$ of the embedded space, Roman upper case indices label its spatial coordinates $(1, \ldots, n-1)$, and Greek indices label the coordinates $(0, \ldots, m-1; m > n)$ of the embedding space. A tilde denotes quantities pertaining to the embedding space and an overbar denotes quantities obtained from the n-dimensional component of the higher dimensional metric. We use a prime and an overdot to denote partial differentiation with respect to the coordinates r and y, respectively.

Chapter 2

Embedding Theory

2.1 Introduction

Here we present the embedding results and techniques that will be applied in subsequent chapters. To begin with, in section 2.2 we review the tools of differential geometry that are essential in describing an embedding. Section 2.2.1 contains material on fundamental tensorial quantities defined on pseudo-Riemannian manifolds and the field equations of general relativity. The concepts of extrinsic and intrinsic curvature are discussed in section 2.2.2, and formal definitions of local and global embeddings are given in section 2.2.3. In section 2.2.4 we present the general embedding equations known as the Gauss, Codazzi and Ricci equations, and explain the meanings of intrinsic and energetic rigidity. A brief history of existence theorems for local and global embeddings is provided in section 2.3. In section 2.4 we review the Dahia-Romero $(2002a)$ theorem for embedding into Einstein spaces, and in section 2.5 we present some examples of known solutions: the embedding of Einstein spaces into Einstein spaces, and the embedding of Ricci flat cosmic strings into Ricci flat bulks. Appendix A is complementary to this chapter and provides further information on global embeddings.

2.2 Differential geometry

Einstein's theory of general relativity has laid the foundations for much of cosmology and astrophysics. It provides significant tools that are useful in studying embeddings. The theory presented here can be found in texts by Bredon (1997); Choquet-Bruhat et al. (1982); Nakahara (1990); Hawking and Ellis (1973); Stephani (2004); Hobson et al. (2006); Goenner (1980) and Eisenhart (1926).

2.2.1 Fundamentals

A n-dimensional C^k (respectively, C^∞) differentiable manifold is a second countable Hausdorff space M together with a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that:

- each chart is a homeomorphism $\phi_\alpha : U_\alpha \longrightarrow U'_\alpha \subset \mathbb{R}^n$ where U_α is open in M and U'_α is open in $\mathbb{R}^n$,

- each $p \in M$ is in the domain of some chart; i.e. the U_α cover M,

- for any two charts $\phi : U \longrightarrow \mathbb{R}^n$ and $\psi : V \longrightarrow \mathbb{R}^n$, the map
  $\phi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \phi(U \cap V)$ is C^k (resp., C^∞), and

- the collection of charts is maximal.

We define coordinates for a point $p \in U_\alpha \subset M$ by $x^a = u^a \circ \phi_\alpha : U_\alpha \longrightarrow \mathbb{R}$, $a = 1, \ldots, n$, where $u^a : \mathbb{R}^n \longrightarrow \mathbb{R}$. For any point p in a smooth manifold M, $T_p M$ denotes the tangent space of M at p, and is the vector space of all tangent vectors to M at p.

A n-dimensional differentiable manifold M is essentially a topological space that locally resembles n-dimensional Euclidean space $\mathbb{R}^n$, and on which points can be assigned the real coordinates $(x^1, x^2, \ldots, x^n)$. A $(n-1)$-dimensional submanifold $(n \geq 3)$ of M is known as a hypersurface. In general relativity theory we are concerned with pseudo-Riemannian manifolds endowed with the line element

$$ds^2 = g_{ab}(x^c)dx^a dx^b,$$

that measures the invariant infinitesimal distance ds between neighbouring points. A strictly Riemannian manifold has a positive definite line element. The metric tensor $\mathbf{g}$ having components g_{ab} describes the local geometry of the manifold. This symmetric tensor can be used to raise or lower indices of tensors e.g. $u_a = g_{ab}u^b$; satisfies the property $g_{ab}g^{bc} = \delta_a^c$, the Kronecker tensor; and has a vanishing covariant derivative — see equation (2.2.4). In this thesis we employ the signature convention $(-+++)$ for a four-dimensional spacetime. A pseudo-Euclidean (flat) manifold $(\mathbb{R}^n)$ has a metric of the form

$$ds^2 = \sum_{a=1}^{n} \varepsilon_a (dx^a)^2, \quad \varepsilon_a = \pm 1.$$

If $\varepsilon_a = 1$ for all a, then the space is strictly Euclidean. The 4D Minkowski spacetime of special relativity is an example of a pseudo-Euclidean manifold, and is represented by $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ in cartesian coordinates or $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ in spherical coordinates.

Curvature is an important property of a pseudo-Riemannian manifold and it can be described using the notions of extrinsic or intrinsic curvature. From the n-dimensional metric g_{ab} we may derive the following measures of intrinsic curvature of the manifold: the symmetric connection, Riemann tensor, the symmetric Ricci tensor, and the Ricci scalar, which are given respectively by:

$$\Gamma^a_{\ bc} = \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}); \tag{2.2.1}$$

$$R^a_{\ bcd} = \Gamma^a_{\ bc,d} - \Gamma^a_{\ bd,c} + \Gamma^a_{\ ed}\Gamma^e_{\ bc} - \Gamma^a_{\ ec}\Gamma^e_{\ bd}; \tag{2.2.2}$$

$$R_{ab} = \Gamma^d_{\ ad,b} - \Gamma^d_{\ ab,d} + \Gamma^d_{\ eb}\Gamma^e_{\ ad} - \Gamma^e_{\ ab}\Gamma^d_{\ ed}; \tag{2.2.3}$$

$$R = g^{ab}R_{ab}.$$

(Note that we use the above conventions about sign consistently in this thesis.) Here a comma indicates partial differentiation with respect to a coordinate. We use a semi-colon or a nabla to denote covariant differentiation. The covariant derivative of a type (r, s) tensor $\mathbf{T}$ is a type $(r, s+1)$ tensor $\nabla \mathbf{T}$ and its components are:

$$
\begin{aligned}
\nabla_d T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s; d} = {} & T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s, d} \\
& + \Gamma^{a_1}{}_{cd} T^{ca_2 \ldots a_r}{}_{b_1 \ldots b_s} + \cdots + \Gamma^{a_r}{}_{cd} T^{a_1 \ldots c}{}_{b_1 \ldots b_s} \quad (2.2.4) \\
& - \Gamma^{c}{}_{b_1 d} T^{a_1 \ldots a_r}{}_{cb_2 \ldots b_s} - \cdots - \Gamma^{c}{}_{b_s d} T^{a_1 \ldots a_r}{}_{b_1 \ldots c}.
\end{aligned}
$$

By convention, a Ricci flat manifold is one that has vanishing Ricci tensor but non-zero Riemann tensor. For a Riemann flat manifold, both the Riemann and Ricci tensors vanish. The historical relationship between intrinsic and extrinsic curvature has been that of complementary discussions as the community struggled to get to grips with pseudo-Riemannian geometry. General relativity is usually formulated from an intrinsic perspective. We defer a detailed discussion of extrinsic curvature to section 2.2.2.

Suppose f is a function between two manifolds M and N. The function f is a *homeomorphism* if

- f and its inverse f^{-1} are continuous, and

- f is bijective i.e. one-to-one and onto.

Then M is said to be homeomorphic to N. Furthermore, f is a C^k (C^∞) *diffeomorphism* if it is a homeomorphism with f and f^{-1} C^k (C^∞) differentiable. Homeomorphic spaces may be deformed from one to the other in a continuous manner, while diffeomorphic spaces may be deformed into each other smoothly. Consider pseudo-Riemannian manifolds (M, g) and $(N, \hat{g})$. A diffeomorphism $f : M \longrightarrow N$ is an *isometry* if it is metric preserving:

$$
\hat{g}_{f(p)}(f_*(V), f_*(W)) = g_p(V, W), \quad \forall \; V, W \in T_p M, \quad \forall \; p \in M,
$$

where $f_* : T_p M \longrightarrow T_p N$ is the differential map. This property can be expressed in coordinate form as

$$
\frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \hat{g}_{ab}(f(p)) = g_{ij}(p),
$$

where x^i and y^a are the coordinates of p and $f(p)$, respectively. A diffeomorphism $f : M \longrightarrow N$ is a *conformal transformation* if

$$
\hat{g}_{f(p)}(f_*(V), f_*(W)) = e^{2u(p)} g_p(V, W), \quad \forall \; V, W \in T_p M, \quad \forall \; p \in M,
$$

where u is a function on M. In coordinate form, the above condition means that

$$
\frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \hat{g}_{ab}(f(p)) = e^{2u(p)} g_{ij}(p).
$$

This type of diffeomorphism preserves the metric up to a scale. A pseudo-Riemannian manifold is *conformally flat* if it can be mapped to flat space by a conformal transformation. Two metrics h and $\hat{h}$ on M are *conformally related* if $\hat{h}_p = e^{2u(p)} h_p$, $p \in M$, for some function u, called the conformal factor, on M.

The Lie derivative is an essential tool in the study of the symmetries of a spacetime (see chapter 5) and it is defined as follows: the Lie derivative of a type (r, s) tensor $\mathbf{T}$ with respect to a vector field $\mathbf{X}$ is a type (r, s) tensor $\mathbf{L_X T}$ with components:

$$
\begin{aligned}
L_{\mathbf{X}} T^{a_1 \dots a_r}{}_{b_1 \dots b_s} &= T^{a_1 \dots a_r}{}_{b_1 \dots b_s, c} X^c \\
&\quad - T^{ca_2 \dots a_r}{}_{b_1 \dots b_s} X^{a_1}{}_{,c} - \dots - T^{a_1 \dots c}{}_{b_1 \dots b_s} X^{a_r}{}_{,c} \\
&\quad + T^{a_1 \dots a_r}{}_{cb_2 \dots b_s} X^c{}_{,b_1} + \dots + T^{a_1 \dots a_r}{}_{b_1 \dots c} X^c{}_{,b_s}.
\end{aligned}
$$

Another useful tensor is the Weyl tensor or conformal curvature tensor, which is invariant under conformal transformations, and is given by

$$
\begin{aligned}
C_{abcd} &= R_{abcd} - \frac{1}{n-2} \left(g_{ac} R_{bd} + g_{bd} R_{ac} - g_{bc} R_{ad} - g_{ad} R_{bc} \right) \\
&\quad + \frac{R}{(n-1)(n-2)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right).
\end{aligned}
$$

The classification of the Weyl tensor is relevant to the study of gravitational fields (Stephani et al. 2003). The Weyl curvature plays a significant role in the dynamics of brane-world models, where the bulk Weyl tensor can transmit non-local effects, such as tidal and gravitational wave effects, onto the brane (Maartens 2001; Maartens and Koyama 2010).

The most important formulae in general relativity are Einstein's field equations given by

$$
G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab} + \Lambda g_{ab},
$$

that relate spacetime geometry with matter and energy. The quantity T_{ab} is the energy-momentum or matter tensor, Λ is the cosmological constant, and G is the gravitational constant. The Einstein tensor G_{ab} has vanishing divergence, which implies energy-momentum conservation $T^{ab}{}_{;b} = 0$. The field equations may be solved to yield spacetime metrics. However, this is often a non-trivial task due to the non-linearity, and hence complexity, of the equations. Moreover, this problem becomes even more difficult when more than four dimensions are considered. Thus, exact solutions cannot always be determined.

In this study we are particularly interested in Einstein spaces. A n-dimensional Einstein space has the Ricci tensor and Ricci scalar

$$
R_{ab} = \frac{2\Lambda}{2-n} g_{ab},
$$
$$
R = \frac{2n\Lambda}{2-n},
$$

where Λ is the cosmological constant. The Einstein tensor is $G_{ab} = \Lambda g_{ab}$ and so Einstein spaces are empty or vacuum ($T_{ab} = 0$). The case $\Lambda = 0$ corresponds to a Ricci flat space $R_{ab} = 0$.

2.2.2 Extrinsic and intrinsic curvature

The concept of the curvature of a manifold is significant in general relativity and it can be perceived in two complementary ways: namely, intrinsic and extrinsic curvature. The intrinsic perspective of a manifold is confined to the manifold itself, whereas the extrinsic curvature of a manifold is dependent on how it is embedded in a higher dimensional space. Thus, in embedding one space into another, the extrinsic curvature will provide a description of the embedded space in relation to the embedding space. A mathematical approach to determining the curvature of a manifold is to consider the metric connection (2.2.1) and the Riemann tensor (2.2.2) defined in section 2.2.1. The Riemann tensor is a useful measure of intrinsic curvature, and a manifold is said to be intrinsically flat if $R^a{}_{bcd}$ vanishes. Extrinsic curvature can be expressed in terms of the connection $\Gamma^a{}_{bc}$. As an example (Stephani 2004), the extrinsic curvature of a three-dimensional space in a four-dimensional spacetime with metric

$$ds^2 = -\phi^2 dt^2 + g_{AB} dx^A dx^B,$$

where $\phi = \phi(t, x, y, z)$ and A, B label spatial coordinates, is given by

$$\Omega_{AB} \equiv -\frac{1}{2\phi} \frac{\partial g_{AB}}{\partial t} = -\phi \, \Gamma^0{}_{AB}.$$

Pseudo-Euclidean spaces are intrinsically and extrinsically flat. If a seemingly curved space can be transformed into a pseudo-Euclidean space globally, then the space must be intrinsically flat. To develop intuition, let us consider a cylinder with open ends whose surface is represented in cylindrical coordinates (z, ϕ) by

$$ds^2 = dz^2 + a^2 d\phi^2,$$

where a is the fixed radius. By making the coordinate transformation $x = z$, $y = a\phi$, the metric can be written as the two-dimensional Euclidean metric

$$ds^2 = dx^2 + dy^2.$$

This indicates that the surface of the cylinder is intrinsically flat, although it appears curved in three-dimensional space. With respect to its embedding space $ds^2 = dz^2 + r^2 d\phi^2 + dr^2$, it has the non-zero extrinsic curvature component

$$\Omega_{\phi\phi} = -r.$$

In a more physical sense, we observe that the cylinder can be built from a flat sheet without any distortion. This cannot be done for a spherical surface, which is both extrinsically and intrinsically curved. These notions of curvature play a key role in the embedding equations with the extrinsic curvature providing a geometrical relation between the embedded and embedding spaces — see section 2.4.

2.2.3 Definition of an embedding

Definition 1. *Suppose M is a n-dimensional analytic manifold with metric g_{ij} and N is a $(n + k)$-dimensional analytic manifold with metric $\tilde{g}_{\mu\nu}$. Then a function f : $U \subset M \longrightarrow N$, with U an open coordinated neighbourhood in M, is a local isometric embedding (Goenner 1980) if:*

1. *f is a homeomorphism onto its image,*

2. *the differential map $f_* : T_pM \longrightarrow T_{f(p)}N$ is injective (one-to-one) $\forall\, p \in U$, and*

3. *$g_p(V, W) = \tilde{g}_{f(p)}(f_*(V), f_*(W)), \;\; \forall\; V, W \in T_pM, \;\; \forall\; p \in U$.*

The last condition means that the embedding is isometric at all points of U. The embedding is analytic (or C^m, C^∞) if f is analytic (resp. C^m, C^∞). The map f : $M \longrightarrow N$ is a global isometric embedding if and only if the above three conditions hold for all points in M (Goenner 1980).

In coordinate form, f is a local isometric embedding if there exist $n + k$ differentiable functions $y^\alpha = \sigma^\alpha(x^i)$ such that the Jacobian matrix $\left\{ \frac{\partial \sigma^\alpha}{\partial x^i} \right\}$ has rank n and

$$g_{ij}(p) = \frac{\partial \sigma^\alpha}{\partial x^i} \frac{\partial \sigma^\beta}{\partial x^j} \, \tilde{g}_{\alpha\beta}(f(p)) \,,$$

where x^i, y^α denote coordinates of p, $f(p)$, respectively. The above equation is equivalent to the existence of solutions to the Gauss, Codazzi and Ricci equations (defined in section 2.2.4) for local embeddings.

2.2.4 Gauss, Codazzi and Ricci equations

Consider a n-dimensional space V_n with metric

$$ds^2_{(n)} = g_{ij}dx^i dx^j,$$

and a m-dimensional space V_m with metric

$$ds^2_{(m)} = a_{\alpha\beta}dy^\alpha dy^\beta,$$

where $m > n$ and $y^\alpha = y^\alpha(x^i)$. We want to embed V_n (the embedded space) into V_m (the embedding space).

The extrinsic curvature of V_n in V_m has the components (Eisenhart 1926)

$$\Omega^{(\sigma)}_{ij} = a_{\alpha\beta}n^{\beta(\sigma)}y^\alpha_{,i;j} \,,$$

where $n^{\beta(\sigma)}$ are the components of unit normal vectors orthogonal to V_n and each other; $\sigma = n, \ldots, m - 1$; and the terms in brackets are labels and not indices.

A derivation by Eisenhart (1926) produces three equations required for embedding V_n into V_m as a hypersurface. Note that, given V_n, these equations are equivalent to solving the field equations for V_m. The equations are known as the Gauss, Codazzi and Ricci equations and are given, respectively, by

$$R_{hijk} = \sum_{\sigma} e_{(\sigma)}[\Omega_{hj}^{(\sigma)}\Omega_{ik}^{(\sigma)} - \Omega_{hk}^{(\sigma)}\Omega_{ij}^{(\sigma)}] + \bar{R}_{\alpha\beta\gamma\delta}\, y^{\alpha}{}_{,h}y^{\beta}{}_{,i}y^{\gamma}{}_{,j}y^{\delta}{}_{,k}\,, \qquad (2.2.5)$$

$$\Omega_{ij,k}^{(\sigma)} - \Omega_{ik,j}^{(\sigma)} = \sum_{\tau} e_{(\tau)}[t_k^{(\tau\sigma)}\Omega_{ij}^{(\tau)} - t_j^{(\tau\sigma)}\Omega_{ik}^{(\tau)}] + \bar{R}_{\alpha\beta\gamma\delta}\, y^{\alpha}{}_{,i}y^{\gamma}{}_{,j}y^{\delta}{}_{,k}n^{\beta(\sigma)}, \qquad (2.2.6)$$

$$t_{j,k}^{(\tau\sigma)} - t_{k,j}^{(\tau\sigma)} = \sum_{\varrho} e_{(\varrho)}[t_j^{(\varrho\tau)}t_k^{(\varrho\tau)} - t_k^{(\varrho\tau)}t_j^{(\varrho\tau)}] + g^{lh}[\Omega_{lk}^{(\tau)}\Omega_{hj}^{(\sigma)} - \Omega_{lj}^{(\tau)}\Omega_{hk}^{(\sigma)}]$$
$$- \bar{R}_{\alpha\lambda\mu\nu}\, y^{\mu}{}_{,j}y^{\nu}{}_{,k}n^{\lambda(\sigma)}n^{\alpha(\tau)}. \qquad (2.2.7)$$

In the above, $e_{(\sigma)} = \pm 1$ and $t_j^{(\tau\sigma)}$ represents the twisting of the $n^{\alpha(\sigma)}$ vectors in relation to one another, where $\sigma, \tau = n, \ldots, m-1$, $\sigma \neq \tau$. For codimension one, the twisting vectors $t_j^{(\tau\sigma)}$ vanish. The Gauss (2.2.5) and Codazzi (2.2.6) equations must be solved on the hypersurface V_n and the Ricci equation (2.2.7) must be solved off the hypersurface. For embeddings with codimension one, the Ricci equation is void and the space-space components for the Ricci tensor for V_m are typically used as a propagation equation (Dahia and Romero 2002a). There does not exist any known general solution to these equations, so one must consider particular embedding spaces.

The embedding is described as being intrinsically rigid, if the extrinsic curvature components $\Omega_{ij}^{(\sigma)}$ and the twisting vectors $t_j^{(\tau\sigma)}$ can be written in terms of only the metric g, the Ricci tensor for g, and their derivatives (Goenner 1980). Furthermore, if these intrinsic quantities specify $\Omega_{ij}^{(\sigma)}$ and $t_j^{(\tau\sigma)}$ uniquely, then the embedded space V_n is also intrinsically rigid. For an energetically rigid embedding, the extrinsic curvature and twisting vectors depend on only the metric and Ricci tensor, and if this dependence is unique, then V_n itself is said to be energetically rigid (Goenner 1980). The concept of rigidity can be used to determine the variables upon which the extrinsic curvature depends, which allows one to make suitable assumptions for the extrinsic curvature in order to help solve the Gauss, Codazzi and Ricci equations (Londal 2005). All the explicit embeddings we consider in this thesis are rigid. We shall briefly discuss rigidity and analyze implications for Killing geometry inheritance in section 5.2.1.

2.3 Existence results

The earliest studies of local embeddings focussed on flat embedding spaces. The problem of embedding a n-dimensional Riemannian manifold locally into an Euclidean manifold was first discussed by Schläfli (1871). It was suggested that the dimension of the embedding space should be $\frac{n(n+1)}{2}$. Later, Janet (1926) and Cartan (1927) proved this true in their existence theorem for local isometric embeddings. The indefinite case was treated by Friedman (1961). Embedding locally into Euclidean spaces has been useful as a way to obtain and classify general relativistic solutions, and can provide insight into various properties of spacetimes (Stephani 1967, 1968; Stephani et al. 2003). However, there is no physical reason for preferring flat embedding spaces, and other pseudo-Riemannian manifolds, such as spaces of constant curvature (Rund 1972), have been utilized (Campbell 1926; Magaard 1963; Goenner 1980). The first local existence result for a non-Euclidean bulk was provided by the Campbell-Magaard

theorem, stated by Campbell (1926) and proved by Magaard (1963). The theorem guarantees that a Riemannian manifold has an analytic local isometric embedding into a Ricci flat space where at least one extra dimension is required. It is interesting that the presence of curvature in the bulk reduces the codimension to one. The Campbell-Magaard theorem has led to several generalizations (Anderson and Lidsey 2001; Dahia and Romero 2002a,b; Anderson et al. 2003). Anderson and Lidsey (2001) presented constructions embedding Einstein spaces into Einstein spaces and for the embedding of plane wave backgrounds and Ricci flat spacetimes into five-dimensional spacetimes sourced by massless scalar fields. It was further shown that Einstein and Ricci flat spacetimes may be embedded into spacetimes sourced by self-interacting scalar fields (Anderson et al. 2003). Dahia and Romero (2002a,b) extended the Campbell-Magaard theorem to Einstein embedding spaces, and later to more general pseudo-Riemannian manifolds.

The Dahia-Romero theorems:

- A n-dimensional pseudo-Riemannian manifold can be locally, analytically and isometrically embedded in any $(n+1)$-dimensional Einstein manifold (Dahia and Romero 2002a).

- A n-dimensional pseudo-Riemannian manifold can be locally, analytically and isometrically embedded in a $(n+1)$-dimensional pseudo-Riemannian manifold with a non-degenerate Ricci tensor which is equal, up to a local analytic diffeomorphism, to the Ricci tensor of an arbitrarily given pseudo-Riemannian manifold (Dahia and Romero 2002b).

We discuss the technique for embedding into Einstein spaces in the next section.

The meaning of the second Dahia-Romero theorem is as follows. Let $(\hat{E}, \hat{g})$ be the arbitrarily given $(n+1)$-dimensional pseudo-Riemannian manifold with non-degenerate Ricci tensor $S_{\alpha\beta}$ in a coordinate system z^σ. Denote the n-dimensional embedded space by (M, g), and the $(n+1)$-dimensional local isometric embedding space by $(\tilde{E}, \tilde{g})$ with coordinates x^γ and a point $p \in \tilde{E}$. The Ricci tensor $\tilde{R}_{\alpha\beta}$ for $\tilde{E}$ is equivalent to $S_{\alpha\beta}$:

$$\tilde{R}_{\alpha\beta}(x^\gamma) = \frac{\partial \bar{f}^\mu}{\partial x^\alpha} \frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu}(z^\sigma),$$

where $\bar{f} : \tilde{E} \longrightarrow \hat{E}$ is a local analytic diffeomorphism at p, $z^\sigma = \bar{f}^\sigma(x^\gamma)$, and $\det(\frac{\partial \bar{f}^\mu}{\partial x^\alpha})|_p \neq 0$. So M has a local isometric embedding into $\tilde{E}$, which is 'Ricci equivalent' to $\hat{E}$, and the embedding will be unique for all Ricci equivalent tensors. Note that the embedding of M into $\hat{E}$ is not isometric in general. The diffeomorphism that ensures Ricci equivalence reduces to a coordinate transformation only for Einstein embedding spaces, in which case the embedding into $\hat{E}$ is truly isometric (Amery et al. 2011).

Embedding spaces having singular energy-momentum tensors (Dahia and Romero 2004) and five-dimensional Weyl embedding spaces (Dahia et al. 2008) have been considered, and the existence of harmonic (volume minimizing) locally analytic and isometric embeddings into Ricci flat and Einstein spaces has also been established (Chervon et al. 2004).

Similarly to the local case, early work on global embeddings concentrated on Euclidean embedding spaces and involved quite high codimensions. General results include those by Nash (1954, 1956) and Kuiper (1955) and extensions by Clarke (1970); Greene (1970); Gromov (1970); Greene and Jacobowitz (1971); Gunther (1989) and Gunther (1991). Global embedding theory is useful as a way to find new solutions in general relativity (Stephani 1967, 1968). Classical relativistic applications of global embedding theory also include the maximal analytic extensions of the Schwarzschild solution given by Fronsdal (1959), and of the Reissner-Nordström and Kerr spacetimes by Plazowski (1973), as well as results by Friedman (1965) and Penrose (1965). Global embeddings provide insight into the global features of a manifold, such as causality (Clarke 1970). We note that the systematic analysis of global Euclidean embeddings of exact solutions has not yet been carried out (Stephani et al. 2003).

Global embeddings into non-Euclidean spaces have been studied recently by Katzourakis (2005b) and Moodley and Amery (2012). Essentially, Katzourakis (2005b)

partially proved that the Campbell-Magaard theorem for Ricci flat spaces can be made global and that the bulk has the topology $M \times Y$, where M is the embedded space and Y is a one-dimensional analytic manifold. The product metric for the bulk is valid only for embedded spaces that are Ricci flat or Riemann flat. This is because the local embedding equations yield a product metric only for Ricci flat or Riemann flat embedded spaces — refer to section 2.4 for the technical details. The methodology given by Katzourakis (2005b) can be improved to show that any pseudo-Riemannian manifold has a global isometric embedding, of codimension one, into an Einstein space, and also into a more general pseudo-Riemannian bulk (Moodley and Amery 2012). (Details of these proofs are presented in the appendix for the reader's interest.) These particular constructions of a global embedding rely significantly on the relevant local embeddings. Thus, one first has to determine the local embedding of a chosen spacetime before one can proceed to build its global embedding. In the next section we explain a technique that may be used to determine local isometric Einstein embeddings for spacetimes of interest.

