New Classes of Exact Solutions for Charged Perfect Fluids

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This dissertation is submitted to the School of Mathematics, Statistics and Computer Science, University of KwaZulu–Natal, Durban, in fulfillment of the requirements for the degree of Master of Science.

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As the candidate's supervisor, I have approved this dissertation for submission.

Signed Name Date

Abstract

We investigate techniques to generate new classes of exact solutions to the Einstein–Maxwell field equations which represent the gravitational field of charged perfect fluid spherically symmetric distributions of matter. Historically, a large number of solutions have been proposed but only a small number have been demonstrated to satisfy elementary conditions for physical acceptability. Firstly we examine the case of the constant density and constant electric field charged fluid sphere and show empirically that such configurations of matter are unlikely to exist as basic physical requirements are violated. We then make an ansatz relating the fluid's electric field intensity to one of the gravitational potentials thereby simplifying the system of partial differential equations. This prescription yields an algorithmic process to generate new classes of exact solutions. We present a number of new solutions and comment on their viability as stellar models. Graphical plots generated by symbolic software of the main dynamical and geometrical quantities verify that one of our models is suitable to represent a physically relevant distribution of charged matter in the form of a spherical shell. In particular, positive definiteness of energy density and pressure are guaranteed, a pressure free hypersurface denoting the boundary of the star exists, the sound speed is shown to be sub–luminal and the energy conditions are satisfied everywhere in the interior of the star.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

TR Mthethwa

December 2012

FACULTY OF SCIENCE AND AGRICULTURE

DECLARATION - PLAGIARISM

I, Thulani Richard Mthethwa student number: 209541606, declare that

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Dedication

TO My Wife Ivy and children

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Finally I would like to extend my sincere gratitude to the National Research Foundation for funding my studies.

I have dedicated this dissertation to my wife, Ivy and children Vutlharhi, Ntsako, Felicity and Tertius.

Contents

Chapter 1

Introduction

General relativity is a physical theory describing how matter, energy, time and space interact. It treats space and time as the single unified four dimensional space–time. The geometric theory of gravitation is what is entailed in the General Theory of Relativity. It is a widely accepted description of gravitation in modern physics. The centre piece of the theory is the Einstein equations. They formulate the relationship between the space–time geometry and the properties of matter using a mathematical language.

Einstein's theory of gravity published in 1915, which extended special relativity, and takes into account non inertial frames of reference. General relativity takes the form of field equations describing the curvature of space–time and the distribution of matter. The effects of matter and space-time on each other are what we perceive as gravity. The theory of the space-time continuum already existed but under general relativity Einstein was able to describe gravity as the bending of space-time geometry. He defined a set of field equations which represented the way that gravity behaved in response to matter or space–time. These field equations could be used to represent the gravity of space-time that was at the heart of theory of general relativity. In this dissertation we will discuss the recent advances and developments in methods of looking for a class of solutions of Einstein equations. In general this is a system of ten partial differential equations which were reduced to four equations in six variables due to spherical symmetry. The curvature of space–time is best explained by the Riemann tensor and the matter content by the symmetric energy–momentum tensor. The behaviour of fluids with an electromagnetic field is governed by the Einstein–Maxwell field equations. Solving the coupled Einstein–Maxwell equations is difficult because of the non-linearity of the differential equations. Assumptions have been made by some researchers in generating exact solutions (Castejon–Amanedo and Coley 1992, Maharaj *et al* 1991). In this dissertation we are applying a solution generating algorithm.

In the presence of matter and charge the Einstein–Maxwell field equations form a highly non-linear system of equations. A number of exact solutions have been obtained in the past but only a few have physical relevance. The importance of these exact solutions is seen in astrophysics. Classes of exact solutions have been found for static spherically symmetric gravitational fields. A list of neutral solutions matchable to the Schwarzschild exterior at the boundary is provided by Delgaty and Lake (1998). The importance of charged exact solutions is in the modelling of a charged relativistic star. Ivanov (2002) gives a review of Einstein-Maxwell solutions which may be used to model a charged relativistic star since they match to the Reissner-Nordstrom exterior. The following are some of the fundamental exact solutions that were found to be physically viable:

(a) The first solution to the Einstein field equations was obtained by Schwarzschild. The Schwarzschild interior solution describe the interior gravitational field equations of static spheres of constant energy density. These solutions are used in modeling small stars.

(b) The exterior Schwarzschild solution to the Einstein field equations describes the exterior gravitational field of a static spherically symmetric body.

(c) The exterior gravitational field of a static spherically symmetric charged body is represented by the Reissner-Nordstrom solution. The vanishing of the charge reduces the Reissner-Nordstrom model to the Schwarzschild exterior.

(d) The Kerr solution describes the exterior gravitational field of a rotating axially symmetric gravitational body.

(e) Recently new solutions have been obtained by Komathiraj and Maharaj (2007), Thirukkanesh and Maharaj (2006) and John and Maharaj (2011). These solutions generate new metric potentials.

(f) Hansraj and Maharaj (2006) generalised the neutral solutions of Finch and Skea (1989) to include the effect of the electromagnetic field.

In this dissertation we will be discussing methods and ways of obtaining solutions of the Einstein–Maxwell equations. We will also present perfect fluid charged spheres in general relativity. First and foremost it is necessary for us to understand a charged perfect fluid. Our understanding will lead us to approximate and build a realistic model for a general relativistic star. A charged perfect fluid is defined to be a continuous distribution of matter. The matching of nonstatic charged perfect fluid spheres to the Reissner-Nordstrom exterior metric was pursued by Mahomed *et al* [12] who showed the role of Bianchi identities in restricting the number of solutions. A new class of solutions of the Einstein-Maxwell system was studied and they satisfy the physical criteria by Maharaj and Thirukkanesh (2008).

Chapter 2

Mathematical Preliminaries

2.1 Introduction

We present only those parts of differential geometry and general relativity which are necessary for later chapters. For a more comprehensive treatment the reader is referred to Misner *et al* (1973), Narlikar and Padmanabhan (1986) and Will (1981). In §2.2 we define the connection coefficients, curvature tensor and Einstein tensor. The energy–momentum tensor and the Einstein field equations are introduced. Static spacetimes which are spherically symmetric are analysed in §2.3. The Einstein–Maxwell field equations for static, spherically symmetric charged stars are considered in §2.4. Conditions that have to be satisfied for solutions of the Einstein–Maxwell equations to be physically relevant are listed in §2.4. We also give in this section the Einstein–Maxwell equations for a charged compact object. The field equations are expressed in a form first suggested by Durgapal and Bannerji (1983).

2.2 Differential Geometry

We take space–time M to be a 4–dimensional differentiable manifold endowed with a symmetric, nonsingular metric field $\mathbf{g}$ of signature $(- + + +)$. As the metric tensor field is indefinite the manifold is pseudo–Riemannian. Points in M are labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike and x^1, x^2, x^3 are spacelike. The line element is given by

$$ds^2 = g_{ab} dx^a dx^b \tag{2.2.1}$$

which defines the invariant distance between neighbouring points of a curve in M. The fundamental theorem of Riemannian geometry guarantees the existence of a unique symmetric connection that preserves inner products under parallel transport. This is called the metric connection $\mathbf{\Gamma}$ or the Christoffel symbol of the second kind. The coefficients of the metric connection $\mathbf{\Gamma}$ are given by

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2.2}$$

where commas denote partial differentiation.

The quantity

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{2.2.3}$$

is a $(1,3)$ tensor field and is called the Riemann tensor or the curvature tensor. Upon contraction of the Riemann tensor (2.2.3) we obtain

$$
\begin{aligned}
R_{ab} &= R^{c}{}_{acb} \\
&= \Gamma^{d}{}_{ab,d} - \Gamma^{d}{}_{ad,b} + \Gamma^{e}{}_{ab}\Gamma^{d}{}_{ed} - \Gamma^{e}{}_{ad}\Gamma^{d}{}_{eb} \qquad (2.2.4)
\end{aligned}
$$

where R_{ab} is the Ricci tensor. On contracting the Ricci tensor (2.2.4) we obtain

$$
\begin{aligned}
R &= g^{ab}R_{ab} \\
&= R^{a}{}_{a} \qquad (2.2.5)
\end{aligned}
$$

where R is the Ricci scalar. The Einstein tensor $\mathbf{G}$ is constructed in terms of the Ricci tensor (2.2.4) and the Ricci scalar (2.2.5) as follows:

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab} \qquad (2.2.6)$$

The Einstein tensor has zero divergence:

$$G^{ab}{}_{;b} = 0 \qquad (2.2.7)$$

a property referred to in the literature as the contracted Bianchi identity. This identity is useful when studying the conservation of matter which arises as a consequence of the field equations.

An arbitrary rank two tensor can be decomposed into its symmetric and anti–symmetric parts. Similarly the Riemann tensor (2.2.3) decomposes into the Weyl tensor (or conformal curvature tensor) and parts which involve the Ricci tensor and the curvature scalar. This decomposition is given by

$$\begin{aligned} R_{abcd} &= C_{abcd} - \frac{1}{6} R \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) \\ &+ \frac{1}{2} \left(g_{ac} R_{bd} - g_{bc} R_{ad} + g_{bd} R_{ac} - g_{ad} R_{bc} \right) \end{aligned} \tag{2.2.8}$$

where $\mathbf{C}$ is the Weyl tensor. The Weyl tensor is trace–free,

$$C^{ab}{}_{ad} = 0$$

and inherits all the symmetry properties of the curvature tensor (2.2.3).