2.4   Technique for embedding into Einstein spaces

In this thesis we focus on the Dahia-Romero (2002a) theorem which provides a way to determine an Einstein embedding and essentially involves solving the higher dimensional field equations under certain conditions. Here we briefly review the method for the Einstein embedding. The Ricci equivalent case (Dahia and Romero 2002b) is similar.

Consider a n-dimensional analytic pseudo-Riemannian manifold M with metric

$$ds^2 = g_{ik}(x^j)dx^i dx^k.$$

A theorem given by Dahia and Romero (2002a) proves that M has a local analytic isometric embedding into any $(n+1)$-dimensional Einstein manifold N with metric in Gaussian normal coordinates (without loss of generality):

$$d\tilde{s}^2 = \tilde{g}_{\alpha\beta}(x^j, y)dx^\alpha dx^\beta = \bar{g}_{ik}(x^j, y)dx^i dx^k + \epsilon(\bar{\phi}(x^j, y))^2 dy^2, \qquad (2.4.1)$$
$$\epsilon^2 = 1, \quad \bar{g}_{ik}(x^j, 0) = g_{ik},$$

along the hypersurface Σ_0, defined by $y = 0$. Here y denotes the $(n+1)$-th coordinate and

$$\tilde{R}_{\alpha\beta} = \frac{2\Lambda}{1-n}\tilde{g}_{\alpha\beta}, \quad \Lambda \in \mathbb{R}.$$

The metric component $\bar{g}_{ik}$ is a solution to the field equations for (2.4.1) given by

$$\tilde{R}_{ik} = \bar{R}_{ik} + \epsilon \bar{g}^{jm}(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - 2\bar{\Omega}_{jk}\bar{\Omega}_{im}) - \frac{\epsilon}{\bar{\phi}}\frac{\partial\bar{\Omega}_{ik}}{\partial y} + \frac{1}{\bar{\phi}}\bar{\nabla}_i\bar{\nabla}_k\bar{\phi} = \frac{2\Lambda}{1-n}\bar{g}_{ik}\,, \qquad (2.4.2)$$

$$\tilde{R}_{in} = \bar{\phi}\bar{g}^{jk}(\bar{\nabla}_j\bar{\Omega}_{ik} - \bar{\nabla}_i\bar{\Omega}_{jk}) = 0\,, \qquad (2.4.3)$$

$$\tilde{G}^n{}_n = -\frac{1}{2}\bar{g}^{ik}\bar{g}^{jm}[\bar{R}_{ijkm} + \epsilon(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - \bar{\Omega}_{jk}\bar{\Omega}_{im})] = \Lambda\,, \qquad (2.4.4)$$

subject to the condition $\bar{g}_{ik}(x^j, 0) = g_{ik}(x^j)$. The term $\bar{\Omega}_{ik}$ is the extrinsic curvature of Σ_0 and is defined by

$$\bar{\Omega}_{ik} = -\frac{1}{2\bar{\phi}}\frac{\partial\bar{g}_{ik}}{\partial y}\,, \quad \bar{\Omega}_{ik}(x^j, 0) = \Omega_{ik}\,.$$

The expression $\bar{R}_{ik}$ is the Ricci tensor derived from $\bar{g}_{ik}$. The equations (2.4.3) and (2.4.4) are, respectively, the Codazzi and Gauss equations defined in section 2.2.4. Equation (2.4.2) is referred to as the propagation equation because it is used to propagate off the hypersurface Σ_0 so as to specify the whole bulk.

Taking $\bar{\phi}$ to be arbitrary (it can be set to unity without loss of generality), and appealing to the Cauchy-Kowalewskaja theorem, it can be shown that the above system of equations admits an analytic solution $\bar{g}_{ik}$ that reduces to g_{ik} on Σ_0. Moreover, the Codazzi and Gauss equations (2.4.3) and (2.4.4) need only be solved on the hypersurface Σ_0, since, by a similar argument, solutions to these equations on the hypersurface imply that solutions exist in the bulk. Thus, the local isometric embedding is guaranteed. For a specified g_{ik} and Ω_{ik}, the embedding $f : U \subset M \longrightarrow N$ is unique. However, since there are more independent Ω_{ik} functions than relevant equations, we have more freedom in solving Ω_{ik}, and so different embedding spaces may be generated. By the Dahia-Romero (2002a) theorem, any non-degenerate Einstein embedding space may be used. The above system can be rewritten as

$$\bar{R}_{ik} + \frac{\epsilon}{2}\frac{\partial^2\bar{g}_{ik}}{\partial y^2} + \frac{\epsilon\bar{g}^{jm}}{4}\left(\frac{\partial\bar{g}_{ik}}{\partial y}\frac{\partial\bar{g}_{jm}}{\partial y} - 2\frac{\partial\bar{g}_{im}}{\partial y}\frac{\partial\bar{g}_{jk}}{\partial y}\right) = \frac{2\Lambda}{1-n}\bar{g}_{ik}\,, \qquad (2.4.5)$$

$$g^{jk}(\nabla_j\Omega_{ik} - \nabla_i\Omega_{jk}) = 0\,, \qquad (2.4.6)$$

$$R + \epsilon g^{ik}g^{jm}(\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im}) = -2\Lambda\,. \qquad (2.4.7)$$

So to determine a solution for the bulk metric (2.4.1) (with $\bar{\phi} = 1$), we must solve the propagation equation (2.4.5) for $\bar{g}_{ik}$ and the Codazzi (2.4.6) and Gauss (2.4.7) equations for Ω_{ik} such that

$$\bar{g}_{ik}(x^j, 0) = g_{ik}\,, \quad \text{and} \quad -\frac{1}{2}\frac{\partial\bar{g}_{ik}(x^j, 0)}{\partial y} = \Omega_{ik}\,.$$

The same formalism applies for locally embedding M along some other hypersurface $y = y_0$, $y_0 \in \mathbb{R}$, in N. It can further be shown that an analytic (or C^m, C^∞) global Einstein embedding exists for any analytic (resp. C^m, C^∞) pseudo-Riemannian

embedded space (refer to Appendix A and Moodley and Amery (2012) for a detailed discussion). Note that the global embedding introduces a further lack of uniqueness, since its topology and metric need only be specified near the embedded hypersurface.

Since the Cauchy-Kowalewskaja theorem plays a key role in existence results like the one reviewed here, concerns have been raised regarding physical properties, such as causality and stability, of the embedding space (Anderson 2004; Wesson 2005). In response to this problem, the theory of local Sobolev spaces is used to show that (Dahia and Romero 2005a,b) for any four-dimensional spacetime, there exist initial data sets whose Cauchy development for the Einstein vacuum equations is a five-dimensional vacuum space into which this spacetime may be locally, analytically and isometrically embedded. This ensures that we have causality and stability for both the embedded and embedding spaces. In the same papers (Dahia and Romero 2005a,b), it is shown that perturbations outside the (local) initial hypersurface do not affect the future domain of dependence, and so causality is not violated. This result also holds for embedding spaces with cosmological constants.

We note that taking $\bar{g}_{ik} = g_{ik}$ forces the extrinsic curvature $\bar{\Omega}_{ik}$ to vanish, and the embedding equations (2.4.5)–(2.4.7) become

$$R_{ik} = \frac{2\Lambda}{1-n} g_{ik},$$
$$R = -2\Lambda.$$

From the first equation above we deduce that $R = \frac{2\Lambda n}{1-n}$, and equating this expression for R with -2Λ yields $\Lambda = 0$, so that $R_{ik} = 0$. Thus, $\bar{g}_{ik}$ can equal g_{ik} only if the embedded space is Ricci flat or Riemann flat, in which case the embedding space

$$d\tilde{s}^2 = g_{ik} + \epsilon dy^2,$$

is also Ricci flat or Riemann flat, respectively. This observation is crucial in our analysis of a global embedding construction (see Appendix A and Moodley and Amery (2012)).

2.5   Examples of embeddings

We illustrate the embedding technique with some known examples. First we present the embedding of a n-dimensional Einstein space (M, g) into a $(n + 1)$-dimensional Einstein bulk, which was obtained by Lidsey et al. (1997) and Anderson and Lidsey (2001). Since M has a constant Ricci scalar $R = \frac{2n\omega}{2-n}$, where ω is the four-dimensional cosmological constant, we may set

$$\Omega_{ik} = fg_{ik}, \quad f \in \mathbb{R}.$$

The Codazzi equation (2.4.6) is trivially satisfied and the Gauss equation (2.4.7) specifies

$$f = \pm\sqrt{\frac{-2\epsilon\Lambda}{n(n-1)} + \frac{2\epsilon\omega}{(n-1)(n-2)}}.$$

Note that Λ is the five-dimensional cosmological constant. Next we assume that

$$\bar{g}_{ik} = F(y)g_{ik}\,,$$

where the function $F(y)$ must obey the initial conditions $F(0) = 1$ and $\dot{F}(0) = -2f$. Inserting this ansatz in the propagation equation (2.4.5) leads to the equation

$$\ddot{F} + \frac{n-2}{2}\frac{\dot{F}^2}{F} + \frac{4\epsilon\Lambda}{n-1}F = \frac{4\epsilon\omega}{n-2}\,,$$

which can be solved along with the initial conditions to obtain

$$F(y) = \begin{cases} \left[\cosh\left(\sqrt{\frac{-2\epsilon\Lambda}{n(n-1)}}\,y\right) + \sqrt{1+\frac{R}{2\Lambda}}\sinh\left(\sqrt{\frac{-2\epsilon\Lambda}{n(n-1)}}\,y\right)\right]^2, & \Lambda \neq 0, \\ \left(1 + \sqrt{\frac{-\epsilon R}{n(n-1)}}\,y\right)^2, & \Lambda = 0. \end{cases} \qquad (2.5.1)$$

Thus, a solution for the $(n+1)$-dimensional Einstein space that embeds M is

$$d\tilde{s}^2 = F(y)g_{ik}dx^i dx^k + \epsilon dy^2,$$

with $F(y)$ expressed by (2.5.1). For M that is Ricci flat or Riemann flat, this bulk metric reduces to

$$d\tilde{s}^2 = \exp\left(2\sqrt{\frac{-2\epsilon\Lambda}{n(n-1)}}\right)ds^2 + \epsilon dy^2. \qquad (2.5.2)$$

As another application of the Campbell-Magaard-Dahia-Romero theorems, we reproduce the example provided by Amery et al. (2011) for embedding Ricci flat cosmic string exterior spacetimes into Ricci flat bulks with geometries different to (2.5.2). Consider a 4D cosmic string exterior spacetime M of the form

$$ds^2 = e^{\hat{A}(r)}(-dt^2 + dz^2) + dr^2 + e^{\hat{B}(r)}d\theta^2,$$

and take it to be Ricci flat. The functions $\hat{A}$, $\hat{B}$ must satisfy the 4D vacuum field equations

$$-R_{11} = 0 = R_{00} = \left[-\frac{\hat{A}''}{2} - \frac{(\hat{A}')^2}{2} - \frac{\hat{A}'\hat{B}'}{4}\right]e^{\hat{A}},$$

$$0 = R_{22} = \hat{A}'' + \frac{\hat{B}''}{2} + \frac{(\hat{A}')^2}{2} + \frac{(\hat{B}')^2}{4}, \qquad (2.5.3)$$

$$0 = R_{33} = \left[\frac{\hat{B}''}{2} + \frac{(\hat{B}')^2}{4} + \frac{\hat{A}'\hat{B}'}{2}\right]e^{\hat{B}}.$$

This system admits the solutions (Vilenkin 1981; Kasner 1921):

$$e^{\hat{A}(r)} = a_1, \quad \text{and} \quad e^{\hat{B}(r)} = (a_2 r + a_3)^2, \quad a_i \in \mathbb{R},$$

$$\text{or} \quad e^{\hat{A}(r)} = (b_1 r + b_2)^{4/3}, \quad \text{and} \quad e^{\hat{B}(r)} = (b_1 r + b_2)^{-2/3}, \quad b_i \in \mathbb{R},$$

and so the metric for M has the form of either a conical geometry (Vilenkin 1981):

$$ds^2 = a_1(-dt^2 + dz^2) + dr^2 + (a_2 r + a_3)^2 d\theta^2, \tag{2.5.4}$$

or a special case of the Kasner metric (Vilenkin 1981; Kasner 1921):

$$ds^2 = (b_1 r + b_2)^{4/3}(-dt^2 + dz^2) + dr^2 + (b_1 r + b_2)^{-2/3} d\theta^2. \tag{2.5.5}$$

Note that M is embeddable into the 5D Einstein space (2.5.2) (Anderson and Lidsey 2001) and its Ricci flat ($\Lambda = 0$) embedding is just a stacking $M \times Y$ (Lidsey et al. 1997). However, since solutions for embedding spaces need not be unique, other kinds of geometries may be determined.

Consider the following form for the 5D metric:

$$\bar{g}_{ik} = \text{diag}[-e^{A(y,r)}, e^{A(y,r)}, 1, e^{B(y,r)}], \tag{2.5.6}$$

where $A(y,r) = \hat{A}(r) + \bar{A}(y)$ and $B(y,r) = \hat{B}(r) + \bar{B}(y)$ with initial conditions $\bar{A}(0) = 0 = \bar{B}(0)$. Using the fact that $R_{ik} = 0$, we find that the Ricci tensor $\bar{R}_{ik}$ calculated from (2.5.6) also vanishes. Substituting $\Lambda = 0$, $\bar{R}_{ik} = 0$ and the ansatz (2.5.6) in the propagation equation (2.4.5), it can be shown that $\bar{A}$ and $\bar{B}$ must satisfy equations equivalent to the first and third equations of (2.5.3) with $\hat{A}$, $\hat{B}$, r replaced by $\bar{A}$, $\bar{B}$, y, respectively. Those equations yield solutions of either the conical form:

$$e^{\bar{A}(y)} = \bar{a}_1, \quad \text{and} \quad e^{\bar{B}(y)} = (\bar{a}_2 y + \bar{a}_3)^2, \quad \bar{a}_i \in \mathbb{R}, \tag{2.5.7}$$

or the Kasner form:

$$e^{\bar{A}(y)} = (\bar{b}_1 y + \bar{b}_2)^{4/3}, \quad \text{and} \quad e^{\bar{B}(y)} = (\bar{b}_1 y + \bar{b}_2)^{-2/3}, \quad \bar{b}_i \in \mathbb{R}. \tag{2.5.8}$$

The initial conditions indicate that $\bar{a}_1 = 1$, $\bar{a}_3 = \pm 1$ and $\bar{b}_2 = 1$. The ansatz (2.5.6) implies that we may take

$$\Omega_{ik} = \text{diag}[-ag_{00}, ag_{11}, 0, bg_{33}],$$

where a, b are constants such that

$$\dot{\bar{A}}(0) = -2a, \quad \text{and} \quad \dot{\bar{B}}(0) = -2b. \tag{2.5.9}$$

Using the above expression for Ω_{ik} in the Codazzi-Gauss equations (2.4.6) and (2.4.7) yields the additional constraints

$$a\hat{A}' + \frac{b}{2}\hat{B}' = 0, \quad \text{and} \quad 2a(a + 2b) = 0. \tag{2.5.10}$$

Consider the cosmic string M to be a conical spacetime (2.5.4), and suppose that $\bar{A}$ and $\bar{B}$ also have the conical form (2.5.7). The conditions (2.5.9) and (2.5.10) imply that either $\bar{a}_2 = 0$ or $a_2 = 0$. For $\bar{a}_2 = 0$, the Ricci flat bulk is

$$d\tilde{s}^2 = a_1(-dt^2 + dz^2) + dr^2 + (a_2 r + a_3)^2 d\theta^2 + \epsilon dy^2,$$

which represents a stacking $M_{(r)}^{(\text{conical})} \times Y_{(y)}$, and regains the result of Lidsey et al. (1997). Here the subscript in brackets indicates the functional dependence. For $a_2 = 0$, Amery et al. (2011) obtain

$$d\tilde{s}^2 = a_1(-dt^2 + dz^2) + dr^2 + (a_3)^2(\bar{a}_2 y \pm 1)^2 d\theta^2 + \epsilon dy^2,$$

which embeds a particular conical cosmic string at $y = 0$, and can be viewed as another stacking $N_{(y)}^{(\text{conical})} \times R_{(r)}$. Similarly, taking M to be a conical type cosmic string (2.5.4) with $\bar{A}$ and $\bar{B}$ having the Kasner form (2.5.8) results in an embedding of a special conical geometry into $N_{(y)}^{(\text{Kasner})} \times R_{(r)}$. Other permutations may be similarly considered. As this example shows, various geometries may be possible for the embedding space of a given spacetime, which can be useful in the study of higher dimensional models.

Chapter 3

Gravitational Field of a Four-Dimensional Global Monopole Embedded in a Five-Dimensional Vacuum

3.1 Introduction

During the very early universe, when temperatures were high, the universe is considered to have been in a purely symmetric state. As the universe expanded and cooled, this symmetry started breaking at particular temperatures, and phase transitions occurred (Kibble 1982). Topological defects (Vilenkin and Shellard 1994) can be formed when causally independent regions that undergo phase transitions and then expand to eventually meet don't fit together smoothly. Depending on the types of phase transitions and symmetries involved, different kinds of topological defects can be produced, of which key examples are monopoles, strings, domain walls and textures. The exterior field to a gauge defect (such as a local string) with a δ-function energy-momentum may be simply embedded (Amery et al. 2011; Anderson and Lidsey 2001) — refer to section 2.5 for examples of embeddings of Ricci flat cosmic strings. Global topological defects, however, have non-trivial Ricci curvature which makes them difficult to deal with. Motivated by the Dahia-Romero theorem for Einstein embedding spaces and the lack of results for embeddings of non-vacuum spaces, we consider the embedding of the gravitational field of a four-dimensional global monopole, as it represents one of the simpler spacetimes with non-trivial energy-momentum.

Besides its mathematical interest, the monopole is also of relevance in high-energy physics and cosmology, appearing as a limit in a model of a Kaluza-Klein black hole in Einstein-Gauss-Bonnet gravity (Maeda and Dadhich 2006). Thus, a study of its embedding might provide physical insights into the structure of our universe. In section 3.2 we present the metric of the global monopole exterior and discuss some of its properties. We then proceed in section 3.3 to embed the monopole exterior metric into an Einstein space, obtaining a Riemann flat solution in section 3.4. Moreover, we show how spacetimes of the same form as the monopole metric are the only static spherically symmetric spacetimes that can be embedded into a particular bulk with vanishing Ricci

tensor. In section 3.5 we present an analysis of the Riemann flat embedding result. In particular, we verify that this bulk can be transformed to 5D Minkowski space, and so by applying recent global embedding theorems, the bulk may be taken as globally flat. Further comments are provided in section 3.6, and a summary of our main results is given in section 3.7. The original work presented in this chapter has been written up and will be submitted for publication shortly (Moodley and Amery 2013b).

3.2 The gravitational field of a global monopole

Monopoles are point-like topological defects formed as a result of spontaneous symmetry breaking phenomena in the early universe (Kibble 1976, 1982; Vilenkin 1985). They arise when the vacuum manifold contains surfaces that are not continuously contractible to a point (Vilenkin and Shellard 1994). The simplest global monopole occurs due to the global symmetry breaking $SO(3) \longrightarrow SO(2)$. A metric describing the gravitational field exterior to such a global monopole is given in static spherically symmetric (SSS) form, with coordinates t, r, θ, ϕ, by (Barriola and Vilenkin 1989)

$$ds^2 = -dt^2 + K^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.2.1}$$

where $K = 1 - 8\pi G\eta^2$ is a constant and $\eta \approx 10^{16}$ GeV is a typical grand unification (energy) scale at which monopoles could have formed. To interpret this as the field exterior to a global monopole, we must restrict $r_c < r < \infty$. The non-zero components of the Ricci tensor (2.2.3) for metric (3.2.1) are

$$R_{22} = K - 1, \qquad R_{33} = (K - 1)\sin^2\theta,$$

and the Ricci scalar is

$$R = \frac{2(K - 1)}{r^2}\,.$$

The energy-momentum tensor outside the monopole core has the components

$$T_{00} = \frac{-\eta^2}{r^2}\,, \qquad T_{11} = \frac{\eta^2}{Kr^2}\,, \qquad T_{22} = 0 = T_{33}\,. \tag{3.2.2}$$

Some features of the metric (3.2.1) are that it is not locally flat, that its subspace $\theta = \frac{\pi}{2}$ has a conical geometry with a deficit angle of $8\pi^2 G\eta^2$, and that the monopole exerts no gravitational force on the matter surrounding it. There are issues regarding the stability (or instability) of the global monopole (Vilenkin and Shellard 1994; Goldhaber 1989; Rhie and Bennett 1991; Perivolaropoulos 1992), which are based on the possible collapse of the monopole's gravitational field into a string (Goldhaber 1989). However, there exist counterarguments that such a collapse will not occur (Rhie and Bennett 1991; Perivolaropoulos 1992): Goldhaber's argument ignores the core. Global monopoles are also subject to strong cosmological constraints on their number density (Vilenkin and Shellard 1994). However, they do arise so naturally that they are of interest: their stability if embedded into a higher dimensional model is a natural question, and a first step is to embed the exterior field. We shall find that the embedding yields stability from a five-dimensional perspective — see section 3.5.3.

The global monopole metric has also appeared in the context of Einstein-Gauss-Bonnet (EGB) gravity (Maeda and Dadhich 2006). Einstein-Gauss-Bonnet theory (Dadhich 2005) requires a higher dimensional view of gravity and is based on the notion that gravity is self-interactive. The new field equations involve higher order derivatives of the metric than those used in general relativity, but reduce to the field equations of general relativity in four dimensions. The global monopole metric occurs in the $r \to 0$ limit of a Kaluza-Klein black hole solution in EGB gravity (Maeda and Dadhich 2006), where η corresponds to a function of a Weyl charge. The Gauss-Bonnet term thus weakens the singularity, while Kaluza-Klein modes generate a Weyl charge; the $r \to \infty$ limit is the Reissner-Nordström metric. The n-dimensional bulk is empty, locally homeomorphic to the product of the usual 4D Lorentzian manifold with a $(n-4)$-dimensional space of constant negative curvature, and has compact extra dimensions.

3.3   Embedding the global monopole exterior

The equations (2.4.5)–(2.4.7) for the case of an embedding for the global monopole exterior spacetime g_{ik} into a 5D Einstein space with metric

$$d\tilde{s}^2 = \bar{g}_{ik}(x^j, y)dx^i dx^k + \epsilon dy^2, \quad \epsilon^2 = 1, \quad \tilde{R}_{\alpha\beta} = \frac{-2\Lambda}{3}\tilde{g}_{\alpha\beta}, \qquad (3.3.1)$$

along the $y = 0$ hypersurface, are

$$\frac{\partial^2 \bar{g}_{ik}}{\partial y^2} = -\frac{4\epsilon\Lambda\bar{g}_{ik}}{3} - \frac{\bar{g}^{jm}}{2}\left(\frac{\partial\bar{g}_{ik}}{\partial y}\frac{\partial\bar{g}_{jm}}{\partial y} - 2\frac{\partial\bar{g}_{im}}{\partial y}\frac{\partial\bar{g}_{jk}}{\partial y}\right) - 2\epsilon\bar{R}_{ik}, \qquad (3.3.2)$$

$$0 = g^{jk}(\nabla_j \Omega_{ik} - \nabla_i \Omega_{jk}), \qquad (3.3.3)$$

$$-2\Lambda = R + \epsilon g^{ik}g^{jm}(\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im}), \qquad (3.3.4)$$

with initial conditions

$$\bar{g}_{ik}(t, r, \theta, \phi, 0) = g_{ik}(t, r, \theta, \phi),$$

$$\frac{\partial\bar{g}_{ik}(t, r, \theta, \phi, 0)}{\partial y} = -2\,\Omega_{ik}(t, r, \theta, \phi),$$

where $\Omega_{ik} = \bar{\Omega}_{ik}(x^j, 0)$ and $\bar{\Omega}_{ik} = -\frac{1}{2}\frac{\partial\bar{g}_{ik}}{\partial y}$. By the Dahia-Romero (2002a) theorem, a solution to the above system for $\bar{g}_{ik}$ exists.

We begin to determine the embedding by making the following ansatz for Ω_{ik}:

$$\Omega_{ik} = \text{diag}[a(r)g_{00}, \, b(r)g_{11}, \, c(r)g_{22}, \, c(r)g_{33}],$$

on the $y = 0$ hypersurface. Then equations (3.3.3) and (3.3.4) become

$$- a' - 2c' + \frac{2b}{r} - \frac{2c}{r} = 0 \,,$$

$$- 2\epsilon\Lambda - \frac{2\epsilon(K-1)}{r^2} = 2c^2 + 2ab + 4ac + 4bc \,.$$

Setting $a' = -2b'$, the first equation above implies that $c = \frac{\alpha}{r} + b$, $\alpha \in \mathbb{R}$. Inserting this expression for c and $a = -2b - 2I$, $I \in \mathbb{R}$ into the second equation yields an algebraic equation that can be solved for b. Thus, the Codazzi-Gauss equations (3.3.3) and (3.3.4) admit a solution

$$a(r) = -2b(r) - 2I \,,$$

$$b(r) = -I \pm \left(\frac{\epsilon\Lambda}{3} + \frac{\epsilon(K-1)}{3r^2} + \frac{\alpha^2}{3r^2} + I^2 - \frac{4I\alpha}{3r} \right)^{1/2} \,, \tag{3.3.5}$$

$$c(r) = \frac{\alpha}{r} + b(r) \,,$$

where I and α are integration constants.

Now we make the assumption

$$\bar{g}_{ik} = \mathrm{diag}[-e^{A(y,r)},\ e^{B(y,r)},\ (C(y,r))^2,\ (C(y,r))^2 \sin^2\theta] \,.$$

The initial conditions, which specify the embedded global monopole spacetime, become

$$A(0,r) = 0, \qquad\qquad B(0,r) = -\ln K, \quad C(0,r) = r, \tag{3.3.6}$$

$$\dot{A}(0,r) = 4b(r) + 4I, \qquad \dot{B}(0,r) = -2b(r), \quad \dot{C}(0,r) = -\alpha - b(r)\,r, \tag{3.3.7}$$

where $b(r) = -I \pm \left(\frac{\epsilon\Lambda}{3} + \frac{\epsilon(K-1)}{3r^2} + \frac{\alpha^2}{3r^2} + I^2 - \frac{4I\alpha}{3r} \right)^{1/2}$, and the propagation equation (3.3.2) becomes

$$\ddot{A} + \frac{\dot{A}^2}{2} + \frac{\dot{A}\dot{B}}{2} + \frac{2\dot{A}\dot{C}}{C} + \frac{4\epsilon\Lambda}{3} = \frac{-2\epsilon}{e^B} \left(\frac{A''}{2} + \frac{A'^2}{4} - \frac{A'B'}{4} + \frac{A'C'}{C} \right), \tag{3.3.8}$$

$$\ddot{B} + \frac{\dot{B}^2}{2} + \frac{\dot{A}\dot{B}}{2} + \frac{2\dot{B}\dot{C}}{C} + \frac{4\epsilon\Lambda}{3} = \frac{-2\epsilon}{e^B} \left(\frac{A''}{2} + \frac{A'^2}{4} - \frac{A'B'}{4} - \frac{B'C'}{C} + \frac{2C''}{C} \right), \tag{3.3.9}$$

$$2C\ddot{C} + 2\dot{C}^2 + C\dot{C}\dot{A} + C\dot{C}\dot{B} + \frac{4\epsilon\Lambda}{3}C^2 = \frac{-2\epsilon}{e^B} \left(-e^B + CC'' + C'^2 - \frac{B'CC'}{2} \right.$$
$$\left. + \frac{A'CC'}{2} \right). \tag{3.3.10}$$

In order to determine the Einstein embedding space (3.3.1), one must solve (3.3.8)–(3.3.10) for A, B and C subject to the conditions (3.3.6) and (3.3.7). However, as it

stands, this system of partial differential equations is highly non-linear and very difficult to solve. A general solution to this system is yet to be found. In the next section we show that we can obtain a Riemann flat solution by making further assumptions regarding the unknown functions.

3.4 A Riemann flat solution

Consider the above system (3.3.6)–(3.3.10), and set $A = 0$ and $B = -\ln K$ which trivially satisfies the initial condition (3.3.6). The second condition (3.3.7) for A and B implies that $b(r)$ and I vanish. From the expression for $b(r)$, we deduce that

$$\Lambda = 0, \quad \text{and} \quad \alpha^2 = \epsilon(1 - K).$$

With our assumptions for A and B and the restriction $\Lambda = 0$, the propagation equation (3.3.8) holds trivially. From equation (3.3.9) we have $C'' = 0$, and so

$$C(y, r) = f(y)\, r + g(y),$$

where f and g are unknown functions of y satisfying $f(0) = 1$, $\dot{f}(0) = 0 = g(0)$ and $\dot{g}(0) = -\alpha$. Substituting this function into the third propagation equation (3.3.10), we obtain

$$(2\ddot{f}f + 2\dot{f}^2)\, r^2 + (2\ddot{f}g + 2\ddot{g}f + 4\dot{f}\dot{g})\, r + 2\ddot{g}g + 2\dot{g}^2 = 2\epsilon(1 - Kf^2). \qquad (3.4.1)$$

Consider coefficients of powers of r in equation (3.4.1). We must have

$$\ddot{f}f + \dot{f}^2 = 0,$$
$$\ddot{f}g + \ddot{g}f + 2\dot{f}\dot{g} = 0,$$
$$\ddot{g}g + \dot{g}^2 = \epsilon(1 - Kf^2).$$

We solve the above equations with the initial conditions for f and g to find that $f = 1$ and $g = -\alpha y$, so that

$$C(y, r) = r - \alpha y, \quad \alpha^2 = \epsilon(1 - K).$$

Thus, the metric (3.3.1) of the embedding space is

$$d\tilde{s}^2 = -dt^2 + K^{-1}dr^2 + (r - \alpha y)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2, \qquad (3.4.2)$$

which embeds the global monopole metric at $y = 0$. It can be verified that both the Ricci and Riemann tensors are proportional to $1 - K - \epsilon\alpha^2$, which is zero, and so the bulk is Riemann flat (see section 3.5.1 for an alternate proof of this). We note that this local embedding is simple and natural, and that we may follow the same technique to embed along a hypersurface $y = y_0$, in which case we obtain the Riemann flat metric

$$d\tilde{s}^2 = -dt^2 + K^{-1}dr^2 + (r - \alpha y + \alpha y_0)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2. \qquad (3.4.3)$$

We have obtained the metric (3.4.2) by choosing the simplest functions $A = 0$ and $B = -\ln K$, but the system (3.3.8)–(3.3.10) with initial conditions (3.3.6) and (3.3.7) may admit other solutions with $A \neq 0$ and/or $B \neq -\ln K$. An investigation of these cases is currently underway.

We observe that metric (3.4.3) lies outside the class of SSS embeddings (containing only the Einstein Universe and general Schwarzschild-de Sitter space) discussed in chapter 4 and in work by Moodley and Amery (2011). Following a similar investigation to that carried out by Moodley and Amery (2011) (see chapter 4), we consider what static spherically symmetric spacetimes (Stephani 2004)

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

other than the global monopole metric, may be embedded into a bulk with metric

$$d\tilde{s}^2 = -e^{-B(y,r)}dt^2 + e^{B(y,r)}dr^2 + (r - \sigma y)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2, \qquad (3.4.4)$$

and vanishing Ricci tensor ($\Lambda = 0$). Here $\sigma \in \mathbb{R}$ and $B(y, r)$ is an unknown function satisfying the initial condition

$$B(0, r) = -2\nu(r) = 2\lambda(r).$$

From the ansatz (3.4.4), we calculate the extrinsic curvature at $y = 0$:

$$\Omega_{ik} = \text{diag}\left[-b(r)g_{00},\ b(r)g_{11},\ \sigma r,\ \sigma r \sin^2\theta\right],$$

where $\dot{B}(0, r) = -2b(r)$. With the above expression and $\Lambda = 0$, the Codazzi (2.4.6) and Gauss (2.4.7) equations admit a solution

$$b(r) = \frac{P_1 e^{-2\nu}}{r^2}\,, \quad P_1 \in \mathbb{R},$$

and a constraint equation

$$-\epsilon R = \frac{2\sigma^2}{r^2} - \frac{2P_1^2 e^{-4\nu}}{r^4}\,, \quad P_1 \in \mathbb{R}. \qquad (3.4.5)$$

Using the ansatz (3.4.4) in the propagation equation (2.4.5) (or equivalently (3.3.8)–(3.3.10)), we find that

$$\ddot{B} - \frac{2\sigma\dot{B}}{r - \sigma y} = 0\,, \qquad (3.4.6)$$

$$B'' - B'^2 + \frac{2B'}{r - \sigma y} = 0\,, \qquad (3.4.7)$$

$$1 - e^{-B} + e^{-B}B'(r - \sigma y) = \epsilon\sigma^2, \qquad (3.4.8)$$

Equations (3.4.7) and (3.4.8) along with the initial condition $B(0, r) = -2\nu(r) = 2\lambda(r)$ yield solutions

$$B(y,r) = -\ln\left[(r - \sigma y)^{-1}P_2(y) + 1 - \epsilon\sigma^2\right], \tag{3.4.9}$$

$$\nu(r) = \frac{1}{2}\ln\left(\frac{P_2(0)}{r} + 1 - \epsilon\sigma^2\right) = -\lambda(r), \tag{3.4.10}$$

where P_2 is an arbitrary function of y. From (3.4.10) we compute

$$R = \frac{-2\epsilon\sigma^2}{r^2},$$

and substituting this scalar into the constraint (3.4.5) implies that $P_1 = 0$, so that $\dot{B}(0,r) = -2b = 0$. Now solving the remaining equation (3.4.6) with the initial conditions for B gives $B(y,r) = -2\nu(r)$, and comparing this solution to (3.4.9) shows that $P_2(y) = 0$. Therefore, the only possible static spherically symmetric spacetimes embeddable into the bulk (3.4.4) are of the form

$$ds^2 = -(1 - \epsilon\sigma^2)dt^2 + (1 - \epsilon\sigma^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \sigma \in \mathbb{R}, \tag{3.4.11}$$

and their Riemann flat embedding spaces are given by

$$d\tilde{s}^2 = -(1 - \epsilon\sigma^2)dt^2 + (1 - \epsilon\sigma^2)^{-1}dr^2 + (r - \sigma y)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2. \tag{3.4.12}$$

Hence, the global monopole metric is the canonical example for embeddings of this type. This is unfortunate as one might hope that, say, the Reissner-Nordström metric could be embedded in this fashion.

We now briefly discuss the solution (3.4.2), while we provide a more detailed analysis of its properties in section 3.5. We note that, although the solution (3.4.2) is 'simple', it is not obvious nor is it a trivial product metric ($M \times Y$). Rather it is a very rare embedding of a 4D spacetime (the global monopole exterior) with complicated (i.e. not constant) curvature. Moreover, the solution (3.4.2) represents an embedding into 5D Minkowski space, for which there does not exist an existence result. This challenges perspectives on the meaning of energy-momentum in embedding scenarios. In section 3.5.2 we show that while our bulk is empty, its 4D hypersurface do contain matter, which is of relevance to space-time-matter theory. (This is not necessarily to endorse the space-time-matter programme; we merely note that the issue seems significant.)

Classical general relativity is a theory of metrics, and a metrically driven approach might seek to identify curved (e.g. constant curvature or Ricci flat, with no energy-momentum) embedding solutions, with the aim of obtaining and understanding a fuller picture of the nature of gravity. This could have positive implications for unification models in physics. For example, consider notions such as (Ricci flat) Calabi-Yau manifolds. From this perspective, Minkowski space may seem uninteresting, although there are those who devote much time to studying it (Lindblad and Rodnianski 2004). However, we make several observations. Firstly, a local embedding into Minkowski space need not compromise a global embedding into a Riemannian curved space. One need merely, say, embed into some asymptotically flat space. Less contrived examples

are the object of current study, using the algorithms described in Appendix A and Moodley and Amery (2012). Secondly, whether for logical reasons, or for the sake of simplicity, there are many instances of higher dimensional flat space in high-energy physics, and this raises several possibilities; for instance, the possibility of topological effects (Green et al. 1987; Becker et al. 2007; Vilenkin and Shellard 1994). Finally, the mere existence of embeddings of the type here must be accounted for and understood, whatever one's prejudices about higher dimensional curvature are.

3.5 Properties of the bulk

Here we discuss the features of the embedding space (3.4.2). A similar analysis follows for the embedding space (3.4.3). First we note that $\alpha^2 = \epsilon(1 - K) = \epsilon(8\pi G\eta^2)$. If we insist that α be real, then ϵ must be positive, and so the fifth dimension is space-like. With $\epsilon = 1$, the value of α is approximately $\pm 4.11 \times 10^{-3}$ (using $G = m_{\mathrm{pl}}^{-2}$, where m_{pl} denotes Planck mass, and the grand unification scale $\eta \approx 10^{16}$ GeV). Recall that the 4D monopole metric with $\theta = \frac{\pi}{2}$ has a deficit angle $\Delta = 8\pi^2 G\eta^2$, and so we can write $\alpha^2 = \frac{\epsilon\Delta}{\pi}$.

3.5.1 Hypersurfaces and other coordinate systems

We begin by considering hypersurfaces of the bulk metric. For a surface with $y = c = constant$ we have that

$$ds^2 = -dt^2 + K^{-1}dr^2 + (r - \alpha c)^2(d\theta^2 + \sin^2\theta d\phi^2)$$
$$= -dt^2 + K^{-1}d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2), \quad r > \alpha c,$$

which describes the gravitational field of a 4D global monopole. Since every hypersurface $y = constant$ has the structure of a monopole exterior but are not all identical, the embedding space is 'line-like', but it is not a 'stacking' (i.e., $N \neq M \times Y$ at a metrical level). Surfaces of constant t are described by the metric

$$ds^2 = \epsilon dy^2 + K^{-1}dr^2 + (r - \alpha y)^2(d\theta^2 + \sin^2\theta d\phi^2),$$

which resembles the flat-space metric for a texture given by (Vilenkin and Shellard 1994):

$$ds^2 = -dt^2 + dr^2 + (r - \pi\epsilon t)^2(d\theta^2 + \sin^2\theta d\phi^2), \quad t \gg r,$$

with iy behaving as t for positive ϵ, or with y behaving as t for negative ϵ. Also, a hypersurface of fixed radius $r = c$ has

$$ds^2 = -dt^2 + (\alpha y - c)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2$$
$$= -dt^2 + (1 - K)^{-1}d\bar{y}^2 + \bar{y}^2(d\theta^2 + \sin^2\theta d\phi^2),$$

which resembles a 4D global monopole exterior with $\bar{y} = \alpha y - c$ as a radial coordinate. The other 4D hypersurfaces are

$$ds^2 = -dt^2 + K^{-1}dr^2 + (r - \alpha y)^2 d\theta^2 + \epsilon dy^2,$$
$$ds^2 = -dt^2 + K^{-1}dr^2 + (r - \alpha y)^2 d\varphi^2 + \epsilon dy^2, \quad \varphi \in [\,0,\, 2\pi\,|\sin c\,|\,),$$

for $\phi = constant$ and $\theta = c = constant$, respectively. Furthermore, surfaces with both angular coordinates fixed are three-dimensional Minkowski spaces:

$$ds^2 = -dt^2 + K^{-1}dr^2 + \epsilon dy^2 = -dt^2 + d\bar{r}^2 + \epsilon dy^2.$$

We note that metric (3.4.2) is not invariant under $y \to -y$; the bulk is not symmetric with respect to the fifth dimension.

The 5D line element (3.4.2) can also be written in the non-diagonal forms

$$d\tilde{s}^2 = -\,dt^2 + K^{-1}d\bar{r}^2 + 2\alpha K^{-1}d\bar{r}dy + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2) + (\alpha^2 + \epsilon)dy^2,$$
$$\bar{r} = r - \alpha y, \quad \text{and,}$$

$$d\tilde{s}^2 = -\,dt^2 + d\bar{r}^2 + 2\alpha K^{-1/2}d\bar{r}dy + K\bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2) + (\alpha^2 K^{-1} + \epsilon)dy^2,$$
$$K^{1/2}\,\bar{r} = r - \alpha y,$$

or, with $\hat{r} = K^{-1/2}\,r$, in the diagonal form

$$d\tilde{s}^2 = -dt^2 + d\hat{r}^2 + (K^{1/2}\,\hat{r} - \alpha y)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2.$$

Applying the coordinate transformation

$$R = r - \alpha y, \qquad Y = \alpha(\epsilon K)^{-1/2}\,r + (\epsilon K)^{1/2}\,y, \tag{3.5.1}$$

to the bulk metric (3.4.2) yields the 5D Minkowski spacetime

$$d\tilde{s}^2 = -dt^2 + dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) + dY^2,$$

which reduces to the 4D global monopole exterior when $Y = \alpha(\epsilon K)^{-1/2}R$. This is an alternate verification that the bulk is Riemann flat. Here the angular coordinates have no deficits and t, Y and R can be any real value. We note that the relevance of the 5D Minkowski space to the 4D global monopole metric is a new result. The above transformation involves a rotation of (r, y) to (R, Y) by a small angle ω of magnitude $|\omega| \approx |\alpha| = \sqrt{8\pi G}\,\eta$ for the y axis and a small angle μ of magnitude $|\mu| \approx |\alpha|/K = \sqrt{8\pi G}\eta/(1 - 8\pi G\eta^2)$ for the r axis. The rotation is in the negative (clockwise) direction for positive α and positive (anticlockwise) direction for negative α. Thus, the angles of rotation between the metric (3.4.2) and the Minkowski metric are related to the energy scale η and the deficit angle Δ of the 4D metric. Similarly, the metric (3.4.12) can also be transformed to 5D flat space.

We note that an application of the Campbell-Magaard theorem to embed a three-dimensional conical geometry (Lidsey et al. 1997) implicitly anticipates our results. Consider a hypersurface $\phi = constant$ of the global monopole exterior metric (3.2.1):

$$ds^2 = -dt^2 + K^{-1}dr^2 + r^2 d\theta^2, \quad r > r_{\mathrm{c}}, \qquad (3.5.2)$$
$$= -dt^2 + d\rho^2 + K\rho^2 d\theta^2, \quad \rho > \rho_{\mathrm{c}},$$

which represents a 3D conical spacetime with a radial deficit, and has vanishing Ricci tensor. As we discussed in section 2.5, it is well known that the metric (3.5.2) has a local embedding into the 4D flat space (Lidsey et al. 1997)

$$ds^2 = -dt^2 + K^{-1}dr^2 + r^2 d\theta^2 + \epsilon dy^2.$$

We observe that the 4D metric

$$ds^2 = -dt^2 + K^{-1}dr^2 + (r - \alpha y)^2 d\theta^2 + \epsilon dy^2, \quad r > r_{\mathrm{c}},$$

which is the hypersurface of the bulk (3.4.2) for $\phi = constant$, is also a flat local embedding of the conical geometry (3.5.2) at $y = 0$, and can be transformed to 4D Minkowski space without any deficits. By recent existence theorems (Katzourakis 2005b; Moodley and Amery 2012), both embeddings above can be made global. The analysis of Lidsey et al. (1997), however, is focussed on a local embedding and was hampered by the lack of the perspective offered by the global, and (non-Ricci-flat) local existence theorems. They further showed that the conical spacetime can be globally embedded into 4D Minkowski space, although this is achieved via an adhoc construction. Our results arise from a systematic application of the Dahia-Romero (2002a) and Katzourakis-Moodley-Amery (Katzourakis 2005b; Moodley and Amery 2012) theorems. In this light we see that their discussion of local versus global issues conflates questions of curvature and singularities.

3.5.2 Extrinsic curvature and energy-momentum

The embedding space is intrinsically flat. However, its submanifolds with constant y do admit extrinsic curvature given by $\bar{\Omega}_{ik} = -\frac{1}{2}\frac{\partial \bar{g}_{ik}}{\partial y}$ of which the non-zero components are

$$\bar{\Omega}_{22} = \alpha(r - \alpha y), \qquad \text{and} \qquad \bar{\Omega}_{33} = \alpha(r - \alpha y)\sin^2 \theta.$$

The extrinsic curvature of a manifold is dependent on the space in which it is embedded, hence the y dependence of $\bar{\Omega}_{ik}$. Under the coordinate transformation to flat space these components become

$$\bar{\Omega}_{22} = \alpha R, \qquad \text{and} \qquad \bar{\Omega}_{33} = \alpha R \sin^2 \theta,$$

so that the extrinsic curvature is proportional to the radial coordinate R and α.

The above transformation (3.5.1) makes it obvious that the 5D spacetime is empty ($\tilde{T}_{\mu\nu} = 0$), while the 4D hypersurfaces contain matter. We may compute the energy-momentum tensor of $\bar{g}_{ik}$, the metric induced on a hypersurface of constant y, from $\bar{G}_{ik} = 8\pi G \bar{T}_{ik}$ with

$$\bar{T}_{00} = \frac{-\eta^2}{(r - \alpha y)^2} = \frac{-\eta^2}{R^2}, \qquad \bar{T}_{11} = \frac{\eta^2}{K(r - \alpha y)^2} = \frac{\eta^2}{KR^2}, \qquad \bar{T}_{22} = 0 = \bar{T}_{33}. \quad (3.5.3)$$

At $y = 0$, equation (3.5.3) reduces to the matter tensor (3.2.2) of the global monopole. In the context of induced matter theory (Wesson and Ponce de Leon 1992; Wesson et al. 1996; Overduin and Wesson 1997; Wesson 1999), equation (3.5.3) is a general 4D energy-momentum tensor that depends on the extra dimension y and represents matter induced from the higher dimensional geometry. Space-time-matter or induced matter theory concentrates on 5D vacuum embedding spaces and involves the view that the 5D geometry and the shape of the 4D hypersurfaces explain matter arising in the 4D surfaces. An effective energy-momentum tensor is obtained from the field equations for the 4D component of the 5D metric, and it is generally dependent on the extra dimension. As is the case for our result here, the 5D vacuum field equations contain the 4D field equations with matter.

3.5.3 Global and local geometry

The paradox of a vacuum spacetime (in Minkowski form) containing hypersurfaces with singular energy-momentum may be understood as a consequence of the local embedding of the global monopole exterior for $r > r_{\rm c}$, and/or the global structure of the local embedding space (3.4.2). Firstly, we note that, since we may take (3.4.2) as a global isometric embedding metric on (some suitably defined subset of) $\mathbb{R}^5$ (Moodley and Amery 2012), we have an everywhere analytic and Riemann flat isometric global embedding: the 4D global monopole spacetime is 'analytic enough' to allow globally analytic (3.4.2) as an embedding space, despite it containing hypersurfaces of singular energy-momentum. On the other hand, the topological singularity on the 4D hypersurfaces is weak: the integral of the density over volume vanishes at $r = 0$. Thus, we have an example of a weak topological singularity being 'undone' by the geometry of the embedding.

Secondly, consider the ry-plane with $r > 0$ divided into four regions by the line $y = \frac{r}{\alpha}$, $\alpha > 0$ and the cutoff $r = r_{\rm c}$. This is depicted in Figure 3.1. The constant y and constant r hypersurfaces have singular energy-momentum. For each of the four regions, the ranges of the coordinates R and Y are presented in Table 3.1.
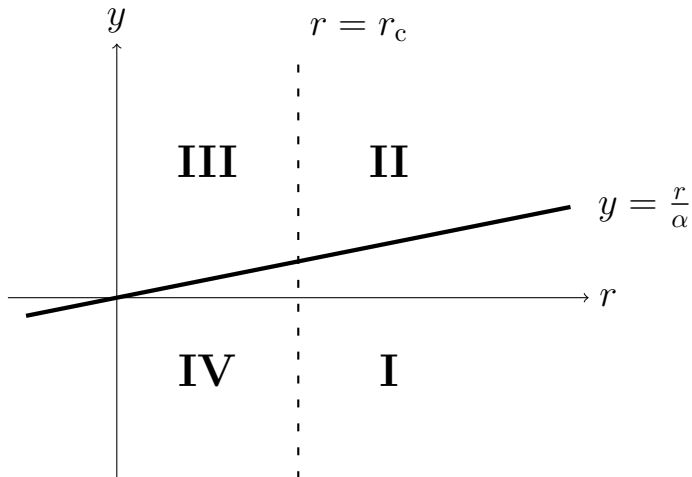
Figure 3.1: Regions I – IV in the ry-plane.

Region	Range of R	Range of Y
I	$[0\,,\infty)$	$(-\infty\,,\infty)$
II	$(-\infty\,,0)$	$\left(r_{\mathrm{c}}/\sqrt{K(1-K)}\,,\infty\right)$
III	$(-\infty\,,0)$	$(0\,,\infty)$
IV	$[0\,,\infty)$	$\left(-\infty\,,r_{\mathrm{c}}/\sqrt{K(1-K)}\,\right]$

Table 3.1: Ranges of R and Y for regions I – IV.

Consider region I where $r > r_{\mathrm{c}}$ and $y \leq \frac{r}{\alpha}$. Here $R \in [0,\infty)$ and $Y \in \mathbb{R}$. This region may be interpreted as the local embedding of the global monopole metric for $r > r_{\mathrm{c}}$ into (all of) 5D Minkowski space. For region II where $r > r_{\mathrm{c}}$ and $y > \frac{r}{\alpha}$, we have $R \in (-\infty,0)$ and $Y \in (r_{\mathrm{c}}/\sqrt{K(1-K)},\infty)$. This region may be given an analogous interpretation to region I by rescaling Y by an additive constant and recalling that a hypersurface of constant r represents another embedding of a global monopole exterior metric, this time for $y > y_{\mathrm{c}} = \frac{r_{\mathrm{c}}}{\alpha}$.

The two regions with $r < r_{\mathrm{c}}$ have a similarly dual interpretation. Note though that the metric (3.2.1) for $r < r_{\mathrm{c}}$ does not represent the space exterior to nor interior to a global monopole. A similar analysis follows for the case $\alpha < 0$, with the roles of regions I and II reversed. The main point of this discussion is that there is no singularity if we consider only $r > r_{\mathrm{c}}$, and that the global embedding space is 5D Minkowski space. Moreover, this is true for arbitrarily small r_{c}. This property facilitates many different global embeddings. The first and most obvious of these is, of course, 5D Minkowski space itself, but the imposition of a deficit angle would yield a 5D cosmic string (Vilenkin and Shellard 1994). For a compactified (Kaluza-Klein) extra dimension one could obtain (Azreg-Aïnou and Clément 1996) static cosmic strings, flux

strings, and 5D generalizations of 4D longitudinal dislocations (Gal'tsov and Letelier 1993; Tod 1994) and 4D spinning cosmic strings (Deser et al. 1984). Azreg-Ainou and Clément (1996) also obtained a Kaluza-Klein Gauss-Bonnet superconducting cosmic string which asymptotes (up to logarithms) to the static cosmic string metric.

Recall the issue of stability of the global monopole exterior noted in section 3.2. Since 5D Minkowski space is stable (Choquet-Bruhat et al. 2006; Choquet-Bruhat 2009; Lindblad and Rodnianski 2004), we may conclude that the 5D embedding of the global monopole exterior yields stability from a 5D perspective / in the bulk.

3.5.4   Comment on Einstein-Gauss-Bonnet gravity scenario

As remarked in section 3.2, we can view our very natural construction as the embedding of the $r \to 0$ limit of a 5D EGB black hole. We may do so because of the product topology assumed in that construction (Maeda and Dadhich 2006). In fact, since all analytic (vacuum) solutions to the EGB equations are locally Minkowskian (any pseudo-Riemannian manifold may be transformed to normal coordinates in which the connections vanish locally (Stephani 2004)), we may also view our result (with appropriate compactifications) as a local embedding of the 4D global monopole exterior into the 5D EGB Kaluza-Klein black hole exterior. It is worth contrasting the 5D flat vacuum embedding in general relativity with the EGB case. In the former case the topology is product at a non-metrical level, but we have a metric that is not expressible as a product metric. In the latter, the presence of the Gauss-Bonnet term gives a freedom in terms of the topologies that may be chosen for 5D vacuum spaces that is not present in pure Einstein gravity — the product $M_{(GM)} \times Y$ in EGB theory is empty, whereas the product $M_{(GM)} \times Y$ in Einstein theory yields a 5D global monopole metric (Banerjee et al. 1996; Rahaman, Ghosh, Kalam and Gayen 2005)

$$d\tilde{s}^2 = -dt^2 + K^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + d\psi^2, \quad \psi \in \mathbb{R},$$

which does not have vanishing energy-momentum (and hence, is not an Einstein space). On the other hand, the flat Einstein embedding is natural and simple. The further investigation of this relationship would involve embedding the $r \to \infty$ limit of the EGB black hole, namely the Reissner-Nordström solution. This investigation is underway.

3.5.5   A six-dimensional Einstein embedding

As a final remark, we observe that the 5D flat space (3.4.3) can be locally embedded into a six-dimensional Einstein space (Anderson and Lidsey 2001) with line element

$$d\tilde{s}^2 = \exp\left(\sqrt{\frac{2\epsilon_z\Lambda_{(6)}}{5}}\,(z - z_0)\right) \times \left(-dt^2 + K^{-1}dr^2\right.$$
$$\left. + (r - \alpha y + \alpha y_0)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2\right) + \epsilon_z dz^2,$$

where $\epsilon_z^2 = 1$ and $\Lambda_{(6)}$ is the 6D cosmological constant, and this space embeds the 4D global monopole metric at $y = y_0$ and $z = z_0$. This may be of relevance to six-dimensional theories such as those of Tye (2008) and Aharony (2000). The EGB black hole (Maeda and Dadhich 2006) may also be contextualized in six dimensions.