The distribution of matter is specified by the energy–momentum tensor $\mathbf{T}$ which is given by

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{2.2.9}$$

for neutral matter. In the above μ is the energy density, p is the isotropic pressure, q_a is the heat flow vector and π_{ab} represents the stress tensor. These quantities are measured relative to a fluid four–velocity $\mathbf{u}$ ($u^a u_a = -1$). The heat flow vector and stress tensor satisfy the conditions

$$\begin{aligned} q^a u_a &= 0 \\ \pi^{ab} u_b &= 0 \end{aligned}$$

In the simpler case of a perfect fluid, which is the case for most cosmological models, the energy–momentum tensor (2.2.9) has the form

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} \tag{2.2.10}$$

The energy–momentum tensor (2.2.9) is coupled to the Einstein tensor (2.2.6) via the Einstein field equations

$$G_{ab} = T_{ab} \tag{2.2.11}$$

We utilise geometric units where the speed of light and the coupling constant are taken to be unity. The field equations (2.2.11) relate the gravitational field to the matter content. This is a system of coupled partial differential equations which are highly nonlinear and consequently difficult to integrate in general. Here we have provided only a brief outline of the results necessary for later work. For a comprehensive treatment of differential geometry applicable to general relativity the reader is referred to de Felice and Clarke (1990), Hawking and Ellis (1973) and Misner *et al* (1973).

2.3 Static Spherically Symmetric Spacetimes

The most general line element for static spherically symmetric spacetimes, in coordinates $(x^a) = (t, r, \theta, \phi)$, is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3.1}$$

where the gravitational potentials ν and λ are functions only of the spacetime coordinate r. It is reasonable to assume that the interior and exterior gravitational fields of an isolated charged star are described by (2.3.1) in the absence of other matter.

For the line element (2.3.1) the non–zero coefficients of the metric connection (2.2.2) are given by

$$\Gamma^0{}_{01} = \nu' \qquad \Gamma^2{}_{12} = \frac{1}{r}$$

$$\Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)} \qquad \Gamma^2{}_{33} = -\sin\theta\cos\theta$$

$$\Gamma^1{}_{11} = \lambda' \qquad \Gamma^3{}_{13} = \frac{1}{r}$$

$$\Gamma^1{}_{22} = -re^{-2\lambda} \qquad \Gamma^3{}_{23} = \cot\theta$$

$$\Gamma^1{}_{33} = -\sin^2\theta\, re^{-2\lambda}$$

where primes denote differentiation with respect to r. These connection coefficients may also be generated via the Euler–Lagrange equations.

It is now possible to evaluate the Ricci tensor (2.2.4), utilising the above connection coefficients, to yield the following non–zero components:

$$R_{00} = e^{2(\nu-\lambda)}\left[\nu'' - \nu'\lambda' + \lambda'^2 + \frac{2\nu'}{r}\right] \tag{2.3.2}$$

$$R_{11} = -\left[\nu'' + \nu'^2 - \frac{2\lambda'}{r} - \nu'\lambda'\right] \tag{2.3.3}$$

$$R_{22} = 1 - [1 + r\nu' - r\lambda]\, e^{-2\lambda} \tag{2.3.4}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.3.5}$$

Then the Ricci tensor components and the definition yield the following form for the Ricci scalar:

$$R = 2\left[\frac{1}{r^2} - \left(\nu'' - \nu'\lambda' + \lambda'^2 + \frac{1}{r^2} + \frac{2\nu'}{r} - \frac{2\lambda'}{r}\right) e^{-2\lambda}\right] \tag{2.3.6}$$

for the spherically symmetric spacetime (2.3.1). The Ricci tensor components (2.3.2) - (2.3.5) and the Ricci scalar (2.3.6) generate the corresponding non–vanishing components of the Einstein tensor (2.2.6). These are given by

$$G_{00} = \frac{e^{2\nu}}{r^2}\left[r\left(1-e^{-2\lambda}\right)\right]' \tag{2.3.7}$$

$$G_{11} = -\frac{e^{2\nu}}{r^2}\left(1-e^{-2\lambda}\right)+\frac{2\nu'}{r} \tag{2.3.8}$$

$$G_{22} = \frac{r^2}{e^{2\lambda}}\left[\frac{\nu'}{r}-\frac{\lambda'}{r}+\nu''-\nu'\lambda'+\lambda'^2\right] \tag{2.3.9}$$

$$G_{33} = \sin^2\theta\, G_{22} \tag{2.3.10}$$

for the line element (2.3.1).

2.4 Einstein–Maxwell Field Equations

The differential equations governing models of spherically symmetric charged stars comprise the coupled Einstein–Maxwell field equations. This system is given by

$$\begin{aligned} G_{ab} &= T_{ab} \\ &= M_{ab} + E_{ab} \end{aligned} \tag{2.4.1}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.4.2}$$

$$F^{ab}{}_{;b} = J^a \tag{2.4.3}$$

where $\mathbf{T}$ is the total energy–momentum tensor, $\mathbf{M}$ is the (uncharged) energy–momentum tensor, $\mathbf{E}$ is the contribution of the electromagnetic field, $\mathbf{F}$ is the electromagnetic field tensor and $\mathbf{J}$ is the four–current density.

The electromagnetic contribution $\mathbf{E}$ to the total energy–momentum tensor is given by

$$E_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \tag{2.4.4}$$

where $\mathbf{F}$ is skew–symmetric. Note that $\mathbf{F}$ can be decomposed in the following way

$$F^{ab} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

where $\mathbf{E} = (E^1, E^2, E^3)$ is the electric field and $\mathbf{B} = (B^1, B^2, B^3)$ is the magnetic field. The four–current density can be written as

$$J^a = \sigma u^a \tag{2.4.5}$$

where σ is the proper charge density. The electromagnetic field tensor $\mathbf{F}$ is defined in terms of the four–potential $\mathbf{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b} \tag{2.4.6}$$

Note that the four–potential $\mathbf{A}$ is not uniquely determined by Maxwell's equations but is constrained by gauge transformations.

In chapter 3 we model general relativistic charged spheres. In many physical situations it is reasonable to assume that the interior of the charged star is described by the line element

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.4.7}$$

where the functions $\nu(r)$ and $\lambda(r)$ are gravitational potentials. The line element (2.4.7) is the most general for static spherically symmetric spacetimes. When considering static, spherically symmetric models it is customary to choose the four–potential as

$$A_a = (\phi(r), 0, 0, 0)$$

This choice is made by Humi and Mansour (1984), Pant and Sah (1979) and Patel and Mehta (1995) amongst others. The above choice is the simplest choice possible and has the advantage of generating only one non–vanishing component, and its skew–symmetric counterpart, of the electromagnetic field tensor; this component is given by

$$F_{01} = -\phi'(r) \tag{2.4.8}$$

where we have utilised (2.4.6). The corresponding contravariant component has the form

$$F^{01} = e^{-2(\nu+\lambda)}\phi'(r) = e^{-(\nu+\lambda)}E(r)$$

where we have put

$$E(r) = e^{-(\nu+\lambda)}\phi'(r) \tag{2.4.9}$$

following the treatment of Herrera and Ponce de Leon (1985). The quantity $E(r)$ may be interpreted as the electrostatic field intensity. The components of the electromagnetic field energy tensor (2.4.4) are given by

$$E_{ab} = \mathrm{diag}\left(\frac{1}{2}e^{2\nu}E^2,\, -\frac{1}{2}e^{2\lambda}E^2,\, \frac{1}{2}r^2E^2,\, \frac{1}{2}\sin^2\theta\, r^2E^2\right) \tag{2.4.10}$$

where we have used (2.4.8). The energy–momentum tensor $\mathbf{M}$ for the comoving fluid velocity vector $u^a = e^{-\nu}\delta^a_0$ has the form

$$M_{ab} = \mathrm{diag}\left(e^{2\nu}\rho,\, e^{2\lambda}p,\, r^2p,\, \sin^2\theta\, r^2p\right) \tag{2.4.11}$$

for uncharged matter.

We are now in a position to generate the Einstein–Maxwell field equations for the spherically symmetric spacetime (2.4.7). Using (2.4.8) it is easy to verify that (2.4.2) is identically satisfied. The field equation (2.4.3) is identically satisfied for $a = 1, 2, 3$ but when $a = 0$ we obtain the condition

$$e^{-\lambda}\left(r^2E\right)' = r^2\sigma \tag{2.4.12}$$

The Einstein equations (2.4.1) may be expressed as the system

$$\left[r(1-e^{-2\lambda})\right]' = r^2\rho + \frac{1}{2}r^2E^2 \tag{2.4.13}$$

$$-\left(1-e^{-2\lambda}\right) + 2\nu' re^{-2\lambda} = pr^2 - \frac{1}{2}r^2E^2 \tag{2.4.14}$$

$$e^{-2\lambda}\left[\frac{\nu'}{r} - \frac{\lambda'}{r} + \nu'' - \nu'\lambda' + \nu'^2\right] = p + \frac{1}{2}E^2 \tag{2.4.15}$$

for the static spherically symmetric spacetime (2.4.7). The conservation laws $T^{ab}{}_{;b} = 0$ reduce to the equation

$$p' + (\rho + p)\nu' = \frac{E}{r^2}\left[r^2E\right]' \tag{2.4.16}$$

which can be used in the place of one of the field equations in the system (2.4.13) - (2.4.15).