3.6 Further remarks

We note that since metric (3.2.1) is only applicable away from the core, it would be interesting to consider the embedding of a four-dimensional global monopole with core mass M and/or a cosmological constant term λ, which has the metric (Rahaman, Mandal and Gayen 2005)

$$
ds^2 = -\left(1 - 8\pi G\eta^2 - \frac{M}{r} - \frac{\lambda}{3}r^2\right)dt^2 + \left(1 - 8\pi G\eta^2 - \frac{M}{r} - \frac{\lambda}{3}r^2\right)^{-1}dr^2 \\
+ r^2(d\theta^2 + \sin^2\theta d\phi^2),
$$

although this task would be even more complicated. The possible embedding of a global monopole in this fashion could facilitate a programme contrasting the known strong constraints on cosmological global monopole abundances with string abundances in 5D embedded Einstein theories or black hole abundances in 5D embedded EGB theories. It would similarly be interesting to consider embeddings of the gravitational fields of global cosmic strings (Vilenkin and Shellard 1994) such as a linearized global string exterior with metric (Harari and Sikivie 1988)

$$
ds^2 = \left(1 - 4\pi G\nu^2 \ln\frac{r}{r_c}\right)(-dt^2 + dz^2) + dr^2 + r^2\left(1 - 8\pi G\nu^2 k - 8\pi G\nu^2 \ln\frac{r}{r_c}\right)d\theta^2,
$$

where $\nu < 10^{17}$ GeV is a symmetry breaking scale, r_c is the radius of the string core, and $\pi G\nu^2 k$ is the energy per unit length within the core. These points will be considered in future work.

3.7 Summary

In this chapter we considered the local isometric embedding of the gravitational field exterior to a 4D global monopole defect, given by metric (3.2.1), into a 5D Einstein bulk (3.3.1). We produced a Riemann flat solution (3.4.2) that embeds the global monopole exterior along the hypersurface $y = 0$. It is further demonstrated that the monopole metric is the typical static spherically symmetric spacetime that can be embedded into a 5D space with form (3.4.4) and zero Ricci tensor. We studied the geometry of the embedding space (3.4.2). We verified that the metric can be transformed into 5D Minkowski space, and there is no inherited deficit angle. Applying global existence theorems (Katzourakis 2005*b*; Moodley and Amery 2012), we observed that the monopole exterior can be embedded into a globally flat space, and noted that various cosmic string solutions may also be produced. Since 5D Minkowski space is

stable, this potentially alleviates the stability problem for the 4D monopole exterior metric, provided we regard gravity as being five-dimensional. Although the bulk is empty, its hypersurfaces of constant y contain matter. This idea is relevant to induced matter theory (Wesson 1999). We also discussed our solution in the context of Einstein-Gauss-Bonnet theory, where it can be viewed as a local embedding of the monopole exterior into the 5D EGB Kaluza-Klein black hole (Maeda and Dadhich 2006).

The embedding of the global monopole exterior into a general Einstein space remains unsolved and is the subject of ongoing work. Thus, even with specific choices for the lower dimensional space, solutions for the embedding spaces are difficult to find. In the next section, we modify our approach to tackle this embedding problem.

Chapter 4

An Investigation of Embeddings for Spherically Symmetric Spacetimes into Einstein Manifolds

4.1   Introduction

In this chapter we concentrate on local isometric embeddings of four-dimensional spherically symmetric spacetimes into five-dimensional Einstein manifolds. Spherically symmetric spacetimes provide useful models of stars and black holes etc., and so their embeddings will be of relevance in astrophysics and cosmology. We begin in section 4.2 by describing the metric and Ricci tensor components of a spherically symmetric spacetime. As observed in the previous chapter for the case of the global monopole exterior, difficulties often arise in solving the five-dimensional embedding equations for given four-dimensional spaces. This problem motivates us to investigate embedded spaces that admit bulks of a specific type. So, in section 4.3, we consider an Einstein space of a particular form where the metric components are separable with respect to the extra dimension, and we obtain restrictions for the possible embedded spacetimes. In section 4.4 we show that the general Schwarzschild-de Sitter spacetime and the Einstein Universe are the only spherically symmetric spacetimes that can be embedded into an Einstein space of this form, and we discuss their five-dimensional solutions. We also consider another form for an Einstein bulk in section 4.5 that reduces to a special case of the first bulk form. In section 4.6 we focus on metrics describing four-dimensional spherically symmetric spacetimes in retarded time coordinates, and we examine embeddings of the general Vaidya-de Sitter model into specific five-dimensional Einstein bulks. A summary of our main results is provided in section 4.7. The original studies presented in sections 4.3 and 4.4 have been published (Moodley and Amery 2011), and the physical interpretation of a solution presented in section 4.4.2 is being written up for publication (Okelola et al. 2013). The results contained in section 4.5 have been submitted for publication (Amery and Moodley 2012), and the results contained in section 4.6 are being prepared for publication (Moodley and Amery 2013a).

4.2 Embedding spherically symmetric spacetimes

We apply the technique reviewed in section 2.4 to embed a four-dimensional spherically symmetric (SS) spacetime g_{ik} which has the line element (Stephani 2004):

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

The non-zero components of the Ricci tensor (2.2.3) calculated from this metric are:

$$R_{00} = e^{2(\nu-\lambda)}(-\nu'' - \nu'^2 + \nu'\lambda' - \frac{2}{r}\nu') + \lambda_{tt} + \lambda_t^2 - \nu_t\lambda_t\,,$$

$$R_{01} = -\frac{2}{r}\lambda_t\,,$$

$$R_{11} = \nu'' + \nu'^2 - \nu'\lambda' - \frac{2}{r}\lambda' - e^{2(\lambda-\nu)}(\lambda_{tt} + \lambda_t^2 - \nu_t\lambda_t)\,,$$

$$R_{22} = re^{-2\lambda}\nu' - re^{-2\lambda}\lambda' + e^{-2\lambda} - 1\,,$$

$$R_{33} = R_{22}\sin^2\theta\,.$$

The equations (2.4.5)–(2.4.7) for embedding a 4D spherically symmetric spacetime g_{ik} into a 5D Einstein space with metric

$$d\tilde{s}^2 = \bar{g}_{ik}(x^j, y)dx^i dx^k + \epsilon dy^2\,,\quad \epsilon^2 = 1,\quad \tilde{R}_{\alpha\beta} = \frac{-2\Lambda}{3}\tilde{g}_{\alpha\beta}\,, \tag{4.2.1}$$

along the $y = 0$ hypersurface, are

$$\frac{\partial^2 \bar{g}_{ik}}{\partial y^2} = -\frac{4\epsilon\Lambda\bar{g}_{ik}}{3} - \frac{\bar{g}^{jm}}{2}\left(\frac{\partial \bar{g}_{ik}}{\partial y}\frac{\partial \bar{g}_{jm}}{\partial y} - 2\frac{\partial \bar{g}_{im}}{\partial y}\frac{\partial \bar{g}_{jk}}{\partial y}\right) - 2\epsilon\bar{R}_{ik}\,, \tag{4.2.2}$$

$$0 = g^{jk}(\nabla_j\Omega_{ik} - \nabla_i\Omega_{jk})\,, \tag{4.2.3}$$

$$-2\Lambda = R + \epsilon g^{ik}g^{jm}(\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im})\,, \tag{4.2.4}$$

with initial conditions

$$\bar{g}_{ik}(t, r, \theta, \phi, 0) = g_{ik}(t, r, \theta, \phi)\,, \tag{4.2.5}$$

$$\frac{\partial \bar{g}_{ik}(t, r, \theta, \phi, 0)}{\partial y} = -2\,\Omega_{ik}(t, r, \theta, \phi)\,, \tag{4.2.6}$$

where $\Omega_{ik} = \bar{\Omega}_{ik}(x^j, 0)$ and $\bar{\Omega}_{ik} = -\frac{1}{2}\frac{\partial \bar{g}_{ik}}{\partial y}$. Although the Dahia-Romero (2002a) theorem guarantees that a solution to the above system for $\bar{g}_{ik}$ exists, these equations are complex and the general solution is not yet known. For the case of the embedding of a 4D global monopole metric, which is static and spherically symmetric, we were able to obtain a 5D solution (3.4.2) that is Riemann flat. A solution for an Einstein space that embeds the monopole is still to be determined. Thus, even for specific SS spacetimes, it is very difficult to solve the embedding equations. So, we consider another approach: we assume that the 5D Einstein space (4.2.1) has a particular form, and we investigate what SS spacetimes may embed into it.

4.3   Bulk form 1

We proceed by making a fairly simple assumption

$$\bar{g}_{ik} = \text{diag}[A(y)g_{00},\, B(y)g_{11},\, C(y)g_{22},\, D(y)g_{33}]\,, \tag{4.3.1}$$

for the 5D metric, where the unknown functions A, B, C and D depend on y only and each metric component is separable in y. This type of metric for the embedding space includes warped geometries such as those of the Randall-Sundrum (1999a; 1999b) and related (see Maartens and Koyama (2010)) brane-world scenarios and induced matter theory (Wesson 2002), and allows for an energetically rigid embedding (Londal 2005; Amery et al. 2011). The initial conditions (4.2.5) and (4.2.6) become

$$A(0) = B(0) = C(0) = D(0) = 1\,, \tag{4.3.2}$$

$$\dot{A}(0) = -2\,\Omega_{00}g^{00}, \qquad \dot{B}(0) = -2\,\Omega_{11}g^{11},$$
$$\dot{C}(0) = -2\,\Omega_{22}g^{22}, \qquad \dot{D}(0) = -2\,\Omega_{33}g^{33}. \tag{4.3.3}$$

Condition (4.3.3) implies that the extrinsic curvature must have the form

$$\Omega_{ik} = \text{diag}[ag_{00},\, bg_{11},\, cg_{22},\, dg_{33}]\,,$$

where a, b, c and d are constants. We substitute the above expression into the Codazzi (4.2.3) and Gauss (4.2.4) equations to obtain $c = d$ and

$$0 = (a - b)\,\lambda_t\,, \tag{4.3.4}$$

$$0 = (b - a)\,\nu' + (b - c)\,\frac{2}{r}\,, \tag{4.3.5}$$

$$-2\Lambda = R + 2\epsilon\left(ab + 2ac + 2bc + c^2\right)\,, \tag{4.3.6}$$

where equation (4.3.6) indicates that the Ricci scalar R of the embedded space must be constant.

The components of the Ricci tensor $\bar{R}_{ik}$ calculated from (4.3.1) are

$$\bar{R}_{00} = \frac{A}{B}R_{00} + \left(1 - \frac{A}{B}\right)\left(\lambda_{tt} + \lambda_t{}^2 - \nu_t\lambda_t\right)\,,$$

$$\bar{R}_{01} = R_{01}\,,$$

$$\bar{R}_{11} = R_{11} + \left(1 - \frac{B}{A}\right)\left(\lambda_{tt} + \lambda_t{}^2 - \nu_t\lambda_t\right)e^{2(\lambda-\nu)}\,,$$

$$\bar{R}_{22} = \frac{C}{B}R_{22} + \frac{C}{B} - 1\,,$$

$$\bar{R}_{33} = \frac{D}{B}R_{33} + \left(\frac{D}{B} - \frac{D}{C}\right)\sin^2\theta = \frac{D}{C}\bar{R}_{22}\sin^2\theta\,.$$

The propagation equation (4.2.2) for $i = 0$ and $k = 1$ gives

$$0 = \bar{R}_{01},$$

which implies that

$$\lambda_t = 0.$$

Thus, equation (4.3.4) is satisfied. With $\lambda_t = 0$ and the above components of $\bar{R}_{ik}$, the other components of the propagation equation (4.2.2) are

$$\ddot{A} + \frac{\dot{A}}{2}\left(-\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\frac{A}{B}R_{00}g^{00}, \tag{4.3.7}$$

$$\ddot{B} + \frac{\dot{B}}{2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}B = -2\epsilon R_{11}g^{11}, \tag{4.3.8}$$

$$\ddot{C} + \frac{\dot{C}}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}C = -2\epsilon\frac{C}{B}R_{22}g^{22} - 2\epsilon\left(\frac{C}{B} - 1\right)\frac{1}{r^2}, \tag{4.3.9}$$

$$\ddot{D} + \frac{\dot{D}}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}D = -2\epsilon\frac{D}{B}R_{33}g^{33} - 2\epsilon\left(\frac{D}{B} - \frac{D}{C}\right)\frac{1}{r^2}. \tag{4.3.10}$$

Since the left-hand sides of equations (4.3.7)–(4.3.10) depend on y only, we should have

$$R_{00}g^{00} = e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2}{r}\nu'\right) = \alpha_1, \tag{4.3.11}$$

$$R_{11}g^{11} = e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' - \frac{2}{r}\lambda'\right) = \alpha_2, \tag{4.3.12}$$

$$\frac{C}{B}R_{22}g^{22} + \left(\frac{C}{B} - 1\right)\frac{1}{r^2} = \frac{C}{B}e^{-2\lambda}\left(\frac{\nu'}{r} - \frac{\lambda'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \alpha_3(y), \tag{4.3.13}$$

$$\frac{D}{B}R_{33}g^{33} + \left(\frac{D}{B} - \frac{D}{C}\right)\frac{1}{r^2} = \frac{D}{B}e^{-2\lambda}\left(\frac{\nu'}{r} - \frac{\lambda'}{r} + \frac{1}{r^2}\right) - \frac{D}{C}\frac{1}{r^2} = \alpha_4(y), \tag{4.3.14}$$

where α_1 and α_2 are constants, and α_3 and α_4 are functions of y. Comparing equations (4.3.13) and (4.3.14), we deduce that $\alpha_4 = \frac{D}{C}\alpha_3$, and so these equations are equivalent.

Subtracting equation (4.3.11) from equation (4.3.12), we obtain

$$\nu' = -\lambda' - \frac{(\alpha_2 - \alpha_1)}{2}re^{2\lambda}, \tag{4.3.15}$$

and inserting this expression into (4.3.13) leads to a first-order linear differential equation for $e^{-2\lambda}$ that admits the solution

$$\lambda(r) = -\frac{1}{2} \ln \left[\frac{B}{C} + \frac{\alpha_5(y)}{r} + \left(\frac{B}{C} \frac{\alpha_3}{3} + \frac{\alpha_2 - \alpha_1}{6} \right) r^2 \right].$$

Using the fact that $\lambda(r)$ has no y dependence, and applying initial condition (4.3.2), we determine that

$$B = C, \quad \text{and} \quad \dot{\alpha}_3 = 0 = \dot{\alpha}_5.$$

With $B = C$, the condition (4.3.3) implies that $b = c$, and equations (4.3.8) and (4.3.9) imply that $R_{11}g^{11} = R_{22}g^{22}$. So, by equations (4.3.12) and (4.3.13), we have $\alpha_3 = \alpha_2$. Furthermore, equations (4.3.11)–(4.3.14) show that the Ricci scalar is

$$R = \alpha_1 + 3\alpha_2.$$

Substituting (4.3.15) and

$$\lambda(r) = -\frac{1}{2} \ln \left(1 + \frac{\alpha_5}{r} + \frac{3\alpha_2 - \alpha_1}{6} r^2 \right)$$

into (4.3.11) and simplifying the result yields

$$\frac{(\alpha_2 - \alpha_1)(-2\alpha_1 r^3 + 3\alpha_5)}{\frac{3\alpha_2 - \alpha_1}{2} r^3 + 3r + 3\alpha_5} = 0.$$

For the above equation to hold, we require either

$$\alpha_1 = \alpha_2, \quad \text{or} \quad \alpha_1 = 0 = \alpha_5.$$

In each case, the solution for $\lambda(r)$ can be substituted into (4.3.15), which can then be integrated to provide a solution for $\nu(t, r)$. The resulting spacetimes satisfy (4.3.11)–(4.3.14), and are the only 4D spherically symmetric spacetimes that may be embedded into a 5D Einstein space with $\bar{g}_{ik}$ given by (4.3.1). Note that for each of the embedded spaces we still need to solve the Codazzi-Gauss equations (4.3.5) and (4.3.6), and the propagation equations (4.3.7)–(4.3.10) subject to the initial conditions (4.3.2) and (4.3.3), in order to determine the bulk metric explicitly. We consider each case in the next section.

4.4   Solutions for bulk form 1

4.4.1   Case I: $\alpha_1 = \alpha_2$

With $\alpha_1 = \alpha_2$, we have

$$\lambda(r) = -\frac{1}{2} \ln \left(1 + \frac{\alpha_5}{r} + \frac{\alpha_1}{3} r^2 \right),$$

and

$$\nu(t, r) = -\lambda(r) + g(t),$$

where $\alpha_1, \alpha_5 \in \mathbb{R}$ and $g(t)$ is an arbitrary function. So,

$$ds^2 = -e^{2g(t)}\left(1 + \frac{\alpha_5}{r} + \frac{\alpha_1}{3}r^2\right)dt^2 + \frac{dr^2}{1 + \frac{\alpha_5}{r} + \frac{\alpha_1}{3}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.4.1)$$

This solution is known as the general Schwarzschild-de Sitter spacetime, which represents a Schwarzschild black hole in a universe with a four-dimensional cosmological constant given by $\omega = -\alpha_1$. This space is a vacuum and has Ricci scalar $R = 4\alpha_1$.

Now we solve the Codazzi-Gauss equations (4.3.5) and (4.3.6) to obtain

$$a = b = c = \pm\sqrt{\frac{-\epsilon\Lambda - 2\epsilon\alpha_1}{6}},$$

and so the initial condition (4.3.3) becomes

$$\dot{A}(0) = \dot{B}(0) = \dot{D}(0) = \pm\sqrt{\frac{-2\epsilon\Lambda - 4\epsilon\alpha_1}{3}}. \quad (4.4.2)$$

The propagation equations (4.3.7)–(4.3.10) for this case are

$$\ddot{A} + \frac{\dot{A}}{2}\left(-\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} + \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\frac{A}{B}\alpha_1, \quad (4.4.3)$$

$$\ddot{B} + \frac{\dot{B}}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}B = -2\epsilon\alpha_1, \quad (4.4.4)$$

$$\ddot{D} + \frac{\dot{D}}{2}\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} - \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}D = -2\epsilon\frac{D}{B}\alpha_1. \quad (4.4.5)$$

As the Schwarzschild-de Sitter spacetime is an Einstein space, a solution for the Einstein embedding is already known (Lidsey et al. 1997; Anderson and Lidsey 2001), and it can be obtained as follows. We may set $A = B = D$ so that the equations (4.4.3)–(4.4.5) reduce to a single equation

$$\ddot{A} + \frac{\dot{A}^2}{A} + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\alpha_1,$$

which has the solution

$$A(y) = \begin{cases} \left[\cosh\left(\sqrt{\frac{-\epsilon\Lambda}{6}}\,y\right) + \sqrt{1 + \frac{2\alpha_1}{\Lambda}}\sinh\left(\sqrt{\frac{-\epsilon\Lambda}{6}}\,y\right)\right]^2, & \Lambda \neq 0, \\ \left(1 + \sqrt{\frac{-\epsilon\alpha_1}{3}}\,y\right)^2, & \Lambda = 0, \end{cases} \quad (4.4.6)$$

that satisfies the conditions (4.3.2) and (4.4.2). Hence, the 5D Einstein embedding space for the general Schwarzschild-de Sitter spacetime (4.4.1) is

$$d\tilde{s}^2 = A(y)ds^2 + \epsilon dy^2,$$

with $A(y)$ given by (4.4.6). We note that other solutions for the embedding of this spacetime into bulks of the form (4.3.1) may be possible.

In Case II we shall see that the bulk metric cannot take the form $\bar{g}_{ik} = A(y)g_{ik}$. So we conclude that the general Schwarzschild-de Sitter model is the only spherically symmetric spacetime that admits a 5D Einstein bulk with $d\tilde{s}^2 = A(y)ds^2 + \epsilon dy^2$.

4.4.2 Case II: $\alpha_1 = 0 = \alpha_5$

When $\alpha_1 = 0 = \alpha_5$, we have

$$\lambda(r) = -\frac{1}{2}\ln\left(1 + \frac{\alpha_2}{2}r^2\right),$$

and

$$\nu(t) = g(t),$$

where $\alpha_2 \in \mathbb{R}$ and $g(t)$ is an arbitrary function. So,

$$ds^2 = -e^{2g(t)}dt^2 + \left(1 + \frac{\alpha_2}{2}r^2\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (4.4.7)$$

which has Ricci scalar $R = 3\alpha_2$. The scale factor $e^{2g(t)}$ can be absorbed into a redefined time coordinate. For $\alpha_2 < 0$, the metric describes the Einstein Universe in which the scale factor is a constant, so the universe does not expand or contract. The Einstein Universe has a cosmological constant $\omega = \frac{-\alpha_2}{2}$. Note that unlike in Case I, the 4D spacetime (4.4.7) is not an Einstein space.

Since $\nu' = 0$, the general solution to the Codazzi-Gauss equations (4.3.5) and (4.3.6) is given by

$$c = b = \frac{-a}{2} \pm \frac{1}{2}\sqrt{a^2 - \frac{4\epsilon\Lambda}{3} - 2\epsilon\alpha_2},$$

where $a \in \mathbb{R}$. The initial condition (4.3.3) becomes

$$\dot{A}(0) = -2a, \quad \dot{B}(0) = \dot{D}(0) = a \pm \sqrt{a^2 - \frac{4\epsilon\Lambda}{3} - 2\epsilon\alpha_2}. \qquad (4.4.8)$$

The functions A, B and D must satisfy the propagation equations

$$\ddot{A} + \frac{\dot{A}}{2}\left(-\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} + \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}A = 0, \qquad (4.4.9)$$

$$\ddot{B} + \frac{\dot{B}}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}B = -2\epsilon\alpha_2, \qquad (4.4.10)$$

$$\ddot{D} + \frac{\dot{D}}{2}\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} - \frac{\dot{D}}{D}\right) + \frac{4\epsilon\Lambda}{3}D = -2\epsilon\frac{D}{B}\alpha_2, \qquad (4.4.11)$$

with initial conditions (4.3.2) and (4.4.8). Here we can set $B = D$ so that (4.4.11) is equivalent to (4.4.10). Then the above system becomes

$$\ddot{A} + \frac{\dot{A}}{2}\left(-\frac{\dot{A}}{A} + \frac{3\dot{B}}{B}\right) + \frac{4\epsilon\Lambda}{3}A = 0\,, \tag{4.4.12}$$

$$\ddot{B} + \frac{\dot{B}}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{4\epsilon\Lambda}{3}B = -2\epsilon\alpha_2\,. \tag{4.4.13}$$

Note that taking $A = B$ yields $\alpha_2 = 0$, which corresponds to a special subcase of Case I with $\alpha_5 = 0$. In general we cannot have $A = B$ here.

Consider $\Lambda = 0$. By setting $A = 1$, equation (4.4.12) holds trivially and condition (4.4.8) implies that $a = 0$. Equation (4.4.13) becomes

$$\ddot{B} + \frac{\dot{B}^2}{2B} = -2\epsilon\alpha_2\,,$$

which admits the solution

$$B(y) = \left(1 + \sqrt{\frac{-\epsilon\alpha_2}{2}}\,y\right)^2\,,$$

that satisfies the initial conditions (4.3.2) and (4.4.8). Hence, the spacetime (4.4.7) can be embedded into a 5D space with metric

$$d\tilde{s}^2 = -e^{2g(t)}dt^2 + \left(1 + \sqrt{\frac{-\epsilon\alpha_2}{2}}\,y\right)^2\left(\frac{dr^2}{1 + \frac{\alpha_2}{2}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) + \epsilon dy^2. \tag{4.4.14}$$

By applying the coordinate transformation

$$e^{g(t)}dt = dT\,, \quad \sqrt{\frac{-\epsilon\alpha_2}{2}}\,r = \frac{R}{Y}\left(1 + \frac{\epsilon R^2}{Y^2}\right)^{-1/2}\,,$$

$$1 + \sqrt{\frac{-\epsilon\alpha_2}{2}}\,y = \sqrt{\frac{-\epsilon\alpha_2}{2}}\,Y\left(1 + \frac{\epsilon R^2}{Y^2}\right)^{1/2}\,,$$

the metric (4.4.14) can be written in the Minkowski form

$$d\tilde{s}^2 = -dT^2 + dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dY^2\,,$$

and so the embedding space is flat. Thus, for $\alpha_2 < 0$, we regain the result obtained by Wesson (1994) for the embedding of the Einstein Universe into a flat space. However, there the method was to start with the 5D Minkowski metric and transform it to a space that embeds the Einstein Universe along a $y = constant$ hypersurface. Here we consider a more general 5D metric, directly solve the field equations to find the Einstein Universe as the only possible non-vacuum spherically symmetric embedded space, and obtain a Minkowski bulk solution in the special case $\Lambda = 0$.

When $\Lambda = \frac{-3}{2}\alpha_2$, Lie symmetry analysis of the system (4.4.9)–(4.4.11) yields a hitherto unknown solution:

$$A(y) = \left[\cosh\left(\sqrt{\frac{-2\epsilon\Lambda}{3}}\,y\right) - a\sqrt{\frac{-3\epsilon}{2\Lambda}}\,\sinh\left(\sqrt{\frac{-2\epsilon\Lambda}{3}}\,y\right)\right]^2, \quad a \in \mathbb{R}, \qquad (4.4.15)$$

$$B(y) = C(y) = D(y) = 1.$$

The details of this analysis appear elsewhere (Okelola 2011; Okelola et al. 2013). So, the spacetime (4.4.7) can be embedded into a 5D Einstein bulk with cosmological constant $\Lambda = \frac{-3}{2}\alpha_2$ and metric

$$d\tilde{s}^2 = -A(y)\,e^{2g(t)}dt^2 + \frac{dr^2}{1 + \frac{\alpha_2}{2}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2, \qquad (4.4.16)$$

where $A(y)$ is given by (4.4.15). The solution to equations (4.4.9)–(4.4.11) for $\Lambda \neq \frac{-3}{2}\alpha_2$ is yet to be determined. Note, however, that for any given 5D cosmological constant Λ, we may choose $\alpha_2 = \frac{-2}{3}\Lambda$, and conversely.

4.5 Bulk form 2

In this section we consider the following ansatz for the bulk metric (4.2.1):

$$\bar{g}_{ik} = A(y, r)g_{ik}, \qquad (4.5.1)$$

where A is an unknown function of y and r, and $A(0, r) = 1$. We are interested in determining what 4D spherically symmetric spacetimes admit a 5D Einstein embedding space of this metric form. The propagation equation (4.2.2) becomes

$$\left(\ddot{A} + \frac{\dot{A}^2}{A} + \frac{4\epsilon\Lambda}{3}A\right)g_{ik} = -2\epsilon\bar{R}_{ik}. \qquad (4.5.2)$$

For $i = 0$ and $k = 1$ we have $\bar{R}_{01} = 0$. From metric (4.5.1) and the Ricci tensor components for a SS spacetime, we calculate that

$$\bar{R}_{01} = R_{01} - \frac{A'}{A}\lambda_t = -\lambda_t\left(\frac{2}{r} + \frac{A'}{A}\right).$$

So $\bar{R}_{01} = 0$ implies that either $\lambda_t = 0$ or $\frac{A'}{A} = -\frac{2}{r}$. The latter equation can be solved to obtain $A(y, r) = \frac{G(y)}{r^2}$, but this cannot satisfy the initial condition $A(0, r) = \frac{G(0)}{r^2} = 1$. Thus, we must have $\lambda_t = 0$.

Setting $i = 0 = k$ and $i = 2 = k$ in equation (4.5.2) yields

$$\ddot{A} + \frac{\dot{A}^2}{A} + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\bar{R}_{00}g^{00},$$

$$\ddot{A} + \frac{\dot{A}^2}{A} + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\bar{R}_{22}g^{22},$$

respectively, and indicates that $\bar{R}_{00}g^{00} = \bar{R}_{22}g^{22}$. Using metric (4.5.1) and $\lambda_t = 0$, we determine that

$$\bar{R}_{00}g^{00} = e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2}{r}\nu'\right) + e^{-2\lambda}\left(\frac{A''}{2A} + \frac{3A'}{2A}\nu' - \frac{A'}{2A}\lambda' + \frac{A'}{rA}\right),$$

$$\bar{R}_{22}g^{22} = e^{-2\lambda}\left(\frac{\nu'}{r} - \frac{\lambda'}{r} + \frac{1}{r^2} - \frac{e^{2\lambda}}{r^2}\right) + e^{-2\lambda}\left(\frac{A''}{2A} + \frac{A'}{2A}\nu' - \frac{A'}{2A}\lambda' + \frac{2A'}{rA}\right).$$

After equating the above two equations and simplifying we obtain

$$\nu'' + \nu'^2 - \nu'\lambda' + \frac{\nu'}{r} + \frac{\lambda'}{r} + \frac{e^{2\lambda} - 1}{r^2} = \left(-\nu' + \frac{1}{r}\right)\frac{A'}{A}.$$

We deduce that $\frac{A'}{A} = H_1(r)$, where H_1 is a function of r only. This equation admits the solution $A = H_2(y)\exp(\int H_1(r)dr)$ which, however, cannot satisfy the initial condition $A(0, r) = H_2(0)\exp(\int H_1 dr) = 1$ unless $H_1 = 0$. This implies that $A' = 0$. Thus, $\bar{g}_{ik} = A(y)g_{ik}$ and we regain the result obtained in section 4.4.1. In fact, we can describe the general Schwarzschild-de Sitter model as the only possible spherically symmetric embedded space for a 5D Einstein bulk $\bar{g}_{ik} = A(y, r)g_{ik}$, and a solution for A is given by (4.4.6).

4.6 Embedding spherically symmetric spacetimes in retarded time coordinates

The usual metric for 4D spherically symmetric spacetimes

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (4.6.1)$$

can be written in the form

$$ds^2 = \sigma(v, r)dv^2 - 2\mu(v, r)dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (4.6.2)$$

via the relations

$$dv = f^{-1}e^\nu dt - f^{-1}e^\lambda dr,$$
$$\sigma = -f^2, \quad \mu = fe^\lambda,$$

where v represents retarded time and f is an integrating factor such that dv is an exact differential (Lindquist et al. 1965). The class of exact solutions with form (4.6.2) includes the general Vaidya-de Sitter model (Vaidya 1953, 1966; Mallett 1985):

$$ds^2 = -\mu^2(v)\left(1 - \frac{2m(v)}{r} - \frac{\omega}{3}r^2\right)dv^2 - 2\mu(v)dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (4.6.3)$$

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which describes the exterior region of a radiating star with cosmological constant ω. The mass term m is an arbitrary function of v, and the energy-momentum tensor is given by $T^{ij} = \rho u^i u^j$, where u^i is a null vector having direction radially outward and ρ denotes the energy density of the radiation. A more generalized version of this model is the monopole-de Sitter charged Vaidya spacetime (Wang and Wu 1999; Bonnor and Vaidya 1970) given by

$$ds^2 = -\left(K - \frac{2m(v)}{r} - \frac{\omega}{3}r^2 + \frac{4\pi e^2(v)}{r^2}\right)dv^2 - 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (4.6.4)$$

where $K = 1 - 8\pi G\eta^2$, $\eta \approx 10^{16}$ GeV, and $e(v)$ denotes an electric charge.

With m constant, the general Vaidya-de Sitter spacetime (4.6.3) is empty (i.e. it is an Einstein space) and can be written in the spherically symmetric form (4.6.1) as the general Schwarzschild-de Sitter spacetime

$$ds^2 = -e^{2g(t)}\left(1 - \frac{2m}{r} - \frac{\omega}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{\omega}{3}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Thus, the general Vaidya-de Sitter model with $m \in \mathbb{R}$ can be embedded into a 5D Einstein bulk $d\tilde{s}^2 = A(y)ds^2 + \epsilon dy^2$ with $A(y)$ given by (4.4.6). By our analysis of the embedding of the general Schwarzschild-de Sitter model (refer to section 4.4.1), we deduce that the $m = constant$ general Vaidya-de Sitter solution is the only spherically symmetric spacetime of type (4.6.2) that is embeddable into a 5D Einstein bulk of the form $\bar{g}_{ik} = A(y)g_{ik}$.

We further investigate other metric forms for Einstein embeddings of SS spacetimes type (4.6.2). We consider

$$\bar{g}_{ik} = A(y,v)g_{ik}, \quad (4.6.5)$$

$$\bar{g}_{ik} = A(y)g_{00}\delta_i^0\delta_k^0 + A(y)g_{22}\delta_i^2\delta_k^2 + A(y)g_{33}\delta_i^3\delta_k^3 + B(y)g_{01}\delta_i^0\delta_k^1 + B(y)g_{10}\delta_i^1\delta_k^0, \quad (4.6.6)$$

where $A(0,v) = 1$ in (4.6.5) and $A(0) = 1 = B(0)$ in (4.6.6), and follow the same technique employed in sections 4.3 and 4.5. For both metric ansatzen, the propagation equation (4.2.2) with $i = 1 = k$ implies that $\mu' = 0$. Then, for metric (4.6.5), the propagation equation for $i = 0$, $k = 1$ and $i = 2 = k$ becomes

$$\ddot{A} + \frac{\dot{A}^2}{A} + \frac{4\epsilon\Lambda}{3}A = \frac{\epsilon\sigma''}{\mu^2} + \frac{2\epsilon\sigma'}{r\mu^2} + \frac{2\epsilon}{r\mu}\frac{A_v}{A},$$

$$\ddot{A} + \frac{\dot{A}^2}{A} + \frac{4\epsilon\Lambda}{3}A = \frac{2\epsilon}{r^2} + \frac{2\epsilon\sigma'}{r\mu^2} + \frac{2\epsilon\sigma}{r^2\mu^2} + \frac{4\epsilon}{r\mu}\frac{A_v}{A}.$$

Setting the right-hand sides of these two equations equal and simplifying the result, we obtain

$$\frac{r\sigma''}{2\mu} - \frac{\sigma}{r\mu} - \frac{\mu}{r} = \frac{A_v}{A}.$$

Since the left-hand side of the above equation does not depend on y, we must have $\frac{A_v}{A} = g(v)$, a function of v only. Solving this constraint with the condition $A(0, v) = 1$ yields $\frac{\partial A}{\partial v} = 0$, and so $A = A(y)$. For the metric (4.6.6), the propagation equation (4.2.2) with $i = 2 = k$ reduces to

$$\ddot{A} + \frac{\dot{A}\dot{B}}{B} + \frac{4\epsilon\Lambda}{3}A = 2\epsilon\left[\frac{A^2}{B^2}\left(\frac{\sigma'}{r\mu^2} + \frac{\sigma}{r^2\mu^2}\right) + \frac{1}{r^2}\right],$$

which shows that

$$\frac{A^2}{B^2}\left(\frac{\sigma'}{r\mu^2} + \frac{\sigma}{r^2\mu^2}\right) + \frac{1}{r^2} = h(y),$$

where h is a function of y only. Taking $y = 0$ in the above expression and using the initial conditions $A(0) = 1 = B(0)$, we observe that $\frac{\sigma'}{r\mu^2} + \frac{\sigma}{r^2\mu^2} = h(0) - r^{-2}$. Then the above expression can be rewritten as

$$\frac{A^2}{B^2}h(0) - h(y) = \left(\frac{A^2}{B^2} - 1\right)\frac{1}{r^2},$$

which only makes sense when $A = B$ and $h(y) = h(0)$. Thus, the analyses for the metrics (4.6.5) and (4.6.6) show that we regain the case $\bar{g}_{ik} = A(y)g_{ik}$. Hence, the $m = constant$ general Vaidya-de Sitter model is the only spherically symmetric spacetime of type (4.6.2) that can be embedded into a 5D Einstein bulk of type (4.6.5) or (4.6.6).

The investigation of more complicated bulk metric forms provides a programme of further research. It would be especially interesting to embed the monopole-de Sitter charged Vaidya spacetime (4.6.4), and to study exact solutions for embeddings of collapse and radiation. This is, however, a highly non-trivial task and may require improved techniques in solving non-linear partial differential equations.

4.7   Summary

In this chapter we studied local isometric embeddings of 4D spherically symmetric spacetimes, in both the usual metric form (4.6.1) and the retarded time coordinate form (4.6.2), into 5D Einstein bulks. We emphasized that the embedding equations (4.2.2)–(4.2.4) are very difficult to solve, even with specific choices of non-vacuum SS spacetimes. This problem led us to modify our approach, and to investigate what SS spacetimes can embed into Einstein bulks with particular metric forms. We chose the bulk metrics (4.3.1) and (4.5.1) for embeddings of SS spacetimes type (4.6.1), and the bulk metrics (4.6.5) and (4.6.6) for embeddings of SS spacetimes in retarded time coordinates. The embedding analysis with the metric form (4.3.1) showed that the only 4D solutions are the general Schwarzschild-de Sitter spacetime (4.4.1) and the Einstein Universe (4.4.7). The Einstein embedding space $d\bar{s}^2 = A(y)ds^2 + \epsilon dy^2$ (where $A(y)$ is given by (4.4.6)) of the general Schwarzschild-de Sitter spacetime is already known, and we explained how it is obtained. We determined that the Einstein Universe can be embedded into the 5D flat bulk (4.4.14), and verified that the 5D metric

can be transformed into Minkowski form. This regains the result by Wesson (1994), although a different method was used there. Furthermore, the Einstein Universe has an embedding into the 5D Einstein bulk (4.4.16) with cosmological constant $\Lambda = \frac{-3}{2}\alpha_2$, where $A(y)$ is given by (4.4.15). This solution is a new result, and was obtained with collaborators via Lie analysis techniques. For the metric form (4.5.1), we recovered the case of embedding the general Schwarzschild-de Sitter spacetime into a bulk with $\bar{g}_{ik} = A(y)g_{ik}$. The embedding analyses with the bulk metrics (4.6.5) and (4.6.6) yielded the general Vaidya-de Sitter model (4.6.3) with constant mass as the only 4D solution. Its Einstein embedding space, which is well known, is $d\tilde{s}^2 = A(y)ds^2 + \epsilon dy^2$, where $A(y)$ is specified by (4.4.6). In the next chapter, we consider symmetry inheritance properties in embedding theory.

Chapter 5

Killing Geometry in Higher Dimensions

5.1   Introduction

So far we have been concerned with finding solutions for embedding spacetimes into higher dimensions. It is also important to study the properties of embedding spaces for both mathematical interest and astrophysical/cosmological consequences. Since static spherically symmetric spacetimes provide good first descriptions of astrophysical objects such as black holes and stars, an analysis of their embedding properties allows for a confrontation between higher dimensional cosmological models and astrophysics (Amery et al. 2011). The conformal and Killing geometries of a spacetime can be useful for various reasons such as providing insight into symmetries and conservation laws, which can improve our understanding of a spacetime's structure. It is therefore of interest to investigate any relationship between the conformal geometries of embedded and embedding spaces. In this chapter we shall study the (conformal) Killing geometry of a 5D Einstein space that embeds a static spherically symmetric (SSS) spacetime along a hypersurface. We begin in section 5.2 with a review of important concepts, providing definitions of various types of conformal geometry in section 5.2.1 and noting some results for decomposable spaces in section 5.2.2. In section 5.3 we discuss the methodology used to obtain the conformal Killing vectors for the embedded SSS space and its Einstein bulk, and we highlight the equations to be solved for five-dimensional Killing vectors. We demonstrate that there are 5D Killing vectors inheriting the four-dimensional ones in section 5.4. These vectors are shown, in section 5.5, to be the only hypersurface-like Killing vectors of the general embedding space. Further comments regarding the general conformal geometry of the bulk are also made in section 5.5, and a summary of our main results is provided in section 5.6. The original results contained in this chapter have been published in Amery et al. (2011).

5.2 Review

5.2.1 Definition and properties of conformal Killing vectors

A conformal Killing vector (CKV) $\mathbf{X}$ of a metric space is defined by the action of the Lie derivative on the metric tensor $\mathbf{g}$ (Stephani et al. 2003):

$$\mathcal{L}_{\mathbf{X}} g_{ij} = 2\psi(\mathbf{x}) g_{ij} \,, \tag{5.2.1}$$

where ψ is the conformal factor. If $\psi = 0$, then $\mathbf{X}$ is a proper Killing vector (KV), and if $\psi_{,a} = 0 \neq \psi$, then $\mathbf{X}$ is called a homothetic Killing vector (HKV). Applying the Lie derivative, equation (5.2.1) can be written as

$$X_{i;j} + X_{j;i} = 2\psi(\mathbf{x}) g_{ij} \,. \tag{5.2.2}$$

Conformal Killing vectors generate isometries along null geodesics and are useful in simplifying the field/embedding equations, in the classification of spacetimes (Stephani et al. 2003), and in applications to, for instance, perturbation theory (Katz et al. 1997; Amery and Shellard 2003) and singularity theorems (Hawking and Ellis 1973; Joshi 1993). Homothetic Killing vectors scale distances by a constant factor, preserve the null geodesic affine parameter, and are related to self-similarity (Stephani et al. 2003). Proper Killing vectors characterize the (continuous) symmetry properties of pseudo-Riemannian spaces in an invariant fashion: they generate first integrals along time-like geodesics via Noether's theorem, and may be used to investigate the physical properties of a spacetime, via the structure of their isometry group (Stephani et al. 2003).

Recall the concept of intrinsic rigidity explained in section 2.2.4. The Killing geometry of a intrinsically rigid manifold is related to the extrinsic curvature Ω_{ij}, when embedding into some higher dimensional space, in an interesting way:

$$\mathcal{L}_{\mathbf{X}} \Omega_{ij} = 0,$$

for a Killing vector $\mathbf{X}$ (Goenner 1980; Londal 2005). This property follows easily from the fact that the extrinsic curvature depends only on the metric and its derivatives, by the commutativity of the Lie and partial derivatives, and by the definition $\mathcal{L}_{\mathbf{X}} g_{ij} = 0$. For SSS spacetimes, the above condition can be used to show that the extrinsic curvature in the bulk must depend on r and y only (Londal 2005).

5.2.2 Decomposable spaces

Some results have been obtained for the conformal Killing geometry of 'decomposable metrics' (Apostolopolous and Carot 2005). A $(m+n)$-decomposable space is a product space $P = M \times N$ endowed with a decomposable metric

$$\tilde{g}_{\mu\nu}(x^k, x^\Pi) = g_{ij}(x^k)\delta_\mu^i \delta_\nu^j + \hat{g}_{\Upsilon\Xi}(x^\Pi)\delta_\mu^\Upsilon \delta_\nu^\Xi \,,$$

where $\mathbf{g}$ and $\hat{\mathbf{g}}$ are metrics corresponding to the m- and n-dimensional spaces M and N, respectively. In Gaussian coordinates, the $(n + 1)$-dimensional embedding space

(2.4.1) is decomposable if and only if $\frac{\partial \bar{g}_{ij}}{\partial y}$ vanishes, so that $\bar{g}_{ij}(x^k, y) = g_{ij}(x^k)$. The decomposable space P has the following inheritance properties:

- Killing vectors for M and N are also Killing vectors for P;

- Suppose M and N have HKVs $H_i(x^k)$ and $\hat{H}_\Upsilon(x^\Pi)$ with conformal factors b and $\hat{b}$. Then $H_\mu = \hat{b} H_i \delta_\mu^i + b \hat{H}_\Upsilon \delta_\mu^\Upsilon$ is a HKV for P with conformal factor $b\hat{b}$. So P admits a HKV if and only if both M and N do;

- P will possess a proper CKV with conformal factor ψ if M and N admit gradient CKVs $\psi_{,i}$ and $\psi_{,\Upsilon}$, respectively.

A $(m + 1)$-dimensional decomposable space is a generalization of a 'stacking' embedding $M \times AdS_{(1)}$, which is a typical description of, for example, black strings (Horowitz and Strominger 1991; Gregory and Laflamme 1993, 1994). However, there are many spacetimes of interest that are not decomposable, such as in the brane-world scenarios (Randall and Sundrum 1999a,b) which involve more general warped Lorentzian manifolds — $(M, \mathbf{g})$ with $M = O \times S$, $\mathbf{g} = \mathbf{g}_1 \otimes Y \mathbf{g}_2$ for submanifolds $(O, \mathbf{g}_1)$, $(S, \mathbf{g}_2)$, and Y a function defined on O. There the metric may be transformed so that it is conformally related to a decomposable metric. This can be useful since conformal transformations map CKVs to CKVs (Stephani et al. 2003). However, KVs are generally mapped to CKVs, so one still has to explicitly solve for the Killing geometry (Amery et al. 2011). Furthermore, the Einstein bulk we shall consider in this chapter is generally not conformal to a decomposable metric, unless the embedded space is Ricci flat.

5.3 Methodology

We focus on embeddings for 4D SSS spacetimes with metric

$$g_{ij} = \text{diag}[-e^{2\nu(r)}, e^{2\lambda(r)}, r^2, r^2 \sin^2\theta], \tag{5.3.1}$$

into a 5D Einstein bulk of the form

$$d\tilde{s}^2 = \bar{g}_{ij} dx^i dx^j + \epsilon dy^2,$$
$$\bar{g}_{ij} = A(y, r) g_{00} \delta_i^0 \delta_j^0 + B(y, r) g_{CD} \delta_i^C \delta_j^D, \tag{5.3.2}$$

where, in general, A and B are functions of y and r that are unlikely to be separable. The embedding space must satisfy the initial conditions $A(0, r) = 1 = B(0, r)$, $\dot{A}(0, r) g_{00} = -2\,\Omega_{00}$ and $\dot{B}(0, r) g_{CD} = -2\,\Omega_{CD}$, where a solution to the Codazzi-Gauss equations (2.4.6) and (2.4.7) for Ω_{ij} has been obtained (Londal 2005) and includes the solution that appears in chapter 4 in the case $A = B$.

We may compute the non-zero connection coefficients (2.2.1) for metric (5.3.2):

$$\tilde{\Gamma}^4{}_{ij} = -\frac{\epsilon}{2}\bar{g}_{ij,4} \, ,$$

$$\tilde{\Gamma}^i{}_{4j} = \frac{1}{2}\bar{g}^{ik}\bar{g}_{kj,4} \, , \tag{5.3.3}$$

$$\tilde{\Gamma}^i{}_{jk} = \bar{\Gamma}^i{}_{jk} = \frac{1}{2}\bar{g}^{in}\left(\bar{g}_{jn,k} + \bar{g}_{nk,j} - \bar{g}_{jk,n}\right) \, ,$$

and for metric (5.3.1):

$$\begin{array}{lll}
\Gamma^0{}_{01} = \nu', & \Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)}, & \Gamma^1{}_{11} = \lambda', \\
\Gamma^1{}_{22} = -re^{-2\lambda}, & \Gamma^1{}_{33} = -r\sin^2\theta \, e^{-2\lambda}, & \Gamma^2{}_{12} = 1/r, \\
\Gamma^2{}_{33} = -\sin\theta\cos\theta, & \Gamma^3{}_{13} = 1/r, & \Gamma^3{}_{23} = \cot\theta.
\end{array} \tag{5.3.4}$$

Suppose that $\mathbf{X}^{(5)} = (X_0^{(5)}, X_1^{(5)}, X_2^{(5)}, X_3^{(5)}, X_4^{(5)})$ is a CKV for the embedding spacetime (5.3.2) with conformal factor $\psi(y, x^c)$, and that $\mathbf{X}^{(4)} = (X_0^{(4)}, X_1^{(4)}, X_2^{(4)}, X_3^{(4)})$ is a CKV for the embedded spacetime (5.3.1) with conformal factor $\varphi(x^c)$, where the bracketed superscript is a label, not a tensorial index. The defining equations for a CKV corresponding to spaces (5.3.2) and (5.3.1) are

$$X_{\mu;\nu}^{(5)} + X_{\nu;\mu}^{(5)} = X_{\mu,\nu}^{(5)} + X_{\nu,\mu}^{(5)} - 2\tilde{\Gamma}^\sigma{}_{\mu\nu}X_\sigma^{(5)} = 2\psi(y, x^c)\tilde{g}_{\mu\nu} \, , \tag{5.3.5}$$

$$X_{i;j}^{(4)} + X_{j;i}^{(4)} = X_{i,j}^{(4)} + X_{j,i}^{(4)} - 2\Gamma^k{}_{ij}X_k^{(4)} = 2\varphi(x^c)g_{ij} \, , \tag{5.3.6}$$

respectively.

The conformal geometry of a SSS spacetime is well known (Maartens et al. 1995, 1996). The conformal Killing vectors satisfy the equations

$$X_{0,0}^{(4)} - \Gamma^1{}_{00}X_1^{(4)} = \varphi g_{00} \, ,$$

$$X_{0,C}^{(4)} + X_{C,0}^{(4)} - 2\Gamma^0{}_{01}X_0^{(4)}\delta_C^1 = 0 \, ,$$

$$X_{C,D}^{(4)} + X_{D,C}^{(4)} - 2\Gamma^E{}_{CD}X_E^{(4)} = 2\varphi g_{CD} \, ,$$

and the Killing vectors (when $\varphi = 0$) were found to be

$$Y_i^{(0)} = (e^{2\nu}, 0, 0, 0), \tag{5.3.7}$$

$$Y_i^{(1)} = (0, 0, 0, r^2\sin^2\theta), \tag{5.3.8}$$

$$Y_i^{(2)} = (0, 0, r^2\sin\phi, r^2\sin\theta\cos\theta\cos\phi), \tag{5.3.9}$$

$$Y_i^{(3)} = (0, 0, -r^2\cos\phi, r^2\sin\theta\cos\theta\sin\phi). \tag{5.3.10}$$

Using the metric connections (5.3.3), we rewrite (5.3.5) as

$$X_{4,4}^{(5)} = \epsilon\psi(y,x^c)\,, \tag{5.3.11}$$

$$X_{4,i}^{(5)} + X_{i,4}^{(5)} - \bar{g}^{lk}\bar{g}_{ki,4}X_l^{(5)} = 0\,, \tag{5.3.12}$$

$$X_{i,j}^{(5)} + X_{j,i}^{(5)} - \bar{g}^{kn}(\bar{g}_{in,j} + \bar{g}_{nj,i} - \bar{g}_{ij,n})X_k^{(5)} + \epsilon\bar{g}_{ij,4}X_4^{(5)} = 2\psi(y,x^c)\bar{g}_{ij}\,. \tag{5.3.13}$$

The system (5.3.11)–(5.3.13) must be solved to obtain a CKV for the five-dimensional bulk (5.3.2). Just like the case for the embedding equations observed in earlier chapters, the higher dimensional conformal Killing equations are complex and difficult to solve using standard techniques. We therefore shall restrict our attention to five-dimensional KVs (i.e. when $\psi = 0$).

Setting $\psi = 0$, equation (5.3.11) yields

$$X_{4,4}^{(5)} = 0, \tag{5.3.14}$$

so that $X_4^{(5)}$ depends on x^i only. Applying metric (5.3.2), we rewrite equation (5.3.12) as the four equations

$$X_{0,4}^{(5)} + X_{4,0}^{(5)} = \frac{\dot{A}}{A}X_0^{(5)}\,, \tag{5.3.15}$$

$$X_{C,4}^{(5)} + X_{4,C}^{(5)} = \frac{\dot{B}}{B}X_C^{(5)}\,, \tag{5.3.16}$$

and equation (5.3.13) as the ten equations

$$2X_{0,0}^{(5)} - \frac{2A}{B}\Gamma^1{}_{00}X_1^{(5)} + \frac{A'}{B}g^{11}g_{00}X_1^{(5)} + \epsilon\dot{A}g_{00}X_4^{(5)} = 0\,, \tag{5.3.17}$$

$$X_{0,C}^{(5)} + X_{C,0}^{(5)} - 2\Gamma^0{}_{01}X_0^{(5)}\delta_C^1 - \frac{A'}{A}X_0^{(5)}\delta_C^1 = 0\,, \tag{5.3.18}$$

$$X_{C,D}^{(5)} + X_{D,C}^{(5)} - 2\Gamma^E{}_{CD}X_E^{(5)} - \frac{1}{B}\left(B_{,D}\delta_C^E + B_{,C}\delta_D^E - B_{,F}g^{EF}g_{CD}\right)X_E^{(5)} + \epsilon\dot{B}g_{CD}X_4^{(5)} = 0\,, \tag{5.3.19}$$

where $\Gamma^k{}_{ij}$ is specified by (5.3.4). The set of equations (5.3.14)–(5.3.19) provide the five-dimensional KVs for the bulk (5.3.2) embedding a SSS spacetime.

5.4 Inheritance of four-dimensional Killing geometry

Here we describe how the Killing geometry (5.3.7)–(5.3.10) of a SSS spacetime is inherited by the five-dimensional Einstein space (5.3.2) into which it is embedded. This is to be expected since we have considered an energetically rigid embedding (Amery et al. 2011).

5.4.1 Decomposable bulk metric

A four-dimensional Ricci flat spacetime has an embedding into a Ricci flat bulk (5.3.2) with $A = B = 1$ (Londal 2005). So

$$d\tilde{s}^2 = g_{ij}(x^k)dx^i dx^j + \epsilon dy^2,$$

and the embedding space is decomposable. The results obtained by Apostolopolous and Carot (2005) are applicable here — see section 5.2.2. The four-dimensional Killing geometry is trivially inherited by the bulk: $\mathbf{X}^{(5)} = (\mathbf{X}^{(4)}, 0)$ with $\mathbf{X}^{(4)}$ given by (5.3.7)–(5.3.10).