The field equations may be expressed in a variety of equivalent forms which for particular applications makes the integration process simpler. We utilise the following transformation, which has been used by Durgapal and Bannerji (1983), Durgapal and Fuloria (1983) and Finch and Skea (1989), to generate new solutions in the case of neutral matter. It is convenient to introduce a new coordinate x and two metric functions $y(x)$ and $Z(x)$ defined as follows

$$x = Cr^2$$

$$Z(x) = e^{-2\lambda(r)}$$

$$A^2 y^2(x) = e^{2\nu(r)}$$

where A and C are constants. For this transformation the Einstein–Maxwell field equations (2.4.12) - (2.4.16) assume the form

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C} \tag{2.4.17}$$

$$\frac{Z-1}{x} + \frac{4Z\dot{y}}{y} = \frac{p}{C} - \frac{E^2}{2C} \tag{2.4.18}$$

$$4x^2 Z\ddot{y} + 2x^2\dot{Z}\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2 x}{C}\right) y = 0 \tag{2.4.19}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2 \tag{2.4.20}$$

where dots represent differentiation with respect to x. We shall use this form of the field equations to generate new solutions in chapter 3. Note that when $E = 0$ the system (2.4.17) - (2.4.20) reduces to the case of neutral matter.

The exterior gravitational field for a static, spherically symmetric charged distribution is governed by the Reissner–Nordstrom (Reissner 1916, Nordstrom 1918) solution. The Reissner–Nordstrom exterior line element has the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2.4.21)$$

where M and Q are associated with the mass and charge of the sphere respectively. For the Reissner–Nordstrom solution (2.4.21) the radial electric field is

$$E = \frac{Q}{r^2}$$

and consequently the proper charge density is

$$\sigma = 0$$

by (2.4.12). Thus $\mathbf{J} = \mathbf{0}$ which is consistent with the exterior. Observe that upon setting $Q = 0$ in (2.4.21) we regain the exterior Schwarzschild (1916a) solution.

We now consider briefly the conditions that have to be satisfied for solutions of the Einstein–Maxwell system to be physically admissible. The system (2.4.17) - (2.4.20) admits an infinite number of exact solutions as there are more variables than equations. Unfortunately many of the solutions reported in the literature correspond to unrealistic distributions of charged matter. It is desirable to isolate those solutions which are physically reasonable as these can then be used to model charged stars. Often the following constraints are imposed on solutions of the Einstein–Maxwell system in order to obtain models of stellar configurations that are physically plausible:

(a) Positivity and finiteness of pressure and energy density everywhere in the interior of the star including the origin and boundary:

$$0 \leq p < \infty \qquad 0 < \rho < \infty$$

(b) The pressure and energy density should be monotonic decreasing functions of the coordinate r. The pressure vanishes at the boundary $r = R$:

$$\frac{dp}{dr} \leq 0 \qquad \frac{d\rho}{dr} \leq 0 \qquad p(R) = 0$$

(c) Continuity of gravitational potentials across the boundary of the star. The interior line element should be matched smoothly to the exterior Reissner–Nørdstrom line element at the boundary:

$$e^{2\nu(R)} = e^{-2\lambda(R)} = 1 - \frac{2M}{R} + \frac{Q^2}{R^2}$$

(d) The principle of causality must be satisfied, i.e., the speed of sound should be everywhere less than the speed of light in the interior:

$$0 \leq \frac{dp}{d\rho} \leq 1$$

(e) Continuity of the electric field across the boundary:

$$E(R) = \frac{Q}{R^2}$$

(f) The metric functions $e^{2\nu}$ and $e^{2\lambda}$ and the electric field intensity E should be positive and non–singular everywhere in the interior of the star.

(g) The energy conditions: (weak, strong, dominant) should be satisfied.

(h) The redshift should correspond to observed stars.

Regardless of the non-linearity of the Einstein-Maxwell equations, various exact solutions for static and spherically symmetric metric are available in the literature (Stephani *et al* 2003). Five different types of exact solutions for static cases were obtained by Tolman (1939). The solution obtained by Vaidya and Tikekar (1982) corresponds to relativistic spherical stars. The class of exact solutions discussed by Durgapal (1982), and also Durgapal and Fuloria (1983) are valid for neutron stars. The conditions (a) to (h) indicated above are not satisfied by all the solutions throughout the interior of the star. For example the Tolman V and VI solutions are not physically viable as they correspond to singular solutions, as they have infinite values of central density. Additionally, some of the above conditions may be overly restrictive. For example, observational evidence suggests that in particular stars the energy density ρ is not a strictly monotonically decreasing function (Shapiro and Teukolsky 1983). With the exception of Tolman's V and VI, all the other solutions above are regular with finite density at the centre.

There are a number of solutions which are singular at the stellar centre in the literature. Such solutions are to be matched to other solutions valid for the core. Such solutions have to be treated as an envelope of the core and are to be matched to another solution valid for the core. For example the solutions by Herrera and Ponce de Leon (1985), Pant and Sah (1979), Tikekar (1984) and Whitman and Burch (1981) all suffer the drawback of a singularity at the stellar centre. The solution by Bannerjee and Santos (1981) becomes singular at a point in the interior of the distribution. Some solutions presented are regular at the centre but are not stable. For example the solution by Maartens and Maharaj (1990) violates the positivity of pressure condition; these solutions should not be rejected as negative pressures may have occurred in the early universe and thus such models may be acceptable in cosmology. Bonnor (1960), Bonnor and Wickramasuriya (1973) and Raychaudhuri (1975) showed that it is possible to generate realistic solutions with vanishing pressure. In such charged dust distributions the Coulombic repulsion is the force responsible for holding the matter in equilibrium in the absence of isotropic particle pressure. De and Raychaudhari (1968) have verified that in order to guarantee the equilibrium of a static charged dust sphere the relation $\sigma = \pm\rho$ must be satisfied. Other configurations of spherically symmetric distributions include the presence of anisotropic pressures. Such cases were examined by Maharaj and Maartens (1989) and Ruderman (1972) in the case of neutral spheres, and by Herrera and Ponce de Leon (1985) and Maartens and Maharaj (1990) in the presence of charge.

Chapter 3

Charged Incompressible Fluid Spheres

3.1 Introduction

The Einstein–Maxwell system of field equations form a highly nonlinear system of equations. This system of partial differential equations (2.4.17) – (2.4.20) admits an infinite number of solutions. So far only a small number of physically acceptable exact solutions have been found. For a charged static spherically symmetric fluid of with non-zero density the system (2.4.17) – (2.4.20) comprises four equations in six unknowns. This results in the non uniqueness of solutions. This means that two of the matter or geometric variables have to be selected at the outset and then the remaining four unknowns must be obtained through the integration of the system of field equations. The first two equations, (2.4.17) and (2.4.18), may be taken to be definitions of the energy density and pressure, while (2.4.20) is taken as a definition of the proper charge density σ. Therefore equation (2.4.19) which follows from the condition of pressure isotropy constitutes the master field equation which is written in terms of three items: the two metric potentials Z and y as well as the electric field intensity E. Any two of these may be selected *a priori* and then the field equations must be integrated to obtain the remaining unknown.

The most common approach has been to regard equation (2.4.17) as a second order ordinary differential equation in y and then specifying Z and E. This approach has been followed by Hansraj and Maharaj (2006) and Thirukannesh and Maharaj (2009). The former have electrified the neutral solution of Finch and Skea (1989), while the latter researchers have generalised the work of Hansraj and Maharaj (2006). A comprehensive list of approaches has been documented by Ivanov (2002). This collection lists a variety of initial choices of two functions that have historically been made. Our interest lies in investigating the physically important case of a sphere of constant proper charge density and constant energy density. Regretably in the former case, no exact solution results on account of the intractability of the master field equation. On the other hand setting the electric field intensity to a constant value together with a constant energy density has not been studied before. We are able to solve the resulting system in general. Finally we examine our solutions for physical plausibility to check if it conforms to reality in terms of the simplest tests.

3.2 Historical Perspectives

It is interesting to consider the history of the uniform density fluid sphere.

- Karl Schwarzschild (1916b) studied uniform density fluid spheres. He produced the first model of a static perfect fluid which is still used to model gravitational behaviour of black holes.
- Non-static isotropic perfect fluid spheres were discussed by Bonnor and Faulkes (1967), Thomson and Whitrow (1967) and Rao (1973).
- Studies carried out on static fluid spheres with uniform density by Knutsen (1983) were found to be conformally flat like the Schwarzschild (1916) solution - the implication is that the Weyl tensor vanishes and that the metric may be trivially related to the Minkowski metric of flat spacetime.
- Non–static spherically symmetric fluids surrounded by empty space were investigated by Sharif and Iqbal (2001).
- Ponce de Leon (1986) investigated non-static spherically symmetric fluid surrounded by empty space. Such solutions were discovered to be representing expanding and contracting configurations that approach the static Schwarzschild solution asymptotically. The work provides oscillating solutions.
- The recent work of Durgapal and Fuloria (2010) on constant density fluid spheres of the non–static kind considers the baryonic conservation law and the condition

of no heat transfer.

- Extended theories of gravity have been explored by researchers in the past. One approach is the Einstein–Gauss–Bonnet theory of gravity, which is a theory in higher dimensions. This theory can be reduced to the Einstein standard theory of general relativity for four spacetime dimensions. The validity of the Schwarzschild density fluid sphere in the Einstein–Gauss–Bonnet gravity was proved by Dadhich *et al* (2010). It is also valid in higher dimensions as well.