5.4.2 Case $A(y,r) = B(y,r)$

Consider the case when $A(y,r) = B(y,r)$. Our analysis in chapter 4 shows that the only possible embedded SSS spacetime is the general Schwarzschild-de Sitter model, and the five-dimensional Einstein embedding space has the form $\bar{g}_{ij} = A(y)g_{ij}$ with $A(y)$ specified by (4.4.6). The Killing equations (5.3.14)–(5.3.19) become

$$\dot{X}_4^{(5)} = 0, \tag{5.4.1}$$

$$\dot{X}_{i,4}^{(5)} + X_{4,i}^{(5)} = \frac{\dot{A}}{A} X_i^{(5)}, \tag{5.4.2}$$

$$X_{i,j}^{(5)} + X_{j,i}^{(5)} - 2\Gamma_{ij}^k X_k^{(5)} + \epsilon \dot{A} g_{ij} X_4^{(5)} = 0. \tag{5.4.3}$$

Set $X_4^{(5)} = 0$. Equation (5.4.1) is trivially satisfied. Equation (5.4.2) reduces to a separable differential equation which admits the solutions $X_i^{(5)} = f_i(x^c)A(y)$, where the f_i are unknown functions. Substituting this expression for $X_i^{(5)}$ into (5.4.3) implies that

$$f_{i,j} + f_{j,i} - 2\,\Gamma_{ij}^k f_k = 0,$$

which is equivalent to the defining equation (5.3.6) for a 4D KV ($\varphi = 0$). So we must have $f_i = X_i^{(4)}$, a Killing vector of the four-dimensional SSS spacetime. Hence,

$$\mathbf{X}^{(5)} = (A(y)\,\mathbf{X}^{(4)},\, 0), \tag{5.4.4}$$

where $A(y)$ is given by (4.4.6) and $\mathbf{X}^{(4)} \in \{\mathbf{Y}^{(0)}, \mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}, \mathbf{Y}^{(3)}\}$, the 4D KVs (5.3.7)–(5.3.10). This proves the inheritance of the 4D Killing geometry by the five-dimensional Einstein embedding space.

5.4.3 Case $R = R(r)$

Now we investigate the inheritance properties of the Einstein embedding space (5.3.2) for the general case in which R may depend on r and $A(y,r)$ may not equal $B(y,r)$. Suppose that

$$\mathbf{X}^{(5)} = (P(x^i, y)\,\mathbf{X}^{(4)},\, H(x^i, y)),$$

where $\mathbf{X}^{(4)}$ is a KV (5.3.7)–(5.3.10) of the embedded SSS spacetime, and P and H are unknown functions. Substituting the above expression into the system (5.3.14)–(5.3.19), we obtain the following set of equations that must be solved for P and H:

$$\dot{H} = 0\,,$$

$$H_{,0} + \dot{P}X_0^{(4)} - \frac{\dot{A}}{A}PX_0^{(4)} = 0\,, \qquad H_{,C} + \dot{P}X_C^{(4)} - \frac{\dot{B}}{B}PX_C^{(4)} = 0\,,$$

$$P_{,0}X_0^{(4)} + \frac{\epsilon\dot{A}}{2}g_{00}H = 0\,, \qquad P_{,C}X_0^{(4)} + P_{,0}X_C^{(4)} - \frac{A'}{A}PX_0^{(4)}\delta_C^1 = 0\,,$$

$$P_{,D}X_C^{(4)} + P_{,C}X_D^{(4)} - \frac{B'}{B}\left(\delta_D^1\delta_C^2 + \delta_C^1\delta_D^2\right)PX_2^{(4)}$$

$$- \frac{B'}{B}\left(\delta_D^1\delta_C^3 + \delta_C^1\delta_D^3\right)PX_3^{(4)} + \epsilon\dot{B}g_{CD}H = 0\,. \qquad (5.4.5)$$

Here we have used the fact that for all four KVs of a SSS spacetime, $X_1^{(4)} = 0$ and $X_{0,0}^{(4)} = 0$. Setting $C = 1 = D$ in equation (5.4.5) yields $H = 0$. Thus, any inherited KVs are hypersurface-like. Then the above system becomes

$$\left(\dot{P} - \frac{\dot{A}}{A}P\right)X_0^{(4)} = 0\,, \qquad\qquad \left(\dot{P} - \frac{\dot{B}}{B}P\right)X_C^{(4)} = 0\,,$$

$$P_{,0}X_0^{(4)} = 0\,, \qquad P_{,C}X_0^{(4)} + P_{,0}X_C^{(4)} - \frac{A'}{A}PX_0^{(4)}\delta_C^1 = 0\,,$$

$$P_{,2}X_2^{(4)} = P_{,3}X_3^{(4)} = 0\,, \qquad\qquad P_{,2}X_3^{(4)} + P_{,3}X_2^{(4)} = 0\,,$$

$$\left(P' - \frac{B'}{B}P\right)X_2^{(4)} = 0\,, \qquad\qquad \left(P' - \frac{B'}{B}P\right)X_3^{(4)} = 0\,.$$

For $\mathbf{X}^{(4)} = \mathbf{Y}^{(0)} = (e^{2\nu}, 0, 0, 0)$ we have

$$\dot{P} = \frac{\dot{A}}{A}P\,, \qquad P' = \frac{A'}{A}P\,, \qquad \text{and} \qquad P_{,0} = P_{,2} = P_{,3} = 0.$$

These equations are easily solved to obtain $P(y, r) = k_0 A(y, r)$, $k_0 \in \mathbb{R}$. Thus,

$$\mathbf{X}^{(5)} = (k_0\, A(y, r)\mathbf{Y}^{(0)}, 0)\,, \quad k_0 \in \mathbb{R}. \qquad (5.4.6)$$

For any of the three Killing vectors $\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}, \mathbf{Y}^{(3)}$ we have

$$\dot{P} = \frac{\dot{B}}{B}P\,, \qquad P' = \frac{B'}{B}P\,, \qquad \text{and} \qquad P_{,0} = P_{,2} = P_{,3} = 0,$$

which yields the solution $P(y, r) = k_1 B(y, r)$, $k_1 \in \mathbb{R}$, and so

$$\mathbf{X}^{(5)} = (k_1\, B(y, r)\mathbf{X}^{(4)}, 0)\,, \quad k_1 \in \mathbb{R}, \qquad (5.4.7)$$

with $\mathbf{X}^{(4)} \in \{\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}, \mathbf{Y}^{(3)}\}$.

We have thus provided an explicit demonstration of the inherited Killing geometry for a 5D Einstein bulk (5.3.2) embedding a SSS spacetime. Note that with $A = B$ we regain the results of section 5.4.2.

5.5 General considerations

5.5.1 Remarks on general Killing geometry of the bulk

Here we consider the 5D embedding space (5.3.2) with $R = R(r)$ in general, and we investigate the existence of Killing vectors other than the inherited SSS ones covered in section 5.4.

First we observe that by setting $X_i^{(5)} = 0$, equations (5.3.17) and (5.3.19) imply that either $X_4^{(5)} = 0$ or $\dot{A} = 0 = \dot{B}$. In the latter case, the initial conditions of the embedding further show that $A = 1 = B$, and this corresponds to the case of a decomposable bulk embedding a Ricci flat spacetime — see section 5.4.1. There the remaining Killing equations indicate that $X_4^{(5)}$ is a constant, so that $\mathbf{X}^{(5)} = (0, 0, 0, 0, k)$, $k \in \mathbb{R}$, is a Killing vector. In general we must have $X_4^{(5)} = 0$, and so there does not exist any Killing vector in the y-direction. This is sensible as such a KV would result in y-independence for the extrinsic curvature Ω_{ij} and hence $\tilde{g}_{\mu\nu}$, which contradicts the general form (5.3.2) for the bulk metric, and yields a non-Einsteinian stacking (Amery et al. 2011).

Now we consider $X_4^{(5)} = 0$ and $X_i^{(5)}$ not all vanishing. From equation (5.3.19) with $C = 1 = D$ we deduce that

$$\frac{X_{1,1}^{(5)}}{X_1} = \lambda' + \frac{B'}{2B},$$

which gives the solution

$$X_1^{(5)} = f(y, t, \theta, \phi) \, e^\lambda \, B^{1/2}.$$

Using the above expression for $X_1^{(5)}$ in equation (5.3.16) with $C = 1$ shows that

$$\frac{\dot{f}}{f} = \frac{\dot{B}}{2B}$$
$$\Rightarrow \quad f(y, t, \theta, \phi) = g(t, r, \theta, \phi) \, (B(y, r))^{1/2}.$$

For $gB^{1/2}$ to be independent of r, B must be separable, and with $B(0, r) = 1$, this means that $B = B(y)$. However, this is not generally true, so g must vanish, forcing $X_1^{(5)} = 0$. Similarly, taking $X_1^{(5)} = 0$ in equation (5.3.19) with $C = 1 = D$ yields $X_4^{(5)} = 0$ or $B = 1$. Hence, for the general $R = R(r)$ case, $X_1^{(5)} = 0 \iff X_4^{(5)} = 0$.

Next we present a full analysis of the five-dimensional Killing vectors that are hypersurface-like. This is motivated by the fact that all the inherited SSS KVs investigated in section 5.4 are hypersurface-like. We take $X_4^{(5)} = 0$, which immediately implies that $X_1^{(5)} = 0$ by the previous argument, and we proceed by choosing $\mathbf{X}^{(5)}$ to

consist of only one non-zero component, then two non-zero components, and so on. We list only those cases yielding Killing vectors. Consider $X_\mu^{(5)} = X_0^{(5)} \delta_\mu^0$. The Killing equations (5.3.15)–(5.3.19) are simplified to

$$\dot{X}_0^{(5)} = \frac{\dot{A}}{A} X_0^{(5)} ,$$

$$X_{0,1}^{(5)} = 2\nu' X_0^{(5)} + \frac{A'}{A} X_0^{(5)} ,$$

$$X_{0,0}^{(5)} = X_{0,2}^{(5)} = X_{0,3}^{(5)} = 0 ,$$

and we obtain the solution $X_0^{(5)} = k_0 \, e^{2\nu} A(y, r)$, $k_0 \in \mathbb{R}$, which was already presented in section 5.4.3 as $\mathbf{X}^{(5)} = (k_0 \, A(y, r) \mathbf{Y}^{(0)}, \, 0)$. Similarly, taking $X_\mu^{(5)} = X_3^{(5)} \delta_\mu^3$ leads to the solution $\mathbf{X}^{(5)} = (k_1 \, B(y, r) \mathbf{Y}^{(1)}, \, 0)$, $k_1 \in \mathbb{R}$.

Now take $X_\mu^{(5)} = X_0^{(5)} \delta_\mu^0 + X_3^{(5)} \delta_\mu^3$. Substituting this into equations (5.3.15)–(5.3.19) and solving the system yields the Killing vector

$$\mathbf{X}^{(5)} = (k_0 \, e^{2\nu} A(y, r), \, 0, \, 0, \, k_1 \, r^2 \sin^2 \theta B(y, r), \, 0)$$
$$= (k_0 \, A \mathbf{Y}^{(0)} + k_1 \, B \mathbf{Y}^{(1)}, \, 0) , \quad k_0, k_1 \in \mathbb{R} .$$

The case $X_\mu^{(5)} = X_2^{(5)} \delta_\mu^2 + X_3^{(5)} \delta_\mu^3$ is not as straightforward as the previous ones. Here the Killing equations become

$$\dot{X}_2^{(5)} = \frac{\dot{B}}{B} X_2^{(5)}, \qquad\qquad\qquad\qquad \dot{X}_3^{(5)} = \frac{\dot{B}}{B} X_3^{(5)},$$

$$X_{2,1}^{(5)} = \frac{2}{r} X_2^{(5)} + \frac{B'}{B} X_2^{(5)}, \qquad X_{3,1}^{(5)} = \frac{2}{r} X_3^{(5)} + \frac{B'}{B} X_3^{(5)}, \qquad (5.5.1)$$

$$X_{3,3}^{(5)} = -\sin\theta\cos\theta X_2^{(5)}, \qquad X_{2,3}^{(5)} + X_{3,2}^{(5)} = 2\cot\theta X_3^{(5)},$$

with $X_{2,0}^{(5)} = X_{3,0}^{(5)} = X_{2,2}^{(5)} = 0$. The first four equations in (5.5.1) give

$$X_2^{(5)} = f_1(\phi) \, r^2 B(y, r), \quad \text{and} \quad X_3^{(5)} = h_1(\theta, \phi) \, r^2 B(y, r) .$$

Inserting these functions into the last two equations of (5.5.1) leads to:

$$h_1(\theta, \phi) = -\sin\theta\cos\theta \int f_1 d\phi + h_2(\theta) , \qquad\qquad (5.5.2)$$

$$\frac{\partial f_1}{\partial \phi} + \frac{\partial h_1}{\partial \theta} - 2\cot\theta h_1 = 0 , \qquad\qquad (5.5.3)$$

and using equation (5.5.2) in (5.5.3), we obtain

$$\frac{\partial f_1}{\partial \phi} + \int f_1 d\phi = -\frac{\partial h_2}{\partial \theta} + 2\cot\theta h_2 . \qquad\qquad (5.5.4)$$

In equation (5.5.4), the terms on the left depend only on ϕ and the terms on the right depend only on θ, so we may set both sides equal to a constant k (which turns out to be zero) and proceed to solve for f_1, h_2 and h_1. We find that

$$f_1(\phi) = f_2 \cos \phi + f_3 \sin \phi \,,$$
$$h_2(\theta) = h_3 \sin^2 \theta \,,$$
$$h_1(\theta, \phi) = -\sin \theta \cos \theta \, (f_2 \sin \phi - f_3 \cos \phi) + h_3 \sin^2 \theta \,,$$

where f_2, f_3, h_3 are arbitrary constants. Thus, the Killing vector is given by

$$\mathbf{X}^{(5)} = (0,\, 0,\, f_1 r^2 B,\, h_1 r^2 B,\, 0)$$
$$= (f_3 \, B\mathbf{Y}^{(2)} - f_2 \, B\mathbf{Y}^{(3)} + h_3 \, B\mathbf{Y}^{(1)},\, 0)\,, \quad f_2, f_3, h_3 \in \mathbb{R}.$$

Finally, we consider $X_\mu^{(5)} = X_0^{(5)} \delta_\mu^0 + X_2^{(5)} \delta_\mu^2 + X_3^{(5)} \delta_\mu^3$, which is similar to the above scenario. We obtain

$$X_0^{(5)} = k_0 \, e^{2\nu} A(y, r)\,,$$
$$X_2^{(5)} = (k_2 \cos \phi + k_3 \sin \phi)\, r^2 B(y, r)\,,$$
$$X_3^{(5)} = \left[-\sin \theta \cos \theta \, (k_2 \sin \phi - k_3 \cos \phi) + k_1 \sin^2 \theta \right] r^2 B(y, r)\,,$$

where $k_\alpha \in \mathbb{R}$, so that

$$\mathbf{X}^{(5)} = (k_0 \, A\mathbf{Y}^{(0)} + k_3 \, B\mathbf{Y}^{(2)} - k_2 \, B\mathbf{Y}^{(3)} + k_1 \, B\mathbf{Y}^{(1)},\, 0).$$

The above analysis indicates that there are no hypersurface-like Killing vectors other than those inheriting the embedded SSS geometry. This is reasonable since we have imposed the SSS form for the $y = constant$ hypersurfaces. The computation of the Killing geometry with $X_4^{(5)} \neq 0$ is a much more complex task and remains to be carried out, although in general one cannot expect any new Killing vectors since we have not specified the bulk geometry.

5.5.2 Remarks on conformal geometry of the bulk

Here we provide a partial analysis of the 5D conformal geometry with $\psi \neq 0$.

Consider the equations (5.3.11)–(5.3.13) defining a CKV ($\psi \neq 0$) in the 5D embedding space (5.3.2), and take $X_i^{(5)} = 0$. Equation (5.3.12) implies that $X_{4,i}^{(5)} = 0$, so $X_4^{(5)}$ depends on y only. Using equation (5.3.11) in (5.3.13) we get

$$\frac{\dot{A}}{2A} = \frac{\dot{X}_4^{(5)}}{X_4^{(5)}}\,, \qquad (5.5.5)$$

$$\frac{\dot{B}}{2B} = \frac{\dot{X}_4^{(5)}}{X_4^{(5)}}\,,$$

which indicates that $\dot{A}/A = \dot{B}/B$. Solving this equation and applying initial conditions, we deduce that $A = B$. Recall from section 4.5 that when $A = B$, the embedding metric has the form

$$d\tilde{s}^2 = A(y)ds^2 + \epsilon dy^2, \qquad (5.5.6)$$

with $A(y)$ given by (4.4.6) and ds^2 the general Schwarzschild-de Sitter spacetime. Noting that $X_{4,i}^{(5)}$ vanishes, we then solve (5.5.5) to obtain

$$X_4^{(5)} = p\,(A(y))^{1/2}\,, \quad p \in \mathbb{R},$$

and from (5.3.11) we compute the conformal factor $\psi = \epsilon \dot{X}_4^{(5)} = \frac{\epsilon p}{2}\dot{A}A^{-1/2}$. Hence,

$$\mathbf{X}^{(5)} = \left(0,\, 0,\, 0,\, 0,\, p\,(A(y))^{1/2}\right), \quad p \in \mathbb{R}, \qquad (5.5.7)$$

where $A(y)$ is specified by (4.4.6), is a CKV with conformal factor $\frac{\epsilon p}{2}\dot{A}A^{-1/2}$ for the bulk (5.5.6). We note that there does not exist a CKV in the y-direction for the general case $A(y, r) \neq B(y, r)$.

Next, we observe that the sum of the CKV (5.5.7) and an inherited SSS KV also yields a CKV for the embedding space (5.5.6). This can be verified by taking $\mathbf{X}^{(5)} = (A(y)\,\mathbf{X}^{(4)},\, p\,(A(y))^{1/2})$ and $\psi = \frac{\epsilon p}{2}\dot{A}A^{-1/2}$, where $\mathbf{X}^{(4)}$ is a SSS KV (5.3.7)–(5.3.10), in equations

$$\dot{X}_4^{(5)} = \epsilon\psi,$$

$$X_{4,i}^{(5)} + \dot{X}_i^{(5)} = \frac{\dot{A}}{A}X_i^{(5)},$$

$$X_{i,j}^{(5)} + X_{j,i}^{(5)} - 2\Gamma_{ij}^k X_k^{(5)} + \epsilon\dot{A}g_{ij}X_4^{(5)} = 2\psi A g_{ij}\,.$$

Noting that $X_{4,i}^{(5)} = 0$ and $X_i^{(5)} = A(y)X_i^{(4)}$, we see that the first two equations are easily satisfied. In the last equation, $\epsilon\dot{A}g_{ij}X_4^{(5)} = 2\psi A g_{ij}$, and

$$X_{i,j}^{(5)} + X_{j,i}^{(5)} - 2\Gamma_{ij}^k X_k^{(5)} = A\left[X_{i,j}^{(4)} + X_{j,i}^{(4)} - 2\Gamma_{ij}^k X_k^{(4)}\right] = 0,$$

since $X_i^{(4)}$ is a 4D Killing vector. Thus,

$$\mathbf{X}^{(5)} = \left(A(y)\,\mathbf{X}^{(4)},\, p\,(A(y))^{1/2}\right), \quad p \in \mathbb{R}, \tag{5.5.8}$$

is a CKV for the bulk (5.5.6) with conformal factor $\frac{ep}{2}\dot{A}A^{-1/2}$.

The calculation of other conformal Killing vectors for the bulk (5.5.6) and the full conformal geometry for the general case $A \neq B$ involves more complex equations and represents a continuing programme of research. We intend (with collaborators) to employ the techniques of Lie symmetry analysis to attack these problems, both to establish the conformal Killing geometry of a bulk with an embedded SSS hypersurface as a problem in its own right, and also as an experimental platform from which we hope to draw insight into the relationship between Lie symmetry reduction and CKV inheritance (Amery et al. 2011).

5.6   Summary

In this chapter, we investigated the conformal Killing geometry when embedding 4D static spherically symmetric spacetimes into a 5D Einstein bulk with form (5.3.2). We presented the equations (5.3.11)–(5.3.13) that must be solved to obtain a CKV in the bulk. First we restricted our treatment to KVs, and showed that the Killing vectors of the embedded SSS spacetime are inherited by the bulk. For the case $A = B$, we obtained the 5D Killing vector (5.4.4), where $A(y)$ is given by (4.4.6) and $\mathbf{X}^{(4)}$ are the 4D Killing vectors (5.3.7)–(5.3.10). The general case $R = R(r)$ yielded the Killing vectors (5.4.6) and (5.4.7), where $\mathbf{X}^{(4)}$ are the 4D KVs $\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}, \mathbf{Y}^{(3)}$ in the latter solution. Next we considered the general Killing geometry, and found that the Killing vectors inheriting the embedded SSS geometry are the only five-dimensional Killing vectors that are hypersurface-like, which is a reasonable result since we had imposed the SSS form for the $y = constant$ hypersurfaces. We also showed that in general there does not exist any 5D Killing vector in the y-direction. Finally, we discussed the general conformal geometry of the bulk. For the existence of a CKV in the y-direction, we found that we must have $A = B$, which corresponds to the embedding of the general Schwarzschild-de Sitter spacetime. In that case we obtained the 5D conformal Killing vector (5.5.7) with conformal factor $\frac{ep}{2}\dot{A}A^{-1/2}$, where $A(y)$ is specified by (4.4.6). For the same case, the solution (5.5.8), which is a sum of the CKV (5.5.7) and an inherited SSS KV, is also a 5D conformal Killing vector with conformal factor $\frac{ep}{2}\dot{A}A^{-1/2}$. The computation of the bulk Killing geometry with non-zero fifth component as well as the full 5D conformal geometry remains to be carried out, and provides an avenue for future work.

Chapter 6

Conclusion

The primary objective in this thesis was to apply the method of embeddings to investigate solutions in five-dimensional gravity. This research is motivated by the importance of embedding theory in the study of higher dimensional models, and is also of purely mathematical interest. In the past, embedding into Euclidean spaces has been useful in spacetime classification and in obtaining new solutions to the field equations. Embedding into curved pseudo-Riemannian spaces has become a promising field of research in recent years. The Dahia-Romero (2002a; 2002b) theorems guarantee the existence of embeddings into Einstein and more general pseudo-Riemannian spaces. However, general solutions to the embedding equations are not yet known. Thus, we are encouraged to explicitly obtain solutions to the higher dimensional field equations for particular cases of interest. Here we sought to isometrically embed four-dimensional spherically symmetric spacetimes into five-dimensional Einstein manifolds. We focussed on spherically symmetric spacetimes since they are relevant in astrophysics and cosmology, and on Einstein spaces because of their role in higher dimensional particle physics, as well as their relative geometric simplicity.

In chapter 2 we presented preliminary definitions and concepts that were essential in carrying out this research. We reviewed the technique provided by Dahia and Romero (2002a) for isometric embeddings of pseudo-Riemannian manifolds into Einstein spaces. We began with a 5D Einstein bulk having the line element (2.4.1):

$$d\tilde{s}^2 = \bar{g}_{ik}(x^j, y)dx^i dx^k + \epsilon(\bar{\phi}(x^j, y))^2 dy^2, \quad \epsilon^2 = 1, \ \ \bar{g}_{ik}(x^j, 0) = g_{ik}\,,$$

in Gauss-normal form with $\bar{\phi} = 1$, where $\bar{g}_{ik}$ are unknown functions of all five coordinates, and considered a 4D metric g_{ik} to be embedded along the hypersurface $y = 0$ in the bulk. The procedure is to solve a form of the five-dimensional field equations known as the Gauss, Codazzi and propagation equations such that when $y = 0$ the higher dimensional metric reduces to the four-dimensional one. The extrinsic curvature tensor is utilized in the Codazzi and Gauss equations, which, according to the Dahia-Romero theorem, only need to be solved at $y = 0$. This simplifies matters. A solution to the Codazzi-Gauss equations for 4D static spherically symmetric spacetimes has been obtained (Londal 2005). The propagation equation, however, is significantly more complicated, and a general solution to this equation for general spherically symmetric spacetimes remains unknown.

We present a summary of the tasks carried out and results produced in this thesis.

Chapter 3: Few solutions for embeddings are known, mostly dealing with embedded spaces having vanishing energy-momentum. We began to tackle the problem of embedding spacetimes with non-trivial energy-momentum by choosing to embed the gravitational field exterior to a four-dimensional global monopole defect, since it is one of the simpler spherically symmetric spacetimes having non-zero energy-momentum. We have determined a solution (3.4.2):

$$d\tilde{s}^2 = -dt^2 + K^{-1}dr^2 + (r - \alpha y)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2, \ \ \alpha^2 = \epsilon(1 - K),$$

for the global monopole exterior embedded locally and isometrically into a five-dimensional Riemann flat manifold along the hypersurface $y = 0$. Moreover, it is demonstrated that for static spherically symmetric spacetimes embedded at $y = 0$ into a particular 5D bulk (3.4.4):

$$d\tilde{s}^2 = -e^{-B(y,r)}dt^2 + e^{B(y,r)}dr^2 + (r - \sigma y)^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2, \ \ \sigma \in \mathbb{R},$$

with vanishing Ricci tensor, the global monopole exterior is the typical solution for the embedded space. We noted that the five-dimensional solution is simple but not trivial, since it is not an obvious solution nor is it of a trivial product form $M \times Y$. A detailed analysis of the above bulk metric is performed. If we want to avoid a complex value for α, then the fifth dimension must be space-like. It is verified that the five-dimensional metric can be transformed to Minkowski space via rotations that are related to the energy scale η and the deficit angle Δ of the 4D metric. There is no inherited deficit angle. This observation as well as the fact that a global embedding can be constructed from the local embeddings (Moodley and Amery 2012) indicates that the bulk obtained may be taken as globally flat. However, we noted that various cosmic strings may also be constructed. We note that 5D Minkowski space is not a new solution. However, its relevance to the 4D global monopole metric was previously unknown. The stability of 5D Minkowski space has obvious implications for discussions about the stability of the 4D global monopole: if gravity is 5D the problem is alleviated. Stability via embeddings may be a profitable avenue of research.

We have investigated further properties of the higher dimensional metric in order to gain a better understanding of the solution. We presented other forms of the metric, considered some of its submanifolds and their extrinsic curvature, and computed the energy-momentum tensor for its four-dimensional component that depends on the extra dimension. This tensor reduces to the matter tensor of the global monopole when $y = 0$. This notion is discussed in the context of induced matter theory (Wesson 1999). We also noted that since the hypersurfaces of constant y all correspond to different global monopoles, the embedding space appears to have a 'line-like' structure, but is not an actual 'stacking' with topology $N = M \times Y$. This facilitated a comparison with the $r \to 0$ limit of the EGB black hole (Maeda and Dadhich 2006) in which we observe how the Gauss-Bonnet term increases the number of permissible vacuum topologies. Furthermore, our (locally Minkowski) metric, with appropriate compactifications, can

be viewed as a local embedding of the 4D global monopole exterior into the 5D EGB Kaluza-Klein black hole. We noted that, by a repeated application of the Dahia-Romero theorem, a six-dimensional vacuum embedding of the global monopole exterior may be constructed. The results obtained in this chapter may also be relevant to other higher dimensional models (Randall and Sundrum 1999a,b; Tye 2008), and are being prepared for publication.

Chapter 4: We considered the general problem of embedding a spherically symmetric spacetime. The complexity of the propagation equations for embedding the global monopole metric shows just how difficult it is to find general solutions to the embedding equations, even for specific four-dimensional non-vacuum spaces. This motivated us to restrict the bulk metric to a particular form and to investigate what spherically symmetric spacetimes g_{ik} may embed into it. To begin with, we chose a 5D metric of the form (4.3.1):

$$d\tilde{s}^2 = A(y)g_{00}dt^2 + B(y)g_{11}dr^2 + C(y)g_{22}d\theta^2 + D(y)g_{33}d\phi^2 + \epsilon dy^2,$$

whose components are separable with respect to the extra dimension y. A metric of this type ensures energetic rigidity and has been used in the Randall-Sundrum (1999a; 1999b) and other (Maartens and Koyama 2010) brane-world scenarios as well as induced matter theory (Wesson 2002). We determined that the general Schwarzschild-de Sitter space and the Einstein Universe are the only 4D spherically symmetric spacetimes that may embed into a 5D Einstein space with this particular form. As the general Schwarzschild-de Sitter spacetime is an Einstein space, its Einstein embedding is already known (Lidsey et al. 1997; Anderson and Lidsey 2001) and we discussed how it is obtained. In the case of the Einstein Universe, we obtained an embedding space (4.4.14):

$$d\tilde{s}^2 = -e^{2g(t)}dt^2 + \left(1 + \sqrt{\frac{-\epsilon\alpha_2}{2}}\,y\right)^2\left(\frac{dr^2}{1 + \frac{\alpha_2}{2}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) + \epsilon dy^2,$$

that is Riemann flat and verified that the metric can be transformed to Minkowski form. This coincides with a result obtained by Wesson (1994) for the embedding of the Einstein Universe into a flat space, although a different method was used there. We obtained a new solution (4.4.16):

$$d\tilde{s}^2 = -\left[\cosh\left(\sqrt{\frac{-2\epsilon\Lambda}{3}}\,y\right) - a\sqrt{\frac{-3\epsilon}{2\Lambda}}\sinh\left(\sqrt{\frac{-2\epsilon\Lambda}{3}}\,y\right)\right]^2 e^{2g(t)}dt^2$$
$$+ \frac{dr^2}{1 + \frac{\alpha_2}{2}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \epsilon dy^2, \quad a \in \mathbb{R},$$

for the particular case $\Lambda = \frac{-3}{2}\alpha_2$. However, the embedding metric for other non-flat cases remains to be solved. We also concluded that the general Schwarzschild-de Sitter

spacetime is the only possible embedded spherically symmetric spacetime with bulk $d\tilde{s}^2 = A(y)ds^2 + \epsilon dy^2$, since the bulk metric for the Einstein Universe cannot have $A = B = C = D$. These studies have been published in Moodley and Amery (2011). For both embedding scenarios, additional solutions may be obtained by assuming different functional relationships between A, B, C and D. This is the subject of ongoing work.

Next we chose a 5D metric of the form (4.5.1):

$$d\tilde{s}^2 = A(y, r)ds^2 + \epsilon dy^2,$$

and demonstrated that A depends on y only, which regains the previously considered case. Thus, we concluded that the general Schwarzschild-de Sitter spacetime is the only spherically symmetric spacetime that may be embedded into a vacuum bulk of this metric form. This result has been submitted for publication. We also considered embeddings of spherically symmetric spacetimes written in retarded time coordinates into 5D vacuum bulks with metric forms (4.6.5) and (4.6.6):

$$d\tilde{s}^2 = A(y, v)ds^2 + \epsilon dy^2,$$
$$d\tilde{s}^2 = A(y)g_{00}dv^2 + A(y)g_{22}d\theta^2 + A(y)g_{33}d\phi^2 + 2B(y)g_{01}dvdr + \epsilon dy^2.$$

The analyses for both ansatzen yielded the general Vaidya-de Sitter model with constant mass as the only possible embedded space, and its embedding space is known to be $d\tilde{s}^2 = A(y)ds^2 + \epsilon dy^2$ with $A(y)$ given by (4.4.6). This work is being prepared for publication.

Chapter 5: Our intention in this chapter was to investigate the relationship between the conformal geometry of the embedded and embedding spaces. We considered 4D static spherically symmetric spacetimes embedded into a 5D Einstein bulk with the general form (5.3.2):

$$d\tilde{s}^2 = \left(A(y, r)g_{00}\delta_i^0\delta_j^0 + B(y, r)g_{CD}\delta_i^C\delta_j^D \right) dx^i dx^j + \epsilon dy^2,$$

where A, B remain unsolved functions. We demonstrated that the 4D Killing geometry $\mathbf{X}^{(4)}$, which is well known, is inherited by the bulk Killing geometry $\mathbf{X}^{(5)}$. In the case $A = B$, which corresponds to the embedding of the general Schwarzschild-de Sitter model into a vacuum bulk with $A = A(y)$ given by (4.4.6), it was shown that $\mathbf{X}^{(5)} = (A(y)\mathbf{X}^{(4)}, 0)$. For the general case in which the 4D Ricci scalar R may depend on r, we found the solutions (5.4.6) and (5.4.7):

$$\mathbf{X}^{(5)} = (k_0 A(y, r)\mathbf{Y}^{(0)}, 0),$$
$$\mathbf{X}^{(5)} = (k_1 B(y, r)\mathbf{X}^{(4)}, 0), \quad \mathbf{X}^{(4)} \in \{\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}, \mathbf{Y}^{(3)}\},$$

where k_0, $k_1 \in \mathbb{R}$, and the $\mathbf{Y}^{(j)}$ are the Killing vectors (5.3.7)–(5.3.10) of a static spherically symmetric spacetime. It was further determined that these Killing vectors inheriting the embedded SSS geometry are the only five-dimensional Killing vectors that are hypersurface-like, which is a reasonable result since we had imposed the SSS form for the $y = constant$ hypersurfaces. We also investigated the existence of 5D

Killing vectors in the y-direction, and found that there can be no such vectors in general. In fact, it was proved that for the 5D Killing vector to have a non-zero fifth component, its second component must be non-zero as well. As our final task, we discussed the general conformal geometry of the bulk. A five-dimensional conformal Killing vector in the y-direction exists only for the case $A = B$, which applies to the embedding of the general Schwarzschild-de Sitter model. For that case we obtained the solution (5.5.7):

$$\mathbf{X}^{(5)} = (0,\, 0,\, 0,\, 0,\, p\,(A(y))^{1/2}), \quad p \in \mathbb{R},$$

with conformal factor $\frac{\epsilon p}{2}\dot{A}A^{-1/2}$, and noted that the sum of this vector with any of the hypersurface-like Killing vectors is also a conformal Killing vector, given by (5.5.8):

$$\mathbf{X}^{(5)} = (A(y)\,\mathbf{X}^{(4)},\, p\,(A(y))^{1/2}), \quad p \in \mathbb{R},$$

with conformal factor $\frac{\epsilon p}{2}\dot{A}A^{-1/2}$. The results of this chapter have been published (Amery et al. 2011).

This programme of study has shown that embedding techniques can be useful as a way to find new five-dimensional exact solutions which can then be applied in astrophysics and cosmology.

We note that for non-vacuum embedded spaces, we encounter challenges in solving the higher dimensional field equations due to their highly non-linear nature. Thus, much work needs to be done to tackle this problem. We comment on remaining issues of interest and possible directions of future research in this field.

- The embedding of the global monopole exterior metric into an Einstein space is yet to be determined. In this general case, we were able to obtain a solution to the Codazzi and Gauss equations, given by (3.3.5). However, the propagation equations (3.3.8)–(3.3.10) with the initial conditions (3.3.6) and (3.3.7) remain unsolved. This problem is currently under investigation. As this system of equations is highly non-linear, a deeper analysis of such equations may also lead to new ideas and/or methods in solving non-linear differential equations.

- The embedding of the Einstein Universe into an Einstein bulk with arbitrary value for Λ remains to be solved. The existence of other solutions with $A \neq B$ for embedding the general Schwarzschild-de Sitter spacetime is also a point of interest.

- The computation of the full conformal and Killing geometry for rigid vacuum embeddings of static spherically symmetric spacetimes remains to be carried out. Again, the key challenge here is in solving non-linear equations. However, this could be made easier if explicit solutions for A and B were first determined.

- Future work would include investigating what spherically symmetric spacetimes, in the usual coordinates or retarded time coordinates, may embed into five-dimensional Einstein spaces of other metric forms.

- Further spacetimes that would be interesting to embed include the global monopole interior with core mass and/or a cosmological constant, global cosmic strings, the Reissner-Nordström black hole, and the general Vaidya-de Sitter spacetime with non-constant mass term.

- It would also be interesting to investigate embeddings into non-Einstein bulks such as asymptotically Einstein spaces, which could be significant to brane-world theory.

Appendix A

Global Embeddings of Pseudo-Riemannian Spaces

A.1 Introduction

In this appendix we provide further information on global embeddings for the reader's interest. Motivated by various higher dimensional theories in high-energy physics and cosmology, we consider the problem of constructing global embeddings of pseudo-Riemannian manifolds into manifolds of higher dimensions. A recent theorem by Katzourakis (2005b) claims that the local existence theorem (Dahia and Romero 2002a) for embedding into Einstein spaces can be made global, and his subsequent papers build upon this result (Katzourakis 2005a, 2006a,b). However, careful analysis reveals that there has been a crucial misunderstanding of the local result: the assumed form of the local embedding space is only valid for Ricci flat embedded spaces. Hence, Katzourakis' result is limited. We elucidate the impact of this misapprehension on the subsequent proof, and amend the given construction so that it applies to all analytic embedded spaces as well as to embedding spaces of arbitrarily specified (non-degenerate) Ricci tensor. We also quantify how similar an arbitrary global space must be to the promoted local embedding space in the vicinity of the embedded hypersurface, in order for it to be a global embedding space. We acknowledge that the article by Katzourakis (2005b) was recently updated and it has been noted that "There has been discovered a misconception which does not allow the solvability of the algebraic systems arising in the proof without further topological assumptions on the manifold" (Katzourakis 2011). We address this issue in our construction of a global embedding space. The work presented here has been submitted for publication (Moodley and Amery 2012) and an earlier version of this result was submitted as the author's M.Sc. dissertation (Moodley 2008). The outline of this appendix is as follows. In section A.2 we provide relevant background material. In section A.3 we concentrate on global embeddings, presenting two theorems: Theorem 1 pertaining to embeddings into Einstein spaces, and its immediate generalization, Theorem 2, pertaining to embeddings into arbitrarily specified pseudo-Riemannian spaces. We also contextualize these theorems as special cases of an appropriate theorem generally applicable to metric spaces. We provide commentary on the proofs and subsequent results in section A.4.

A.2 Background

Here we review some definitions and notions in topology and embedding theory (Bredon 1997; Hurewicz and Wallman 1948; Choquet-Bruhat et al. 1982; Eisenhart 1926; Goenner 1980).

Consider a topological space $\mathcal{M}$ and its cover $\mathcal{V}$. A cover $\mathcal{U}$ of $\mathcal{M}$ is a *refinement* of $\mathcal{V}$ if each element in $\mathcal{U}$ is a subset of some element in $\mathcal{V}$. A cover $\mathcal{U}$ is said to be *locally finite* if each point $p \in \mathcal{M}$ has a neighborhood which meets, non-trivially, only a finite number of members of $\mathcal{U}$. A space $\mathcal{M}$ is *paracompact* if any open cover of $\mathcal{M}$ has an open locally finite refinement. All metric spaces are paracompact. A fundamental theorem of dimension theory states that if $\mathcal{M}$ is a n-dimensional manifold, then every open cover $\mathcal{V}$ of $\mathcal{M}$ has a refinement $\mathcal{U}$ such that no point of $\mathcal{M}$ lies in more than $n+1$ elements of $\mathcal{U}$. For every locally finite cover $\mathcal{V} = \{V_i\}$ of a paracompact manifold $\mathcal{M}$, there exists a partition of unity $\{g_i\}$ subordinate to this refinement consisting of a family of differentiable functions $g_i : \mathcal{M} \longrightarrow \mathbb{R}$ such that: (i) $0 \leq g_i \leq 1$ on $\mathcal{M}$ for all i; (ii) $g_i(p) = 0$ for all $p \notin V_i$; and (iii) $\Sigma_i \, g_i(p) = 1$ for all $p \in \mathcal{M}$. In this appendix we utilize 'Bell functions' satisfying the first two criteria above, but not the normalization condition — see section A.3.2, Step 3.

Our theorems concern local and global isometric embeddings which were defined earlier in section 2.2.3. Recall the Dahia-Romero theorems (stated in section 2.3):

- *A n-dimensional pseudo-Riemannian manifold can be locally, analytically and isometrically embedded in any $(n+1)$-dimensional Einstein manifold (Dahia and Romero 2002a);*

- *A n-dimensional pseudo-Riemannian manifold can be locally, analytically and isometrically embedded in an arbitrary $(n+1)$-dimensional pseudo-Riemannian manifold with a non-degenerate Ricci tensor which is equal, up to a local analytic diffeomorphism, to the Ricci tensor of a given pseudo-Riemannian manifold (Dahia and Romero 2002b).*

Significant for our global constructions, is the observation that all the components of the local embedding space metric may have functional dependence on all the variables.

A.3 Global embeddings

A.3.1 The problem

Katzourakis (2005*b*) posits a global isometric embedding of an arbitrary n-dimensional pseudo-Riemannian space $\mathcal{M}$ into a $(n + 1)$-dimensional Einstein space $\mathcal{E} := \mathcal{M} \times \mathcal{Y}$, where $\mathcal{Y}$ is a one-dimensional analytic manifold. Repeated application of the theorem embeds $\mathcal{M}$ into a space with codimension greater than one. As a corollary, it is claimed that any analytic product manifold of the form $\mathcal{E}^{(n+d)} = \mathcal{M}^{(n)} \times \mathcal{Y}^{(d)}$, $d \geq 1$ admits an Einstein metric, and so is an Einstein space. The proof for these results rests on the assumption that the local Einstein embedding has the form $\mathcal{M} \times \mathcal{Y}$. However, the Dahia and Romero (2002*a*) result generally equips the embedding space with metric $\tilde{g}_{\mu\nu}$ all of *the components of which depend (differently) on the $(n + 1)$th coordinate; the product*

structure $\mathcal{M} \times \mathcal{Y}$ is a local embedding iff $\mathcal{M}$ is Ricci flat. As an explicit counterexample, consider a static spherically symmetric spacetime $g_{ik} = \text{diag}(-e^{2\nu(r)}, e^{2\lambda(r)}, r^2, r^2\sin^2\theta)$, where the Ricci curvature scalar is a non-trivial function of r only, and set the embedding metric to be $\tilde{g}_{\alpha\beta} = \text{diag}(g_{ik}, \phi(y))$ in the local embedding equations for an Einstein embedding space. This misunderstanding is the crucial limitation of Katzourakis' result. Next, we amend the construction so that it successfully applies to any pseudo-Riemannian embedded space.

A.3.2 Global isometric embedding theorems

Theorem 1. *Any n-dimensional real analytic pseudo-Riemannian manifold $(\mathcal{M}, \mathbf{g}_{\mathcal{M}})$ has a global isometric analytic embedding into a $(n+1)$-dimensional Einstein manifold $(\mathcal{E}, \mathbf{g}_{\mathcal{E}})$.*

Overview of the proof: Before proceeding to the full development of the proof, we outline the methodology used. We note that we proceed to keep the global embedding space as free as possible, constraining its structure where necessary and assuring ourselves that existence is guaranteed. This allows us to cast some light on the non-uniqueness of the global embedding. First consider a global embedding space $\mathcal{E}$ with an Einsteinian metric structure containing a hypersurface with the same topology as $\mathcal{M}$ — up to the metrical level. There certainly exists such a space: the promoted local embedding space — the local embedding metric applied to the subspace of $\mathbb{R}^{n+1}$ having the same coordinate restrictions and identifications as $\mathcal{M}$. This is 'topologically' $\mathcal{M} \times \mathcal{Y}$, albeit with a different metrical structure. For spacetimes with non-zero Ricci tensor this is necessarily more subtle than a product topology: one has to manually insert the embedded manifold into the global embedding manifold. Note that $\mathcal{E}$ is paracompact, so one may consider a 'partition of unity', though in an unnormalized fashion, to construct the 'Bell' functions that are essential in specifying the global analytic metric. Paracompactness further implies the existence of a locally finite (maximum d) cover from which one may construct several more locally finite covers. Ultimately one constructs two types of covers, both cases having as domain precisely those subsets of the original patches on which the 'Bell' functions are strictly positive, but the one case having N 'copies' of each domain, being distinguished by different coordinate systems. The arbitrariness of N allows one to 'sew' together the patches (and metrics) by means of a finite number of finite linear systems of equations in a large (and unspecified) number of arbitrary functions $\psi_{\alpha\beta}^{(i_a)}$. By choosing the number of these functions to be sufficiently large, the existence of solutions to this metasystem is guaranteed for a non-trivial class of $(n+1)$-dimensional global Einstein spaces, so long as there do not exist inconsistent rows in the metasystem. If there exist up to $(d-1)(n+1)(n+2)/2$ inconsistent rows in the metasystem, new variables may be introduced so that the new equations are consistent and the whole system admits solutions. Each new equation means that one metric component on one global patch is a linear combination of the local isometric embedding metric components. The greater number of inconsistent rows that we have, the more the global embedding space is metrically constrained by the local embedding space near the hypersurface. If there are more than $(d-1)(n+1)(n+2)/2$ inconsistent rows, our construction fails. However, the

promoted local embedding space is an obvious example of a space that satisfies our system: existence is guaranteed.

Detailed proof: The proof is separated into five steps, loosely matching those of the original proof (Katzourakis 2005b): Step 1 contains a new construction, the effects of which manifest in the sequel; Steps 2 and 3 are essentially unchanged, with slightly different notation and some commentary; and Steps 4 and 5 are, of necessity, technical developments of the corresponding steps in the original. Two of Katzourakis' steps (asserting a locally Einstein metric and the isometric condition) are subsumed in Steps 1 and 4.

Step 1: The bulk $\mathcal{E}$ containing $\mathcal{M}$. We assume that there exists an $\bar{\mathcal{E}}$ which is an arbitrary $(n+1)$-dimensional real analytic pseudo-Riemannian space. Recall that $\bar{\mathcal{E}}$ is paracompact and Hausdorff since it is a metric space (Hurewicz and Wallman 1948; Choquet-Bruhat et al. 1982; Bredon 1997). We shall further insist that $\bar{\mathcal{E}}$ is an Einstein space globally, and hence locally. Thus, we have a vacuum solution to Einstein's field equations defining a global metric over $\mathbb{R}^{n+1}$, modulo possible purely topological alterations such as holes and identifications. Since to specify the metric is to specify an induced topology and differential structure, our argument is principally about this process. However, in order to proceed with the embedding we need to also insist that $\bar{\mathcal{E}}$ is at a purely topological level 'like' $\mathcal{M}$ on some hypersurface. As noted above (in the overview), existence is guaranteed. The class of such Einstein spaces may be very large. The promoted local embedding space alone may have arbitrary pure topological alterations (e.g. holes) in the bulk. Denote an open cover of $\bar{\mathcal{E}}$ by $\mathcal{U} := \{U_i \mid i \in I\}$ where I may be infinite. Since $\bar{\mathcal{E}}$ is paracompact, there exists a locally finite refinement of this cover, given by

$$\bar{\mathcal{Q}} := \{\bar{Q}_j \mid j \in \bar{J}\},$$

where $\bar{J} \subseteq I$, and such that no point of $\bar{\mathcal{E}}$ lies in more than $n+2$ of its elements. Denote the coordinates of a point $p \in \bar{\mathcal{E}}$ by $(z^1(p), \ldots, z^n(p), y(p))$. We construct a set $\mathcal{Q}'$ from $\bar{\mathcal{Q}}$ by excising all points having coordinates with $y = c$ (c is some fixed constant), which is the $y = c$ hypersurface Σ_c. Any patch of $\bar{\mathcal{Q}}$ that includes points on the $y = c$ plane is split into two open patches not containing those points with $y = c$. Thus, we have that $\mathcal{Q}'$ is a cover for $\bar{\mathcal{E}} \setminus \Sigma_c$, the complement of the hypersurface Σ_c in $\bar{\mathcal{E}}$. Any point in $\bar{\mathcal{E}} \setminus \Sigma_c$ that is covered by the maximum of $n+2$ elements of $\bar{\mathcal{Q}}$ will still be covered by $n+2$ elements of $\mathcal{Q}'$. We then specify another (locally) Einstein space $\mathcal{E}$ via its cover

$$\widetilde{\mathcal{Q}} := \{\widetilde{Q}_j \mid j \in J\},$$

as the union of $\mathcal{Q}'$ and the $(n+1)$-dimensional patches generated through the application of the local (Einsteinian) embedding theorem (Dahia and Romero 2002a) to a locally finite cover for $\mathcal{M}$. We denote these additional patches by $\mathcal{M}'$ which covers all points on the $y = c$ plane. Locally this procedure yields at most $n+1$ additional patches since $\mathcal{M}$ is a n-dimensional manifold. Thus, each element of $\mathcal{E}$ lies in at most $d = (n+1) + (n+2) = 2n+3$ elements of the cover $\widetilde{\mathcal{Q}}$. Note that the cover $\widetilde{\mathcal{Q}}$ may be refined further, subject to the fact that we would like to retain the $(n+1)$-dimensional patches generated by the local embeddings. However, the finitude of d is sufficient to

proceed with our proof. Note also that, on every patch in $\widetilde{\mathcal{Q}}$, the above construction guarantees that there exists a (local) Einstein metric. Finally, observe that (modulo the existence of a global analytic metric) local isometry implies global isometry, since the $y = c$ plane is covered only by the patches induced by the local embeddings. To prove that this class of spaces contains a non-trivial subset of spaces providing a global embedding, we need to show that the metric matches across all intersections of patches. That the patches themselves 'work' is standard — the metric induces a topology on the subset of $\mathbb{R}^{n+1}$ having the coordinate restrictions/identification of $\mathcal{M}$ (for a non-compact embedding), and having also a S^1 topology for the extra dimension (for a Kaluza-Klein embedding). To proceed with specifying our metric we first need some machinery: we define several additional covers in Step 2.

Step 2: The covers $\mathcal{W}$, $\mathcal{W}_{\mathbb{B}}$ and $\mathcal{Q}$. At each $p \in \mathcal{E}$, we have $p \in \widetilde{Q}_j$ for some $j \in J$. For every $\widetilde{Q}_j$ we construct N additional distinct neighborhoods $(W_{i_a}, \chi_{(i_a)})$, $1 \leq a \leq N$, such that the W_{i_a} cover the same domain $\widetilde{Q}_j$ in $\mathcal{E}$, but are distinguished by their different coordinate functions $\chi_{(i_a)} : W_{i_a} \longrightarrow \mathbb{R}^{n+1}$, $1 \leq a \leq N$. Here $N \in \mathbb{N}$ is large but finite and unspecified for now, and each $\chi_{(i_a)} = (x^1_{(i_a)}, \ldots, x^\alpha_{(i_a)}, \ldots, x^{n+1}_{(i_a)})$, where $x^\alpha_{(i_a)} = \tau^\alpha \circ \chi_{(i_a)} : W_{i_a} \longrightarrow \mathbb{R}$, $\tau^\alpha : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$. Since $\mathcal{E}$ is a pseudo-Riemannian manifold, these neighborhoods on $\mathcal{E}$ can be chosen distinct with geodesic coordinates, and an arbitrarily large but finite number of such patches may be constructed by choosing different initial directions at p.

We form a N-element class of patches at each $p \in \mathcal{E}$:

$$[W_j] := \left\{ (W_{i_a}, \chi_{(i_a)}) \mid a = 1, \ldots, N \mid \operatorname{dom}(W_{i_a}) = \operatorname{dom}(\widetilde{Q}_j) \right\},$$

to obtain the locally finite cover:

$$\begin{aligned} \mathcal{W} :&= \{[W_j] \mid j \in J\} \\ &= \{(W_{i_a}, \chi_{(i_a)}) \mid a = 1, \ldots, N \mid \operatorname{dom}(W_{i_a}) = \operatorname{dom}(\widetilde{Q}_j) \mid j \in J\}. \end{aligned}$$

Here $\operatorname{dom}(\widetilde{Q}_j)$ denotes the domain of the patch $\widetilde{Q}_j$. Now, using a Euclidean transfer, we can identify each of the N distinct points $\chi_{(i_1)}(p), \ldots, \chi_{(i_N)}(p)$ as the origin $0 \in \mathbb{R}^{n+1}$. Consider the intersection $\bigcap_{a=1}^{N} \chi_{(i_a)}(W_{i_a})$ which is an open set in $\mathbb{R}^{n+1}$. Within this set lies an open $(n+1)$-dimensional ball $\mathbb{B}(0, R_i)$ of maximum radius $R_i > 0$. Choosing any $r_i < R_i$, $\mathbb{B}(0, r_i) \subseteq \bigcap_{a=1}^{N} \chi_{(i_a)}(W_{i_a})$, invert $\mathbb{B}(0, r_i)$ via one of the coordinates, say the first, $\chi_{(i_1)}|_{\mathbb{B}(0, r_i)}$, so that each W_{i_a} contains an analytically diffeomorphic copy of the ball $\mathbb{B}(0, r_i)$: $\chi_{(i_1)}^{-1}(\mathbb{B}(0, r_i)) \subseteq W_{i_a}$, $a = 1, \ldots, N$. We denote $\chi_{(i_1)}^{-1}(\mathbb{B}(0, r_i))$ by $W_{\mathbb{B}i_a}$. At each $p \in \mathcal{E}$, form a N-element class of inversed balls, each with different induced coordinates:

$$[\mathbb{B}_j] := \left\{ (W_{\mathbb{B}i_a}, \chi_{(i_a)}|_{W_{\mathbb{B}i_a}}) \mid a = 1, \ldots, N \mid \operatorname{dom}(W_{\mathbb{B}i_a}) = \operatorname{dom}(\widetilde{Q}_j)|_{W_{\mathbb{B}i_a}} \right\},$$

to obtain another locally finite cover

$$\begin{aligned} \mathcal{W}_{\mathbb{B}} :&= \{[\mathbb{B}_j] \mid j \in J\} \\ &= \{(W_{\mathbb{B}i_a}, \chi_{(i_a)}|_{W_{\mathbb{B}i_a}}) \mid a = 1, \ldots, N \mid \operatorname{dom}(W_{\mathbb{B}i_a}) = \operatorname{dom}(\widetilde{Q}_j)|_{W_{\mathbb{B}i_a}} \mid j \in J\}. \end{aligned}$$

Finally, we restrict each element of $\widetilde{\mathcal{Q}}$ on its corresponding inversed ball to obtain the locally finite cover

$$\mathcal{Q} \; := \; \{(Q_j, \{y_{(j)}^A\}_{1 \le A \le n+1}) \; | \; Q_j = \widetilde{Q}_j|_{\chi_{(i_1)}^{-1}(\mathbb{B}(0,r_i))} \; | \; j \in J\},$$

where $y_{(j)}^A : Q_j \longrightarrow \mathbb{R}$. Note that $Q_j \subseteq \widetilde{Q}_j$, and hence is locally Einsteinian by the construction in Step 1.