3.3 Constant Charge Density

The simplest and most physically interesting generalisation of the constant density Schwarzschild sphere is the incompressible fluid sphere with a constant charge density σ. We set

$$\frac{\sigma^2}{C} = k$$

where k is a constant and put

$$Z = 1 + x$$

which is known to yield the Schwarzschild interior solution (Hansraj 2010). Then equation (2.4.20) assumes the form

$$\left(x\dot{E} + E\right)^2 = \frac{kx}{4(1+x)} \tag{3.3.1}$$

Equation (3.3.1) can then be rearranged as

$$E = \pm\frac{1}{x}\left(\int\sqrt{\frac{kx}{4(1+x)}}dx\right) \tag{3.3.2}$$

which in turn may be integrated to yield

$$E = \pm \frac{\sqrt{k}\left(\sqrt{x(1+x)} - \arcsin\sqrt{x}\right) + 2c_1}{2x} \tag{3.3.3}$$

where c_1 is an integration constant. When the expression for $\frac{E^2}{C}$ is substituted into (2.4.19) we obtain:

$$\begin{aligned} 16Cx^3(1+x)\ddot{y} + 2x^2\dot{y} - [k(\sqrt{x(x+1)} - \arcsin\sqrt{x})^2 \\ \pm 4C_1\sqrt{k}(\sqrt{x(x+1)} - \arcsin\sqrt{x} + 4C_1^2)]y = 0 \end{aligned} \tag{3.3.4}$$

Unfortunately the solution to the above differential equation may not readily be found. This illustrates the difficulty of obtaining a model with a constant charge density and constant energy density. This is not to say that constant charged densities are not possible, however, if they exist they may be obtained by assumptions other than a Schwarzschild-like potential $Z = 1 + x$.

3.4 Constant Electric Field Intensity

While our investigation into a constant charge sphere did not yield useful results, we may examine the situation of a sphere of constant electric field intensity. That is we set

$$\frac{E^2}{C} = \alpha$$

in equation (2.4.19). Then the equation (2.4.19) becomes

$$4x(1+x)\ddot{y} + 2x\dot{y} - \alpha y = 0 \tag{3.4.1}$$

On introducing the change of variable $v = 1 + x$, equation (3.4.1) has the form

$$v(1-v)y'' + \frac{1}{2}(1-v)y' + \frac{\alpha}{4}y = 0 \tag{3.4.2}$$

This is the hypergeometric differential equation

$$x(1-x)y'' + [c - (a+b+1)x]\, y' - aby = 0$$

in Gaussian form. Casting (3.4.2) in the form of this differential equation requires setting

$$c = \frac{1}{2}, \qquad a + b = -\frac{1}{2}, \qquad ab = -\frac{\alpha}{4}.$$

and consequently we obtain

$$\{(a,b)\} = \left\{ \left(\frac{-1 \pm \sqrt{1+4\alpha}}{4}, \frac{-1 \mp \sqrt{1+4\alpha}}{4} \right) \right\} \tag{3.4.3}$$

The general solution has the form

$$y = A_2F_1(a,b;c;x) + Bx^{(1-c)}{}_2F_1(a-c+1, b-c+1; 2-c; x) \tag{3.4.4}$$

where ${}_2F_1(a,b;c;x)$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

provided that $c \notin \{0, -1, -2, ...\}$. The Pochhammer symbol is defined by

$$(x)_n = \begin{cases} 1 & \text{if } n = 0 \\ x(x+1)...(x+n-1) & \text{if } n > 0 \end{cases}.$$

The general solution presented above is complicated and not suitable for physical analysis. It is worthwhile noting that for particular values of the parameter α we are able to obtain closed form solutions which are amenable to further investigation. From a survey of the literature these solutions appear to be new. We list some special solutions hereunder:

α	$y(x)$
2	$C_1x - C_2\left(\sqrt{1+x} - x\tanh^{-1}\sqrt{1+x}\right)$
6	$C_1x\sqrt{1+x} - C_2\left(1+3x-3x\sqrt{1+x}\tanh^{-1}\sqrt{1+x}\right)$
12	$C_1x(5x+4) - C_2\left((15x+2)\sqrt{1+x} - 3x(5x+4)\tanh^{-1}\sqrt{1+x}\right)$
20	$C_1x(7x+4)(\sqrt{1+x} - C_2\left(6+95x+105x^2-(60x+105x^2)\sqrt{1+x}\tanh^{-1}\sqrt{1+x}\right)$

From the above results it is easy to conjecture that closed form solutions exist if α has the form $n(n+1)$. In this case, the general solution is given by

$$y(x) = C_1x_2\mathrm{F}_1\left[\frac{1-n}{2}, \frac{2+n}{2}, 2, -x\right] + C_2\mathrm{MG}\left[\left\{\{\}, \left\{\frac{2-n}{2}, \frac{3+n}{2}\right\}\right\}, \{\{0,1\}, \{\}\}, -x\right]$$

where the symbol ${}_2F_1$ has been defined previously, and C_1 and C_2 are constants. Additionally MG is the MeijerG function defined by

$$\mathrm{MeijerG}\quad [\{\{a_1, ..., a_n\}, \{a_{n+1}, ..., a_p\}\}, \{\{b_1, ..., b_m\}, \{b_{m+1}, ..., bq\}\}, z]$$

$$= \quad \frac{1}{2\pi i}\int_{\gamma_L} \frac{\Pi_{j=1}^{m}\Gamma\left(b_j - s\right)\Pi_{j=1}^{n}\Gamma\left(1 - a_j + s\right)}{\Pi_{j=n+1}^{p}\Gamma\left(a_j + s\right)\Pi_{j=m+1}^{q}\Gamma\left(1 - b_j - s\right)} x^s ds$$

3.5 Physical Properties

We now consider the physical properties of the above solution. The energy density (2.4.17) and electric field intensity (2.4.20) are given respectively by

$$\rho = -\left(3 + \frac{\alpha}{2}\right) C \tag{3.5.1}$$

$$E^2 = \alpha C \tag{3.5.2}$$

for a constant density spherical distribution with constant electric field. Positivity of energy density and the positivity of the right hand side of (3.5.2) demand that

$$\left(3 + \frac{\alpha}{2}\right) C < 0 \tag{3.5.3}$$

$$\alpha C > 0 \tag{3.5.4}$$

must be simultaneously satisfied. We consider the cases in turn:

- (i) If $C > 0$ then $\alpha > 0$ by the condition (3.5.4). But $\alpha > 0$ in (3.5.3) forces $C < 0$ and a contradiction arises. Therefore the case $C > 0$ is discarded. This is not unusual as the same behaviour is evident in the celebrated Schwarzschild interior solution in the form presented by Maharaj and Mkhwanazi (1996) as well as John and Maharaj (2011).

- (ii) If $C < 0$ then by (3.5.4) we must have $\alpha < 0$. Additionally for $C < 0$, the condition (3.5.3) results in $\alpha > -6$. An additional constraint is imposed on α

by (3.4.3) that is $-\frac{1}{4} < \alpha < 0$ to guarantee that the metric potential function y remains real valued. Harmonising these requirements we make the conclusion that $C < 0$ and $-\frac{1}{4} < \alpha < 0$ is compatible only with a positive energy density and a well behaved electric field intensity.

Therefore the closed form solutions referred to in the previous section, where α took on positive values, have little merit of being considered as realistic distributions of charged matter. This does not mean that charged spheres with constant energy density and electric field do not exist. It may still be possible to generate series solutions to the hypergeometric equation (3.4.1) for the case $C < 0$ and $-6 < \alpha < 0$.

3.6 An Approximate Solution

We take terms up to order 6 in the series expansion in order to study the profiles of some of the dynamical quantities. The potential $y(x)$ is given by

$$\begin{aligned} y(x) &= 1 + \frac{3x}{32} + \frac{35x^2}{2048} + \frac{455x^3}{65536} + \frac{31395x^4}{8388608} + \frac{1881607x^5}{805306368} + \frac{905052967x^6}{566935683072} \\ &+ \sqrt{x}\left(1 + \frac{x}{32} + \frac{99x^2}{10240} + \frac{10659x^3}{2293760} + \frac{159885x^4}{58720256} + \frac{479655x^5}{268435456} + \frac{281877255x^6}{223338299392}\right) \end{aligned} \tag{3.6.1}$$

while the pressure p obtainable via (2.4.18) is given by

$$\begin{aligned}
p &= 1+\frac{\alpha}{2}+\Big(26(1+x)\Big(19842748907520+3720515420160x^{\frac{1}{2}}+1860257710080x \\
&+ 1356437913600x^{\frac{3}{2}}+959195381760x^{2}+826579353600x^{\frac{5}{2}}+645458558976x^{3} \\
&+ 594103910400x^{\frac{7}{2}}+486255052800x^{4}+463627964800x^{\frac{9}{2}}+390017073600x^{5} \\
&+ 380122246140x^{\frac{11}{2}}+325568229525x^{6}\Big)\Big) \\
&\quad /(257955735797760x^{\frac{1}{2}}+257955735797760x+24183350231040x^{\frac{3}{2}} \\
&+ 8061116743680x^{2}+4408423219200x^{\frac{5}{2}}+2493907992576x^{3}+1790921932800x^{\frac{7}{2}} \\
&+ 1198708752384x^{4}+965418854400x^{\frac{9}{2}}+702368409600x^{5}+602716354240x^{\frac{11}{2}} \\
&+ 460929268800x^{6}+411799099985x^{\frac{13}{2}}+325568229525x^{7})
\end{aligned} \tag{3.6.2}$$

We plot the potential function $y(x)$ in Figure 3.1 and the pressure $p(x)$ in Figure 3.2 for the value $\alpha = -\frac{3}{16}$. This value also has the advantage of ensuring that the function in (3.4.3) remains real-valued. It may be observed that the function $y(x)$ is a smooth singularity-free function. The other pleasing feature is the absence of a singularity at the stellar centre corresponding to $x = 0$ that is $r = 0$. Additionally the pressure is always positive in the specified domain. A drawback of the pressure profile is that the pressure does not vanish for any value of the radial coordinate x. This means that the solution cannot be interpreted as an isolated stellar model. However, it may be applicable to cosmological scenarios and be part of a core-envelope model. It should also be recalled that the causality criterion is violated as a consequence of the constant energy density. Further empirical testing with a variety of α values corroborate the fact that no pressure–free hypersurface exists in general. Therefore we are led to conclude that realistic relativistic perfect fluid spheres with constant energy density and constant electric fields are unlikely to exist. The gravitational field associated with such an object is incompatible with a constant energy density and constant electric field intensity.

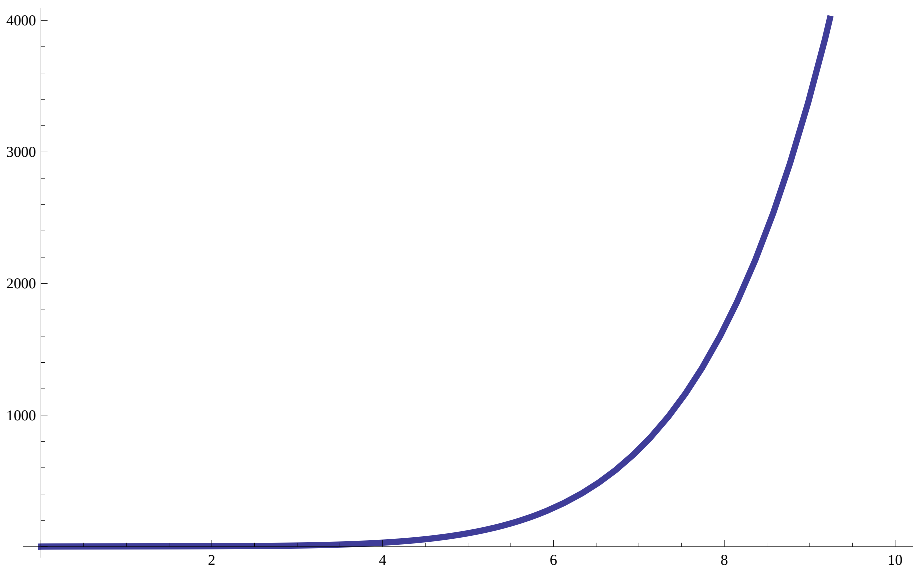

Figure 3.1: Plot of Potential $y(x)$ versus radial coordinate x

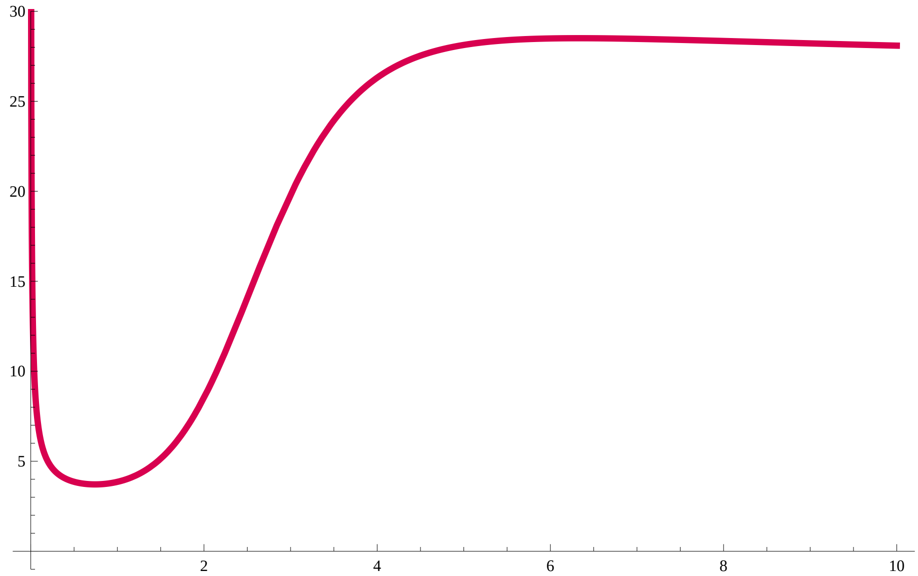

Figure 3.2: Plot of Pressure p versus radial coordinate x

Chapter 4

Solution Generating Algorithm

4.1 Introduction

Thus far we have sought new exact solutions of the Einstein–Maxwell field equations by specifying functional forms for two of the physical variables. In chapter 3 we nominated the energy density of the static field intensity. It must be noted that this is not the only approach that may be followed. A completely different attack is to impose some physical considerations on the system of equations. For example, an equation relating the pressure and energy density, referred to as an equation of state may be utilised. However, only limited success has been reported in the simplest case of a barotropic equation of state of linear type where the pressure is proportional to the energy density. Another physically interesting case is the polytropic equation of state; however, few solutions of this type have been discovered thus far. An alternative approach is to postulate a functional relationship between two other quantities, namely the electric field intensity E and the metric potential Z. We are able to report new solutions based on this assumption.

This is clearly the most common approach to solving coupled Einstein Maxwell field equations found in the literature. For example the Hansraj and Maharaj (2006) model, generalising the Finch and Skea (1989) neutral solution, was found by calculating a form for one of the gravitational potentials and also for the electric field intensity.

Numerical methods have been utilised by Nduka (1976) and Singh and Yadav (1978) in order to electrify the Kuchowicz (1968) neutral solution. An approximately linear equation of state is obtained. Ivanov (2002) explained in detail the difficulty of imposing an equation of state in isotropic static spherical models. The field equations turn out to be not readily solvable. If we relax the condition of pressure isotropy to allow for both tangential and radial pressures, then the problem becomes simpler. Maharaj and Mafa Takisa (2012) reported new anisotropic models with a quadratic equation of state. These models contained other anisotropic models, and when the anisotropy parameter is set to zero, the known isotropic models of De Sitter and Einstein were regained.

4.2 A Solution Generating Algorithm

The most common approach in seeking solutions to the field equations (2.4.17) - (2.4.20) has been to prescribe acceptable forms for two of the unknown functions independently. The choice of these two items is often arbitrary and is usually based on the fact that the second order differential equation (2.4.19) is integrable. For example see the analysis of Thirukkanesh and Maharaj (2009).

An alternative approach requires a functional dependence of one quantity on another, and thereafter prescribing one other variable. For instance, in order to generalise Newtonian behaviour with an equation of state, it is reasonable to expect relativistic phenomena to also display a functional dependence between the pressure and the energy density. As remarked earlier very little success can be reported using the method. This underscores the difficulty in studying such configurations of matter. Imposing physically acceptable criteria such as an equation of state makes the field equations difficult to solve, Therefore our approach has been to specify two of the geometrical or dynamical quantities so that the governing field equations may be integrated. On the successful discovery of the new exact solution, we then consider whether the model corresponds to physically acceptable matter. For example the charged analogue of the Finch–Skea (1985) stars produced by Hansraj and Maharaj (2006) was shown to possess an isotropic equation of state - albeit not linear.

We elect to specify a functional relationship between the electric field intensity $E(x)$ and the metric component $Z(x)$. It is reasonable to expect that the strength of the electric field inside a charged star is position dependent. We invoke the prescription

$$\dot{Z}x - Z + 1 = \frac{E^2}{C}x \tag{4.2.1}$$

The advantage of this choice is that it forces the second order differential equation in y to degenerate into a first order equation. It is worth observing that equation (2.4.19) has often been utilised for its value as a second order differential equation in $y(x)$, however, it may also be rearranged as a first order differential equation in Z of the

Riccati type. Such equations possess their own complications. With the choice (4.2.1), equation (2.4.19) becomes

$$2Z\ddot{y} + \dot{Z}\dot{y} = 0 \tag{4.2.2}$$

which is essentially of first order after a transformation. Introducing the change of variable $\dot{y} = V$, we get

$$\frac{\dot{V}}{V} = -\frac{\dot{Z}}{2Z} \tag{4.2.3}$$

which is immediately integrated to yield

$$V = KZ^{-\frac{1}{2}} \tag{4.2.4}$$

where K is a constant of integration. Now using $V = \dot{y}$ we obtain the expression

$$Z = \frac{1}{K(\dot{y})^2} \tag{4.2.5}$$

which suggests that if we prescribe y then Z will follow trivially. Finally on rearranging (4.2.5) we obtain

$$y = \int \frac{K}{\sqrt{Z}} dx + L \tag{4.2.6}$$

where L is another integration constant.

Now there are three distinct approaches in algorithmically constructing Einstein-Maxwell solutions based on the ansatz (4.2.1):

- From (4.2.6) above we have obtained y in terms of Z. This means that if we pick suitable forms for Z then after integration we should be able to establish the function $y(x)$. Then with the help of (4.2.1) we may obtain the form for the electric field intensity E. Thereafter the energy density ρ, the pressure p and the proper charge density σ may be found using (2.4.17), (2.4.18) and (2.4.20) respectively.

- Alternatively using (4.2.5) we may choose a form for y and then find Z. On substituting into (4.2.1) we may obtain E. The remaining quantities may then be found as outlined above. A distinct advantage of this approach is that no integrations are called for. Therefore an infinite number of choices can be made for the metric potential y that will allow for the complete solution of the Einstein field equations.

- A third approach is to specify the electric field intensity in equation (4.2.1). Then the metric function Z must be obtained by solving this linear differential equation (4.2.1). Finally we may substitute the Z into (4.2.6) to establish y. This method has the obvious drawback that, firstly, it may not be possible to find Z for a particular choice of E as the integration of Z may not be possible to complete. Secondly, even if we are able to find Z then there is still no guarantee that (4.2.6) can be integrated to yield y.

4.3 Specifying E

We explore the option of specifying the charge distribution and consider a variety of choices for the electric field quantity $\frac{E^2}{C}$. Then the algorithm operates as follows: We begin by nominating a form for the electric field intensity E and then solve equation

(4.2.1) for Z. Thereafter the form for Z is substituted into (4.2.6) to give the metric potential y. The choice of E should allow for both Z and y to be obtained explicitly.