While all three covers are locally finite, $\mathcal{W}$ and $\mathcal{W}_{\mathbb{B}}$ each contain more elements than $\mathcal{Q}$: any $p \in \mathcal{E}$ is covered by a maximum of d elements of $\mathcal{Q}$, and a maximum of Nd elements of $\mathcal{W}$ (resp. $\mathcal{W}_{\mathbb{B}}$). We also observe that, even though $W_{\mathbb{B}i_a} \subseteq W_{i_a}, \forall i_a$, and a maximum of Nd elements of each cover contain any given point p, these bounds need not be realized. On the other hand, any given point p lies in some $Q_j \subseteq \widetilde{Q}_j$, and hence lies in at least N of the W_{i_a}'s (resp. $W_{\mathbb{B}i_a}$'s). This observation shall be used later in our counting arguments — see Step 5. The cover $\mathcal{W}$ is used to specify $\mathbf{g}_{\mathcal{E}}$ globally, and the cover $\mathcal{Q}$ to evaluate it locally. This dual perspective gives rise to a system of equations that must be satisfied to ensure the existence of the global metric — see Step 5. The 'Bell' functions of Step 3 are defined such that they are non-zero (and strictly positive) only on elements of $\mathcal{W}_{\mathbb{B}}$.

Step 3: The family $\{f_{i_a}\}$ of smooth 'Bell' functions. Since $\mathcal{E}$ is a paracompact manifold with a locally finite cover $\mathcal{W}$, there exists a 'partition of something' subordinate to this cover (Katzourakis 2005b) — this is standard topology (Hurewicz and Wallman 1948; Choquet-Bruhat et al. 1982; Bredon 1997). Hence, we are assured the existence of a family $\{f_{i_a}\}$ of C^∞ non-negative functions on $\mathcal{E}$, with properties:

- $f_{i_a} \in C^\infty(\mathcal{E} \longrightarrow \mathbb{R} \cap [0, \infty))$;

- $\mathrm{supp}(f_{i_a}) \subseteq W_{i_a}, \; \forall \; [W_i] \in \mathcal{W}$;

- $\underset{\substack{i_a \in J \\ 1 \le a \le N}}{\Sigma} f_{i_a}(p) > 0, \; \forall \; p \in \mathcal{E}$;

- f_{i_a} is real analytic within the (open) set of points $p \in \mathcal{E}$ on which it is strictly positive, denoted by $\{f_{i_a} > 0\}$, and equal to $W_{\mathbb{B}i_a}$;

- $f_{i_a}|_{\{f_{i_a} > 0\}} = f_{i_a}|_{\mathrm{int}(\mathrm{supp}(f_{i_a}))} \in C^\infty(\mathcal{E} \cap \{f_{i_a} > 0\} \longrightarrow \mathbb{R} \cap (0, \infty))$.

Note that the supports of the f_{i_a}'s (as well as their interior) form a locally finite cover of $\mathcal{E}$, and that they are non-zero precisely on the inversed balls and zero elsewhere.

Step 4: The global smooth and locally analytic metric $\mathbf{g}_{\mathcal{E}}$. We have assumed $\mathcal{E}$ to consist of real analytic Einstein patches. It remains to specify the global metric appropriately. Consider the cover $\mathcal{W}$ and the idea of 'sewing' together the W_{i_a} patches to obtain the global metric. For each i_a let $\psi_{\alpha\beta}^{(i_a)} \in C^\infty(W_{i_a} \longrightarrow \mathbb{R})$, $\alpha, \beta \in \{1, \dots n+1\}$ be $(n+1)(n+2)/2$ symmetric analytic functions on W_{i_a}, and consider the coordinate functions $x_{(i_a)}^\alpha : W_{i_a} \longrightarrow \mathbb{R}$ where $\alpha = 1, \dots, n+1$. Define

$$\mathbf{g}_{\mathcal{E}}(U, V) \; := \; \left(\sum_{\alpha, \beta \, \in \, \{1, \dots, n+1\}} \; \sum_{\substack{i_a \in J \\ 1 \le a \le N}} \breve{\psi}_{\alpha\beta}^{(i_a)} \, dx_{(i_a)}^\alpha \otimes dx_{(i_a)}^\beta \right) (U, V),$$

where we have set $\breve{\psi}^{(i_a)}_{\alpha\beta} = f_{i_a}\,\psi^{(i_a)}_{\alpha\beta} \in C^\infty(\mathcal{E} \longrightarrow \mathbb{R})$, the f_{i_a} are the 'Bell' functions, and $U, V \in T\mathcal{E}$. The composition of the analytic functions f_{i_a} and $\psi^{(i_a)}_{\alpha\beta}$ is analytic which implies that $\mathbf{g}_\mathcal{E}$ is real analytic. Next, we employ the cover $\mathcal{Q}$ to evaluate the metric locally. Taking any $Q_j \in \mathcal{Q}$ with coordinates $y^\sigma_{(j)}$, $1 \leq \sigma \leq n+1$, we have

$$\mathbf{g}_\mathcal{E}|_{Q_j} = \sum_{\tau,\sigma} \left(\sum_{\alpha,\beta\,\in\,\{1,\dots,n+1\}} \sum_{\substack{i_a \in J \\ 1 \leq a \leq N}} \breve{\psi}^{(i_a)}_{\alpha\beta}\, \frac{\partial x^\alpha_{(i_a)}}{\partial y^\tau_{(j)}} \frac{\partial x^\beta_{(i_a)}}{\partial y^\sigma_{(j)}} \right)\Bigg|_{Q_j} dy^\tau_{(j)} \otimes dy^\sigma_{(j)}\,,$$

having used $dx^\alpha_{(i_a)} = \frac{\partial x^\alpha_{(i_a)}}{\partial y^\tau_{(j)}}\, dy^\tau_{(j)}$. Thus, the components of $\mathbf{g}_\mathcal{E}|_{Q_j}$ are

$$[g^{(j)}_\mathcal{E}]_{\tau\sigma} = \left(\sum_{\alpha,\beta\,\in\,\{1,\dots,n+1\}} \sum_{\substack{i_a \in J \\ 1 \leq a \leq N}} \breve{\psi}^{(i_a)}_{\alpha\beta}\, \frac{\partial x^\alpha_{(i_a)}}{\partial y^\tau_{(j)}} \frac{\partial x^\beta_{(i_a)}}{\partial y^\sigma_{(j)}} \right)\Bigg|_{Q_j}\,,\quad 1 \leq \tau,\sigma \leq n+1\,,\ \ (\text{A.3.1})$$

which are all analytic functions on Q_j.

Consider the metric evaluated at a point $p \in W_{i_a}$. Note that the f_{i_a} are only non-zero on the $W_{\mathbb{B}i_a} \subseteq W_{i_a}$, which implies that the metric has zero contribution from the W_{i_a} containing p, outside of the corresponding $W_{\mathbb{B}i_a}$. Note also that the sums over $i_a \in J$, $1 \leq a \leq N$, have at least N terms for each α, β — see Step 2. The evaluation of the global metric in the locally Einsteinian cover $\mathcal{Q}$ shall be employed to generate (finite) systems of equations that must be solved in order for the global metric to have the desired local properties. Recall (from Step 1) that $[g^{(j)}_\mathcal{E}]_{\tau\sigma}$ is known and Einsteinian for all patches Q_j. This local specification of $\mathcal{E}$ ensures that $\mathbf{g}_\mathcal{E}$ is well defined, and is utilized in the next step.

Step 5: The existence of functions $\psi^{(i_a)}_{\alpha\beta}$. As yet we have not completely specified the global embedding since we need to ensure that where patches overlap, their metrics coincide, and that the functions $\psi^{(i_a)}_{\alpha\beta}$ do indeed exist on $\mathcal{E}$. Since the f_{i_a} are defined on W_{i_a} but are non-zero only inside $W_{\mathbb{B}i_a}$, and since $\mathrm{dom}(W_{\mathbb{B}i_a}) = \mathrm{dom}(Q_j)$, the components (A.3.1) of $\mathbf{g}_\mathcal{E}|_{Q_j}$ can be rewritten as

$$[g^{(j)}_\mathcal{E}]_{\tau\sigma} = \sum_{\alpha,\beta\,\in\,\{1,\dots,n+1\}} \sum_{\Upsilon} \left\{ f_{i_a}\, \frac{\partial x^\alpha_{(i_a)}}{\partial y^\tau_{(j)}} \frac{\partial x^\beta_{(i_a)}}{\partial y^\sigma_{(j)}} \right\}\Bigg|_{Q_j} \psi^{(i_a)}_{\alpha\beta}\,, \qquad (\text{A.3.2})$$

where $1 \leq \tau, \sigma \leq n+1$, and $\Upsilon = \{i_a \in J \mid 1 \leq a \leq N \mid \mathrm{dom}(W_{i_a}) \cap \mathrm{dom}(Q_j) \neq \emptyset\}$. Note that the $\psi^{(i_a)}_{\alpha\beta}$ are the only unknown functions in the above relation since $[g^{(j)}_\mathcal{E}]_{\tau\sigma}$ is specified. The components of (A.3.2) yield $(n+1)(n+2)/2$ equations on every Q_j. Fix $\tau, \sigma \in \{1, \dots, n+1\}$, and let

$$[g^{(j)}_\mathcal{E}]_{\tau\sigma} = \Phi_{(j)}\,, \quad \text{and} \quad \left\{ f_{i_a}\, \frac{\partial x^\alpha_{(i_a)}}{\partial y^\tau_{(j)}} \frac{\partial x^\beta_{(i_a)}}{\partial y^\sigma_{(j)}} \right\}\Bigg|_{Q_j} = \Theta^{\alpha\beta}_{(i_a)(j)}\,. \qquad (\text{A.3.3})$$

Then (A.3.2) becomes a linear functional equation with analytic coefficients, where the $\psi_{\alpha\beta}^{(i_a)}$ are linearly independent since the patches are distinguished by different coordinate functions:

$$\Phi_{(j)} \;=\; \sum_{\alpha,\beta \,\in\, \{1,\ldots,n+1\}} \; \sum_{\Upsilon} \; \{\Theta_{(i_a)(j)}^{\alpha\beta} \, \psi_{\alpha\beta}^{(i_a)}\} \,. \tag{A.3.4}$$

From (A.3.3), the positivity of $f_{i_a}|_{Q_j}$, and the fact that for any given τ and σ at least one of the $\partial x_{(i_a)}^\alpha / \partial y_{(j)}^\tau$ and one of the $\partial x_{(i_a)}^\beta / \partial y_{(j)}^\sigma$ are non-zero , we may conclude that (A.3.4) evaluated at a point will have at least N variables $\psi_{\alpha\beta}^{(i_a)}$ with non-zero coefficients — see Step 2.

Recall that any point in $\mathcal{E}$ lies in a maximum of d elements of the cover $\mathcal{Q}$. So consider the domain in $\mathcal{E}$ that consists of the maximum d overlaps of the Q_j's: $Q_{j_1} \cap \cdots \cap Q_{j_d} \;=\; \cap_{r=1}^d Q_{j_r}$. We wish to show that there exist solutions $\psi_{\alpha\beta}^{(i_a)}$ on this intersection. Thus, (A.3.4) must be solved on $Q_{j_1},\ldots,Q_{j_d}$ simultaneously. So on $\cap_{r=1}^d Q_{j_r}$ we have one system consisting of d equations:

$$(\Sigma 1) \quad \begin{cases} \Phi_{(j_r)} \;=\; \sum_{\alpha,\beta} \sum_{i_a=1}^d \sum_{a=1}^N \; \{\Theta_{(i_a)(j_r)}^{\alpha\beta} \, \psi_{\alpha\beta}^{(i_a)}\} \,, \\[2mm] 1 \le r \le d. \end{cases}$$

We proceed to extend the $\psi_{\alpha\beta}^{(i_a)}$ solutions on $Q_{j_1} \cap \cdots \cap \widehat{Q}_{j_s} \cap \cdots \cap Q_{j_d}$, then $Q_{j_1} \cap \cdots \cap \widehat{Q}_{j_s} \cap \cdots \cap \widehat{Q}_{j_k} \cap \cdots \cap Q_{j_d}$, and so on until we have solutions on each of the $Q_{j_1},\ldots,Q_{j_d}$, and so on $\cup_{r=1}^d Q_{j_r}$. Here the 'overhat' denotes the omission of the expression underneath. On $Q_{j_1} \cap \cdots \cap \widehat{Q}_{j_{s_1}} \cap \cdots \cap \widehat{Q}_{j_{s_k}} \cap \cdots \cap Q_{j_d}$ we have a system of $d-k$ equations:

$$(\Sigma(k+1)) \quad \begin{cases} \Phi_{(j_r)} \;=\; \sum_{\alpha,\beta} \sum_{\substack{i_a=1 \\ [\mathbb{B}_i] \not\subseteq X}}^d \sum_{a=1}^N \; \{\Theta_{(i_a)(j_r)}^{\alpha\beta} \, \psi_{\alpha\beta}^{(i_a)}\} \,, \\[4mm] 1 \le r \le d, \;\; r \notin S, \;\; S = \{s_1,...,s_k\}, \;\; X = \bigcup_{t=1}^k Q_{j_{s_t}} \,, \end{cases}$$

and there are $\binom{d}{d-k}$ such systems. Recall that all these systems have at least N independent variables, where N is finite but arbitrarily large.

For each choice of τ and σ, the total number of systems is $\Sigma_{t=1}^d \binom{d}{t}$, and the total number of equations that must be solved is $M = \Sigma_{t=1}^d t \binom{d}{t}$. This procedure must be carried out $(n+1)(n+2)/2$ times to solve for all $\tau,\sigma \in \{1,\ldots,n+1\}$. So setting $N = ((n+1)(n+2)/2)M + 1$, we have more independent variables than equations, so there will exist solutions $\psi_{\alpha\beta}^{(i_a)}$ on $\cup_{r=1}^d Q_{j_r}$ (and we have shown that the metric $\mathbf{g}_\mathcal{E}|_{\cup Q_j}$ exists); so long as there does not exist any rows of the form $(0,\ldots,0 \mid \alpha)$, $\alpha \ne 0$, in the augmented matrix.

For points away from the inserted hypersurface $y = c$ we have intersections of patches belonging only to $\mathcal{Q}'$. There the equations should be easily solved for the $\psi_{\alpha\beta}^{(i_a)}$, since the patches in $\mathcal{Q}'$ are generated from the initial manifold $\bar{\mathcal{E}}$ for which a global metric is already defined. Consider intersections that include patches in $\mathcal{M}'$ (the set of local embedding patches). There might arise situations where a system has

inconsistent rows, so that a linear combination of such rows yields a row of the form $(0, \ldots, 0 \mid \alpha)$, $\alpha = \left\{ [g_{\mathcal{E}}^{(j_r)}]_{\tau\sigma} \right\}_\rho \neq 0$. Here ρ denotes some j_r with at least one Q_{j_r} being a local embedding patch, and $\left\{ \ldots \right\}_\rho$ denotes a linear combination labelled by the ρ-rows involved. Note that the reason for the linear combination $\left\{ \ldots \right\}_\rho$ is that the problem row $(0, \ldots, 0 \mid \alpha)$ arose through some linear combination of rows in the row-reduction process. If there exists such a row, we may add a new variable to the system so that the equation becomes consistent, and couples to none of the others; so long as this new equation has a valid meaning in our context. (This is standard linear algebra — see, for example, Strang (2003), Anton and Rorres (2010) and Cullen (1990).) This is carried out as follows. For one of the (maximum d) intersecting patches (labelled by j_r^*) belonging to $\mathcal{Q}'$, let a run through to $N+1$ for some fixed α^*, β^*. The system under consideration will consist of a new variable $\psi_{\alpha^*\beta^*}^{(i_{N+1})}$ and a new equation

$$\left\{ [g_{\mathcal{E}}^{(j_r)}]_{\tau\sigma} \right\}_\rho = \sum_{\alpha,\beta} \sum_{\substack{i_a \\ 1 \leq a \leq N}} \left\{ \Theta_{(i_a)(j_r)}^{\alpha\beta} \right\}_\rho \psi_{\alpha\beta}^{(i_a)} + \Theta_{(i_{N+1})(j_r^*)}^{\alpha^*\beta^*} \psi_{\alpha^*\beta^*}^{(i_{N+1})},$$

where the $\left\{ \Theta_{(i_a)(j_r)}^{\alpha\beta} \right\}_\rho$ for all α, β and $1 \leq a \leq N$, vanish as before. Thus, the function $\psi_{\alpha^*\beta^*}^{(i_{N+1})}$ is equal to a linear combination of the local embedding metrics up to a coordinate change. So the new equation has a valid meaning in our context. This amounts to saying that we can choose one metric component on one global isometric embedding patch equal to a linear combination of the local components, up to a coordinate change. For our purposes we note that we have complete freedom to specify the global embedding space metric and choosing it such that it is like the local one forces a result. However, many global spaces may be metrically 'close' to this one — having the requisite metric at $y = c$. Trivial examples include the global space with the promoted local space up to some $y = y_c$ and then matched to some other $(n+1)$-dimensional vacuum solution.

We can repeat this procedure for more than one inconsistency to create at most $(d-1)(n+1)(n+2)/2$ new variables. If there are more inconsistent equations than this maximum, then this construction fails. If there are no more problems than this maximum then the existence of solutions on the whole system is guaranteed. The number of times we have to carry out this procedure of introducing new variables gives us a measure of how 'close' the global embedding space is to the promoted local embedding space near the $y = c$ hypersurface.

Finally, we need to ensure that one may extend the $\psi_{\alpha\beta}^{(i_a)}$ on the whole of $\mathcal{E}$. For any $p \in \mathcal{E}$ there exists a maximum of d patches Q_j covering p. Choose any one of these patches, say Q_{j_0}, and take any other point q in Q_{j_0}. Now q may lie in a maximum of d patches $Q_{j'}$ including Q_{j_0}. So we have two unions of patches $\cup Q_j$ and $\cup Q_{j'}$ that are overlapping on Q_{j_0}. Their corresponding metrics $\mathbf{g}_{\mathcal{E}}|_{\cup Q_j}$ and $\mathbf{g}_{\mathcal{E}}|_{\cup Q_{j'}}$ must coincide on the intersection $(\cup Q_j) \cap (\cup Q_{j'}) \subseteq Q_{j_0}$; i.e. the equation

$$\Phi_{(j_0)} \; = \; \sum_{\substack{\alpha,\beta \, \in \, \{1,...,n+1\}}} \; \sum_{\substack{i_a=1 \\ [\mathbb{B}_i] \subseteq Q_{j_0}}}^{d} \; \sum_{a=1}^{N} \; \{\Theta^{\alpha\beta}_{(i_a)(j_0)} \; \psi^{(i_a)}_{\alpha\beta}\} \, ,$$

must hold on both the systems solved on $\cup Q_j$ and $\cup Q_{j'}$; which it does, by construction. This implies that there exist solutions $\psi^{(i_a)}_{(\alpha\beta)}$ on $(\cup Q_j) \; \cup \; (\cup Q_{j'})$. By considering all such overlapping unions, we have that the $\psi^{(i_a)}_{(\alpha\beta)}$ exist on the whole of $\mathcal{E}$. This implies that the patches of the global space are appropriately 'sewn' together, and the global metric $\mathbf{g}_\mathcal{E}$ is fully specified.

The isometry condition follows by construction: recall that an embedding is globally isometric if it is isometric at all points of the embedded space. Hence, by construction, there exists a global isometric analytic embedding of M into $\mathcal{E}$. $\square$

The above result may be extended to arbitrarily given pseudo-Riemannian embedding spaces:

Theorem 2. *Any n-dimensional real analytic pseudo-Riemannian manifold $(\mathcal{M}, \mathbf{g}_\mathcal{M})$ has a global isometric analytic embedding into an arbitrarily specified 'Ricci equivalent' $(n+1)$-dimensional pseudo-Riemannian manifold $(\mathcal{E}, \mathbf{g}_\mathcal{E})$.*

We state this theorem without proof since the methodology is essentially the same as for Theorem 1, except that we must take $\bar{\mathcal{E}}$ to be the arbitrarily specified pseudo-Riemannian space of the second Dahia-Romero theorem (2002b). The patches of the local embedding space, which are 'Ricci equivalent' (up to local analytic diffeomorphism) to $\bar{\mathcal{E}}$, are inserted along the excised $y = c$ hypersurface in $\bar{\mathcal{E}}$. The global isometric embedding space $\mathcal{E}$ is only 'Ricci equivalent' (up to global analytic diffeomorphism) to $\bar{\mathcal{E}}$ near the $y = c$ hypersurface, but elsewhere it is exactly $\bar{\mathcal{E}}$. Note that generally $\bar{\mathcal{E}}$ does not satisfy the isometry condition required in order to be a local and hence global isometric embedding space for $\mathcal{M}$.

We may present the results yet more generally as:

Theorem 3. *If any n-dimensional real analytic metric space has a local isometric analytic embedding into some specified $(n+1)$-dimensional metric space, then there exists a global isometric analytic embedding into that space.*

The proof of Theorem 3 is essentially the 'sewing' argument presented above. In light of this, we may consider Theorem 2 and Theorem 1 to be corollaries, in which the conditional statement in Theorem 3 is guaranteed by the Dahia-Romero results.

A.4   Discussion

The principal differences between Katzourakis' proof and ours arise because of the more complicated structure of the local embeddings, and lie in the technicalities of the specification of the bulk cover and metric, as well as the counting arguments demonstrating the existence of the global metric. However, the central idea is the same: the specification of a global metric on $\mathcal{W}$ via the introduction of arbitrary many analytic maps from patches in $\mathcal{W}$ to $\mathbb{R}$, and its evaluation on the local patches so as to generate

a metasystem of equations that may be solved by choosing sufficiently many functions. Unlike Katzourakis, we do not claim every Einstein space may be a global embedding for a given local embedding space — (in Step 5) we quantify how 'close' it must be to the promoted local embedding space, near the hypersurface.

Katzourakis begins by specifying $\mathcal{E}$ to have the topology $\mathcal{M} \times \mathcal{Y}$ resembling that of the (erroneously assumed) local embedding space. Due to its product structure, $\mathcal{E}$ inherits the properties of being real analytic, Hausdorff and paracompact from $\mathcal{M}$ and $\mathcal{Y}$ via their property of being metric spaces. It also induces a natural cover consisting of product charts, from which one may form the locally finite refinements required to specify the global metric. The product structure implies that Katzourakis need only posit unspecified functions $\psi^{(i_a)}$ for one component of the metric. Crucially, it also ensures that every neighborhood of $\mathcal{E}$ is of precisely the (Einsteinian) form of the $(n+1)$-dimensional neighborhoods induced by the local embedding theorems. We have not assumed as much for the topology of $\mathcal{E}$, so we must proceed through a more elaborate construction for the bulk — we must manually embed $\mathcal{M}$ in $\mathcal{E}$ to generate the (Einsteinian) local embedding patches — and necessarily require more functions, and hence more systems of equations. (For each of the $(n+1)(n+2)/2$ symmetric metric components.) Note that the required number of variables may be made smaller by first placing the global metric in Gaussian normal form.

Katzourakis (2005a, 2006a,b) has also considered situations in which the (Einstein) bulk contains differential topological singularities, or in which several pseudo-Riemannian manifolds are globally and analytically embedded into the bulk, or where there is a combination of both. In all three scenarios, the given proofs are limited in a fashion similar to his original result. For example, m pseudo-Riemannian manifolds $\mathcal{M}_1^{(n_1)}, \ldots, \mathcal{M}_k^{(n_k)}, \ldots, \mathcal{M}_m^{(n_m)}$ with dimension $n_1, \ldots, n_k, \ldots, n_m$, are embedded as $\mathcal{E} = \mathcal{M} \times \mathcal{Y}$, where $\mathcal{M} = \mathcal{M}_1^{(n_1)} \times \cdots \times \mathcal{M}_k^{(n_k)} \times \cdots \times \mathcal{M}_m^{(n_m)}$ so that the branes are disjoint submanifolds in the bulk. This embedding space is Einsteinian for only Ricci flat $\mathcal{M}$, and so each brane must be Ricci flat. An analysis of the application of our construction to these contexts is underway, and further work is also motivated by the physical interest in cases with singular brane energy-momentum.

We observe that our theorems demonstrate that the work in reducing the codimension is done locally. Our construction of the global embedding is not necessarily unique from a topological view: the metrical formulation in general relativity constrains the global topology, but does not completely specify it (Lachieze-Rey and Luminet 1995; Rebouças 2005). Moreover, at a metrical level, we have not only choice for the embedding metric, but also the freedom to specify the functions $\psi_{\alpha\beta}^{(i_a)}$ in many different ways: we have requested only that the number of functions be sufficiently large. There may be only one global embedding if, for all Einstein spaces except the promoted local embedding space, the number of inconsistent rows is larger than can be dealt with by introducing new variables (if the original Einstein space $\bar{\mathcal{E}}$ is 'too different' from the local embedding space near $y = c$). However, in general there is a lack of metrical uniqueness. In a sense, these considerations avoid concerns that the local results do not ensure a well posed initial value problem or the non-occurrence of singularities, since, while such properties may be present in other constructions, here we deal only with analytic manifolds embedded, via one particular construction, into analytic manifolds. Note that the preceding caveats don't compromise the existence result: the

embedding has the same existential level as any solution to the field equations, and any local embedding may be directly promoted to a global one.

Future work should consider embedded spacetimes that possess singularities, such as the Schwarzschild interior spacetime. Locally one can simply avoid the singularity, but the global situation is more problematic as we need to take the singularity into account when embedding $\mathcal{M}$ in $\mathcal{E}$. Of course, one could take the point of view that singularities are unphysical and/or censored from our experience. In this case the existence of analytic global isometric embeddings for say, the Schwarzschild exterior spacetime, is clearly a boon to the study of the astrophysical effects of higher dimensions. Finally, we note that it would be interesting to relate topological invariants for the global embedding space to those for the local embedding space and the embedded space.

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