4.3.1 The case $\frac{E^2}{C} = \alpha$

We commence by examining the simplest case $\dfrac{E^2}{C} = \alpha$ where α is a real parameter. Note that this implies a constant electric field - however this is not the same as our work in Chapter 3 as a different prescription on Z was used there. Equation (4.2.1) becomes

$$\dot{Z}x - Z + (1 - \alpha x) = 0 \tag{4.3.1}$$

which has the solution

$$Z(x) = C_1 x + 1 + \alpha x \log x \tag{4.3.2}$$

where C_1 is a constant. Substituting this form of Z into (4.2.6) we obtain

$$y = \int \frac{K}{\sqrt{C_1 x + 1 + \alpha x \log x}} dx + L \tag{4.3.3}$$

Unfortunately, for the form of Z used. the resulting integral is not obtainable in closed form. We therefore do not pursue this solution - although some progress can be made using numerical integration. Our search is for exact solutions. Nevertheless, we have commented on this case to indicate that the simplest case of a constant electric field intensity is non-trivial in general.

Note that if we set the arbitrary constant C_1 to zero, then the integration in (4.3.3) may be achieved. The potential function y has the form

$$y(x) = \frac{\sqrt{\pi}}{\sqrt{a}} e^{-1/a} \text{Erfi}\left[\frac{\sqrt{1 + a \log x}}{\sqrt{a}}\right]$$

where Erfi represents the imaginary error function on complex values z defined by $\text{Erfi}(z) = -i\text{Erf}(iz)$ where $\text{Erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}dt$. Since it is not possible to decompose the error function into elementary functions, we abandon further treatment of the case.

4.3.2 The case $\frac{E^2}{C} = \alpha x$

With the above choice of E, equation (4.2.1) becomes

$$\dot{Z}x - Z = \alpha x^2 - 1 \tag{4.3.4}$$

where α is an arbitrary constant, and it is solved by

$$Z = 1 + C_3 x + \alpha x^2 \tag{4.3.5}$$

where C_3 is a constant of integration.

Substituting this form of Z into equation (4.2.6) we obtain

$$y = \int \frac{K}{\sqrt{1 + C_3 x + \alpha x^2}} dx + C_2 \tag{4.3.6}$$

and performing the integration yields

$$y = \frac{1}{\sqrt{\alpha}} \log\left(C_3 + 2\alpha x + 2\sqrt{\alpha\left(1 + C_3 x + \alpha x^2\right)}\right) + C_2 \tag{4.3.7}$$

which establishes the remaining geometric potential.

The energy density ρ is given by

$$\frac{\rho}{C} = -3C_3 - \frac{11}{2}\alpha x \tag{4.3.8}$$

with the aid of equation (4.1.3) and its rate of change has the form

$$\frac{d\rho}{Cdx} = -\frac{11}{2}\alpha \tag{4.3.9}$$

which is constant. Note that our prescription $E^2 = \alpha C^2 r^2$ forces $\alpha > 0$, and the density is a monotonically decreasing function outwards from the centre.

The pressure p has the form

$$\frac{p}{C} = C_3 + \frac{3\alpha x}{2} + \frac{4\sqrt{\alpha} f_1^2(x)}{f_1(x)\log(C_3 + 2\alpha x + 2\sqrt{\alpha(1 + C_3 x)\log(c_3 + 2\alpha x + 2\sqrt{\alpha} f_1(x)))}} \tag{4.3.10}$$

where we have introduced the function $f_1(x) = \sqrt{1 + C_3 x + \alpha x^2}$ to simplify the expressions. The rate of change of pressure is given by

$$\begin{aligned} \frac{dp}{dx} &= \frac{4\alpha}{\log(C_3 + 2\alpha x + 2\sqrt{\alpha}(f_1(x))^2)} \\ &+ 2\sqrt{\alpha}\left(\frac{(C_3 + 2\alpha x)(1 + C_3 x) - \alpha^2(2 + C_3 x)\log(C_3 + 2\alpha x^2)\sqrt{\alpha} f_1(x)}{(f_1(x))^3 \log(C_3 + 2\alpha x + 2\sqrt{\alpha} f_1)}\right) \end{aligned}$$

while the adiabatic sound speed index has the form

$$\frac{dp}{d\rho} = \frac{\alpha\left[(8 + 3\alpha)\sqrt{\alpha} f_1(x) + 4(C_3 + 2\alpha x)\log\left[C_3 + 2\alpha x + 2\sqrt{\alpha} f_1(x)\right]\right]}{44\sqrt{\alpha} f_1(x)}$$

The following quantities

$$\frac{\rho - p}{C} = -4C_3 - 7\alpha x - \frac{4\left[\sqrt{\alpha}f_1(x)\log(C_3 + 2\alpha x + 2\sqrt{\alpha(1 + C_3x + \alpha x^2)})\right]}{\alpha} \tag{4.3.11}$$

$$\frac{\rho + p}{C} = -2C_3 - 4\alpha x + \frac{4\left[\sqrt{\alpha}f_1(x)\log(C_3 + 2\alpha x + 2\sqrt{\alpha(1 + C_3x + \alpha x^2)})\right]}{\alpha} \tag{4.3.12}$$

$$\frac{\rho + 3p}{C} = -\alpha x + \frac{12\left[\sqrt{\alpha}f_1(x)\log(C_3 + 2\alpha x + 2\sqrt{\alpha(1 + C_3x + \alpha x^2)})\right]}{\alpha} \tag{4.3.13}$$

are useful when studying the energy conditions. These are all required to be positive within the stellar interior.

4.4 Specifying Z

We next consider exact solutions that may be derived by nominating the metric potential Z at the outset. Of course in view of the reciprocal of $\sqrt{Z}$ in the integrand of (4.2.6), we expect only a small number of choices will result in a complete resolution of y.

4.4.1 $Z = (1 + x)^n$

This choice will lead to a Schwarzschild sphere in the absence of charge for the case $n = 1$ as was demonstrated by Hansraj (2010). In other words this prescription will produce charged analogues of the Schwarzschild interior solution. This form also has the property of yielding solutions that are regular at the stellar centre. We obtain the electric field intensity in the form

$$\frac{E^2}{C} = \frac{1 + nx(1+x)^{n-1} - (1+x)^n}{x} \tag{4.4.1}$$

with the help of (4.2.1). With the choice of $Z = (1+x)^n$, we obtain

$$y = \frac{2}{2-n}(1+x)^{1-\frac{n}{2}} + C_4 \tag{4.4.2}$$

after integration using (4.2.6). Then the system of Einstein-Maxwell field equations is completely solvable. The density ρ and pressure p are given by

$$\frac{\rho}{C} = \frac{1}{2}\left(\frac{1 - (1+x)^n - 5nx(1+x)^{n-1}}{x}\right) \tag{4.4.3}$$

$$\frac{p}{C} = \frac{2x(2-n+nx)(1+x)^{n-1} + (1+x)^n - 1}{2x} \tag{4.4.4}$$

respectively. The rate of change of the energy density ρ and pressure p have the forms

$$\begin{aligned} \frac{d\rho}{Cdx} = {} & \frac{-5(-1+n)nx(1+x)^{-2+n} - 6n(1+x)^{-1+n}}{2x} \\ & - \frac{1 - 5nx(1+x)^{-1+n} - (1+x)^n}{2x^2} \end{aligned} \tag{4.4.5}$$

$$\begin{aligned} \frac{dp}{Cdx} = {} & \frac{1}{2x^2(1+x)^2}\big(1 - (1+x)^n + 2n^2x^3(1+x)^n \\ & + x\left(2 - 2(1+x)^n + n(1+x)^n\right) + x^2\left(1 - (5 - 9n + 2n^2)(1+x)^n\right)\big) \end{aligned} \tag{4.4.6}$$

which allows us to compute the sound speed parameter $\frac{dp}{d\rho}$. This is given by

$$\frac{dp}{d\rho} = x^2 \left[\frac{\frac{1}{2} - (2n^2 - 2(2+nx) + (1+x)^n + (\frac{7}{2}x - \frac{3}{2}nx - nx^2 - 3n^2x^3)(1+x)^{n-2}}{-5nx^2(n-1)(1+x)^{n-2} - nx(1+x)^{n-1} + (1+x)^n - 1} \right] \tag{4.4.7}$$

Finally the following quantities are useful to study the energy conditions

$$\frac{\rho + p}{C} = \frac{1 + (1+x)^{n-1} \left[5nx - x^3(3 - 2n - 3nx)\right]}{2x} \tag{4.4.8}$$

$$\frac{\rho - p}{C} = \frac{1 + (1+x)^{n-1} \left[5nx + x^3(3 - 2n + 2nx)\right]}{2x} \tag{4.4.9}$$

$$\frac{\rho + 3p}{C} = \frac{1 + (1+x)^{n-1} \left[5nx - 3x^3(3 - 2n + 2nx)\right]}{2x} \tag{4.4.10}$$

Since the above solution does not readily lend itself to an analytical treatment, we opt to generate plots of the dynamical and geometric quantities, with the help of Mathematica (Wolfram 2010) to obtain a qualitative view on the acceptability of these solutions to represent physical matter.

From an investigation of Figure 4.1, it is evident that the energy density is a positive, smooth and monotonically decreasing function. This is pleasing, however, it is noted that the density is singular at the stellar centre $r = 0$. What this means is that this model may, at best, depict a two–fluid situation where the interior is composed of a different fluid with the requisite zero central density and is surrounded by our fluid solution. There should then exist an interface between the fluids across which both match. Additionally, the pressure profile (Figure 4.2) is also positive and smooth and most importantly the pressure vanishes for a finite radius corresponding approximately to $x = 5$ geometric units. This hypersurface of zero pressure identifies the boundary of the star. This is an important requirement, the absence of which would indicate that our solution could only be used to model a cosmological fluid. The sound speed parameter (Figure 4.3) is found to be less than unity everywhere in the interior except for a small section near the centre. However, we have explained that in a two–fluid model, the centre may be excluded. In other words we have generated a charged star model which displays causal behaviour in a spherical shell. Finally, it may be observed from Figure 4.4 that all the energy requirements are satisfied everywhere within the star. Accordingly, our model may have merit in representing a realistic charged shell distribution of finite inner and outer radius.

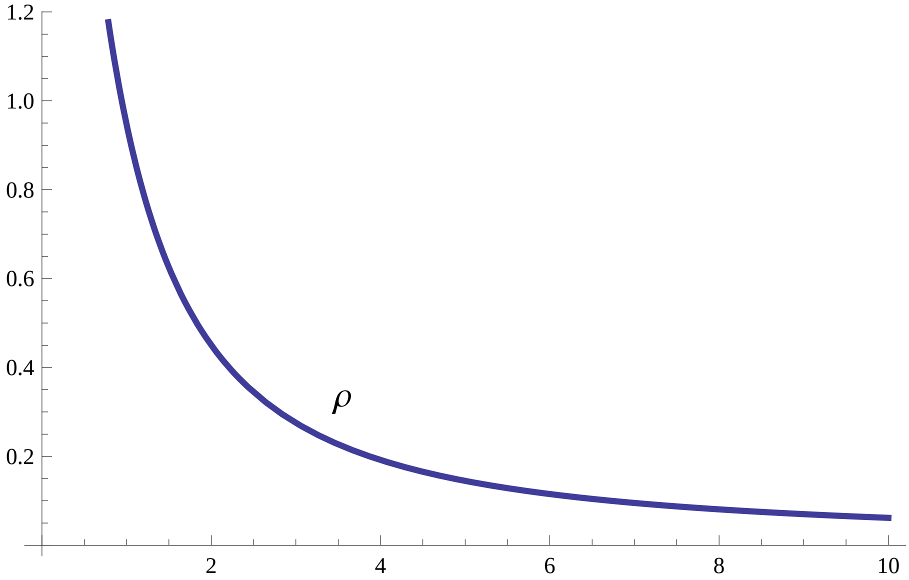

Figure 4.1: Plot of energy density versus radial coordinate

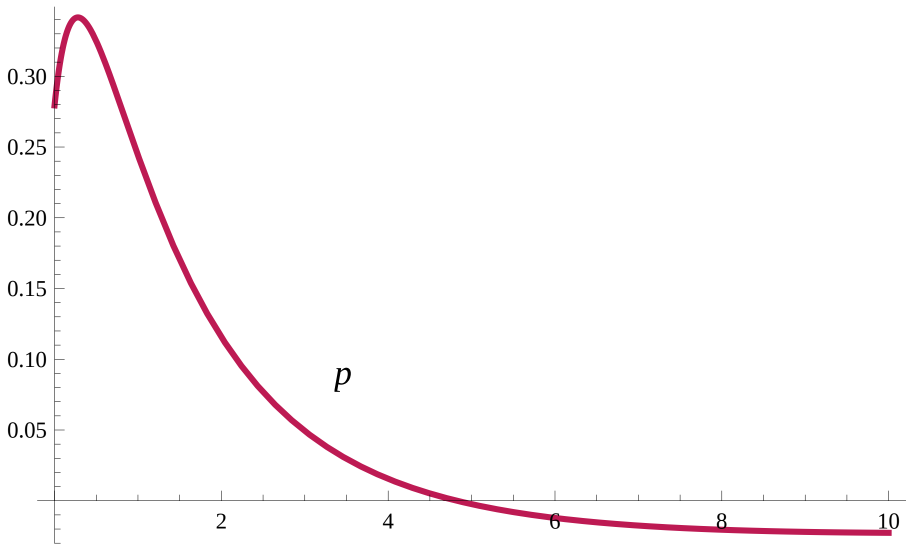

Figure 4.2: Plot of pressure versus radial coordinate

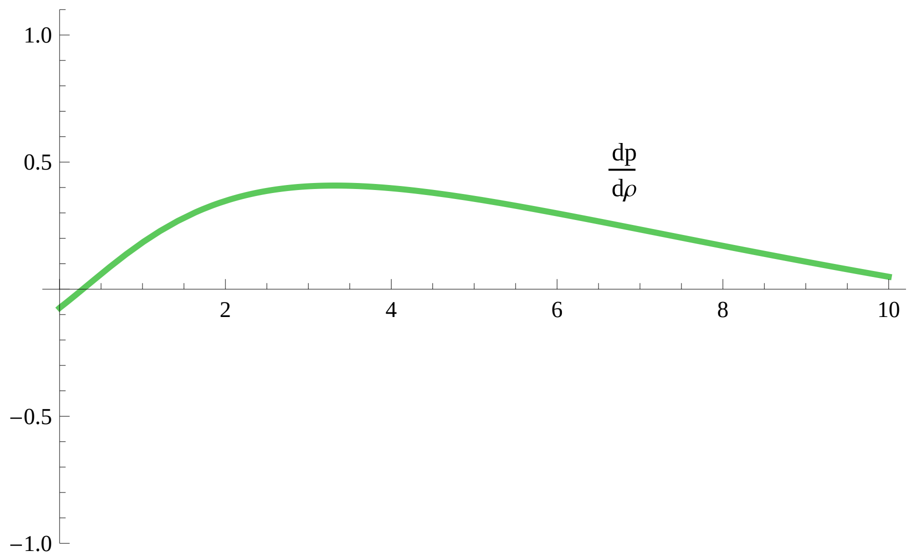

Figure 4.3: Plot of sound speed versus radial coordinate

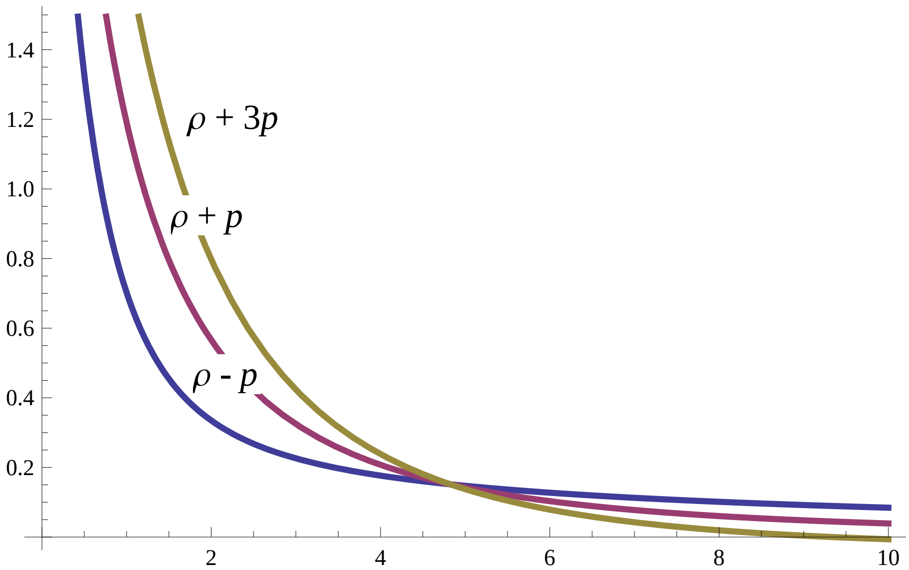

Figure 4.4: Plot of $\rho - p, \rho + p, \rho + 3p$ versus radial coordinate

4.4.2 The choice $Z = (1+x^2)^n$

We substitute $Z = (1+x^2)^n$ in (4.2.8). This gives

$$y = x_2F_1\left[\frac{1}{2}, \frac{n}{2}, \frac{3}{2}. - x^2\right] + L \tag{4.4.11}$$

where ${}_2F_1$ is the hypergeometric series. For certain values of n, simple closed form solutions result. We tabulate some particular exact solutions in Table 4.1.

From Table 4.1 we conjecture that the form for y, when n is an odd integer, involves a polynomial of order $(n-2)$ in the numerator and a term $(1+x^2)^{\frac{n-2}{2}}$ in the denominator. The solutions appear to follow a pattern for certain values of n. It is remarkable that these solutions are simple rational polynomial type functions. They have ostensibly not been previously found using other approaches in the literature.

We now consider briefly the particular case where $n = -2$. From the table, when $n = -2$, we have $y = x + \dfrac{x^3}{3} + C_5$ and the electric field intensity is given by

$$\frac{E^2}{C} = \frac{6x + 3x^3 + x^5}{(1+x^2)^3}. \tag{4.4.12}$$

The density and pressure are given by

$$\frac{\rho}{C} = \frac{14x + 3x^3 + x^5}{2(1+x^2)^3} \tag{4.4.13}$$

$$\frac{p}{C} = \frac{12 + 39x^2 - 6x^3 + 17x^4 + 3x^5 + 2x^6 + x^7}{x(3+x)(1+x^2)^3} \tag{4.4.14}$$

respectively.

Table 4.1: Exact solutions

Z=$(1+x^2)^n$	y
$n=1$	$\sinh^{-1} x + C$
$n=2$	$\sqrt{1+x}\ \arctan x + C$
$n=3$	$\dfrac{x}{\sqrt{1+x^2}} + C$
$n=4$	$\dfrac{x + \arctan x + x^2 \arctan x}{2(1+x^2)} + C$
$n=5$	$\dfrac{x(3+2x^2)}{3(1+x^2)^{\frac{3}{2}}} + C$
$n=6$	$\dfrac{x(5+3x^2)+3}{8(1+x^2)} + C$
$n=7$	$\dfrac{x(15+20x^2+8x^4}{15(1+x^2)^5} + C$
$n=9$	$\dfrac{x(35+70x^2+56x^4+16x^6)}{35(1+x)^{\frac{7}{2}}} + C$
$n=11$	$\dfrac{x(315+840x^2+1008x^4+576x^6+128x^8)}{315(1+x^2)^{\frac{9}{2}}} + C$
$n=-1$	$\dfrac{1}{2}(x\sqrt{1+x^2} + \arcsin x + C$
$n=-2$	$x + \dfrac{x^3}{3} + C$
$n=-3$	$\dfrac{1}{8}[\sqrt{1+x^2}(5+2x^2) + 3\arcsin x] + C$

The rate of change of the energy density ρ and pressure p have the form

$$\frac{1}{C}\frac{d\rho}{dx} = \frac{14 - 16x^2 - 4x^4 - x^6}{2(1+x^2)^4} \tag{4.4.15}$$

$$\frac{1}{C}\frac{dp}{dx} = \frac{36 + 24x + 135x^2 + 132x^3 + 438x^4 + 92x^5 + 84x^6 + 60x^7 + 10x^8 + 4x^9 + x^{10}}{x^2(3+x)^2(1+x^2)^4} \tag{4.4.16}$$

which will allow us to compute the sound–speed parameter which is given by

$$\frac{dp}{d\rho} = \frac{2(36 + 24x + 135x^2 + 132x^3 + 438x^4 + 92x^5 + 84x^6 + 60x^7 + 10x^8 + 4x^9 + x^{10})}{x^2(3+x)(-14 + 61x^2 + 4x^4 + x^6)} \tag{4.4.17}$$

Finally the following expressions are useful when considering the energy conditions:

$$\frac{\rho + p}{C} = \frac{24 + 120x^2x^3 + 43x^4 + 9x^5 - 7x^6 + 3x^7}{2x(3+x)(1+x^2)^3} \tag{4.4.18}$$

$$\frac{\rho - p}{C} = \frac{24 + +36x^2 - 26x^3 + 25x^4 + 3x^5 + x^6 + x^7}{2x(3+x)(1+x^2)^3} \tag{4.4.19}$$

$$\frac{\rho + 3p}{C} = \frac{72 + 276x^2 - 22x^3 + 111x^4 + 21x^5 + 15x^6 + 7x^7}{2x(3+x)(1+x^2)^3} \tag{4.4.20}$$

4.4.3 The choice $Z = 1 + x^n$

We substitute $Z = 1 + x^n$ in (4.2.8). Then the general solution for y is given by

$$y = x_2F_1\left[\frac{1}{2}, \frac{1}{n}, \frac{n+1}{n}, -x^2\right] + L \tag{4.4.21}$$

in terms of the hypergeometric function ${}_2F_1$. Observe that the case $n = 1$ in the present function Z corresponds to $n = 1$ for the previously treated $Z = (1+x)^n$ case in section 4.4.1. In the case $n = 2$ the present function Z is equivalent to the choice $n = 1$ in the case $Z = (1 + x^2)^n$ in section 4.4.2. Hence we do not consider these again. Empirical testing for exact models with $n = 3, 4, 5, ...$ suggests that closed form solutions do not emerge. The forms for y turn out to involve elliptic functions. Accordingly we confine our attention to those cases which produce elementary forms for y on integration. In addition, fractional values of n appear to yield closed form solutions as do negative integral values of n. We tabulate some exact solutions that we have generated in Table 4.2.

It is not surprising that these solutions all appear to be novel. They follow essentially because of our prescription of linking the electric field intensity to the gravitational potential Z. This approach has not previously been attempted. Therefore we find that the solutions reported in the previous two sections are new.

Table 4.2: Exact solutions

Z=$(1+x^n)$	y
$n=-1$	$y=\sqrt{x(x+1)}-\sinh^{-1}\sqrt{x}$
$n=-2$	$y=\sqrt{x^2+1}$
$n=\frac{1}{2}$	$y=\frac{4}{3}(2-2\sqrt{1+\sqrt{x}})+\sqrt{x(1+\sqrt{x})}$
$n=\frac{1}{3}$	$\frac{2}{3}\sqrt{1+x^{\frac{1}{3}}}(8+4x^{\frac{1}{2}}+3x^{\frac{2}{3}})$
$n=\frac{1}{4}$	$y=\frac{8}{35}\sqrt{1+x^{\frac{1}{4}}}(-16+8x^{\frac{1}{4}}-6\sqrt{x}+5x^{\frac{3}{4}})$

4.5 Specifying $y(x)$

Using $Z=\dfrac{\gamma}{\dot{y}^2}$ we can nominate any form for y and then obtain Z via (4.2.5) and finally $\dfrac{E^2}{C}$ with the help of (4.2.1). The major advantage here is that the are no integrations to be performed. This means that **any** analytic function y will allow the complete integration of the Einstein field equations for this scheme. Recall the algorithm works subject to $\frac{E^2}{C}x=\dot{Z}x-Z+1$ which we prescribed. We give a simple example to illustrate this trivial method of finding new exact solutions.

If we take $y=1+x$ for example then we obtain $Z=\gamma$ a constant using (4.2.5). Finally we get $\dfrac{E^2}{C}=\dfrac{1-\gamma}{x}$ with the aid of (4.2.1). The density ρ and pressure p are given by

$$\frac{\rho}{C} = \frac{1-\gamma}{2x} \tag{4.5.1}$$

$$\frac{p}{C} = \frac{9\gamma x + \gamma - x - 1}{2x(1+x)} \tag{4.5.2}$$

respectively. The rates of change of these dynamical quantities are given by

$$\frac{d\rho}{Cdx} = \frac{-(1-\gamma)}{2x^2}$$

$$\frac{dp}{Cdx} = \frac{x^2(9\gamma-1)-(\gamma-1)(x+2)}{4x^2(1+x)^2}$$

which in turn allows us to obtain the expression

$$\frac{dp}{d\rho} = \frac{(1+x)^2 - \gamma\,(1+2x+9x^2)}{(\gamma-1)(1+x)^2}$$

which represents the adiabatic sound speed index. The expressions for the energy conditions are represented by the following equations:

$$\begin{aligned} \rho - p &= \frac{1-5\gamma x-\gamma}{x(1+x)} \\ \rho + p &= \frac{4\gamma-1}{x+1} \\ \rho + 3p &= \frac{-(1+x-\gamma-13\gamma x)}{x(x+1)} \end{aligned}$$

We do not pursue any physical study of these quantities. We present them merely to illustrate the ease of finding new exact solutions using the method we have proposed. It is patently obvious that all the expressions above are singular at the stellar centre so if this model were to correspond to realistic matter there will have to exist another core fluid which has finite density and pressure.

Chapter 5

Conclusion

Our study commenced with a recollection of fundamental ideas in differential geometry required for a study of fluids in the presence of a gravitational field. The gravitational field is known to influence the geometry of the spacetime manifold and so aspects of Riemannian geometry are required. This was presented in chapter 2. Additionally, we derived the differential equations governing models of spherically symmetric charged stars which comprise the coupled Einstein-Maxwell field equations. Through the use of certain transformations the canonical equations were recast in a more palatable form. It was observed that the field equations have more variables than equations and consequently these equations have an infinite number of exact solutions. It is not trivial to actually find solutions and historically a number of approaches have been attempted. The reason why this study is still pursued despite the large number of exact solutions that are reported, is that new solutions are not all physically reasonable. To isolate those that are physically admissible, we impose some conditions which models must satisfy. The physically admissible solutions are used to model a charged star as the dynamical and geometric variables may all be determined explicitly. This also allows us the opportunity of studying how such fluid models behave if certain parameters are altered.

In chapter 3 our investigation reveals that it is a complicated task to model a constant charge density and a constant energy density fluid sphere. This is the simplest generalisation of the constant density fluid sphere analysed by Schwarzschild (1916b), however it has not been thoroughly investigated from our survey of the literature. Additionally it does not appear as a special case of solutions already reported. The study of the incompressible constant electric field fluid sphere has resulted in a solution given in terms of the hypergeometric function (3.4.4). The drawback is that an analytical treatment of hypergeometric functions is very difficult and we could not find any special cases that yielded solutions in terms of elementary functions which were physically relevant. Closed form solutions do emerge for certain values for a parameter and these are tabulated. However, these solutions violate simple requirements for physical acceptability. To obtain a qualitative idea of the behaviour of our solutions we used truncated series representations of the associated hypergeometric function. These produced a pressure profile that was positive but which lacked a vanishing pressure hypersurface. This means that the solution may have cosmological interest only and is unsuitable for astrophysical applications.

Another approach of solving the Einstein field equations is to specify any two of the dynamical or geometrical functions independently at the outset. This has been pursued by a large number of researchers with good success. In contrast we proposed a functional dependence of the electric field intensity on one of the gravitational potentials. This idea may be derived from the other more physically important approach of declaring an equation of state relating the pressure and the energy density from the beginning. This last mentioned approach has not yielded many useful results, however, our prescription was shown to result in a variety of new exact solutions in an algorithmic manner. There were three approaches to running our algorithm. Prudent choices of the remaining variable has allowed us to generate wide classes of new solutions. We have investigated some for physical acceptability and models of realistic charged shells were produced satisfying the requirements for physical admissibility.

Despite the fact that this problem ranks as amongst the oldest in general relativity theory it continues to generate interest by virtue of the fact there are so few models of charged matter that are physically viable. The challenge is to produce a comprehensive model that satisfies observable characteristics of real stars. We need to investigate in greater depth aspects such as the mass–radius ratio, the mass–charge ratio to see if they conform to the limits already established. Additionally the value of the surface redshift needs further study. In future, it will be interesting to incorporate the effects of pressure anisotropy in the presence of an electromagnetic field. Additionally we anticipate pursuing non-static generalisations of some well known static models such as the physically reasonable Finch–Skea (1989) model which has been demonstrated to be consistent with the astrophysical theory of Walecka (1975).

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